COHOMOLOGY WITH CHAINS AS COEFFICIENTS

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Introduction

THE cohomology theory described in the title is obtained by replacing the usual coefficient group by an arbitrary chain complex. This theory satisfies all of the Eilenberg–Steenrod axioms for a cohomology theory except the dimension axiom. There are other theories with this property, and among these our theory is probably the least extraordinary. That is to say, its definition and techniques are very similar to those of the usual cohomology theory.

The theory is justified by its applications. The first of these is a very simple construction† of the Künneth isomorphism

$$\kappa : H^*(X \times Y;\, G) \to H^*(X;\, H^*(Y;\, G)).$$

With this construction, κ turns out to be natural with respect to maps of X, and this increases notably the utility of κ. This amount of naturality is best possible: examples show that κ cannot in general be chosen to be natural with respect to maps of Y or homomorphisms of G.

The second application is to generalize known results on Eilenberg–MacLane complexes to arbitrary css-abelian groups. This generalization depends heavily on work of Dold–Kan (**3**), and is related to other work of Dold (**4**).

The third application uses the earlier results to give a generalization of the suspension of a cohomology operation. This generalization involves an arbitrary css-complex Y rather than, as with the suspension, the 1-sphere S^1.

This paper is in two parts. Part I deals only with chain complexes, while Part II deals with the cohomology of css-complexes. An appendix gives computations of Künneth isomorphisms and an example of the Künneth suspension.

I am deeply indebted to Dr M. G. Barratt for his encouragement to develop this work, and for his criticism of several versions of it.

† *Note added in proof.* A similar construction has been given by W. Shih in his paper, 'Homologie des espaces fibres', *Publications Mathématiques Inst. Haute École Sci.* 13 (1962) 93–174.

PART I. CHAIN COMPLEXES

1. Preliminaries

1.1. A *chain complex* A is a sequence

$$\cdots \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots \quad (-\infty < n < \infty)$$

of abelian groups and homomorphisms such that $\partial_{n-1}\partial_n = 0$. The sequence $\{\partial_n\}$ is called the *differential*: we often omit the subscript n on ∂_n, and also write ∂ as ∂^A if we wish to emphasize the complex to which ∂ belongs.

The cycles $Z(A)$, boundaries $B(A)$, and homology $H(A)$ are defined as usual.

A *chain map* $f: A \to B$ *of degree* p of chain complexes A, B is a sequence $f_n : A_n \to B_{n+p}$ of homomorphisms such that $\partial^B f = (-1)^p f \partial^A$. Two such chain maps f, g are *chain homotopic*, written $f \simeq g$, if there is a sequence $\{D_n : A_n \to B_{n+p+1}\}$ of homomorphisms such that $f - g = \partial^B D + (-1)^p D \partial^A$.

A chain map $f: A \to B$ of degree p induces a map $H(f): H(A) \to H(B)$ of homology. If $f \simeq g$, then $H(f) = H(g)$.

A chain map of degree 0 is called simply a chain map. The group of chain homotopy classes of chain maps $A \to B$ is written $\langle A, B \rangle$.

1.2. The tensor product $A \otimes B$ of chain complexes is defined as usual. For the *hom product* $A \pitchfork B$ we take the following definition, which is now standard.

An element of $(A \pitchfork B)_p$ is a map $A \to B$ of degree p of graded groups. Hence

$$(A \pitchfork B)_p = \prod_n \operatorname{Hom}(A_n, B_{n+p}),$$

where $\prod$ denotes direct product. The differential δ is given by

$$(\delta f)(a) = \partial^B f(a) - (-1)^p f \partial^A(a), \quad f \in (A \pitchfork B)_p, \; a \in A.$$

The p-cycles $Z_p(A \pitchfork B)$ are then simply the chain maps $f: A \to B$ of degree p, and $f \simeq 0$ if and only if $f \in B_p(A \pitchfork B)$. It follows that $H_0(A \pitchfork B) = \langle A, B \rangle$.

Both the tensor and hom product are functors of two chain complexes. Let $f: A \to A'$, $g: B \to B'$ be chain maps of degree p, q respectively. Let $f \otimes g : A \otimes A' \to B \otimes B'$, $f \pitchfork g : A' \pitchfork B \to A \pitchfork B'$ be defined by

$$(1.3) \qquad \begin{aligned} (f \otimes g)(a \otimes a') &= (-1)^{qr} fa \otimes ga', \quad a \in A_r, \; a' \in A'; \\ (f \pitchfork g)(h) &= (-1)^{p(q+r)} ghf, \quad h \in (A \pitchfork B)_r. \end{aligned}$$

Then $f \otimes g$ and $f \pitchfork g$ are chain maps of degree $p+q$.

1.4. The products $\otimes, \pitchfork$ are connected by an *exponential* (or *associativity*) *law*. Let A, B, C be chain complexes, and let the *exponential map*

$$\mu : (A \otimes B) \pitchfork C \to A \pitchfork (B \pitchfork C)$$

be defined by

$$\mu(f)(a)(b) = f(a \otimes b), \quad f \in (A \otimes B) \pitchfork C, \ a \in A, \ b \in B.$$

1.5. LEMMA. *The exponential map is a natural chain isomorphism of degree* 0.

The proof is elementary and is omitted.

1.6. A chain complex F is *free* if each F_n is a free abelian group. The following realization lemma is standard (cf. ((**4**) § 3)).

1.7. LEMMA. *Let G be any graded group. There is a free chain complex F and an isomorphism $H(F) \to G$.*

For any chain complex A, free chain complex F, and map $\varphi : H(F) \to H(A)$ of graded groups, there is a chain map $f : F \to A$ such that $H(f) = \varphi$.

1.8. The *suspension* of a chain complex A is the chain complex sA such that

$$(sA)_n = A_{n-1}, \quad \partial^{sA} = -\partial^A.$$

Then $s : A \to sA$ given by $s(a) = a \in (sA)_n$ is a chain isomorphism of degree $+1$. The p-fold suspension s^p is defined inductively by $s^0 =$ identity, $s^p = ss^{p-1}$.

The chain complex which is G in dimension 0 and is 0 otherwise is written $s^0 G$, or, when no confusion will arise, simply as G. Thus Z will denote the additive group of integers, and also the chain complex $s^0 Z$.

The inverse of s (the functor and the map) is written s^-; and s^{-p} is the inverse of s^p.

1.9. Let A, B be chain complexes. We make the identification

$$A \pitchfork sB = s(A \pitchfork B) \tag{1.10}$$

by means of the map

$$A \pitchfork sB \xrightarrow{1 \pitchfork s^-} A \pitchfork B \xrightarrow{s} s(A \pitchfork B).$$

The natural chain isomorphism of degree 0

$$\sigma' : (sA) \pitchfork B \to A \pitchfork s^- B \tag{1.11}$$

is the composition

$$(sA) \pitchfork B \xrightarrow{s^- \pitchfork 1} A \pitchfork B \xrightarrow{1 \pitchfork s^-} A \pitchfork s^- B.$$

This isomorphism involves a sign, while the identification (1.10) does not.

2. The Künneth isomorphism

The following definition is fundamental for this paper.

2.1. Definition. Let K, A be chain complexes. The *cohomology of K with coefficients in A* is the graded group

$$H^*(K; A) = H(K \pitchfork A).$$

This cohomology is a functor of K and A: a chain map $f: K \to K'$ induces $f^*: H^*(K'; A) \to H^*(K; A)$, and a chain map $g: A \to A'$ induces $g_*: H^*(K; A) \to H^*(K; A')$.

2.2. Proposition. *Let K be a free chain complex and $f: A \to A'$ a chain map such that $H(f)$ is an isomorphism. Then $f_*: H^*(K; A) \to H^*(K; A')$ is an isomorphism.*

This is Satz 3.1 of (**4**), and is proved in (**4**) by standard techniques.

2.3. Let L, A, A' be chain complexes, and suppose we are given an isomorphism

$$\lambda: H^*(L; A) \to H(A').$$

For the rest of this paper, we make the convention

2.4. *If $A' = L \pitchfork A$ or $A' = H^*(L; A)$, then λ is to be the identity.* With all this as given, we have

2.5. Definition. A *Künneth isomorphism of type* $(L, A; A')$ *associated with* λ is an isomorphism

$$\kappa: H^*(K \otimes L; A) \to H^*(K; A')$$

which is defined for each free chain complex K, is natural with respect to maps of K, and which reduces to λ under the obvious identifications if $K = Z$. It is shown below that such a κ always exists.

Special cases of this definition are isomorphisms

$$\kappa: H^*(K \otimes L; A) \to H^*(K; H^*(L; A)), \tag{2.6}$$

$$\kappa: H^*(K; A) \to H^*(K; H(A)), \tag{2.7}$$

natural with respect to maps of the free chain complex K, and reducing to the identity if $K = Z$.

It is not claimed that κ is uniquely determined by λ. Indeed we will show in § 3 that κ of (2.7) is in general not natural with respect to automorphisms of A, and it is easy to deduce from this that κ is not unique. This non-naturality shows also that the notion of cohomology with chains as coefficients is not reducible to cohomology with coefficients in a graded group—structure is lost in passing from $H^*(K; A)$ to $H^*(K; H(A))$.

We now prove existence of Künneth isomorphisms. Let L, A, A' be chain complexes and

$$\lambda : H^*(L; A) \to H(A')$$

any homomorphism.

2.8. Theorem. *For each free chain complex K there is a homomorphism*

$$\kappa : H^*(K \otimes L; A) \to H^*(K; A')$$

natural with respect to maps of K and reducing to λ if $K = Z$. Further, κ is an isomorphism if λ is.

Proof. The exponential map μ induces an isomorphism

$$H(\mu) : H^*(K \otimes L; A) \to H^*(K; L \pitchfork A).$$

Let F be a free chain complex and $f : F \to L \pitchfork A$ a chain map such that $H(f)$ is an isomorphism. Let $g : F \to A'$ be a chain map such that $H(g) = \lambda H(f)$. Such F, f, g exist by 1.7.

By 2.2, $f_* : H^*(K; F) \to H^*(K; L \pitchfork A)$ is an isomorphism. Hence

$$\kappa = g_* f_*^{-1} H(\mu) : H^*(K \otimes L; A) \to H^*(K; A')$$

is well defined. Clearly κ is natural with respect to maps of K.

If $K = Z$ then, under the identifications $Z \otimes X = X$ and $Z \pitchfork X = X$, μ reduces to the identity and

$$\kappa = g_* f_*^{-1} H(\mu) = \lambda H(f) H(f)^{-1} = \lambda.$$

Finally, if λ is an isomorphism so is $H(g) = \lambda H(f)$. Hence, by 2.2, $g_* : H^*(K; F) \to H^*(K; A')$ is an isomorphism, and therefore so also is κ.

2.9. Remark. There is a natural filtration on $K \pitchfork A$ by dimension of K. This filtration determines a spectral sequence $\{E_r\}_{r \geqslant 2}$ such that if K is free,

$$E_\infty = E_2 = H^*(K; H(A)).$$

Further E_∞ is the graded group $\mathrm{Gr}\, H^*(K; A)$ associated with the induced filtration on $H^*(K; A)$. (However, we must define Gr using direct product rather than direct sum.) So from (2.7) we deduce an isomorphism

$$H^*(K; A) \to \mathrm{Gr}\, H^*(K; A)$$

natural with respect to maps of K.

3. Examples

This section contains examples of non-naturality of the Künneth isomorphism. For completeness, we include a well-known example of non-naturality of the splitting map of the Universal Coefficient Theorem.

3.1. Example. κ of (2.7) is not natural with respect to automorphisms of A.

Let K have only two generators a, b in dimensions $0, 1$ respectively, with $\partial b = 2a$. Let A have generators c, d, d' in dimensions $0, 1, 1$ respectively, with $\partial d = \partial d' = 2c$. Let $\tau : A \to A$ be the automorphism which interchanges d and d'.

It is easily checked that τ induces a non-trivial automorphism of $H^*(K; A)$ and a trivial automorphism of $H^*(K; H(A))$. Hence there is no isomorphism $H^*(K; A) \to H^*(K; H(A))$ commuting with τ_*.

3.2. Example. Let G be an abelian group. We prove that there is no isomorphism

$$\kappa : H^*(K \otimes L; G) \to H^*(K; H^*(L; G))$$

natural with respect to homomorphisms of G.

Suppose to the contrary that for all complexes K, L, with K free, κ may be constructed to be natural with respect to maps of G.

Consider the functor $\chi(E, F, G)$ of abelian groups E, F, G defined by

$$\chi(E, F, G) = \operatorname{Hom}(E, \operatorname{Ext}(F, G)) + \operatorname{Ext}(E, \operatorname{Hom}(F, G)).$$

Let K, L be free resolutions of E, F respectively (**2**); then

$$H^0(K; H^{-1}(L; G)) = \operatorname{Hom}(E, \operatorname{Ext}(F, G)),$$
$$H^{-1}(K; H^0(L; G)) = \operatorname{Ext}(E, \operatorname{Hom}(F, G)).$$

So κ induces an isomorphism

$$\kappa_1 : H^{-1}(K \otimes L; G) \to \chi(E, F, G)$$

natural with respect to maps of G. Similarly, there is an isomorphism

$$\kappa_2 : H^{-1}(L \otimes K; G) \to \chi(F, E, G)$$

natural with respect to maps of G. But there is a natural isomorphism $T^* : H^*(K \otimes L, G) \to H^*(L \otimes K; G)$. So we deduce an isomorphism

$$\varphi(E, F, G) : \chi(E, F, G) \to \chi(F, E, G)$$

natural with respect to maps of G. The following example shows this to be impossible.

Let $E = Z_{pq}$ (where $p | q$), let $F = Z_p = G_2$, $G_1 = Z$, and let $\theta : G_1 \to G_2$ be epi. Since $\operatorname{Hom}(E, G_1) = \operatorname{Hom}(F, G_1) = 0$, we have

$$\chi(E, F, G_1) = \operatorname{Hom}(E, \operatorname{Ext}(F, G_1)),$$
$$\chi(F, E, G_1) = \operatorname{Hom}(F, \operatorname{Ext}(E, G_1)).$$

Now $\theta_* : \operatorname{Ext}(F, G_1) \approx \operatorname{Ext}(F, G_2)$, and hence

$$\theta_{**} : \chi(E, F, G_1) \approx \chi(E, F, G_2).$$

But $\theta'_* : \operatorname{Ext}(E, G_1) \to \operatorname{Ext}(E, G_2)$ has kernel $p \operatorname{Ext}(E, G_1)$; hence

$$\theta'_{**} : \chi(F, E, G_1) \to \chi(F, E, G_2)$$

has kernel $\operatorname{Hom}(F, p \operatorname{Ext}(E, G_1))$, which is non-zero since $p \mid q$. So

$$\varphi(E, F, G_2)\, \theta_{**} \neq \theta'_{**}\, \varphi(E, F, G_1),$$

contrary to hypothesis.

3.3. Example. The Künneth isomorphism of (2.6) does not in general preserve cup products.† For let $K = L$ be the singular chain complex of the Klein bottle. In $H^*(K \otimes L; Z)$ there is a (-3)-dimensional class whose cup products with two linearly independent (-1)-dimensional classes are non-trivial ((**9**) 186); no such (-3)-dimensional class exists in $H^*(K; H^*(L; Z))$.

3.4. Example. Let K be a free chain complex, and G an abelian group. The universal coefficient theorem ((**2**) II § 3) states that there is a natural short exact sequence for each n,

$$\operatorname{Ext}(H_{n-1}(K), G) \to H^{-n}(K; G) \to \operatorname{Hom}(H_n(K), G), \tag{3.5}$$

and that this sequence splits non-naturally with respect to maps of K. Before giving an example, we remark that the splitting map is obtained by choosing a chain map $K \to Z(K)$ which is a left-inverse to the inclusion $Z(K) \to K$. Hence the splitting map does not depend on choices with respect to G, and is natural with respect to maps of G.

Let $G = Z$, let K have generators a, b in dimension 0, 1 respectively, with $\partial b = 2a$, and let L have one generator c in dimension 1. Let $f: K \to L$ be defined by $f(b) = c$. We put $n = 1$ in (3.5) and obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & Z \arrow[r, "\approx"] \arrow[d, "f^*"] & Z \arrow[d] \\ Z_2 \arrow[r, "\approx"'] & Z_2 \arrow[r] & 0 \end{tikzcd}$$

in which f^* is epi. Clearly there is no natural splitting map.

PART II. *FD*-COMPLEXES

4. The normalization functor

An *FD-complex* is simply a css-abelian group (**8**), and an *FD-map* $f: A \to B$ of *FD*-complexes A, B is a css-homomorphism. The abelian group of *FD*-maps is written $\mathscr{F}(A, B)$; the category of *FD*-complexes and *FD*-maps is written $\mathscr{F}\mathscr{D}$.

† This contradicts 9.4.16 of (**7**).

The FD-complex $K(q)$ ($q = 0, 1, 2, \ldots$) is the free FD-complex (free with respect to both the FD-operators and the group structure) on one generator δ^q of dimension q.

The *cartesian product* (**3**) of FD-complexes A, B is the FD-complex $A \times B$ with $(A \times B)_q = A_q \otimes B_q$ and with basic FD-operators $\partial_i \otimes \partial_i$, $s_i \otimes s_i$.

An *FD-homotopy* (**3**) $F : f_0 \simeq f_1$ of FD-maps $f_0, f_1 : A \to B$ is an FD-map $F : K(1) \times A \to B$ such that

$$F(s_0{}^q \partial_i \delta^1, a_q) = f_i(a_q) \quad (a_q \in A_q;\ i = 0, 1;\ q = 0, 1, \ldots).$$

The group of FD-homotopy classes of FD-maps $A \to B$ is written $\langle A, B \rangle$.

This notation coincides with what we have used for chain complexes. Because of the close relation between FD-complexes and chain complexes, we write $\mathscr{F}(A, B)$ also for the group of chain maps $A \to B$ when A, B are chain complexes. As shown in 1.2, $\mathscr{F}(A, B) = Z^0(A; B)$, where $Z^*(A; B) = Z(A \pitchfork B)$ is the group of cocycles of A with coefficients in B.

The category of chain complexes and chain maps is written $\mathscr{C}$. The full subcategory consisting of chain complexes A such that $A_i = 0$, $i < 0$, is written $\mathscr{C}_0$.

We recall J. C. Moore's definition (**8**) of a normalization functor $N : \mathscr{F}\mathscr{D} \to \mathscr{C}_0$. This functor is shown in (**3**) to be equivalent to the normalization functor of Eilenberg–MacLane (**5**).

4.1. Definition. For any A in $\mathscr{F}\mathscr{D}$, the chain complex $N(A)$ in $\mathscr{C}_0$ is defined by

$$N(A)_q = \bigcap_{i>0} \operatorname{Ker}(\partial_i : A_q \to A_{q-1}),$$

with differential $\partial_0 | N(A)$. If $f : A \to B$ is an FD-map, then $fN(A) \subseteq N(B)$, and f defines a chain map $N(f) : N(A) \to N(B)$. So N becomes an additive functor $\mathscr{F}\mathscr{D} \to \mathscr{C}_0$.

In particular, the chain complex $N(q)$ is defined by $N(q) = NK(q)$.

Any FD-complex A may be regarded as a css-complex by forgetting about the group structure. Then A becomes a Kan complex, so its homotopy groups are defined. The following is proved in (**8**).

4.2. Theorem. *There is a natural isomorphism of graded groups*

$$\xi : \pi_*(A) \to HN(A).$$

The next propositions give additional information on this isomorphism. (These are probably well known, but do not seem to be in the literature.)

4.3. Proposition. *The functor $N : \mathscr{F}\mathscr{D} \to \mathscr{C}_0$ is exact.*

Proof. Let $A' \xrightarrow{i} A \xrightarrow{j} A''$ be a short exact sequence in $\mathscr{F}\mathscr{D}$. Let $i' = N(i)$, $j' = N(j)$. Certainly i' is mono and $j'i' = 0$.

Let a in $N(A)_q$ be such that $j'(a) = 0$. Since $N(A) \subseteq A$, there is by exactness an a' in A'_q such that $i(a') = a$. By the definition of N, and since i is mono, $a' \in N(A')_q$. This proves exactness at $N(A)$.

Let $a'' \in N(A'')_q$; then $\partial_i a'' = 0$, $i > 0$. By Proposition 1 of Exposé 1 of (**1**), any epimorphism of css-groups is a Kan fibre map. Hence there is an a in A_q such that $j(a) = a''$ and $\partial_i(a) = 0$, $i > 0$; thus $a \in N(A)_q$, and hence j' is epi.

4.4. Proposition. *Let $E : A' \to A \to A''$ be a short exact sequence in $\mathscr{F}\mathscr{D}$. The isomorphism ξ of* (4.2) *gives an isomorphism between the homotopy exact sequence of E and the homology exact sequence of $N(E) : N(A') \to N(A) \to N(A'')$.*

Proof. The only non-trivial part is commutativity with the boundary operators. This is easily checked from the definitions of the boundary operators and of ξ. We omit the details.

Remark. Theorem 4.2 is proved in (**8**) for arbitrary css-groups. The two previous propositions also generalize to the non-commutative case.

5. The Dold–Kan functor

As stated in Theorem 4.2, the homotopy groups of an FD-complex A may be determined from NA. An important fact about FD-complexes is that *A itself may be recovered from NA*. This result, which is due to Dold–Kan (**3**), is stated more precisely below. For certain applications, it is convenient to generalize slightly their definition.

5.1. Definition. The *Dold–Kan functor*† $R : \mathscr{C} \to \mathscr{F}\mathscr{D}$ is defined on each C in $\mathscr{C}$ by

$$R(C)_q = \mathscr{F}(N(q), C), \quad q = 0, 1, 2, \ldots,$$

with the obvious FD-operators and values on maps of C. The restriction $R \mid \mathscr{C}_0$ is written R_0 (R_0 is the functor considered in (**3**)).

5.2. Theorem. *There are natural equivalences*

$$\Phi : NR_0 \to 1, \quad \Psi : R_0 N \to 1,$$

the respective identity functors.

Let $C, C' \in \mathscr{C}_0$, $A, A' \in \mathscr{F}\mathscr{D}$. An easy consequence of Theorem 5.2 is

5.3. Corollary. *The functors N, R_0 induce isomorphisms*

$$N : \mathscr{F}(A, A') \to \mathscr{F}(NA, NA'),$$

$$R_0 : \mathscr{F}(C, C') \to \mathscr{F}(R_0 C, R_0 C').$$

† This functor is denoted by a Gothic K in (**3**). As the letter K is already overused in this subject, we have preferred to translate $\mathfrak{K}$ as R.

5.4. THEOREM. *The functors N, R_0 preserve homotopy* ((**3**) 2.6) *and so induce isomorphisms of groups of homotopy classes*

$$N : \langle A, A' \rangle \to \langle NA, NA' \rangle,$$
$$R_0 : \langle C, C' \rangle \to \langle R_0 C, R_0 C' \rangle.$$

5.2, 5.3, 5.4 are due to Dold–Kan (**3**).

These results are not valid for the functor R. To describe the properties of R we need an additional functor.

5.5. DEFINITION. The additive functor $T : \mathscr{C} \to \mathscr{C}$ is defined by

$$T(C) = \begin{cases} C_q, & q > 0, \\ Z_0(C), & q = 0, \ C \in \mathscr{C}, \\ 0, & q < 0, \end{cases}$$

with differential and maps induced by those of C. The natural inclusion $T(C) \to C$ is a natural transformation $t : T \to 1$. We also regard T as a functor $\mathscr{C} \to \mathscr{C}_0$.

5.6. PROPOSITION. *For each C in $\mathscr{C}$, $R(t) : RT(C) \to R(C)$ is an isomorphism.*

Proof. Since $t : T(C) \to C$ is mono, so also is $R(t)$.

Any chain map $N(q) \to C$ factors uniquely through $T(C)$ (since $N(q) \in \mathscr{C}_0$). Hence $R(t)$ is epi.

5.7. PROPOSITION. *The functor $NR : \mathscr{C} \to \mathscr{C}_0$ is naturally equivalent to T.*

Proof. By 5.6, $R(t) : RT \to R$ is a natural equivalence, and hence so also is $NR(t) : NRT \to NR$. But $NRT = NR_0 T$, which is naturally equivalent to T by 5.2.

5.8. REMARK. In (**5**), Eilenberg–MacLane construct complexes $K(\pi, m)$ belonging to $\mathscr{F}\mathscr{D}$ for each integer $m \geqslant 0$ and abelian group π by setting (in our notation)

$$K(\pi, m)_q = Z^{-m}(N(q); \pi).$$

Now $Z^{-m}(N(q); \pi) = Z^0(N(q); s^m \pi) = \mathscr{F}(N(q), s^m \pi)$. Hence there is a natural isomorphism of FD-complexes

$$K(\pi, m) \approx R(s^m \pi);$$

that is, the Dold–Kan functor may be regarded as a generalization of the Eilenberg–Maclane complex in which the abelian group is replaced by a chain complex.

We now consider how known properties of Eilenberg–MacLane complexes generalize to the Dold–Kan functor.

5.9. PROPOSITION. *R is a left-exact, and R_0 an exact, functor.*

Proof. For each $q \geqslant 0$, $\mathscr{F}(N(q), C)$ is a left-exact functor in the chain complex C of $\mathscr{C}$. Hence R and R_0 are left-exact.

Now let $j: B \to C$ be an epimorphism of chain complexes, where $B, C \in \mathscr{C}_0$. The proof is completed by proving that $Rj: RB \to RC$ is epimorphic.

Let $D = \operatorname{Coker}(Rj)$. The functor N is exact; so by Theorem 5.2 there is a commutative diagram with exact rows

$$\begin{tikzcd} NRB \arrow[r, "NRj"] \arrow[d, "\Phi"', "\approx"] & NRC \arrow[r, "Nf"] \arrow[d, "\Phi"', "\approx"] & ND \arrow[r] \arrow[d] & 0 \\ B \arrow[r] & C \arrow[r] & 0 & \end{tikzcd}$$

in which $f: RC \to D$ is the projection. By the 5-lemma, $ND = 0$, and therefore $D \approx RND = 0$.

5.10. REMARK. The previous proposition suggests determining the right-derived functors $R^m R: \mathscr{C} \to \mathscr{F}\mathscr{D}$. However, these are not very interesting, for there is a natural isomorphism

$$R^m R(C) \to K(H_{-m}(C), 0), \quad C \in \mathscr{C}, \quad m > 0$$

(we omit the proof); thus $R^m R(C)$ is essentially just a discrete abelian group.

There is a well-known functor $\bar{W}: \mathscr{F}\mathscr{D} \to \mathscr{F}\mathscr{D}$ which assigns a classifying complex to each FD-complex (8). This functor is such that $\bar{W}K(\pi, m)$ is naturally isomorphic to $K(\pi, m+1)$. More generally we have

5.11. PROPOSITION. *There are natural equivalences*

$$sN \to N\bar{W}, \quad RsN \to \bar{W}.$$

Proof. Let $A \in \mathscr{F}\mathscr{D}$, and let $f: sN(A) \to N\bar{W}(A)$ be defined by

$$f(a) = [a, -\partial_0 a, 0, \ldots, 0], \quad a \in (sN(A))_{q+1}.$$

Then f is a natural isomorphism of chain complexes, and so defines a natural equivalence $f: sN \to N\bar{W}$. Hence $R(f): RsN \to RN\bar{W}$ is a natural equivalence, and so also is $\Phi \circ R(f): RsN \to \bar{W}$.

To conclude this section, we give a short proof of a well-known theorem of J. C. Moore.

5.12. THEOREM. *For any A in $\mathscr{F}\mathscr{D}$, there is a (non-natural) css-homotopy equivalence*

$$h: A \to \prod_{r=0}^{\infty} K(\pi_r(A), r).$$

Proof. Let F in $\mathscr{C}_0$ be a free chain complex, and let

$$NA \xleftarrow{f} F \xrightarrow{g} \pi_*(A)$$

be chain maps inducing isomorphisms in homology, where $\pi_*(A)$ is regarded as a chain complex with trivial differential.

We make the identifications

$$RN(A) = A, \quad R(\pi_*(A)) = \prod_{r=0}^{\infty} K(\pi_r(A), r) = B, \quad \text{say.}$$

Let $F' = R(F)$, $f' = R(f) : F' \to A$, $g' = R(g) : F' \to B$. Then f', g' induce isomorphisms of homotopy groups (by 4.2, 5.2) and so are css-homotopy equivalences ((8) I, Appendix C). Let $f'' : A \to F'$ be a homotopy inverse of f'; then $h = g'f'' : A \to B$ is a homotopy equivalence.

6. Css-complexes

We deal with the category $\mathscr{X}$ of css-complexes with base point, as this is the more convenient for discussing suspension. The set of maps $X \to Y$ in $\mathscr{X}$ is written Map(X, Y).

The *smash product* of complexes X, Y belonging to $\mathscr{X}$ is

$$X \divideontimes Y = X \times Y / (X \times * \cup * \times Y),$$

where $*$ denotes the base point of a complex.

The complex Δ^q is the free css-complex on one generator δ^q in dimension q. This complex has no base point, and it is convenient to define $\Delta^q \divideontimes X$ for any X in $\mathscr{X}$ by

$$\Delta^q \divideontimes X = \Delta^q \times X / \Delta^q \times *.$$

With this definition, a homotopy rel base point is simply a map (in $\mathscr{X}$) $F : \Delta^1 \divideontimes X \to Y$. The set of homotopy classes of maps $X \to Y$ is defined if Y is a Kan complex, and is then written $[X, Y]$.

There is a natural embedding $\mathscr{F}\mathscr{D} \subseteq \mathscr{X}$, the base point of any FD-complex being the subcomplex of zeros. If $A \in \mathscr{F}\mathscr{D}$ and $X \in \mathscr{X}$ then both Map(X, A) and $[X, A]$ obtain an abelian group structure from that of A.

If $X \in \mathscr{X}$, let $C(X)_q$ be the quotient of the free abelian group on X_q by the subgroup generated by $s_0{}^q *$. The css-operators on X induce FD-operators on $C(X) = \bigcup_q C(X)_q$, and a map $f : X \to Y$ induces $C(f) : C(X) \to C(Y)$. Thus C becomes a functor $\mathscr{X} \to \mathscr{F}\mathscr{D}$. Clearly $C(\Delta^1 \divideontimes X) = K(1) \times C(X)$ and $C(X \divideontimes Y) = C(X) \times C(Y)$ if $X, Y \in \mathscr{X}$.

Now let $A \in \mathscr{F}\mathscr{D}$, $X \in \mathscr{X}$.

6.1. Proposition. *There is a natural isomorphism of groups*

$$D : \mathrm{Map}(X, A) \to \mathscr{F}(C(X), A).$$

Further, D is homotopy-preserving and induces an isomorphism

$$D : [X, A] \to \langle C(X), A \rangle.$$

If $f : X \to A$, then $D(f)$ is obtained by extending f linearly from the generators. The proof of the proposition is obvious.

7. Cohomology and cohomology operations

7.1. Definition. Let $X \in \mathscr{X}$. The *normalized chain complex* of X is a free chain complex in $\mathscr{C}_0$ defined by

$$C_N(X) = NC(X).$$

Let $A \in \mathscr{C}$. The *cohomology of X with coefficients in A* is the graded group

$$H^*(X; A) = H^*(C_N(X); A).$$

The graded group of cocyles of X with coefficients in A is

$$Z^*(X; A) = Z^*(C_N(X); A).$$

The following theorem generalizes a well-known theorem on maps into an Eilenberg–MacLane complex.

7.2. Theorem. *Let $X \in \mathscr{X}$, $A \in \mathscr{F}\mathscr{D}$. There is a natural isomorphism*

$$\gamma' : \mathrm{Map}(X, A) \to Z^0(X; NA)$$

inducing a natural isomorphism

$$\gamma : [X, A] \to H^0(X; NA).$$

Proof. γ' is the composite of the natural isomorphisms (5.3, 6.1)

$$\mathrm{Map}(X, A) \xrightarrow{D} \mathscr{F}(C(X), A) \xrightarrow{N} \mathscr{F}(C_N(X), NA) = Z^0(X; NA).$$

Each of these maps is homotopy-preserving, so γ' induces an isomorphism $\gamma : [X, A] \to H^0(X; NA)$.

7.3. Proposition. *Let $X \in \mathscr{X}$, $B \in \mathscr{C}$. The map $t_* : H^0(X; TB) \to H^0(X; B)$ is an isomorphism.*

Proof. This follows easily from the fact that $C_N(X) \in \mathscr{C}_0$.

7.4. Corollary. *There is a natural isomorphism*

$$\gamma : [X, RB] \to H^0(X; B).$$

Proof. $[X, RB] \approx H^0(X; NRB) \approx H^0(X; TB) \approx H^0(X; B)$.

7.5. Definition. Let $A \in \mathscr{F}\mathscr{D}$, $B \in \mathscr{C}$. We define fundamental classes $\omega(A)$ in $H^0(A; NA)$, $\omega(B)$ in $H^0(RB; B)$ by

$$\omega(A) = \gamma(\iota_1), \quad \omega(B) = \gamma(\iota_2),$$

where ι_1 in $[A, A]$ and ι_2 in $[RB, RB]$ are the homotopy classes of the identity maps.

In the case $A = K(\pi, m)$ we have

$$NA \approx s^m \pi, \quad H^0(A\,; NA) \approx H^{-m}(K(\pi, m)\,; \pi),$$

and $\omega(A)$ corresponds under this isomorphism to the classical fundamental class.

An immediate consequence of naturality of the maps γ is that they are given by

$$\gamma[f] = f^* \omega(A), \quad [f] \in [X, A],$$
$$\gamma[g] = g^* \omega(B), \quad [g] \in [X, RB]. \tag{7.6}$$

7.7. Definition. Let $A, B \in \mathscr{C}$. *A cohomology operation of type* (A, B) is a natural transformation

$$\theta : H^0(\ ; A) \to H^0(\ ; B),$$

where these cohomology functors are taken as functors from $\mathscr{X}$ to the category of abelian groups and set maps. The set of operations of type (A, B) is written $\mathrm{Op}(A, B)$, and this set is given the structure of an abelian group by addition of values.

These operations do generalize the classical cohomology operations. Let G, H be abelian groups, and m, n positive integers. Then classically (**10**) a cohomology operation of type $(G, m\,; H, n)$ is a natural transformation

$$\theta : H^{-m}(\ ; G) \to H^{-n}(\ ; H),$$

where the cohomology functors are functors as above, and we have used our conventions as to grading. However, the natural identifications $H^{-m}(\ ; G) = H^0(\ ; s^m G)$, $H^{-n}(\ ; H) = H^0(\ ; s^n H)$ imply that such an operation is exactly an operation in our sense of type $(s^m G, s^n H)$.

The following theorem is therefore a generalization of a famous theorem of Serre (Theorem 1 of § 4 of (**10**)).

7.8. Theorem. *Let $A, B \in \mathscr{C}$. There is a natural isomorphism*

$$\Theta : \mathrm{Op}(A, B) \to H^0(RA\,; B).$$

Proof. The proof is the same as that of (**10**), and we merely indicate it here.

For θ in $\mathrm{Op}(A, B)$, we set $\Theta(\theta) = \theta(\omega(A))$. An inverse Θ' to Θ is defined as follows. Let $\varphi \in H^0(RA\,; B)$, and let $x \in H^0(X\,; A)$. The composition of the homotopy classes $\gamma^{-1}(\varphi)$ in $[RA, RB]$ and $\gamma^{-1}(x)$ in $[X, RA]$ is an element y of $[X, RB]$, and we set $\Theta'(\varphi)(x) = \gamma(y) \in H^0(X\,; B)$.

The next proposition shows that abstractly the structure of $\mathrm{Op}(A, B)$ depends only on $H(A)$ and $H(B)$.

7.9. Proposition. *For $A, B \in \mathscr{C}$, there is a (non-natural) isomorphism*

$$\Lambda : \mathrm{Op}(A, B) \to \mathrm{Op}(H(A), H(B)).$$

Proof. By (2.6) there are isomorphisms for each X in $\mathscr{X}$,

$$\lambda_1 : H^0(X;A) \to H^0(X;H(A)), \quad \lambda_2 : H^0(X;B) \to H^0(X;H(B)),$$

which are natural with respect to maps of X. Hence for any θ in $\mathrm{Op}(A,B)$, an operation $\Lambda(\theta)$ in $\mathrm{Op}(H(A),H(B))$ may be defined by the requirement that for each X in $\mathscr{X}$ the following diagram is commutative:

$$\begin{tikzcd} H^0(X;A) \arrow[r, "\lambda_1"] \arrow[d, "\theta"'] & H^0(X;H(A)) \arrow[d, "\Lambda(\theta)"] \\ H^0(X;B) \arrow[r, "\lambda_2"'] & H^0(X;H(B)). \end{tikzcd}$$

This obviously defines an isomorphism $\Lambda : \mathrm{Op}(A,B) \to \mathrm{Op}(H(A),H(B))$.

7.10. PROPOSITION. *Cohomology operations preserve zero; i.e. if* $\theta \in \mathrm{Op}(A,B)$ *and* $x = 0 \in H^0(X;A)$, *then* $\theta(x) = 0 \in H^0(X;B)$.

Proof. Let $\theta' = \gamma^{-1}\,\Theta(\theta) \in [RA, RB]$. The class $\gamma^{-1}(x)$ in $[X, RA]$ is the class of the constant map, and hence so is $\theta' \circ \gamma^{-1}(x)$ in $[X, RB]$. Therefore $0 = \gamma(\theta' \circ \gamma^{-1}(x)) = \theta(x) \in H^0(X;B)$.

8. Suspension and Künneth suspension

Classically, the suspension homomorphism σ assigns to each operation θ of type $(G, m; H, n)$ an operation $\sigma(\theta)$ of type $(G, m-1; H, n-1)$. This notion generalizes easily to a homomorphism

$$\sigma : \mathrm{Op}(A,B) \to \mathrm{Op}(s^- A, s^- B)$$

for any A, B in $\mathscr{C}$. We shall present a further generalization, the *Künneth suspension*, whose construction uses in an essential way the naturality of the Künneth isomorphism of § 2.

The Künneth suspension has an application to function complexes, generalizing the application of σ to loop spaces, to be dealt with elsewhere.

We first place the Künneth isomorphism of § 2 in the context of css-complexes. This requires

8.1. THEOREM (Eilenberg–Zilber (**6**)). *Let* $X, Y \in \mathscr{X}$. *There is a natural chain homotopy equivalence*

$$\Delta : C_N(X) \otimes C_N(Y) \to C_N(X \divideontimes Y).$$

Let Y in $\mathscr{X}$ and A, A' in $\mathscr{C}$ be such that there is an isomorphism

$$\lambda : H^*(Y;A) \to H(A').$$

Let κ be a Künneth isomorphism of type $(C_N(Y), A; A')$ associated with λ as in § 2. We recall that if $A' = C_N(Y) \pitchfork A$ or $A' = H^*(Y;A)$, then λ is assumed to be the identity.

8.2. Definition. Let $X \in \mathscr{X}$. The composition

$$H^*(X \divideontimes Y;A) \xrightarrow{\Delta^*} H^*(C_N(X) \otimes C_N(Y);A) \xrightarrow{\kappa} H^*(X;A')$$

is called a *Künneth isomorphism of type* $(Y,A;A')$ *associated with* λ. This is an isomorphism, also written κ, *natural with respect to maps of* X.

As a special case, let $Y = S^1$, the 1-sphere defined by $S^1 = \Delta^1/\dot{\Delta}^1$. Then $C_N(S^1) = sZ$, and by 1.9 there is a chain isomorphism

$$(sZ) \pitchfork A \xrightarrow{\sigma'} Z \pitchfork s^- A = s^- A.$$

For any X in $\mathscr{X}$, let σ be the composition

$$H^*(X \divideontimes S^1;A) \xrightarrow{H(\mu)\Delta^*} H^*(X;C_N(S^1) \pitchfork A) \xrightarrow{H(\sigma')} H^*(X;s^- A).$$

Then σ is a Künneth isomorphism of type $(S^1,A;s^-A)$ called the *suspension isomorphism*. It is natural with respect to maps of both X and A.

Now let κ_1, κ_2 be Künneth isomorphisms of types $(Y,A;A')$, $(Y,B;B')$ respectively, where $Y \in \mathscr{X}$ and $A, A', B, B' \in \mathscr{C}$.

8.3. Definition. The *Künneth suspension homomorphism* is the homomorphism

$$\boldsymbol{\kappa} : \mathrm{Op}(A,B) \to \mathrm{Op}(A',B')$$

such that for each θ in $\mathrm{Op}(A,B)$, and X in $\mathscr{X}$, the following diagram is commutative:

$$\begin{tikzcd} H^0(X \divideontimes Y;A) \arrow[r, "\kappa_1"] \arrow[d, "\theta"'] & H^0(X;A') \arrow[d, "\boldsymbol{\kappa}(\theta)"] \\ H^0(X \divideontimes Y;B) \arrow[r, "\kappa_2"'] & H^0(X;B'). \end{tikzcd}$$

As a special case, let $Y = S^1$ and let κ_1, κ_2 be suspension isomorphisms. Then $\boldsymbol{\kappa}$ is the *suspension homomorphism*

$$\boldsymbol{\sigma} : \mathrm{Op}(A,B) \to \mathrm{Op}(s^-A, s^-B).$$

Another example is when $A' = H^*(Y;A)$, $B' = H^*(Y;B)$. Then $\boldsymbol{\kappa}$ is a homomorphism

$$\mathrm{Op}(A,B) \to \mathrm{Op}(H^*(Y;A);\ H^*(Y;B)).$$

There remains the question of computing $\boldsymbol{\kappa}$. This is most conveniently done by computing the composition

$$\mathrm{Op}(A,B) \xrightarrow{\boldsymbol{\kappa}} \mathrm{Op}(A',B') \xrightarrow{\Theta} H^0(RA';B'),$$

where Θ is as in Theorem 7.8. We first give

8.4. Definition. Let κ be a Künneth isomorphism of type $(Y,A;A')$. The *evaluation class* of κ is the cohomology class e in $H^0(RA' \divideontimes Y;A)$

such that $\kappa(e) = \omega(A')$. The reason for the name given to this class is a connexion with the evaluation map of function complexes to be discussed elsewhere.

Now suppose we are in the situation of Definition 8.3. Let e_1 be the evaluation class of κ_1. Let $\boldsymbol{\kappa}' : \mathrm{Op}(A,B) \to H^0(RA';B')$ be defined by

$$\boldsymbol{\kappa}'(\theta) = \kappa_2\theta(e_1), \quad \theta \in \mathrm{Op}(A,B).$$

Then $\Theta\boldsymbol{\kappa}(\theta) = \boldsymbol{\kappa}(\theta)(\omega(A')) = \kappa_2\theta\kappa_1^{-1}(\omega(A')) = \kappa_2\theta(e_1)$. So $\Theta\boldsymbol{\kappa} = \boldsymbol{\kappa}'$.

In the Appendix we give formulae for Künneth isomorphisms and evaluation classes which are useful in computing Künneth suspensions. We conclude this section by showing that one part of $\boldsymbol{\kappa}$ is reducible to the suspension homomorphism.

Let A, B be chain complexes with trivial differential. Then

$$\mathrm{Op}(A,B) \approx H^0\Big(\prod_{p=0}^{\infty} K(A_p,p);B\Big).$$

Therefore $\mathrm{Op}(A,B)$ contains as a direct summand

$$\prod_{p=0}^{\infty}\sum_{q=0}^{\infty}\mathrm{Op}(s^pA_p, s^qB_q).$$

In particular, there is a projection $\rho_p : \mathrm{Op}(A,B) \to \mathrm{Op}(s^pA_p, s^pB_p)$ $(p \geqslant 0)$, and the latter group is (as is well known) $\mathrm{Hom}(A_p,B_p)$ if $p > 0$, and is the group $\mathrm{Map}(A_0,B_0)$ of functions preserving 0 if $p = 0$.

This projection ρ_p is made explicit as follows. Let $S^p = \Delta^p/\dot{\Delta}^p$ be the p-sphere. Then $C_N(S^p) = s^pZ$ and by 1.9 we have an isomorphism

$$\rho'_p = H(\sigma'^p) : H^0(S^p;A) \to H_0(Z \pitchfork s^{-p}A) = A_p.$$

For any θ in $\mathrm{Op}(A,B)$ we define $\rho_p(\theta) : A_p \to B_p$ by the commutative diagram

$$\begin{array}{ccc} H^0(S^p;A) & \xrightarrow{\theta} & H^0(S^p;B) \\ \rho'_p\Big\downarrow & & \rho'_p \\ A_p & \xrightarrow[\rho_p(\theta)]{} & B_p. \end{array}$$

Let κ_1, κ_2 be Künneth isomorphisms of types $(Y,A;H^*(Y;A))$, $(Y,B;H^*(Y;B))$ respectively. We identify $H^p(Y;A)$ with $H^0(Y;s^{-p}A)$, and $H^p(Y;B)$ with $H^0(Y;s^{-p}B)$. The Künneth suspension is now a homomorphism $\boldsymbol{\kappa} : \mathrm{Op}(A,B) \to \mathrm{Op}(H^*(Y;A), H^*(Y;B))$; so if $\theta \in \mathrm{Op}(A,B)$ then $\rho_p\boldsymbol{\kappa}(\theta)$ is a map $H^0(Y;s^{-p}A) \to H^0(Y;s^{-p}B)$.

8.8. THEOREM. *For each θ in $\mathrm{Op}(A,B)$, and $p \geqslant 0$,*

$$\rho_p\boldsymbol{\kappa}(\theta) = \boldsymbol{\sigma}^p(\theta),$$

the p-fold suspension of θ.

Proof. We consider the following diagram, where $T : S^p \circledast Y \to Y \circledast S^p$ is the twisting map:

$$\begin{array}{ccc}
H^0(Y \circledast S^p; A) & \xrightarrow{\theta} & H^0(Y \circledast S^p; B) \\
{\scriptstyle T^*}\downarrow & & \downarrow{\scriptstyle T^*} \\
H^0(S^p \circledast Y; A) & \xrightarrow{\theta} & H^0(S^p \circledast Y; B) \\
{\scriptstyle \kappa_1}\downarrow & & \downarrow{\scriptstyle \kappa_2} \\
H^0(S^p; H^*(Y; A)) & \xrightarrow{\boldsymbol{\kappa}(\theta)} & H^0(S^p; H^*(Y; B)) \\
{\scriptstyle \rho'_p}\downarrow & & \downarrow{\scriptstyle \rho'_p} \\
H^0(Y; s^{-p} A) & \xrightarrow{\rho_p \boldsymbol{\kappa}(\theta)} & H^0(Y; s^{-p} B).
\end{array}$$

The top square is commutative since θ is an operation, and the other squares are commutative by definition of $\boldsymbol{\kappa}(\theta)$ and $\rho_p\boldsymbol{\kappa}(\theta)$. The theorem is proved if we show that the two vertical compositions are simply σ'^p. This is shown by routine calculations which we omit. (The assumption that each of κ_1, κ_2 is associated with the identity is essential here.)

APPENDIX. COMPUTATIONS

In the first section of this appendix we give formulae for Künneth isomorphisms, and these formulae are used in the second section to obtain formulae for evaluation classes, and so to give an example of a Künneth suspension.

Section 1

In this part, we wish to describe Künneth isomorphisms of type $(L, A; A')$, where L, A, A' are chain complexes. It turns out that the signs are simpler if we regard such a Künneth isomorphism as a map $H^*(L \otimes K; A) \to H^*(K; A')$; that is, if we precede κ by the isomorphism $T^* : H^*(L \otimes K; A) \to H^*(K \otimes L; A)$ induced by the twisting map $T : K \otimes L \to L \otimes K$. We shall accordingly write κ for κT^* throughout this appendix.

The description of κ in general is reduced to that for simple cases by means of the following additivity lemma.

LEMMA (A.1). (i) *Let κ_i be a Künneth isomorphism of type $(L_i, A; A'_i)$, $i = 1, 2$. Then $\kappa_1 + \kappa_2$ is a Künneth isomorphism of type $(L_1 + L_2, A; A'_1 + A'_2)$.* (ii) *Let κ_i be a Künneth isomorphism of type $(L, A_i; A'_i)$, $i = 1, 2$. Then $\kappa_1 + \kappa_2$ is a Künneth isomorphism of type $(L, A_1 + A_2; A'_1 + A'_2)$.*

The proof of the lemma is obvious.

We assume that K and L are free, finitely generated, and bounded, and that $A = s^0 G$, where G is an abelian group. This is the case of most interest in the applications.

We shall use the maps $\delta_n, h_{n,m}, \delta_{n,n}$ of the cohomology spectrum of a chain complex K as defined in (**9**) and (**11**). If G is an abelian group, we write G_n for $Z_n \otimes G$ and identify G_0 with G. The homomorphism $h_{n,m} : H^*(K; Z_m) \to H^*(K; Z_n)$ extends in an obvious way to a homomorphism $h_{n,m} : H^*(K; G_m) \to H^*(K; G_n)$. If $y \in H^*(L; Z_m)$ and $x \in H^*(K; G_n)$, we write $y \times x \in H^*(L \otimes K; G_{(m,n)})$ for the cartesian product of y and x with respect to the usual pairing $Z_m \otimes G_n \to G_{(m,n)}$.

(A.2) By Lemma (A.1) (i), it is sufficient to find formulae for κ when there is an isomorphism

$$\nu : H^*(L;Z) \to s^q Z_t, \quad t \geqslant 0,$$

and this we assume. Let $a_0 = \nu^{-1}(1) \in H^q(L;Z)$. If $t > 0$, the coboundary $\delta_t : H^{q+1}(L;Z_t) \to H^q(L;Z)$ is an isomorphism, and we define b_t in $H^{q+1}(L;Z_t)$ by $\delta_t(b_t) = a_0$.

(A.3) We first consider the case when G has no elements of order t (this includes the case $t = 0$). Then the isomorphism ν determines an isomorphism

$$\nu : H^*(L;G) \to s^q G_t.$$

If κ is a Künneth isomorphism of type $(L, G; s^q G_t)$, then in each dimension κ maps

$$H^m(L \otimes K; G) \to H^m(K; s^q G_t) = H^{m-q}(K; G_t).$$

Let $t > 0$. Since G has no elements of order t, the sequence

$$E : 0 \to G \xrightarrow{t} G \to G_t \to 0$$

is exact. The associated Bockstein coboundary is written δ_E. (If $t = 0$, we take $\delta_E = 0$.)

It is well known that $H^*(L \otimes K; G)$ is the direct sum of subgroups A, B such that (i) A is generated by cartesian products $a_0 \times k_0$, all k_0 in $H^*(K;G)$, (ii) B is generated by the elements $\delta_E(b_t \times k_t)$, with k_t in $H^*(K;G_t)$. Hence it is sufficient to describe κ on elements of these types.

THEOREM (A.4). *There is a Künneth isomorphism κ of type $(L, G; s^q G_t)$ which on $H^*(L \otimes K; G)$ is given by the formulae*

(i) $\kappa(a_0 \times k_0) = h_{t,0}(k_0), \quad k_0 \in H^*(K;G)$,

(ii) $\kappa\delta_E(b_t \times k_t) = k_t, \quad k_t \in H^*(K;G_t)$.

The proof of this theorem is straightforward, and is omitted.

The above formulae show that κ may be chosen to be natural with respect to maps of groups G such that $H^*(L;Z)$ and G have no common torsion. If $H^*(L;Z)$ and G have common torsion, we cannot expect such reasonable formulae. However, when G is finitely generated, we can, by Lemma (A.1) (ii), recover κ from the case $G = Z_n$.

(A.5) Let $\nu : H^*(L;Z) \approx s^q Z_t$ $(t > 0)$; let $n > 0$, and let $C = s^q Z_d + s^{q+1} Z_d$, where $d = (n, t)$. Then $H^*(L;Z_n) \approx C$. Let a_0, b_t be as in (A.2), and let

$$a_i = h_{i,0}(a_0) \in H^*(L;Z_i), \quad b_i = h_{i,t}(b_t) \in H^*(L;Z_i).$$

It is proved in (**9**) that $H^*(L \otimes K; Z_n)$ is generated by the elements $h_{n,i}(a_i \times k_i)$, $h_{n,i}(b_i \times k_i)$, $\delta_{n,n} h_{n,i}(a_i \times k_i)$, for k_i in $H^*(K;Z_i)$, $i \geqslant 0$.

Let κ be a Künneth isomorphism of type $(L, Z_n; C)$. The composition

$$\begin{array}{c} H^m(L \otimes K; Z_n) \\ \downarrow \kappa \\ H^m(K; C) \\ \approx \downarrow \\ H^m(K; s^q Z_d) + H^m(K; s^{q+1} Z_d) \\ \approx \downarrow \\ H^{m-q}(K; Z_d) + H^{m-q-1}(K; Z_d) \end{array}$$

is also written κ.

THEOREM (A.6). *Let α, β be integers such that $\alpha n + \beta t = (n, t) = d$. There is a Künneth isomorphism κ of type $(L, Z_n; C)$ which on $H^*(L \otimes K; Z_n)$ is given by the formulae*

(i) $\kappa h_{n,i}(a_i \times k_i) = h_{d,i} k_i$,

(ii) $\kappa h_{n,i}(b_i \times k_i) = h_{d,n} h_{n,i} k_i + (-1)^q \beta h_{d,0} \delta_i k_i$,

(iii) $\kappa \delta_{n,n} h_{n,i}(a_i \times k_i) = h_{d,t} h_{t,i} k_i + (-1)^{q+1} \alpha h_{d,0} \delta_i k_i$,

for all k_i in $H^*(K; Z_i)$, $i \geqslant 0$.

The proof of the theorem consists in constructing κ by the method of proof of Theorem 2.8, and checking that it has the stated values. The integers α, β enter at the stage of constructing a chain map $F \to C$ inducing an isomorphism in homology, where F is a free chain complex such that $H(F) \approx H^*(L; Z_n)$. We omit further details.

Section 2

In this section we find formulae for the evaluation class in $H^*(Y ⋇ RA'; A)$ when $H^*(Y; Z) \approx s^q Z_t$ $(t \geqslant 0)$ and $A = s^m G$. We suppose T to be finite.

(A.7) We first consider the case in which G has no elements of order t, so that $H^*(Y; G) \approx s^q G_t$. Under the identification (1.10), the Künneth isomorphism of (A.4) determines a Künneth isomorphism κ of type $(Y, s^m G; s^{m+q} G_t)$. Let $X = R(s^{m+q} G_t)$. Let ω, in $H^0(X; s^{m+q} G_t) = H^{-m-q}(X; G_t)$, be the fundamental class. The evaluation class of κ is a class e in $H^{-m}(Y ⋇ X; G)$ such that $\kappa(e) = \omega$. We use the notation of (A.2), (A.3).

THEOREM (A.8). *The evaluation class e of κ is given by*

(i) $e = a_0 \times \omega$ *if* $t = 0$,

(ii) $e = \delta_{\mathrm{E}}(b_t \times \omega)$ *if* $t > 0$.

This follows immediately from Theorem (A.4).

(A.9) We now consider the situation of (A.5). Let $X = R(s^m C)$. Then X is the direct sum $X^0 + X^1$, where $X^0 = R(s^{m+q} Z_d)$, $X^1 = R(s^{m+q+1} Z_d)$. Let $p^0 : X \to X^0$, $p^1 : X \to X^1$ be the projections, and let ω^0 in $H^0(X; s^{m+q} Z_d)$ and ω^1 in $H^0(X; s^{m+q+1} Z_d)$ be the images of the fundamental classes of X^0, X^1 under p^{0*}, p^{1*} respectively. Under the identification

$$H^0(X; s^{m+q} Z_d) + H^0(X; s^{m+q+1} Z_d) = H^0(X; s^m C),$$

we have $\omega^0 + \omega^1 = \omega$, the fundamental class of X.

By (1.10) the Künneth isomorphism of (A.6) determines a Künneth isomorphism κ of type $(Y, s^m Z_n; s^m C)$.

THEOREM (A.10). *The evaluation class e of κ is*

$$e = h_{n,d}(b_d \times \omega^1) + \alpha h_{n,d}(a_d \times \omega^0) + \beta \delta_{n,n} h_{n,d}(k_d \times \omega^0).$$

For, as is easily checked, $\kappa(e) = \omega$.

EXAMPLE (A.11). We use this last formula to compute a Künneth suspension. Let $Y = S^{r-1} \cup_2 e^r$ be the $(r-2)$-fold suspension of the real projective plane, so that $H^*(Y; Z) \approx s^{-r} Z_2$. Let $\theta = Sq^n$, in $\mathrm{Op}(s^m Z_2, s^{m+n} Z_2)$, be the Steenrod square.

Let κ be the Künneth isomorphism of type $(Y, Z_2; s^{-r} Z_2 + s^{-r+1} Z_2)$ given by Theorem (A.6) with $q = -r$, $n = t = d = 2$, $\alpha = 1$, $\beta = 0$. The evaluation class of κ is $e = b_2 \times \omega^1 + a_2 \times \omega^0$ (with the notation of (A.9)) so that

$$Sq^n(e) = b_2 \times Sq^n \omega^1 + a_2 \times Sq^{n-1} \omega^1 + a_2 \times Sq^n \omega^0,$$

by the Cartan formula and the relation $Sq^1(b_2) = a_2$. Hence

$$\kappa\, Sq^n(e) = (Sq^n + Sq^{n-1})\, \omega^1 + Sq^n\, \omega^0.$$

That is, $\kappa\, Sq^n$, as a map of FD-complexes, is given by the diagram

$$\begin{tikzcd}
K(Z_2, m-r) \arrow[d, "Sq^n"'] & \times & K(Z_2, m-r+1) \arrow[d, "Sq^n"] \arrow[dll, "Sq^{n-1}"] \\
K(Z_2, m+n-r) & \times & K(Z_2, m+n-r+1).
\end{tikzcd}$$

We could also have taken $\alpha = 0$, $\beta = 1$ in the above. In fact, since two Künneth isomorphisms are involved, there are four possible answers, namely,

(i) $(Sq^n + Sq^{n-1})\, \omega^1 + Sq^n\, \omega^0$ (as above),

(ii) $(Sq^n + Sq^{n-1})\, \omega^1 + (Sq^n + Sq^{n-1})\, \omega^0$,

(iii) $((n+1)\, Sq^n + Sq^{n-1})\, \omega^1 + ((n+1)\, Sq^{n+1} + Sq^n + Sq^n\, Sq^1)\, \omega^0$,

(iv) $((n+1)\, Sq^n + Sq^{n-1})\, \omega^1 + ((n+1)\, Sq^{n+1} + Sq^n)\, \omega^0$.

(We have used the relation $Sq^1\, Sq^n = (n+1)\, Sq^{n+1}$.) This demonstrates clearly the non-canonical nature of the Künneth suspension.

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