ON THE CONNECTION BETWEEN THE SECOND RELATIVE HOMOTOPY GROUPS OF SOME RELATED SPACES

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The title of this paper is chosen to imitate that of the paper by van Kampen **[10]** which gave some basic computational rules for the fundamental group $\pi_1(Y, \zeta)$ of a based space (an earlier more special result is due to Seifert **[14]**).

In **[1]** results more general than van Kampen's were obtained in terms of fundamental groupoids. The advantage of the use of groupoids is that one obtains an easy description of the fundamental groupoid of a union of spaces even when the spaces and their intersections are not path-connected; in such cases, the computation of the fundamental group is greatly simplified by using groupoids.

To obtain analogous results in dimension 2 we make essential use of a kind of *double groupoid* first described in **[4]**. A major aim is to introduce the *homotopy double groupoid* $\rho(X, Y, Z)$ defined for any triple (X, Y, Z) of spaces such that every loop in Z is contractible in Y. The methods of **[1]** are generalized to give results on $\rho(X, Y, Z)$. We obtain, as algebraic consequences, results on the second relative homotopy group $\pi_2(X, Y, \zeta)$ in the form of computational rules for the crossed module

$$\partial\colon \pi_2(X, Y, \zeta) \to \pi_1(Y, \zeta).$$

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1. Preliminaries on double groupoids

By a *double groupoid* we shall always mean a 'special double groupoid with special connection' as defined in § 3 of **[4]**. We recall this definition, adopting a slightly different notation.

A double groupoid $G = (G_2, G_1, G_0)$ has, in the first place, the structure of a two-dimensional cubical complex. Thus there are face maps $\partial_i^\alpha\colon G_n \to G_{n-1}$ ($\alpha = 0, 1,\ i = 1, 2, \ldots, n,\ n = 1, 2$) and degeneracy maps $\varepsilon_i\colon G_{n-1} \to G_n$ ($i = 1, 2, \ldots, n,\ n = 1, 2$) satisfying the usual cubical relations.

Next, for $n = 1, 2$, the pair (G_n, G_{n-1}) has n groupoid structures each with objects G_{n-1} and arrows G_n. The groupoid 'in the ith direction' has initial and final maps $\partial_i^0, \partial_i^1 \colon G_n \to G_{n-1}$, and its identity elements are the degenerate elements $\varepsilon_i y$ for $y \in G_{n-1}$. The notation we use for these groupoid structures is as follows. Let $a, b \in G_n$ satisfy $\partial_i^1 a = \partial_i^0 b$. If $n = 1$ (and therefore $i = 1$) the composite of the edges a, b is written ab, and the identity edge $\varepsilon_1 y$ ($y \in G_0$) is written e_y, or e. If $n = 2$ and $i = 1$, the composite of the squares a and b is written $a \circ b$, with identity squares $1_y = \varepsilon_1 y$ ($y \in G_1$); we refer to this as 'vertical composition' of squares. If $n = 2$ and $i = 2$, the composite of a and b is written $a+b$, with identities $0_y = \varepsilon_2 y$ ($y \in G_1$); this is 'horizontal composition' of squares. If $a \in G_1$, the inverse of a is written a^{-1}, while if $a \in G_2$, its inverse with respect to $\circ$ and $+$ are written a^{-1} and $-a$ respectively. We write $\odot_y$ for the doubly degenerate square $1_{e_y} = 0_{e_y}$ ($y \in G_0$). We require also that the face maps $G_2 \to G_1$ and the degeneracy maps $G_1 \to G_2$ are morphisms of groupoids in the following sense:

(i) if $a+b$ is defined then $\partial_1^\alpha(a+b) = (\partial_1^\alpha a)(\partial_1^\alpha b)$;

(ii) if $a \circ b$ is defined then $\partial_2^\alpha(a \circ b) = (\partial_2^\alpha a)(\partial_2^\alpha b)$;

(iii) if ab is defined then $0_{ab} = 0_a \circ 0_b$ and $1_{ab} = 1_a + 1_b$.

The vertical and horizontal compositions of squares are related by the *interchange law*, namely, that if $a, b, c, d \in G_2$ then

$$(a+b)\circ(c+d) = (a\circ c)+(b\circ d)$$

whenever both sides are defined. It is convenient to use matrix notation for composition of squares. If $a \in G_2$, a *subdivision* of a is defined to be a rectangular array (a_{ij}) $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)$ of elements of G_2 satisfying

$$\begin{cases} \partial_1^1 a_{i-1,j} = \partial_1^0 a_{i,j} & (2 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n), \\ \partial_2^1 a_{i,j-1} = \partial_2^0 a_{i,j} & (1 \leqslant i \leqslant m,\ 2 \leqslant j \leqslant n), \end{cases}$$

such that

$$(a_{11}+a_{12}+\ldots+a_{1n})\circ(a_{21}+a_{22}+\ldots+a_{2n})\circ\ldots\circ(a_{m1}+a_{m2}+\ldots+a_{mn}) = a.$$

We call a the *composite* of the array (a_{ij}) and write $a = [a_{ij}]$. The interchange law implies that if in the array (a_{ij}) we partition the rows and columns into blocks B_{kl} and compute the composite b_{kl} of each block, then $a = [b_{kl}]$. We call the subdivision (a_{ij}) a *refinement* of (b_{kl}) in this case. Note that $a \circ b$, $a+c$ can also be written $\begin{bmatrix} a \\ b \end{bmatrix}$, $[a, c]$, and that the two sides of the interchange law can be written $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

There is one further element of structure on G, namely a *connection* $\Gamma \colon G_1 \to G_2$ which assigns to each edge $p \in G_1$ a square $\Gamma(p)$ whose edges are $\partial_1^0 \Gamma(p) = \partial_2^0 \Gamma(p) = p$ and $\partial_1^1 \Gamma(p) = \partial_2^1 \Gamma(p) = e_y$, where $y = \partial_1^1 p$. This Γ

satisfies the *transport law*: if pq is defined in G_1 then

$$\Gamma(pq) = \begin{bmatrix} \Gamma(p) & 1_q \\ 0_q & \Gamma(q) \end{bmatrix}. \tag{1}$$

We now define a *thin square* in G to be any element t of G_2 having a subdivision (t_{ij}) in which each t_{ij} is of the form 0_p, 1_p, $\Gamma(p)$, $-\Gamma(p)$, $\Gamma(p)^{-1}$, or $-\Gamma(p)^{-1}$ for some $p = p_{ij}$ in G_1. A square a of G_2 is said to have *commuting boundary* if $(\partial_2^0 a)(\partial_1^1 a) = (\partial_1^0 a)(\partial_2^1 a)$. Since 0_p, 1_p, $\Gamma(p)$ all have commuting boundary, so also does any thin square.

PROPOSITION 1. *Let G be a double groupoid and let $p, q, r, s \in G_1$ satisfy $pq = rs$. Then there is a unique thin square $\Theta \in G_2$ such that $\partial_2^0\Theta = p$, $\partial_1^1\Theta = q$, $\partial_1^0\Theta = r$, and $\partial_2^1\Theta = s$.*

Proof. For any $p, q, r, s \in G_1$ satisfying $pq = rs$ define

$$\Theta = \Theta\begin{pmatrix} & r & \\ p & & s \\ & q & \end{pmatrix} = \Gamma(p) + 1_q - \Gamma(s).$$

Then Θ is thin and, since $pq = rs$, its edges are as stated in the proposition. This Θ satisfies the following laws:

(i) $\Theta\begin{pmatrix} & p & \\ e & & e \\ & p & \end{pmatrix} = 1_p, \quad \Theta\begin{pmatrix} & e & \\ p & & p \\ & e & \end{pmatrix} = 0_p, \quad \Theta\begin{pmatrix} & p & \\ p & & e \\ & e & \end{pmatrix} = \Gamma(p);$

(ii) $\Theta\begin{pmatrix} & r & \\ p & & s \\ & q & \end{pmatrix} + \Theta\begin{pmatrix} & u & \\ s & & v \\ & t & \end{pmatrix} = \Theta\begin{pmatrix} & ru & \\ p & & v \\ & qt & \end{pmatrix};$

(iii) $\Theta\begin{pmatrix} & r & \\ p & & s \\ & q & \end{pmatrix} \circ \Theta\begin{pmatrix} & q & \\ t & & v \\ & u & \end{pmatrix} = \Theta\begin{pmatrix} & r & \\ pt & & sv \\ & u & \end{pmatrix};$

(iv) $-\Theta\begin{pmatrix} & r & \\ p & & s \\ & q & \end{pmatrix} = \Theta\begin{pmatrix} & r^{-1} & \\ s & & p \\ & q^{-1} & \end{pmatrix};$

(v) $\Theta\begin{pmatrix} & r & \\ p & & s \\ & q & \end{pmatrix}^{-1} = \Theta\begin{pmatrix} & q & \\ p^{-1} & & s^{-1} \\ & r & \end{pmatrix}.$

The proofs of (i) and (ii) are trivial. To prove (iii) we observe that, since $q = tuv^{-1}$, the two sides of equation (iii) have the common subdivision

$$\begin{pmatrix} \Gamma(p) & 1_t & 1_u & -1_v & -\Gamma(s) \\ 0_t & \Gamma(t) & 1_u & -\Gamma(v) & 0_v \end{pmatrix}.$$

Equation (iv) follows from (i) and (ii), and (v) follows from (i) and (iii).

Equations (ii)–(v) imply that any square $a \in G_2$ having a subdivision (a_{ij}) in which each a_{ij} is of the form Θ, $-\Theta$, Θ^{-1}, or $-\Theta^{-1}$ is itself of the form $\Theta\begin{pmatrix} & r & \\ p & & s \\ & q & \end{pmatrix}$, where p, q, r, s are the edges of a. From this and (i) we deduce that all thin squares are of the form $\Theta\begin{pmatrix} & r & \\ p & & s \\ & q & \end{pmatrix}$ and are therefore uniquely determined by (three of) their edges.

A *morphism* $f\colon G \to H$ of double groupoids is a triple of functions $f_n\colon G_n \to H_n$ $(n = 0, 1, 2)$ preserving all the structures, including the connection.

PROPOSITION 2. *Let G, H be double groupoids and let $f_2\colon G_2 \to H_2$ be a function satisfying*

(i) $f_2(a \circ b) = f_2(a) \circ f_2(b)$ *whenever* $\partial_1^1 a = \partial_1^0 b$,

(ii) $f_2(a + b) = f_2(a) + f_2(b)$ *whenever* $\partial_2^1 a = \partial_2^0 b$, *and*

(iii) f_2 *maps thin squares to thin squares.*

Then there exist unique functions $f_1\colon G_1 \to H_1$, $f_0\colon G_0 \to H_0$ such that (f_2, f_1, f_0) is a morphism $G \to H$ of double groupoids.

Proof. Condition (i) implies that there is a unique function $f_1^1\colon G_1 \to H_1$ such that (f_2, f_1^1) is a morphism of the vertical groupoid structure. This function satisfies $f_1^1(\partial_1^\alpha a) = \partial_1^\alpha f_2(a)$ for all $a \in G_2$. Putting $a = 1_{pq} = 1_p + 1_q$, we deduce that f_1^1 preserves composition of edges and therefore sends identity edges to identity edges. Similarly, by (ii), there is a unique morphism (f_2, f_1^2) of the horizontal groupoid structure. The function $f_1^2\colon G_1 \to H_1$ satisfies $f_1^2(\partial_2^\alpha a) = \partial_2^\alpha f_2(a)$ for all $a \in G_2$ and also sends identity edges to identity edges. Condition (iii) now implies that f_2 sends

$$\Theta\begin{pmatrix} & r & \\ p & & s \\ & q & \end{pmatrix} \quad \text{to} \quad \Theta\begin{pmatrix} & f_1^1 r & \\ f_1^2 p & & f_1^2 s \\ & f_1^1 q & \end{pmatrix},$$

and therefore $(f_1^2 p)(f_1^1 q) = (f_1^1 r)(f_1^2 s)$ whenever $pq = rs$ in G_1. Since both f_1^1 and f_1^2 send identities to identities, this implies that $f_1^1 = f_1^2 = f_1$, say. Hence $f_2\Gamma(p) = f_2\Theta\begin{pmatrix} & p & \\ p & & e \\ & e & \end{pmatrix} = \Gamma(f_1 p)$, and the rest of the proof is routine.

We now recall from [4] the relationship between double groupoids and crossed modules. A *crossed module* (A, B, ∂) consists of groups A, B, a morphism of groups $\partial: A \to B$, and an action of B on A, written $(a, b) \mapsto a^b$ $(a \in A,\ b \in B)$. These must satisfy the laws (i) $\partial(a^b) = b^{-1}(\partial a)b$, and (ii) $a^{-1}a_1 a = a_1^{\partial a}$ for $a, a_1 \in A$, $b \in B$. A *morphism*

$$(f, g) \colon (A, B, \partial) \to (A', B', \partial')$$

of crossed modules is a pair of morphisms $f \colon A \to A'$, $g \colon B \to B'$ of groups such that $g\partial = \partial' f$ and $f(a^b) = f(a)^{g(b)}$ for $a \in A$, $b \in B$.

Given a double groupoid G and a vertex $x \in G_0$ we define groups A, B by

$$A = \{a \in G_2;\ \partial_1^1 a = \partial_2^0 a = \partial_2^1 a = e_x\},$$
$$B = \{p \in G_1;\ \partial_1^0 p = \partial_1^1 p = x\},$$

and a morphism $\partial \colon A \to B$ by $\partial(a) = \partial_1^0 a$. The action of B on A given by $a^b = -1_b + a + 1_b$ makes (A, B, ∂) a crossed module which we denote by $\gamma(G, x)$. If G has only one vertex, we write $\gamma(G)$ for $\gamma(G, x)$. We quote from Theorem A of [**4**]:

THEOREM A. *The rule $G \mapsto \gamma(G)$ defines an equivalence of categories from the category of double groupoids with one vertex to the category of crossed modules.*

2. The homotopy double groupoid of a triple of spaces

Throughout this section $\mathbf{X} = (X, X_1, X_0)$ will be a triple of spaces, so that X_1 is a subspace of X, and X_0 is a subspace of X_1. We shall construct a double groupoid $\rho(\mathbf{X})$ provided that each loop in X_0 is contractible in X_1.

First we construct $R = (R_2, R_1, R_0)$ where $R_0 = X_0$, R_1 is the set of maps $(I, \dot{I}) \to (X_1, X_0)$, and R_2 is the set of maps $(I^2, \dot{I}^2, \ddot{I}^2) \to (X, X_1, X_0)$, where $\dot{I}^2$ is the set of edges and $\ddot{I}^2$ the set of vertices of the square I^2. Then $R = R(\mathbf{X})$ has the structure of a two-dimensional cubical complex.

The set R_1 has its usual composition of paths in X_1 with end points in X_0. The set R_2 has two similar compositions. In more detail, for positive integers m, n let $\varphi_{m,n} \colon I^2 \to [0, m] \times [0, n]$ be the map $(x, y) \mapsto (mx, ny)$. An *$m \times n$-subdivision* of a square $\alpha \colon I^2 \to X$ is a factorization $\alpha = \alpha' \circ \varphi_{m,n}$; its *parts* are the squares $\alpha_{ij} \colon I^2 \to X$ defined by

$$\alpha_{ij}(x, y) = \alpha'(x + i - 1, y + j - 1).$$

We then say that α is the *composite* of the squares α_{ij}, and we write $\alpha = [\alpha_{ij}]$. Similar definitions apply to paths and cubes.

Such a subdivision determines a cell-structure on I^2 as follows. The intervals $[0, m]$, $[0, n]$ have cell-structures with integral points as 0-cells and the intervals $[i, i+1]$ as closed 1-cells. Then $[0, m] \times [0, n]$ has the product cell-structure which is transferred to I^2 by $\varphi_{m,n}^{-1}$. We call the 2-cell $\varphi_{m,n}^{-1}([i-1, i] \times [j-1, j])$ the *domain* of α_{ij}.

We use the same notation for degenerate squares as in § 1. There is also a 'connection' $\Gamma\colon R_1 \to R_2$ given by

$$\Gamma(\sigma)\colon (x,y) \mapsto \begin{cases} \sigma(x) & \text{if } 0 \leqslant y \leqslant x \leqslant 1, \\ \sigma(y) & \text{if } 0 \leqslant x \leqslant y \leqslant 1. \end{cases}$$

Clearly $\partial_1^0\Gamma(\sigma) = \partial_2^0\Gamma(\sigma) = \sigma$ and $\partial_1^1\Gamma(\sigma) = \partial_2^1\Gamma(\sigma) = \varepsilon_0 y$ where $y = \partial_1^1\sigma$. Also Γ satisfies the 'transport law' (1).

If $\alpha \in R_2$, then α^{-1}, $-\alpha$ denote respectively the elements of R_2 defined by $(x,y) \mapsto \alpha(1-x,y), (x,y) \mapsto \alpha(x,1-y)$.

The double groupoid $\rho = (\rho_2, \rho_1, \rho_0)$ is given as cubical complex by $\rho_i = \pi_0 R_i$ $(i = 0, 1, 2)$ where R_1, R_2 are given the compact-open topology. Thus $\rho_0 = \pi_0 X_0$ and the elements of ρ_1, ρ_2 are respectively homotopy classes of maps $(I, \dot{I}) \to (X_1, X_0)$, $(I^2, \dot{I}^2, \ddot{I}^2) \to (X, X_1, X_0)$. We write $\equiv$ for this relation of homotopy on R_1 and R_2, and call it *f-homotopy* (or filter homotopy), to distinguish it from homotopy of maps $I \to X_1$ or $I^2 \to X$ which we write $\simeq$. The class in ρ_i of an element θ of R_i is written $\bar{\theta}$.

PROPOSITION 3. *Assume the following condition:*

(*) *each loop in X_0 is contractible in X_1.*

Then the operations on $R(\mathbf{X})$ induce on $\rho(\mathbf{X})$ the structure of double groupoid.

Proof. Multiplication in ρ_1 is defined as follows. Let $\bar{\sigma}, \bar{\tau} \in \rho_1$ satisfy $\partial_1^1\bar{\sigma} = \partial_1^0\bar{\tau}$. Then we may choose a path λ in X_0 so that $\psi = [\sigma\ \lambda\ \tau]$ is defined and put $\bar{\sigma}\bar{\tau} = \bar{\psi}$. Under the condition (*), this multiplication is well defined and ρ_1 becomes a groupoid.

We next define addition on ρ_2. Let $\bar{\alpha}, \bar{\beta} \in \rho_2$ satisfy $\partial_2^1\bar{\alpha} = \partial_2^0\bar{\beta}$. Then there is a square H in X_1 with $\gamma = [\alpha\ H\ \beta]$ defined and with $\partial_1^0 H, \partial_1^1 H$ paths in X_0. We let $\bar{\alpha} + \bar{\beta} = \bar{\gamma}$ and prove this addition to be well defined.

Let $\gamma' = [\alpha'\ H'\ \beta']$ be alternative choices. Then there exist f-homotopies $h_t\colon \alpha \equiv \alpha'$, $k_t\colon \beta \equiv \beta'$. Let $K\colon I \times \dot{I}^2 \to X_1$ be given by $(x,y,0) \mapsto H(x,y)$, $(x,y,1) \mapsto H'(x,y)$, $(x,0,t) \mapsto h_t(x,1)$, $(x,1,t) \mapsto k_t(x,0)$. Then

$$K(\dot{I} \times \dot{I}^2) \subset X_0.$$

By (*) there is a map $\{0\} \times I^2 \to X_1$ extending K to five faces of I^3. By retracting I^3 onto these five faces we obtain a further extension $K\colon I^3 \to X_1$. The composite cube $[h\ K\ k]$ is an f-homotopy $\gamma \equiv \gamma'$ as required.

It is now easy to see that this addition makes (ρ_2, ρ_1) a groupoid with initial and final maps $\partial_2^0, \partial_2^1$ and identity elements 0_s, where $s \in \rho_1$. A similar procedure gives the other groupoid structure.

The verification of the remaining laws for a double groupoid is straightforward.

PROPOSITION 4. *Let* $\mathbf{X} = (X, X_1, X_0)$ *be a triple satisfying* (*) *above, and let* $\rho = \rho(\mathbf{X})$.

(i) *If* σ *is a path in* X_0 *then* $\bar{\sigma}$ *is an identity* e_z *in* ρ_1.

(ii) *If* α *is a square in* X_0 *then* $\bar{\alpha} = \odot_z$ *in* ρ_2 *for some* z.

(iii) *An element of* ρ_2 *is thin if and only if it has a representative square lying in* X_1.

The proof is straightforward.

The next proposition is one of the keys to our work. It shows that double groupoids allow a convenient expression for the homotopy addition lemma in dimension 2.

If $h: I^3 \to X$ is a cube in X, then its faces are, as usual, given by $\partial_i^\alpha h = h \circ \eta_i^\alpha$, where $\eta_i^\alpha(x_1, x_2) = (y_1, y_2, y_3)$, the y_j being defined by $y_j = x_j$ for $j < i$, $y_i = \alpha$, and $y_j = x_{j-1}$ for $j > i$. Also let $\tilde{\eta}_1^\alpha(x_1, x_2) = (\alpha, x_2, x_1)$.

PROPOSITION 5 (the homotopy addition lemma). *Let* $\mathbf{X}, \rho$ *be as in Proposition* 4. *Let* h *be a cube in* X *with edges in* X_1 *and vertices in* X_0, *and let the elements* $a_\alpha, b_\alpha, c_\alpha$ *of* ρ_2 *represented by its faces be respectively the classes of* $h \circ \tilde{\eta}_1^\alpha$, $h \circ \eta_2^\alpha$, $h \circ \eta_3^\alpha$ $(\alpha = 0, 1)$. *Then in* ρ_2

$$c_1 = \begin{bmatrix} -\Gamma^{-1} & a_0^{-1} & \Gamma^{-1} \\ -b_0 & c_0 & b_1 \\ -\Gamma & a_1 & \Gamma \end{bmatrix},$$

where each Γ *stands for* $\Gamma(p)$ *for an appropriate edge* p.

Proof. Consider the maps $\varphi_0, \varphi_1: I^2 \to I^3$ defined by

$$\varphi_0 = \begin{bmatrix} -\Gamma^{-1} & (\tilde{\eta}_1^0)^{-1} & \Gamma^{-1} \\ -\eta_2^0 & \eta_3^0 & \eta_2^1 \\ -\Gamma & \tilde{\eta}_1^1 & \Gamma \end{bmatrix}, \quad \varphi_1 = \begin{bmatrix} -\Gamma^{-1} & 1 & \Gamma^{-1} \\ 0 & \eta_3^1 & 0 \\ -\Gamma & 1 & \Gamma \end{bmatrix}.$$

Then φ_0, φ_1 agree on $\dot{I}^2$ and so, since I^3 is convex, are homotopic rel $\dot{I}^2$. Hence $\overline{h \circ \varphi_0} = \overline{h \circ \varphi_1}$ in ρ_2. But $\overline{h \circ \varphi_0}$ is the composite matrix given in the proposition, and $\overline{h \circ \varphi_1} = c_1$.

A map $f: \mathbf{X} \to \mathbf{Y}$ of triples clearly defines a map $\rho(f): \rho(\mathbf{X}) \to \rho(\mathbf{Y})$ of cubical complexes, and $\rho(f)$ is a morphism of double groupoids if $\mathbf{X}, \mathbf{Y}$ satisfy (*) of Proposition 3.

PROPOSITION 6. *If* $f: \mathbf{X} \to \mathbf{Y}$ *is a map of triples such that each of* $f: X \to Y$, $f_1: X_1 \to Y_1$, $f_0: X_0 \to Y_0$ *are homotopy equivalences, then* $\rho(f): \rho(\mathbf{X}) \to \rho(\mathbf{Y})$ *is an isomorphism.*

Proof. This is an immediate consequence of (10.11) of [**9**]. (In fact the maps $R_i(\mathbf{X}) \to R_i(\mathbf{Y})$ are then homotopy equivalences, as is not hard to deduce for $i = 1, 2$ from the coglueing theorem of [**3**].)

From the homotopy double groupoid $\rho(\mathbf{X})$ we obtain, according to the procedure of § 1, a crossed module $\gamma(\rho(\mathbf{X}), \bar{\zeta})$ for each $\bar{\zeta} \in \pi_0 X_0$. It is well known that, for each $\zeta \in X_0$, the homotopy boundary

$$\partial\colon \pi_2(X, X_1, \zeta) \to \pi_1(X_1, \zeta)$$

and the operation of $\pi_1(X_1, \zeta)$ on $\pi_2(X, X_1, \zeta)$ give a crossed module, which we write $\mu(X, X_1, \zeta)$, or $\mu(X, X_1)$ if the base point is clear.

PROPOSITION 7. *For any* $\zeta \in X_0$, *the crossed modules* $\gamma(\rho(\mathbf{X}), \bar{\zeta})$ *and* $\mu(X, X_1, \zeta)$ *are naturally isomorphic.*

Proof. Let (A, B, ∂) be the crossed module of $\rho(\mathbf{X})$ at $\bar{\zeta}$. It is easy to check (using (*) of Proposition 3 again) that B is naturally isomorphic to $\pi_1(X_1, \zeta)$. Now the elements of $\pi_2(X, X_1, \zeta)$ are homotopy classes of maps $(I^2, \{0\} \times I, J^2) \to (X, X_1, \zeta)$ where $J^2 = (\{1\} \times I) \cup (I \times \dot{I})$. Clearly each such map determines an element of $R_2(\mathbf{X})$ and so by passing to homotopy classes we obtain a morphism $\theta\colon \pi_2(X, X_1, \zeta) \to A$. We omit the proof that θ is an isomorphism and commutes with the operations.

3. The union theorem

In this section we write $\mathbf{X}$ for the triple (X, X_1, X_0) of spaces and we assume the condition

$(*)_{\mathbf{X}}$ each loop in X_0 is contractible in X_1.

We suppose we are given a cover $\mathscr{U} = \{U^\lambda\}_{\lambda \in \Lambda}$ of X such that the interiors of the sets of $\mathscr{U}$ cover X. For each $\nu \in \Lambda^n$ we write

$$U^\nu = U^{\nu_1} \cap \ldots \cap U^{\nu_n}$$

and we also set $U^\nu_i = U^\nu \cap X_i$. We write $\mathbf{U}^\nu = (U^\nu, U^\nu_1, U^\nu_0)$ and shall assume that for all $\nu \in \Lambda^2$ each loop in U^ν_0 is contractible in U^ν_1. This, with $(*)_{\mathbf{X}}$, implies that the homotopy double groupoids in the following *ρ-sequence of the cover* are well-defined:

$$\coprod_{\nu \in \Lambda^2} \rho(\mathbf{U}^\nu) \underset{b}{\overset{a}{\rightrightarrows}} \coprod_{\lambda \in \Lambda} \rho(\mathbf{U}^\lambda) \xrightarrow{c} \rho(\mathbf{X}).$$

Here $\coprod$ denotes coproduct in the category of double groupoids, a, b are determined by the inclusions

$$a_\nu\colon U^\lambda \cap U^\mu \to U^\lambda, \quad b_\nu\colon U^\lambda \cap U^\mu \to U^\mu$$

for each $\nu = (\lambda, \mu) \in \Lambda^2$, and c is determined by the inclusion $c_\lambda : U^\lambda \to X$ for each $\lambda \in \Lambda$.

THEOREM B (the union theorem). *Assume the following conditions for every finite intersection U^ν of elements of $\mathscr{U}$:*

$(\ddagger)_0$ *the maps $\pi_0(U_0^\nu) \to \pi_0(U_1^\nu)$ and $\pi_0(U_0^\nu) \to \pi_0(U^\nu)$ are surjective;*

$(\ddagger)_1$ *the map $\pi_1(U_1^\nu, U_0^\nu) \to \pi_1(U^\nu, U_0^\nu)$ is surjective.*

Then, in the above ρ-sequence of the cover, c is the coequaliser of a, b in the category of double groupoids.

Proof. Suppose we are given a morphism

$$f' : \coprod_{\lambda \in \Lambda} \rho(\mathbf{U}^\lambda) \to G$$

of double groupoids such that $f' \circ a = f' \circ b$. We have to show that there is a unique morphism $f : \rho(\mathbf{X}) \to G$ of double groupoids such that $f \circ c = f'$.

Let $p_\lambda : R(\mathbf{U}^\lambda) \to \rho(\mathbf{U}^\lambda)$ be the projection and let $F_\lambda = f' \circ p_\lambda : R(\mathbf{U}^\lambda) \to G$. We first define f on $\rho_2(\mathbf{X})$ and to this end first construct $F : R_2(\mathbf{X}) \to G$.

Suppose that θ in $R_2(\mathbf{X})$ is such that θ lies in some set U^λ of $\mathscr{U}$. Then θ determines uniquely an element θ^λ of $R_2(\mathbf{U}^\lambda)$ and the rule $f' \circ a = f' \circ b$ implies that

$$F(\theta) = F_\lambda(\theta^\lambda)$$

is determined by θ.

Suppose we are given a subdivision $\theta = [\theta_{ij}]$ of an element θ in $R_2(\mathbf{X})$ such that each θ_{ij} is in $R_2(\mathbf{X})$ and also lies in some U^ν, for $\nu \in \Lambda^n$. Then θ_{ij} also lies in some U^λ, with $\lambda \in \Lambda$, and since the composite $[\theta_{ij}]$ is defined it is easy to check, again using $f' \circ a = f' \circ b$, that the elements $F(\theta_{ij})$ compose in G to give an element $g = [F(\theta_{ij})]$, which we write as $F(\theta)$ although a priori it depends on the subdivision chosen.

We next wish to construct $F(\alpha)$ for an arbitrary element α of $R_2(\mathbf{X})$. This construction is based on the following result.

LEMMA 1. *Let $\alpha \in R_2(\mathbf{X})$ and let $\alpha = [\alpha_{ij}]$ be a subdivision of α such that each α_{ij} lies in some U^{ij}, a finite intersection of elements of $\mathscr{U}$. Then there is an f-homotopy $h : \alpha \equiv \theta$, with $\theta \in R_2(\mathbf{X})$, such that, in the subdivision $h = [h_{ij}]$ determined by that of α, each homotopy $h_{ij} : \alpha_{ij} \simeq \theta_{ij}$ satisfies:*

(i) *h_{ij} lies in U^{ij};*

(ii) *θ_{ij} belongs to $R_2(\mathbf{X})$;*

(iii) *if α_{ij} lies in X_1 or in X_0, so also does θ_{ij}.*

Proof. Let K be the cell-structure on I^2 determined by the subdivision $\alpha = [\alpha_{ij}]$. Let $L_m = K^m \times I \cup K \times \{0\}$ and $X_2 = X$. We construct maps $h_m : L_m \to X_2$, for $m = 0, 1, 2$, such that h_m extends h_{m-1}, where $h_{-1} = \alpha$.

Further we construct h_m to satisfy the following conditions, for each m-cell σ of K:

(a_m) $h_m | \sigma \times \{1\}$ is an element of $R_m(\mathbf{X})$;

(b_m) if α maps σ into X_r, then $h_m(\sigma \times I) \subset X_r$;

(c_m) if σ is contained in the domain of α_{ij}, then $h_m(\sigma \times I) \subset U^{ij}$.

The construction of h_m from h_{m-1} is as follows. We consider an m-cell σ of K, and let r be the smallest integer such that α maps σ into X_r. If $r \leqslant m$, then h_{m-1} can be extended to h_m on $\sigma \times I$ by means of a retraction $\sigma \times I \to \sigma \times \{0\} \cup \dot{\sigma} \times I$. If $r > m$, let U^σ be the intersection of all the sets U^{ij} such that σ is contained in the domain of α_{ij}. The restriction of h_{m-1} to the pair $(\sigma \times \{0\} \cup \dot{\sigma} \times I, \dot{\sigma} \times \{1\})$ determines an element of $\pi_m(U^\sigma_r, U^\sigma_{m-1})$. (Here $m \leqslant 1$ and U^σ_{-1} is taken to be $\varnothing$.) By $(\ddagger)_m$, h_{m-1} extends to h_m on $\sigma \times I$ mapping into U^σ_r and such that $\sigma \times \{1\}$ is mapped into U^σ_m.

COROLLARY. *Let $\alpha \in R_2(\mathbf{X})$. Then there is an f-homotopy $h\colon \alpha \equiv \theta$ such that $F(\theta)$ is defined in G_2.*

Proof. Choose a subdivision $\alpha = [\alpha_{ij}]$ such that each α_{ij} lies in some set U^{ij} of $\mathscr{U}$. Then apply Lemma 1.

This element $F(\theta)$ of the corollary we write $F(\alpha, (h_{ij}))$ and prove first that it depends only on α. Accordingly, let $h'\colon \alpha \equiv \theta'$ be an alternative f-homotopy satisfying the conditions of Lemma 1 with respect to a subdivision $\alpha = [\alpha'_{kl}]$ in which each α'_{kl} lies in some set V^{kl} of $\mathscr{U}$. Since any two subdivisions have a common refinement we may assume, without loss of generality, that $[\alpha'_{kl}]$ is a refinement of $[\alpha_{ij}]$.

For each (kl), let $W^{kl} = V^{kl} \cap U^{ij}$ where U^{ij} is such that α'_{kl} is a part of α_{ij}. By Lemma 1 there is an f-homotopy $h^\dagger = [h^\dagger_{kl}]$ from α to $\theta^\dagger$ such that each $h^\dagger_{kl}$ lies in W^{kl}. The f-homotopy $H = \bar{h}' h^\dagger\colon \theta' \equiv \theta^\dagger$ (where $\bar{h}'$ is the reverse of h') has the subdivision $H = [H_{kl}]$ where $H_{kl}\colon \theta'_{kl} \simeq \theta^\dagger_{kl}$ and H_{kl} lies in V^{kl}.

Let θ^*_{ij} be the composite of those $\theta^\dagger_{kl}$ such that α'_{kl} is a part of α_{ij}. Then we also have a subdivision $h^\dagger = [h^*_{ij}]$, where $h^*_{ij}\colon \alpha_{ij} \simeq \theta^*_{ij}$ lies in U_{ij}. So $H^* = \bar{h}^\dagger h\colon \theta^\dagger \equiv \theta$ is an f-homotopy with subdivision $H^* = [H^*_{ij}]$, where $H^*_{ij}\colon \theta^*_{ij} \simeq \theta_{ij}$ is a homotopy lying in U^{ij}.

It will follow from Lemma 3 below that

$$[F(\theta'_{kl})] = [F(\theta^\dagger_{kl})] \quad \text{and} \quad [F(\theta^*_{ij})] = [F(\theta_{ij})].$$

However $[F(\theta^*_{ij})] = [F(\theta^\dagger_{kl})]$, since the latter is a refinement of the former. Hence $[F(\theta'_{kl})] = [F(\theta_{ij})]$ and so $F(\alpha, (h_{ij}))$ depends only on α.

LEMMA 2. *Let $\theta, \theta^* \in R_2$ and suppose we are given an f-homotopy $H\colon \theta \equiv \theta^*$. Let $H = [H_{ij}]$ be a subdivision such that each H_{ij} lies in some set U^{ij} of $\mathscr{U}$. Let $\theta = [\theta_{ij}]$, $\theta^* = [\theta^*_{ij}]$ be the subdivisions of θ, θ^* induced by that* *of H, and suppose that $\theta_{ij}, \theta^*_{ij}$ are in R_2 for all (i, j). Then H is homotopic rel end maps to an f-homotopy $\hat{H}: \theta \equiv \theta^*$ such that for all i, j,*

(i) *$\hat{H}_{ij}$ has its edges in X_1,*

(ii) *$\hat{H}_{ij}$ lies in U^{ij}.*

Proof. The proof is similar to that of Lemma 1. The subdivision $\theta = [\theta_{ij}]$ induces a cell-structure K on I^2, and the homotopy $H \simeq \hat{H}$ is constructed on $K^m \times I \times I \cup K \times \dot{I} \times I \cup K \times I \times \{0\}$ by induction on m.

NOTE. We do not claim that $\hat{H}_{ij}$ is an f-homotopy $\theta_{ij} \equiv \theta^*_{ij}$.

LEMMA 3. *Let $\theta, \theta^*, H, (H_{ij})$ be as in Lemma 2. Then in G_2,*

$$[F(\theta_{ij})] = [F(\theta^*_{ij})].$$

Proof. We replace H by the $\hat{H}: \theta \equiv \theta^*$ given by Lemma 2. Let $F(\theta_{ij}) = c_{ij}$, $F(\theta^*_{ij}) = c^*_{ij}$. Since $\hat{H}_{ij}$ has its edges in X_1 and vertices in X_0, the homotopy addition lemma (Proposition 5) gives, on applying F, a relation in G_2 of the form

$$c^*_{ij} = \begin{bmatrix} -\Gamma^{-1} & a^{-1}_{i-1,j} & \Gamma^{-1} \\ -b_{i,j-1} & c_{ij} & b_{ij} \\ -\Gamma & a_{ij} & \Gamma \end{bmatrix}, \tag{2}$$

where the a's and b's are images in G_2 of certain faces of the $\hat{H}_{ij}$.

The interchange law for G allows us to refine the subdivision $c^* = [c^*_{ij}]$ by the substitution (2) and to compose the parts in any convenient fashion. By cancellation of pairs b_{ij}, $-b_{ij}$ and $a_{ij}, a_{ij}{}^{-1}$, by composing thin elements and absorbing 0's and 1's, and by composing border elements, we can obtain a new subdivision of c^* of the form

$$c^* = \begin{bmatrix} -\Gamma^{-1} & a_0{}^{-1} & \Gamma^{-1} \\ -b_0 & c & b_1 \\ -\Gamma & a_1 & \Gamma \end{bmatrix}, \tag{3}$$

where $c = [c_{ij}]$ and the elements a_i, b_i are composites in G_2 of the images of squares lying on the boundary of $\hat{H}$. Since $\hat{H}$ is an f-homotopy, these squares are in X_1 and so, by Proposition 4(iii), the a_i, b_i are thin. Similarly, Proposition 4(i) implies that each corner element in (3) is $\odot$. It now follows that the a_i are 1's and the b_i are 0's, and therefore $c^* = c$.

With the proof of Lemma 3 we have completed the proof that $F(\alpha, (h_{ij}))$ depends only on α.

LEMMA 4. *$F(\alpha, (h_{ij}))$ depends only on the class of α in ρ_2.*

Proof. Let $K\colon \alpha \equiv \alpha'$ be an f-homotopy. Then there is an $(m \times n \times p)$-subdivision $K = [K_{ijk}]$ such that each K_{ijk} lies in some set of $\mathscr{U}$, say U^{ijk}. Let $\alpha = [\alpha_{ij}]$, $\alpha' = [\alpha'_{ij}]$ be the induced subdivisions of α, α'. A simple induction on p reduces us to the case where $p = 1$, and so we may assume that the subdivision of K has a single layer $K = [K_{ij}]$, each K_{ij} being a homotopy $\alpha_{ij} \simeq \alpha'_{ij}$ lying in U^{ij}. Then we choose $h\colon \alpha \equiv \theta$, $h'\colon \alpha' \equiv \theta'$ as in Lemma 1. Let H be the composite homotopy $\bar{h}kh'\colon \theta \equiv \theta'$. Then by Lemma 3, $[F(\theta_{ij})] = [F(\theta'_{ij})]$.

We have now proved that there is a well-defined map $f\colon \rho_2(\mathbf{X}) \to G_2$, given by $f(\tilde{\alpha}) = F(\alpha, (h_{ij}))$, which satisfies $f \circ c = f'$ at least on 2-dimensional elements of ρ.

The remainder of the proof is straightforward. It is easy to check that f preserves addition and composition of squares, and it follows from Proposition 4 of § 2 and (iii) of Lemma 1 that f preserves thin elements. Now Proposition 2 is used to extend f to a morphism $f\colon \rho(\mathbf{X}) \to G$ of double groupoids, and clearly f satisfies $f \circ c = f'$ and is the only such morphism.

REMARKS. 1. An examination of the above proof shows that condition $(\ddagger)_m$ is required only for 8-fold intersections of elements of $\mathscr{U}$. However, it has been shown by Razak [**13**] that in fact one need only assume $(\ddagger)_0$ for 4-fold intersections and $(\ddagger)_1$ for 3-fold intersections. Further, these conditions are best possible.

2. There is, alternative to $\rho(X, X_1, X_2)$ as defined here, a version in which the homotopies of maps $(I, \dot{I}) \to (X_1, X_0)$, $(I^2, \dot{I}^2, \ddot{I}^2) \to (X, X_1, X_0)$ are taken rel $\dot{I}$, rel $\ddot{I}^2$ respectively. It is this version which includes the groupoid πYZ of [**1, 2**]. Both versions are special cases of the double groupoid $\rho(X \xleftarrow{p} Y \xleftarrow{q} Z)$ discussed in [**13**].

3. Theorem B contains 1-dimensional information which includes most known results expressing the fundamental group of a space in terms of an open cover, but it does not assume that the spaces of the cover or their intersections are path-connected.

Of especial interest (but not essentially easier to prove) is the case of Theorem B in which the cover $\mathscr{U}$ has only two elements; in this case Theorem B gives a push-out of double groupoids. In the applications below we shall consider only path-connected spaces and assume that $Z = \{\zeta\}$ is a singleton. Taking ζ as base point, the double groupoids can then be interpreted as crossed modules to give the following 2-dimensional analogue of the Seifert–van Kampen theorem. We do not know how to prove Theorem C without using double groupoids.

THEOREM C. *Suppose that the commutative diagram of based pairs of spaces*

$$\begin{array}{ccc} (W, W_1) & \xrightarrow{f} & (U, U_1) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \bar{\imath}} \\ (V, V_1) & \xrightarrow[\bar{f}]{} & (X, X_1) \end{array} \tag{4}$$

satisfies one of the two following hypotheses:

HYPOTHESIS $\mathscr{A}$: *the maps* $i, f, \bar{\imath}, \bar{f}$ *are inclusions of subspaces,* $W = U \cap V$, X *is the union of the interiors of the sets* U *and* V, *and* $V_1 = X_1 \cap V$, $U_1 = X_1 \cap U$, $W_1 = X_1 \cap W$;

HYPOTHESIS $\mathscr{B}$: *the maps* $i\colon W \to V$, $i_1\colon W_1 \to V_1$ *are closed cofibrations,* $W_1 = W \cap V_1$, *and* X, X_1 *are the adjunction spaces* $U \cup_f V$, $U_1 \cup_{f_1} V_1$.

Suppose also that all the spaces are path-connected and that the induced maps $\pi_1(V_1) \to \pi_1(V)$, $\pi_1(U_1) \to \pi_1(U)$, $\pi_1(W_1) \to \pi_1(W)$ *are surjective. Then the induced diagram*

$$\begin{array}{ccc} \mu(W, W_1) & \longrightarrow & \mu(U, U_1) \\ \downarrow & & \downarrow \\ \mu(V, V_1) & \longrightarrow & \mu(X, X_1) \end{array} \tag{5}$$

is a push-out of crossed modules.

Proof. In the case where (X, X_1) is a based pair with base point ζ, $\rho(X, X_1, \zeta)$ is abbreviated to $\rho(X, X_1)$. That ρ applied to diagram (4) gives a push-out of double groupoids under Hypothesis $\mathscr{A}$ is simply a special case of the union theorem. That diagram (5) is a push-out is immediate from Theorem A and Proposition 7.

The corresponding result under Hypothesis $\mathscr{B}$ follows from that under Hypothesis $\mathscr{A}$ by standard techniques using mapping cylinders (see a similar proof in [**2**, 8.4.2]).

4. Push-outs of crossed modules

The usefulness of Theorem C lies in the fact that, given a push-out square

$$\begin{array}{ccc} (A_0, G_0, \partial_0) & \longrightarrow & (A_1, G_1, \partial_1) \\ \downarrow & & \downarrow \\ (A_2, G_2, \partial_0) & \longrightarrow & (A, G, \partial) \end{array} \tag{6}$$

in the category $\mathscr{C}$ of crossed modules, one can write down generators and relations for the groups A and G if one knows generators and relations for the groups A_i, G_i $(i = 0, 1, 2)$ and one knows the various actions and maps between them. The computations are conveniently described in terms of *induced crossed modules*.

For a given group G, let $\mathscr{C}_G$ be the category of crossed G-modules (A, G, ∂); the morphisms of $\mathscr{C}_G$ are morphisms of crossed modules inducing the identity on G. As for ordinary modules we shall often refer simply to the crossed G-module A. Let $\lambda\colon G \to H$ be a fixed morphism of groups. If B is a crossed H-module, let A be the pull-back

$$\begin{array}{ccc} A & \xrightarrow{\beta} & B \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial} \\ G & \xrightarrow[\lambda]{} & H \end{array}$$

in the category of groups. Then G acts on $A \subset B \times G$ by the rule $a^g = ((\beta a)^{\lambda g}, g^{-1}(\partial a)g)$ for $g \in G$, $a \in A$, making A into a crossed G-module so that $(\beta, \lambda)\colon (A, G, \partial) \to (B, H, \partial)$ is a morphism of crossed modules. This morphism is universal for morphisms from crossed G-modules to (B, H, ∂) which induce $\lambda\colon G \to H$. Writing $A = \lambda^* B$ we obtain a functor $\lambda^*\colon \mathscr{C}_H \to \mathscr{C}_G$ called *restriction*.

There is also, for any crossed G-module A, an *induced* crossed H-module $C = \lambda_* A$ and a morphism $(\nu, \lambda)\colon (A, G, \partial) \to (C, H, \partial)$ which is universal for morphisms from A to crossed H-modules which induce $\lambda\colon G \to H$. This gives a functor $\lambda_*\colon \mathscr{C}_G \to \mathscr{C}_H$ which is left-adjoint to λ^*. It can be described as follows.

PROPOSITION 8. *Let A be a crossed G-module and let $\lambda\colon G \to H$ be a morphism of groups. Then the induced crossed H-module $C = \lambda_* A$ is generated, as a group, by the set $A \times H$ with defining relations*

(i) $(a_1, h)(a_2, h) = (a_1 a_2, h)$,

(ii) $(a^g, h) = (a, (\lambda g)h)$,

(iii) $(a_1, h_1)^{-1}(a_2, h_2)(a_1, h_1) = (a_2, h_2 h_1^{-1}(\lambda\partial a_1)h_1)$,

for $a_1, a_2, a \in A$, $h_1, h_2, h \in H$, $g \in G$. The morphism $\partial\colon C \to H$ is given by $\partial(a, h) = h^{-1}(\lambda\partial a)h$, the action of H on C by $(a, h_1)^h = (a, h_1 h)$, and the canonical morphism $\nu\colon A \to C$ by $\nu(a) = (a, 1)$.

Proof. One verifies directly that this recipe defines a crossed H-module and that $(\nu, \lambda)\colon (A, G, \partial) \to (C, H, \partial)$ is a morphism of crossed modules with the required universal property.

Some special cases are worthy of note. First, G is itself a crossed G-module with $\partial\colon G \to G$ the identity and action by conjugation; it is the terminal object of $\mathscr{C}_G$. The corresponding induced crossed H-module $\lambda_* G$ is called the *free crossed module* on $\lambda\colon G \to H$. In this case the relations (ii) are consequences of (i) and (iii), so $\lambda_* G$ has generators $G \times H$ and defining relations $(g_1, h)(g_2, h) = (g_1 g_2, h)$ and

$$(g_1, h_1)^{-1}(g_2, h_2)(g_1, h_1) = (g_2, h_2 h_1^{-1}(\lambda g_1) h_1).$$

If G is a free group with free generators $\{x_i\}$ then $\lambda_* G$ is determined by H and the elements $y_i = \lambda(x_i)$ of H, and it coincides with Whitehead's 'free crossed H-module' [**16**, p. 455], as can be seen by comparing the two presentations. We refer to the elements $\nu(x_i) \in \lambda_* G$ as the *free generators* of this crossed module.

Next, when $\lambda\colon G \to H$ is a surjection or an injection the induced crossed module $\lambda_* A$ has a simpler description which can be either deduced from Proposition 8 or proved in a similar way.

PROPOSITION 9. *If $\lambda\colon G \to H$ is a surjection and A is a crossed G-module, then $\lambda_* A = A/[A, K]$, where $K = \operatorname{Ker}\lambda$ and $[A, K]$ denotes the subgroup of A generated by all $a^{-1}a^k$ for $a \in A$, $k \in K$.*

PROPOSITION 10. *If $\lambda\colon G \to H$ is an injection and A is a crossed G-module, let T be a right transversal of $\lambda(G)$ in H, and let B be the free product of groups A_t $(t \in T)$ each isomorphic with A by an isomorphism $a \mapsto a_t$ $(a \in A)$. Let $h \in H$ act on B by the rule $(a_t)^h = (a^g)_u$, where $g \in G$, $u \in T$, and $th = (\lambda g)u$. Let $\delta\colon B \to H$ be defined by $a_t \mapsto t^{-1}(\lambda\, \partial a)t$. Then $\lambda_* A = B/S$ where S is the normal closure in B of the elements $b^{-1}c^{-1}bc^{\delta b}$ $(b, c \in B)$.*

REMARK. Since any $\lambda\colon G \to H$ is the composite of a surjection and an injection, an alternative description of the general $\lambda_* A$ can be obtained by a combination of the two constructions of Propositions 9 and 10.

Now consider an arbitrary push-out square (6) of crossed modules. In order to describe (A, G, ∂), we first note that G is the push-out of the group morphisms $G_1 \leftarrow G_0 \to G_2$. (This is because the forgetful functor $(A, G, \partial) \mapsto G$ from crossed modules to groups has a right adjoint $G \mapsto (G, G, \mathrm{id})$.) The morphisms $\lambda_i\colon G_i \to G$ $(i = 0, 1, 2)$ in (6) can be used to form induced crossed G-modules $B_i = (\lambda_i)_* A_i$. Clearly A is the push-out in $\mathscr{C}_G$ of the resulting G-morphisms $B_1 \leftarrow B_0 \to B_2$ and can be described as follows.

PROPOSITION 11. *Let B_i be a crossed G-module for $i = 0, 1, 2$, and let A be the push-out in $\mathscr{C}_G$ of G-morphisms $B_1 \xleftarrow{\beta_1} B_0 \xrightarrow{\beta_2} B_2$. Let B be the* *push-out of β_1 and β_2 in the category of groups, equipped with the induced morphism $\partial: B \to G$ and the induced action of G on B. Then $A = B/S$, where S is the normal closure in B of the elements $b^{-1}c^{-1}bc^{\partial b}$ for $b, c \in B$.*

In the case when (A_2, G_2, ∂_2) is the trivial crossed module $(0, 0, \mathrm{id})$ the push-out (A, G, ∂) in (6) is the *cokernel* of the morphism

$$(A_0, G_0, \partial_0) \to (A_1, G_1, \partial_1).$$

Cokernels can be described as follows.

PROPOSITION 12. *The cokernel of a morphism $(\beta, \lambda): (A, G, \partial) \to (B, H, \partial)$ is $(B/\bar{A}, H/\bar{G}, \partial)$ where $\bar{G}$ is the normal closure in H of $\lambda(G)$, and $\bar{A}$ is the H-subgroup of B generated by $\beta(A).[B, \bar{G}]$.*

5. Applications to second relative homotopy groups

We illustrate the use of Theorem C for determining $\pi_2(X, X_1)$ in some cases in which the computations are straightforward.

PROPOSITION 13. *Let U, V, W, X be connected based CW-complexes, with W a subcomplex of V and X the adjunction space $X = U \cup_f V$, where $f: W \to U$ is a cellular map. Let U^1, V^1, W^1, X^1 denote the 1-skeletons of U, V, W, X. Then*

$$\begin{tikzcd} \mu(W, W^1) \arrow[r, "f_*"] \arrow[d, "i_*"'] & \mu(U, U^1) \arrow[d, "\bar{i}_*"] \\ \mu(V, V^1) \arrow[r, "\bar{f}_*"'] & \mu(X, X^1) \end{tikzcd}$$

is a push-out of crossed modules.

Proof. Under these assumptions, Hypothesis $\mathscr{B}$ of Theorem C is satisfied and the induced maps $\pi_1(U^1) \to \pi_1(U)$ etc. are all surjective.

COROLLARY. *Let W be a connected subcomplex of the connected CW-complex V and let $X = V/W$. Then $\pi_2(X, X^1) = \pi_2(V, V^1)/N$ where N is the $\pi_1(V^1)$-subgroup of $\pi_2(V, V^1)$ generated by*

$$i_*\pi_2(W, W^1) \quad \textit{and} \quad [\pi_2(V, V^1), i_*\pi_1(W^1)].$$

Proof. Take $U = U^1 = \{*\}$ so that $\mu(U, U^1) = (0, 0, \mathrm{id})$ and $\mu(X, X^1)$ is the cokernel of $i_*: \mu(W, W^1) \to \mu(V, V^1)$. Apply Proposition 12.

We observe that Proposition 13 also throws light on a problem suggested to us by Saunders MacLane: if the connected CW-complex X is the union of connected sub-complexes U, V with connected intersection W, *determine the relationship between the first Postnikov invariants of X, U, V, W.* This invariant for X is an element $k(X)$ of $H^3(\pi_1(X), \pi_2(X))$ which is shown in [**11**] to be the obstruction class associated with the crossed module $\mu(X, X^1)$, where X^1 is the 1-skeleton of X. Proposition 13 shows that $k(X)$ is determined, if not by $k(U)$, $k(V)$, and $k(W)$, then certainly by the morphisms of crossed modules $\mu(U, U^1) \leftarrow \mu(W, W^1) \rightarrow \mu(V, V^1)$.

The following special case of Theorem C is particularly convenient, as it contains and extends a number of known results.

THEOREM D. *Suppose that the commutative square*

$$\begin{array}{ccc} W & \xrightarrow{f} & U \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \bar{\imath}} \\ V & \xrightarrow[\bar{f}]{} & X \end{array}$$

of based spaces satisfies one of the two hypotheses:

HYPOTHESIS $\mathscr{A}$: *the maps $i, f, \bar{\imath}, \bar{f}$ are inclusions of subspaces, $W = U \cap V$, and X is the union of the interiors of U and V;*

HYPOTHESIS $\mathscr{B}$: *the map i is a closed cofibration and X is the adjunction space $U \cup_f V$.*

Suppose also that U, V, W are path-connected and that $i_*: \pi_1(W) \rightarrow \pi_1(V)$ is surjective. Then $\pi_2(X, U)$ is the crossed $\pi_1(U)$-module induced from $\pi_2(V, W)$ by the morphism $f_*: \pi_1(W) \rightarrow \pi_1(U)$.

Proof. Under these conditions we may take

$$X_1 = U_1 = U \quad \text{and} \quad V_1 = W_1 = W$$

in Theorem C. Writing $G = \pi_1(W)$, $H = \pi_1(V)$, $A = \pi_2(V, W)$, and $B = \pi_2(X, U)$ we find that

$$\begin{array}{ccc} (0, G, 0) & \xrightarrow{(0,\lambda)} & (0, H, 0) \\ \downarrow & & \downarrow \\ (A, G, \partial) & \xrightarrow{(\beta,\lambda)} & (B, H, \partial) \end{array}$$

is a push-out of crossed modules, and this is equivalent to the assertion that B is the induced module $\lambda_* A$.

REMARKS. 1. Induced crossed modules arise under more general circumstances. In Theorem C, make the additional assumptions that

$\pi_2(W, W_1) = \pi_2(V, V_1) = 0$; then $\pi_2(X, X_1)$ is the crossed $\pi_1(X_1)$-module induced from $\pi_2(V, V_1)$ by $\bar{f}_*: \pi_1(V_1) \to \pi_1(X_1)$.

2. Theorem D implies the homotopy excision theorem [9, p. 211] in dimension 2. For suppose the based space X is the union of subspaces U, V, with U, V, and $W = U \cap V$ all path-connected. Assume either Hypothesis $\mathscr{A}$ or Hypothesis $\mathscr{B}'$: *U and V are closed and $i: W \to V$ is a cofibration.* Let $\lambda: \pi_1(W) \to \pi_1(U)$ be induced by inclusion. If $\pi_1(V, W) = 0$, then $\pi_1(W) \to \pi_1(V)$ is surjective, and by Theorem D, $\pi_2(X, U) = \lambda_* \pi_2(V, W)$; this gives an algebraic description of the excision map $\varepsilon: \pi_2(V, W) \to \pi_2(X, U)$. If also $\pi_1(U, W) = 0$, then λ is surjective and we obtain from Proposition 9 the surjectivity of ε which is one part of the usual excision theorem; but we can also, by Theorem D and Proposition 9, state the further result that if $K = \operatorname{Ker} \lambda$, then K acts on $A = \pi_2(V, W)$, and $\operatorname{Ker} \varepsilon = [A, K]$. Suppose further that

$$\partial: \pi_2(U, W) \to \pi_1(W)$$

is trivial (for example if $\pi_2(U, W) = 0$); then $\lambda: \pi_1(W) \to \pi_1(U)$ is an isomorphism and hence so also is ε. This is the final part of homotopy excision under hypotheses slightly weaker than the usual ones.

Other uses of Theorem D are illustrated by the following examples.

EXAMPLES. 1. Let A, B, U be path-connected based spaces. Let $X = U \cup_f (CA \times B)$ where CA is the (unreduced) cone on A and f is a map $A \times B \to U$. It follows from Theorem D that $\pi_2(X, U)$ is the crossed $\pi_1(U)$-module induced from $M = (\pi_1(A), \pi_1(A) \times \pi_1(B), i_1)$ by

$$f_*: \pi_1(A) \times \pi_1(B) \to \pi_1(U)$$

where, in the crossed module M, $\pi_1(A)$ acts on itself by conjugation and $\pi_1(B)$ acts trivially on $\pi_1(A)$.

2. Let $A = S^p$, $B = S^q$ $(p, q \geqslant 1)$ in Example 1. Then it is easily shown from the above that (a) for $p \geqslant 2$, $\pi_2(X, U) = 0$, (b) for $p = 1$ and $q \geqslant 2$, $\pi_2(X, U)$ is the free crossed $\pi_1(U)$-module on one generator x which is the class in $\pi_2(X, U)$ of the disc $CA \xrightarrow{i_1} CA \times B \xrightarrow{\bar{f}} X$, and (c) for $p = q = 1$, $\pi_2(X, U) = F/N$ where F is the free crossed $\pi_1(U)$-module on one generator x as in (b) and N is the $\pi_1(U)$-submodule of F generated by $x^{-1}x^y$, where y is the class in $\pi_1(U)$ of the loop $B \xrightarrow{i_2} A \times B \xrightarrow{f} U$. (These results may also be deduced from results of [15, 16].)

3. Returning to Example 1, suppose next that B is a point. Then $X = U \cup_f CA$, $M = (\pi_1(A), \pi_1(A), i)$ and therefore $\pi_2(X, U)$ is the free crossed module on $f_*: \pi_1(A) \to \pi_1(U)$. We know no other method of proving this result.

4. Any space $\bar{U}$ obtained from the path-connected space U by attaching 2-cells is homotopy equivalent, rel U, to a space $X = U \cup_f CA$ where A is a wedge of circles. In this case, $\pi_1(A)$ is a free group, and Example 3 specializes to Whitehead's theorem that $\pi_2(\bar{U}, U)$ is the free crossed $\pi_1(U)$-module with one generator for each 2-cell attached [**16**, p. 493]. (Applications of Whitehead's theorem are given in [**5**, **6**, **8**, **11**, **12**, **16**, **17**] and a simpler proof of a special case of the theorem is given in [**7**].)

5. Let A, U, X be as in Example 3, and suppose that $f_*: \pi_1(A) \to \pi_1(U)$ is surjective with kernel K. An application of Proposition 8 to the conclusion of Theorem D gives $\pi_2(X, U) = \pi_1(A)/[\pi_1(A), K]$, and it follows that there is an exact sequence

$$\pi_2(U) \to \pi_2(X) \to K/[\pi_1(A), K] \to 0. \tag{7}$$

It is easy to deduce from this exact sequence, applied to the case where $A = K(G, 1)$ and $U = K(Q, 1)$ the well-known result that an exact sequence $1 \to K \to G \to Q \to 1$ of groups gives rise to an exact sequence

$$H_2(G) \to H_2(Q) \to K/[G, K] \to H_1(G) \to H_1(Q) \to 0.$$

As another application of (7) we note that if $\pi_1(U) = \pi_2(U) = 0$, then $\pi_2(X) = \pi_1(A)^{ab}$.

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