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CORRIGENDUM TO "GROUPOIDS, THE PHRAGMEN-BROUWER PROPERTY, AND THE JORDAN CURVE THEOREM", J. HOMOTOPY AND RELATED STRUCTURES 1 (2006) 175-183.

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(*communicated by Fred Cohen*)

Abstract

Omar Antolin Camarena pointed out a gap in the proofs in [**BT&G, Bro06**] of a condition for the Phragmen–Brouwer Property not to hold; this note gives some background and the correction.

1. Introduction

This note is concerned with a gap in the proofs in [**BT&G, Bro06**] that if a connected space X does not have the Phragmen-Brouwer Property, then its fundamental group has the group of integers as a retract.

One source of confusion is that there are two forms of this Property, as follows.

Let X be a topological space. We say X *separates* if it has more than one component, and a subspace D of X *separates* X if $X \setminus D$ separates. We say a subset D of X *separates the points a and b of X* if a and b lie in distinct components of $X \setminus D$.

A topological space X is said to have the *Phragmen-Brouwer Property I*, here abbreviated to PBI), if X is connected and the following holds:

PBI): if D and E are disjoint, closed subsets of X, and if a and b are points in $X \setminus (D \cup E)$ which lie in the same component of $X \setminus D$ and in the same component of $X \setminus E$, then a and b lie in the same component of $X \setminus (D \cup E)$.

Thus the PBI) is that: if D and E are disjoint closed subsets of X and a, b are points of X not in $D \cup E$ such that neither D nor E separate a and b, then $D \cup E$ does not separate a and b.

We also consider:

PBI$'$): If D, E are disjoint closed subsets of X such that neither D nor E separate X, then $D \cup E$ does not separate X.

It is proved in [**Wil49**], pp.47-49, that these two properties are equivalent if X is connected and locally connected, and for the convenience of readers we reproduce that proof in Section 3.

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It is of interest and importance to relate these properties to the fundamental group of the space X, see also [**Eil37**, Section3]. Such a relation was also used in [**BT&G, Bro06**] as a contribution to a proof of the Jordan Curve Theorem. To this end, we assume all spaces in the next Section are locally path connected, so that connected and path connected are equivalent. The result required to relate PBI) to properties of the fundamental group or groupoid is given in the next Section.

The gap in the proof was in considering a pushout of groupoids

$$\begin{array}{ccc} C & \xrightarrow{j} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle v} \\ A & \xrightarrow[u]{} & G \end{array} \tag{1}$$

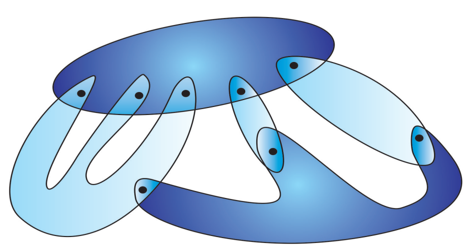
such that C is totally disconnected, and A, B, G are connected. Under these assumptions a vertex group $G(p)$ of G is calculated[1] in [**BT&G**, 8.4.1], [**Bro06**, Proposition 3.3] and from this it is proved that $G(p)$ contains as a retract a free group which is non trivial if C has more than one component. This is sufficient to get the required criterion for PBI$'$). However, in the application to the Jordan Curve Theorem, we really need PBI), as is assumed in the above references, and in this case we do not know that A, B are connected.

So we offer two ways of making the correction. The first is just to use the result already mentioned that properties PBI) and PBI$'$) are equivalent. The second is to give an appropriate retraction result on such a pushout of groupoids in the case where A and B are not necessarily connected. This we give in the next Section in Theorem 2.1.

Partially analogous results on fundamental groups with applications to the Jordan Curve Theorem are given in [**Mun75**, Theorem 63.1], which inspired our results. His proof uses a covering space argument, avoiding the groupoid techniques introduced in [**Bro67**].

2. A result on a pushout of groupoids

The groupoid version of the Seifert-van Kampen Theorem, proved originally in [**Bro67**], and stated as Theorem 2.1 of [**Bro06**], and 6.7.2 of [**BT&G**], allowed for non connected spaces, and produces a pushout of groupoids, rather than the standard form which produces a pushout of groups[2]. So the situation of spaces may be something like a union as follows:



[1] Such a formula for a fundamental group appears without proof in the paper [**Kam33**].
[2] The result for a union of a family of open sets, rather than just two, is given in [**BRa84**]

Some cases of pushouts of groupoids are handled in the above references, but for our purposes we need the following Theorem. Recall that by the *rank* of a connected free groupoid we mean the rank as a free group of any of its vertex groups.

Theorem 2.1. *Let*

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[d, "i"'] & B \arrow[d, "v"] \\ A \arrow[r, "u"'] & G \end{tikzcd} \tag{2}$$

be a pushout of groupoids. We assume C is totally disconnected, that i, j are the identity on objects, and that G is connected. Then G contains as a retract a free groupoid F of rank

$$k = n_C - n_A - n_B + 1,$$

where n_P is the number of components of the groupoid P for $P = A, B, C$.

Further, if C contains distinct objects a, b such that $A(ia, ib), B(ja, jb)$ are nonempty, then F has rank at least 1.

Proof. Let Fr : $\mathsf{DirectedGraphs} \to \mathsf{Groupoids}$ be the free groupoid functor. From [**BT&G**, Chapter 8] it follows that any groupoid G has a retraction $G \to \mathsf{Fr}\, W$ where W is a forest.

Let Z be the set of objects of C (and of A, B and G) regarded as a directed graph with no edges, and pick spanning forests X and Y of the underlying directed graphs of A and B. Then there are retractions

$$C \to \mathsf{Fr}\, Z,\ A \to \mathsf{Fr}\, X,\ B \to \mathsf{Fr}\, Y.$$

By "span" in a category we mean a pair of arrows $U \leftarrow W \to V$; this is the shape of a diagram whose pushout you can take. Then the following diagram in $\mathsf{Groupoids}$ commutes and its rows are spans:

$$\begin{tikzcd} \mathsf{Fr}\, X \arrow[d] & \mathsf{Fr}\, Z \arrow[l] \arrow[r] \arrow[d] & \mathsf{Fr}\, Y \arrow[d] \\ A \arrow[d] & C \arrow[l] \arrow[r] \arrow[d] & B \arrow[d] \\ \mathsf{Fr}\, X & \mathsf{Fr}\, Z \arrow[l] \arrow[r] & \mathsf{Fr}\, Y \end{tikzcd}$$

So the span $\mathsf{Fr}\, X \leftarrow \mathsf{Fr}\, Z \to \mathsf{Fr}\, Y$ is a retract of the span $A \leftarrow C \to B$. This implies that the pushout, say F, of $\mathsf{Fr}\, X \leftarrow \mathsf{Fr}\, Z \to \mathsf{Fr}\, Y$ is a retract of G (which is the pushout of $A \leftarrow C \to B$). Since the span of free groupoids is actually the image under Fr of the obvious span $X \leftarrow Z \to Y$ of graphs, and since Fr is a left adjoint, this pushout F is actually just $\mathsf{Fr}\, W$ where W is the pushout in the category of directed graphs of $X \leftarrow Z \to Y$.

This graph W is connected because G is connected, so, denoting by $e(\mathrm{Q})$ and $v(\mathrm{Q})$ the number of vertices of a graph, the vertex groups in $\mathsf{Fr}\, W$ are free of rank $k = e(W) - v(W) + 1$. We have $v(W) = v(X) = v(Y) = v(Z) = n_C$; and, since Z has no edges, $e(W) = e(X) + e(Y)$. Also, since X

is a spanning forrest we have $e(X) = v(X) - n_A = n_C - n_A$, and similarly, $e(Y) = n_C - n_A$. Putting this all together, the vertex groups in F have rank $(n_C - n_A) + (n_C - n_B) - n_C + 1 = n_C - n_A - n_B + 1$, as claimed.

For the last part of the theorem, we choose X, Y so that the elements α, β of $A(ia, ib), B(jb, ja)$ respectively are parts of $\mathsf{Fr}\, X, \mathsf{Fr}\, Y$ respectively. These map to elements α', β' in F and the element $\alpha'\beta'$ will be nontrivial in F; so F has rank at least 1. $\square$

This following Corollary is an essential part of the proof of the Jordan Curve Theorem. It appears, without the retraction condition, as part of [**Mun75**, Theorem 63.1], and also as [**BT&G**, 9.2.1], [**Bro06**, Proposition 4.1].

Corollary 2.2. *If the space X is path connected and does not have the PBI), then its fundamental group at any point contains the infinite cyclic group as a retract.*

Proof. The proof now follows the methods of [**Bro06**]. If X is path connected and does not have the PBI) then there are disjoint closed subsets D, E of X and points a, b of $X \setminus (D \cup E)$ which lie in the same component of $X \setminus D$ and in the same component of $X \setminus E$, but a and b do not lie in the same component of $X \setminus (D \cup E)$. Let $U = X \setminus D, V = X \setminus E, W = U \cap V$. Then a, b lie in different components of W. Let J be a minimal set containing a, b and one point in each path component of W. The groupoid version of the Seifert-van Kampen Theorem gives as in [**Bro06**, Theorem 2.1], [**BT&G**, 6.7.2][3], a pushout of groupoids to which we can apply Theorem 2.1; the last part of that Theorem gives our conclusion. $\square$

3. Equivalence of forms of the Phragmen-Brouwer Property

The following argument is reproduced for the convenience of readers from [**Wil49**], pp47–49. A stronger result is given in [**Hun74**].

Theorem 3.1. *PBI) implies PBI$'$), and if X is connected and locally connected, then PBI$'$) implies PBI).*

Proof. That PBI) implies PBI$'$) is trivial.

We now assume X is locally connected. This is well known to imply that all components of X are open in X. It follows that:
(*) If F is the boundary of a component C of an open set of X, and $\bar{C} \neq X$, then $X \setminus F = C \cup (X \setminus \bar{C}) J$ is a separation, by which we mean that the equality expresses $X \setminus F$ as a union of sets neither of which meets the closure of the other.

Suppose then X is connected and locally connected and satisfies PBI$'$) but not PBI). Then X contains two points a, b and two disjoint closed sets D, E such that neither of D, E separate a, b but $D \cup E$ does separate a, b. Let C_a be the component of $X \setminus (D \cup E)$ containing a; then b does not belong to C_a.

Let F be the boundary of C_a. Then $X \setminus F = C_a \cup (X \setminus \bar{C}_a)$ is a separation. We define sets F_1, F_2 by $F_1 = D \cap F, F_2 = E \cap F$. Then $F = F_1 \cup F_2$ and neither F_1, F_2 are empty. For if F_1 is empty

[3] Already given in [**Bro67**].

then $F \subseteq E$ and (*) implies that $X \setminus E = C_a \cup (X \setminus C_a \setminus E)$ is a separation; and so E separates a, b, contrary to hypothesis.

It follows by similar reasoning that the component C_b of $X \setminus F$ that contains b has limit points in both F_1 and F_2. Let

$$E_i = C_b \cap F_i, i = 1, 2, E^* = E_1 \cup E_2.$$

Let K_a denote the component of $X \setminus E^*$ which contains C_a. As every limit point of F is a limit point of C_a, so must every point of E^* be a limit point of K_a. Consequently, E^* is the common boundary of the disjoint sets K_a, C_b. To E_1 let us add every component of $X \setminus E^*$ that has its boundary entirely in E_1 and call the resulting set E_1'. Similarly, form a set E_2' relative to E_2.

The set $X \setminus E_2'$ is connected. For let C be a component of this set, and suppose $C \cap E_2 = \emptyset$. Then $C \subseteq X \setminus E^*$. Let C' be the component of $X \setminus E^*$ which contains C. If the boundary E' of C' were entirely in E_1 then we would have

$$C \subseteq C' \subseteq E_1'.$$

If E' were entirely in E_2 then we would have

$$C \subseteq C' \subseteq E_2'$$

and consequently, since $E_2' \subseteq X \setminus E_1'$ we would have

$$C' \subseteq X \setminus E_1' \text{ and } C \cap E_2 \neq \emptyset.$$

Finally, were $E_1; \cap E_1 \neq \emptyset, E' \cap E_2 \neq \emptyset$ we would again have $C' \subseteq X \setminus E_1'$ and hence $C \cap E_2 \neq \emptyset$. Consequently, $C' \cap E_2 \neq \emptyset$. But

$$K_a^* \cup C_b \cup E_2'$$

is a connected subset of $X \setminus E_1'$ and as every component C of $X \setminus E_1'$ has limit points in E_2' the set $X \setminus E_1'$ is connected. Similarly, $X \setminus E_2'$ is connected. But $X \setminus (E_1' \cup E_2')$) is not connected since K_a, K_b are disjoint components of the latter set. This contradicts the fact that X was supposed to satisfy PBI$'$). □

For further information on this area see [**Wil49, GMI89**].

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