CROSSED SEQUENCES, G-GROUPOIDS and DOUBLE GROUPOIDS

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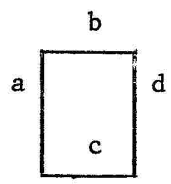
1. Introduction

The results described here arose out of an attempt to construct a 2-dimensional analogue $\rho_2(X)$ of the fundamental groupoid πX of a topological space X and such that $\rho_2(X)$ should have the property of sending reasonable pushouts to pushouts. This property is possessed by the fundamental groupoid and is the foundation of its computability [1]. The proof of this property for πX depends on the fact that πX has good subdivision properties; this suggests that for $\rho_2(X)$ to have this property, it must be defined in terms of rectangles and have two compositions, vertical and horizontal. So $\rho_2(X)$ has to be a double groupoid.

The purpose of this paper is to answer some questions on the algebraic side of this program, namely to show how double groupoids arise and are classified.

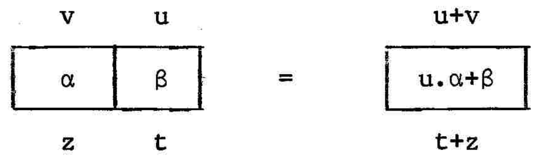
One standard example (due to Ehresmann) of a double groupoid is that of squares in a group L,

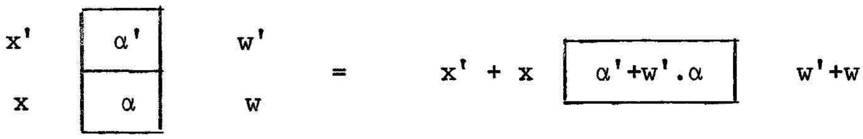
(1.1)



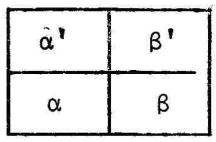
$a,b,c,d \in L$

with the obvious horizontal and vertical compositions. A modification of this example is to consider a sub-group Λ of L and squares as in (1.1) with the condition that $\alpha = b + a - c - d \in \Lambda$. (It is convenient in this paper to write various group operations as addition, even though the groups are not necessarily abelian.) A simple check then gives the following formulae for horizontal and vertical composition





where $u.\alpha = u + \alpha - u$. However we also require the interchange lemma namely that the two ways of evaluating



should give the same answer; for this it is necessary and sufficient that Λ be normal in L.

Even this construction is not general enough for homotopy theory, since one expects a square to have many classes of fillers. So we replace the subgroup Λ of L by a homomorphism $\partial : \Lambda \to L$. A simple check shows that in order to obtain a double groupoid by the above construction the exact sequence

$$C : 0 \to B \to \Lambda \xrightarrow{\partial} L \to Q \to 0$$

where $B = \text{Ker}\,\partial$, $Q = \text{Coker}\,\partial$, must be a crossed sequence in the sense that L operates on Λ with

(1.2) (i) $\partial(x.\alpha) = x + \partial\alpha - x$, $\quad x \in L, \alpha \in \Lambda$

(ii) $\beta + \alpha - \beta = (\partial\beta).\alpha$, $\quad \alpha, \beta \in \Lambda$.

The above construction then gives a functor ρ from the category $\underset{\sim}{C}$ of crossed sequences to a category $\underset{\sim}{DG}$ of double groupoids.

Not all double groupoids are of the form $\rho(C)$. First of all $\rho(C)$ has only one vertex. Secondly, $\rho(C)$ has a <u>connection</u> Γ which is an assignment to each edge a of $\rho(C)$ a 'face'

$$\begin{array}{ccc} & a & \\ 0 & \boxed{\Gamma(a)} & a \\ & 0 & \end{array}$$

satisfying a reasonable 'transport' formula for $\Gamma(b+a)$. Our first main result on double groupoids (Theorem B) is that ρ defines an equivalence between the categories $\underset{\sim}{C}$ and $\underset{\sim}{DG}'$, where $\underset{\sim}{DG}'$ is the category of double groupoids with a preferred connection and exactly one vertex. This is extended (Theorem C) to an equivalence $C \times \underset{\sim}{Ens}* \to \underset{\sim}{DG}^{\wedge}$ where $\underset{\sim}{DG}^{\wedge}$ is the category of connected double groupoids with base point, connection, and tree (the morphisms preserving these structures).

A further aim is a satisfactory homotopy theory for double groupoids. In §7, we give a definition of homotopy (for double groupoids with connection) which allows us to prove in §8 a version of the Whitehead theorem.

An intermediate category between $\underset{\sim}{C}$ and $\underset{\sim}{DG}$, is the category $\underset{\sim}{G}$ of G-groupoids, i.e. of groupoids which are group objects in the category of groupoids. We give here a proof (§3) that the categories $\underset{\sim}{C}$ and $\underset{\sim}{G}$ are equivalent (a result we have learned is due to J. Verdier (unpublished) in 1966) and add to this (§4) that $\underset{\sim}{G}$ has a notion of homotopy so that $\underset{\sim}{C}$ and $\underset{\sim}{G}$ are equivalent 2-categories.

An example of a G-groupoid is the fundamental groupoid πG of a topological group G. The crossed sequence of πG has an associated k-invariant in $H^3(\pi_0 G; \pi_1(G,e))$. In §5 we show that this k-invariant is the first Postnikov invariant of the classifying space B_{SG} of the singular complex of G.

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PART I ; G-groupoids

2. G-groupoids

A G-groupoid is a groupoid G which is a group object in the category of groupoids. Thus G is provided with morphisms $+ : G \times G \to G$, ; $* \to G$ (where $*$ is a singleton) and $u : G \to G$ satisfying the usual axioms for a group. It is convenient to write this group operation as addition (even though it need not be commutative) because of the ease of writing a-b rather than ab^{-1} and because of the clear distinction between the composition operation $\circ$ in G and the group operation +.

The condition that + makes G into a group implies that Ob(G) is a group with zero 0. Then 1_0, the identity in G(0,0) will be the zero for the group operation on arrows. Note that if $a : x \to y$, $a' : x' \to y'$ in G, then $a + a' : x + x' \to y + y'$. Further $-a: -x \to -y$ and $1_{-x} = -1_x$, because the inverse function is a morphism of groupoids.

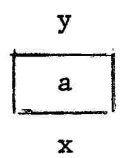
That $+ : G \times G \to G$ is a morphism of groupoids implies the following relation between addition and composition in G.

2.1 The interchange lemma Let $a : x \to y$, $b : y \to z$, $a' : x' \to y'$, $b' : y' \to z'$ be elements of G; then

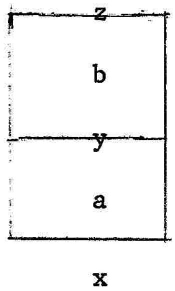
$$b \circ a + b' \circ a' = (b + b') \circ (a + a').$$

For explicit computations with the two structures on G it is often convenient to represent the interchange lemma diagrammatically:

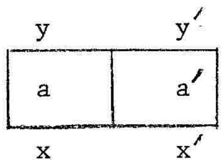
An arrow $a : x \to y$ in G is denoted by



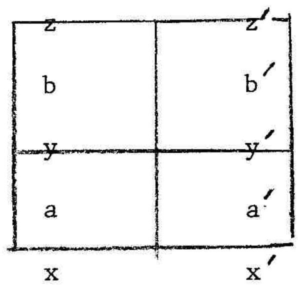
and the composite $b \circ a$ of $b : y \to z$ and $a : x \to y$ is denoted by



while the sum $a' + a$ of $a : x \to y$, $a' : x' \to y'$ is denoted by



The interchange lemma then tells us that the two possible ways of evaluating



give the same arrow $x' + x \to z' + z$.

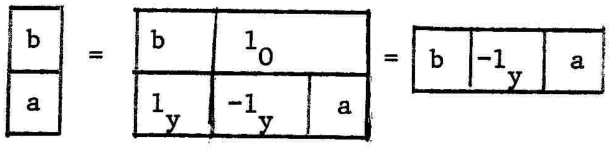
One of the basic uses of the interchange lemma is the following, which expresses composition in terms of +.

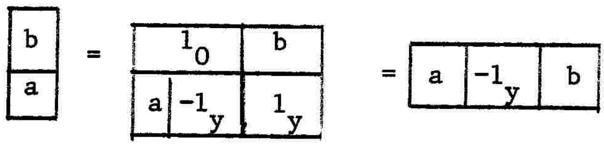
2.2 Proposition Let $a: x \to y$, $b: y \to z$ in G. Then

$$b \circ a = a - 1_y + b = b - 1_y + a .$$

Proof

The proof is expressed by the following diagrams:





2.3 Corollary Let $b \in St_G 0$, $a \in Cost_G 0$. Then

$$b \circ a = b + a = a + b$$

2.4 Corollary Let $a : x \to y$, $b : w \to z$. Then

$$a + b = 1_y + b - 1_w - 1_y + a + 1_w$$

Proof This follows from 2.3 which implies that $b - 1_w$ commutes with $-1_y + a$.

2.5 Corollary If $a : x \to y$, $b : 0 \to z$ then

$$a + b - a = 1_y + b - 1_y$$

Proof This follows from 2.4 on putting $w = 0$.

The following corollaries give further applications of these methods but will not be used elsewhere in this paper.

2.6 Corollary Let $a : x \to y$. Then

$$a^{-1} = 1_x - a + 1_y = 1_y - a + 1_x$$

Proof This follows from 2.2 by evaluating $a^{-1} \circ a$ and $a \circ a^{-1}$.

2.7 Corollary Let $a : 0 \to x$, $b : x \to y$. Then

$$b \circ a + a^{-1} = b.$$

<u>Proof</u> $b \circ a + a^{-1} = a - 1_x + b + 1_o - a + 1_x$

$$= a + (- a + 1_x) + (-1_x + b)$$

since $-a + 1_x$, $-1_x + b$ commute, by 2.3 The result follows.

2.8 <u>Corollary</u> <u>Let $a : y \to 0$, $b : x \to y$. Then</u>

$$a^{-1} + a \circ b = b$$

The proof is similar to that of 2.7.

3. The comparison of $\underset{\sim}{C}$ and $\underset{\sim}{G}$.

Let $\underset{\sim}{G}$ be the category whose objects are G-groupoids and whose morphisms are morphisms of groupoids preserving the group structure. Let $\underset{\sim}{C}$ be the category whose objects are crossed sequences $C : 0 \to B \overset{i}{\to} \Lambda \overset{\partial}{\to} L \to Q \to 0$ and whose morphisms $f : C \to C'$ are commutative diagrams

$$\begin{array}{ccccccccc} 0 \to & B & \overset{i}{\to} & \Lambda & \overset{\partial}{\to} & L & \to & Q & \to 0 \\ & \downarrow f_B & & \downarrow f_\Lambda & & \downarrow f_L & & \downarrow f_Q & \\ 0 \to & B' & \to & \Lambda' & \underset{\partial}{\to} & L' & \to & Q' & \to 0 \end{array}$$

of morphisms such that f_Λ is an operator morphism with respect to f_L , i.e. $f_\Lambda (x. a) = f_L (x). f_\Lambda (a)$, $x \in L$, $a \in \Lambda$. It is often convenient to omit the suffixes on the f's.) In this section we show that the categories $\underset{\sim}{C}$ and $\underset{\sim}{G}$ are equivalent.

Let G be a G-groupoid. Then the group structure on G induces group structures on $Ob(G)$, $\pi_0 G$ (the set of components of G) and on $St_G 0$. It also ensures that the group $G\{0\}$ is abelian.

Let $\partial : St_G 0 \to Ob(G)$ be the final point map. Then ∂ is a homomorphism for +. Further there is an operation of $Ob(G)$ on $St_G 0$ given by

$$x. a = 1_x + a - 1_x$$

for each $x \in Ob(G)$, $a \in St_G 0$. Clearly $\partial (x. a) = x + \partial a - x$, and if $a, b \in St_G 0$, then $a + b - a = (\partial a). b$, by 2.5.

So we have (setting $e = 1_0$) a sequence:

(3.1) $\psi(G) : 0 \to G\{e\} \xrightarrow{i} St_G O \xrightarrow{\partial} Ob\ G \xrightarrow{j} \pi_0\ G \to 0$

in which i is the inclusion, j is the projection.

Exactness is easily verified, and so $\psi(G)$ is a crossed sequence.

Clearly ψ extends to a functor $G \to C$.

Now let

$$C : 0 \to B \to \Lambda \xrightarrow{\partial} L \to Q \to 0$$

be a crossed sequence.

3.2 <u>Definition</u> The G-groupoid $\theta(C)$ has objects the elements of L and $\theta(C)\ (x, y)$ consists of the pairs (y, a) such that $a \in \Lambda$ and $\partial\, a = y - x$. Composition is defined by

$$(z, b) \circ (y, a) = (z, b + a) \quad \text{when } \partial\, b = z - y,$$

and addition by

$$(z, b) + (y, a) = (z + y, z.a + b).$$

It is a simple consequence of the definition of crossed sequence that if $\partial\, b = z - w$, then

$$(z, b) + (y, a) = (z + y, b + w.a)$$

It is readily checked that $\circ$ and $+$ are associative; that

$$(y, a)^{-1} = (x, -a) \qquad \text{if } \partial a = y - x,$$

and $\quad -(y, a) = (-y, (-y).(-a))$.

Verification of the interchange lemma

z	z'
(z,b) y	(z', b') y'
(y,a) x	(y', a') x'

involves proving that

$$z'.(b + a) + b' + a' = z'.b + b' + y'.a + a'.$$

But using condition (1.2) (ii), we have

$$b' + y'.a - b' = (\partial b').(y'.a)$$
$$= z'.a$$

from which the result follows.

Clearly θ extends to a functor $\underset{\sim}{C} \to \underset{\sim}{G}$.
Given a crossed sequence

$$C : 0 \to B \to \Lambda \xrightarrow{\partial} L \to Q \to 0$$

an isomorphism $T_C : C \to \psi\,\theta\,(C)$ is defined to be the identity on L and Q and on Λ by $a \mapsto (\partial a, a)$. It is easily checked that T is a natural equivalence $1_C \to \psi\,\theta$.

Let G be a G-groupoid. An isomorphism $S_G : \theta\,\psi(G) \longrightarrow G$ is defined to be the identity on objects, and on arrows by $(y, a) \mapsto a + 1_{-\partial a+y}$. Clearly S_G is bijective on arrows so it only remains to show that S_G preserves composition and addition.

For composition we have

$$S_G((z,b) \circ (y,a)) = S_G(z, b+a)$$
$$= b + a + 1_x \quad \text{if } \partial a = y - x$$

On the other hand

$$S_G(z, b) \circ S_G(y, a) = (b + 1_y) \circ (a + 1_x)$$
$$= b + 1_y - 1_y + a + 1_x \quad \text{by 2.2}$$
$$= b + a + 1_x$$

For addition, we have that if $\partial a = y - x, \partial b = z - w$,

then $S_G((z,b) + (y,a)) = S_G(z + y, 1_z + a - 1_z + b)$

$= 1_z + a - 1_z + b + 1_{w+x}$

$= b + (- b) + (1_z + a - 1_z) + b + 1_w + 1_x$

$= b + 1_w + a + 1_x$ by 2.5

$= S_G(z,b) + S_G(y,a)$.

Thus $S_G : \theta\psi(G) \to G$ is an isomorphism of G-groupoids, and it is easily seen that S defines a natural equivalence $S : \theta\psi \to 1_{\underset{\sim}{G}}$.

§4 Homotopies in C and G.

In this section we show that $\underset{\sim}{C}$ with its standard notion of homotopy [3] is a 2-category in the sense of [7]. We then show that homotopy can be defined in $\underset{\sim}{G}$ to make it into a 2-category, and the equivalence θ of the previous section extends to an equivalence of 2-categories.

4.1 Definition [3] Let $f,g : C \to C'$ be morphisms of crossed sequences. A homotopy $d : f \simeq g$ is a function $d : L \to \Lambda'$ such that

(i) $d(x+y) = d(x) + f(x).d(y)$ for all x, y in L,

(ii) $\partial' d(x) = g(x) - f(x)$, all x in L,

(iii) $d\partial(a) = g(a) - f(a)$ for all a in Λ.

To show that this gives a structure of 2-category on C, we work through a sequence of lemmas.

4.2 Lemma Let $f,g,h : C \to C'$ be morphisms of crossed sequences. If $d : f \simeq g$, $d_1 : g \simeq h$ then $d_1 + d : f \simeq h$.

Proof The verification of (ii) and (iii) for $d_2 = d_1 + d$ being a homotopy $f \simeq h$ is trivial. As for (i) we find

$$d_2(x+y) = d_1(x) + g(x).d_1(y) + d(x) + f(x).d(y)$$

$$d_2(x) + f(x).d_2(y) = d_1(x) + d(x) + f(x).d_1(y) + f(x).d(y)$$

But
$$d(x) + f(x).d_1(y) - d(x) = \partial' d(x).f(x).d_1(y) = g(x).d_1(y)$$

from which the required equality follows.

4.3 <u>Lemma</u> <u>Let $d : f \simeq g : C \to C'$, $d' : f' \simeq g' : C' \to C''$ be homotopies of morphisms of crossed sequences. Then</u>

(i) <u>$f'd$ is a homotopy $f'f \simeq f'g$,</u>

(ii) <u>$d'g$ is a homotopy $f'g \simeq g'g$,</u>

(iii) <u>$d'g + f'd = g'd + d'f$ as homotopies $f'f \simeq g'g$.</u>

<u>Proof</u> The proofs of (i) and (ii) are trivial. For the proof of (iii) we note that

$$\begin{aligned} d'g(x) + f'd(x) &= d'(\partial'd(x) + f(x)) + f'd(x) \\ &= d'\partial'd(x) + f'\partial'd(x).d'f(x) + f'd(x) \\ &= g'd(x) - f'd(x) + (f'd(x) + d'f(x) - f'd(x)) + f'd(x) \\ &= g'd(x) + d'f(x) \end{aligned}$$

and either side is a homotopy $f'f \simeq g'g$ by 4.2 and 4.3 (i) and (ii).

It follows easily from 4.2 that the homotopies of morphisms $C \to C'$ form a groupoid, which we write $HOM(C,C')$. By 4.3(iii) we have a pairing $HOM(C',C'') \times HOM(C,C') \to HOM(C,C'')$ in which $(d : f \simeq g,\ d' : f' \simeq g') \mapsto d'g + f'd : f'f \simeq g'g$. The associativity and identities for this pairing are easily verified. So we have a 2-category $\underset{\sim}{C}$.

4.4 <u>Definition</u> Let $f,g : G \to G'$ be morphisms of G-groupoids. A <u>G-homotopy</u> $V : f \simeq g$ is a natural transformation of groupoid morphisms such that V preserves addition, i.e.

$$V(x+y) = V(x) + V(y), \qquad \text{all } x,y \text{ in } Ob(G).$$

It is readily verified that the G-homotopies between morphisms $G \to H$ form a groupoid, which we write $HOM(G,H)$. The pairing $HOM(G',G'') \times HOM(G,G') \to HOM(G,G'')$ is defined to be the restriction of the standard pairing for the 2-category $\underset{\sim}{Cat}$. So we have a 2-category $\underset{\sim}{G}$.

Theorem A The categories $\underset{\sim}{C}$ and $\underset{\sim}{G}$ are equivalent 2-categories

For the proof of Theorem A we show that the equivalence $\theta : C \to G$ extends to a (strict) functor $\theta : \underset{\sim}{C} \to \underset{\sim}{G}$ and for two crossed sequences C, C' the morphism of groupoids $\theta : HOM(C,C') \to HOM(\theta(C),\theta(C$ is an isomorphism of groupoids. This makes θ an equivalence of 2-categories in a stronger sense than that of [9].

4.5 Lemma Let $d : f \simeq g : C \to C'$ be a homotopy of morphisms of crossed sequences. Then $\theta(d) : x \mapsto (g(x), d(x))$ is a G-homotopy $\theta(f) \simeq \theta(g)$. If further $d_1 : g \simeq h$, then $\theta(d_1+d) = \theta(d_1) \circ \theta(d)$.

Proof To verify naturality of $\theta(d)$ we have to prove that if $a \in \Lambda$, and $\partial a = y - x$ then

$$(g(y),d(y)) \circ (f(y), f(a)) = (g(y),g(a)) \circ (g(x),d(x))$$

i.e. that $d(y) + f(a) = g(a) + d(x)$. This is an easy consequence of $d(y-x) = d(y) - f(y).(-f(x)).d(x)$.

For additivity of $\theta(d)$ we have that if $x,y \in L = Ob(\theta(C))$

$$\theta(d)(x) + \theta(d)(y) = (g(x),d(x) + (g(y), d(y))$$
$$= (g(x) + g(y), d(x) + f(x).d(y)) \text{ Since } d(x) = g(x)-f(x)$$
$$= \theta(d)\ (x+y).$$

The final part is immediate from the definition of composition in $\theta(C')$.

This lemma implies that θ extends to morphism of groupoids

$$\theta : HOM(C,C'') \to HOM(\theta(C), \theta(C'))$$

which by the results of the last section is bijective on objects. That θ is an isomorphism of groupoids follows from

4.6 Lemma Let $f,g : C \to C'$ be morphisms of crossed sequences and let V be a G-homotopy $\theta(f) \simeq \theta(g)$. Then V is of the form $x \mapsto (g(x), d(x))$ where d is a homotopy $f \simeq g$.

Proof Since $V : f \simeq g$, we have $V(x) = (g(x), d(x))$ where $\partial d(x) = g(x) - f(x)$. The additivity of V implies that $d(x+y) = d(x) + f(x).d(y)$, while naturality of V implies that $d(y) + f(a) = g(a) + d(x)$ when $\partial a = y - x$. On putting $x = 0$, we obtain $d\partial(a) = g(a) - f(a)$.

Finally, to prove that θ is a strict functor of 2-categories, we have to prove θ preserves the pairings of homotopies [8]. This follows easily from the obvious equalities (using the notation of 4.3)

$$\theta(f'd) = \theta(f')\,\theta(d)$$
$$\theta(d'f) = \theta(d')\,\theta(f)\ .$$

There seem to be results related to the above in [4] §1.4. The 'catégories de Picard strictement commutatif' defined there in 1.4.2 are more general than commutative G-groupoids in that associativity is required only up to coherent homotopy, and that for any $x \in Ob(G)$ the functor $G \to G$ such that $y \mapsto x + y$ is not an isomorphism (as in our case) but an equivalence.

§5 The fundamental groupoid of a topological group

A crossed sequence $C : 0 \to B \to \Lambda \to L \to Q \to 0$ determines an obstruction class or k-invariant $k \in H^3(Q\ ;\ B)$ [10, 11] which for free L classifies the crossed sequence up to homotopy equivalences which are the identity on Q and on B. It follows that G-groupoids, and connected double groupoids with base point, also determine a similar 3-dimensional cohomology class.

A particular example of a G-groupoid is the fundamental groupoid πX of a topological group X - the rule $\pi(X \times X) = \pi X \times \pi X$ ([1] p.189) implies that the group structure on X induces a group structure on πX. The crossed sequence derived from πX is

(5.1) $C_X : 0 \to \pi(X,e) \to St_{\pi X}e \overset{\partial}{\to} X \to \pi_0 X \to 0.$

The object of this section is to prove:

<u>Theorem 5.2</u> <u>The k-invariant of the crossed sequence C_X can be identified with the first Postnikov invariant of B_{SX}, the classifying complex of the singular complex of X.</u>

In this theorem, the singular complex SX is a simplicial group and so its classifying complex is a simplicial set.

Let $K = B_{SX}$. By [11], the Postnikov invariant of K is the k-invariant of the crossed sequence

(5.3) $C_K : 0 \to \pi_2(K) \to \pi_2(K,K^1) \to \pi_1(K^1) \to \pi_1(K) \to 0$

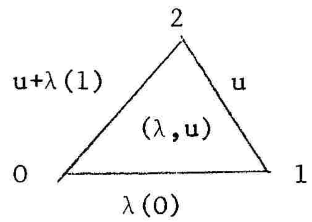
where K^1 is the 1-skeleton of K. The k-invariants of C_K and C_X are related by constructing a morphism $\phi : C_K \to C_X$ of crossed sequences.

The set K_0 is a point, and K_1 consists of the points of X. So $\pi_1(K^1)$ is the free group on $X\backslash\{e\}$, and a morphism $\phi_1 : \pi_1(K^1) \to X$ can be defined by extending the identity map on generators.

Let $\rho_2 = \pi_2(K^2, K^1)$, $\rho_3 = \pi_3(K^3, K^2)$. Then according to [14] adapted here to the simplicial context, or else applied to the geometric realisation, ρ_3 is the free $\pi_1(K^2)$-module on the non-degenerate 3-simplices of K, and there is a morphism $d_2 : \rho_2 \to \rho_1 = \pi_1(K^1)$ such that ρ_2 is the free (ρ_1, d_2) - crossed module on the non-degenerate 2-simplices of K.

Thus to define a morphism $\bar{\phi}_2 : \rho_2 \to St_{\pi X}e$ it is sufficient to specify $\bar{\phi}_2(\delta)$ for each non-degenerate 2-simplex δ of K in such a way that $\partial\bar{\phi}_2(\delta) = \phi_1 d_2(\delta)$.

The elements of K_2 are pairs (λ, u) such that $u \in X$ and λ is a path in X. The edges of (λ, u) are then given by the diagram



where + is given by the group structure in X. We define

$$\bar{\phi}_2(\lambda, u) = -1_{\lambda(0)} + [\lambda]$$

where $[\lambda]$ denotes the class in πX of λ (if (λ, u) is degenerate then $\bar{\phi}_2(\lambda, u)$ is 0). We have to check that $\phi_1 d_2(\lambda, u) = \partial\bar{\phi}_2(\lambda, u) = -\lambda(0)+\lambda(1)$.

At this stage there arises the question of orientation. Because of our convention that we are working with stars in a groupoid, rather than costars, it is necessary to reverse the usual orientation of 2-cells, so that with (λ, u) as above

$$d_2\ (\lambda, u) = -[\lambda(0)] - [u] + [u+\lambda(1)]$$

where $[y]$ for $y \in X$ denotes the corresponding generator of $\pi_1(K^1)$. Hence

$$\begin{aligned} \phi_1 d_2(\lambda, u) &= -\lambda(0) - u + u + \lambda(1) \\ &= \partial\bar{\phi}_2(\lambda, u). \end{aligned}$$

So $\bar{\phi}_2$ extends to a morphism of crossed modules

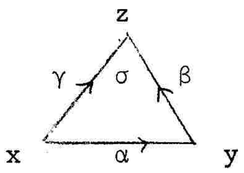
$$\bar{\phi}_2 : \rho_2 \to St_{\pi X} e.$$

There is a morphism $d_3 : \rho_3 \to \rho_2$ such that $\pi_2(K, K^1) = \rho_2 / d_3\rho_3$. So to define $\phi_2 : \pi_2(K, K^1) \to St_{\pi X} e$ we prove that $\bar{\phi}_2$ annihilates $d_3\rho_3$.

Each non-degenerate 3-simplex κ of K determines a free $\pi_1(K^2)$-generator κ of ρ_3 such that with our present conventions

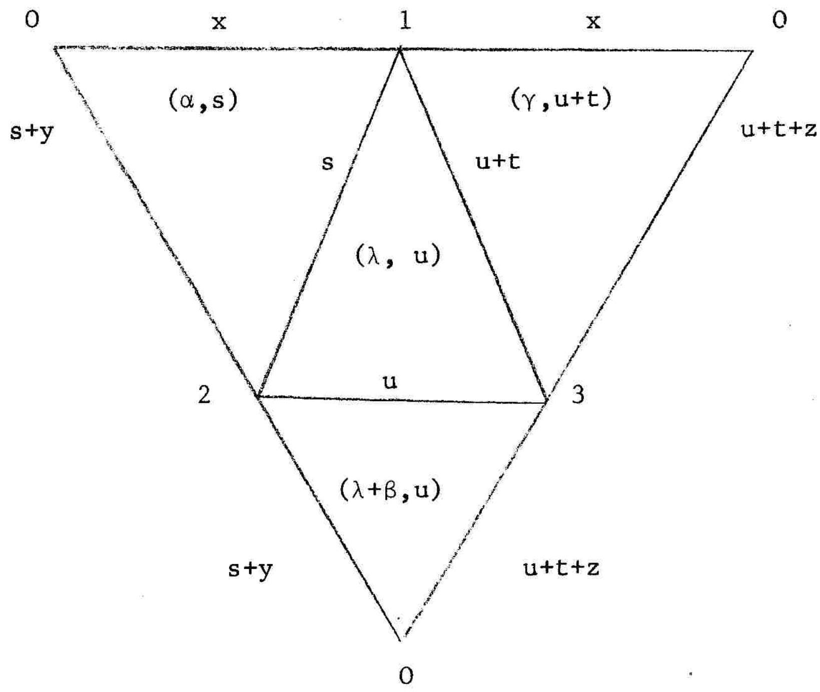
$$d_3(\kappa) = (\partial_3\kappa) + (\partial_1\kappa) - (\partial_2\kappa) - (-a).(\partial_0\kappa)$$

where a is the element of $\pi_1(K^1)$ determined by $\partial_3\partial_2\kappa$.
In the present case $K = B_{SX}$, a 3-simplex is a triple (σ, λ, u) where $u \in X$, λ is a path in X and σ is a singular 2-simplex in X. If the faces of σ are given by



then α, β, γ are paths in X whose classes in X satisfy $[\gamma] = [\beta] \circ [\alpha]$. The formulae for the face operators in B_{SX} now give the following (unfolded) diagram (c.f. [12]).

for the faces of (σ, λ, u), where $\lambda(0)=s$, $\lambda(1)=t$:



So in πX we have

$$\bar{\phi}_2 d_3(\kappa) = \bar{\phi}_2(\alpha, s) + \bar{\phi}_2(\lambda+\beta, u) - \bar{\phi}_2(\gamma, u+t) - (-x).\ \bar{\phi}_2(\lambda,u)$$

$$= -1_x + [\alpha] - 1_{s+y} + [\lambda] + [\beta] - (-1_x + [\gamma]) - (-x).\ (-1_s + [\lambda])$$

$$= -1_x + [\alpha] - 1_y - 1_s + [\lambda] + [\beta] - [\gamma] + 1_x - 1_x - [\lambda] + 1_s + 1_x$$

$$= -1_x + [\alpha] - 1_y - 1_s + [\lambda] + [\beta] - [\beta] + 1_y - [\alpha] - [\lambda] + 1_s + 1_x \quad \text{by 2.2}$$

$$= -1_x + [\alpha] - 1_y + (-1_s + [\lambda]) + (1_y - [\alpha]) - [\lambda] + 1_s + 1_x$$

$$= 1_x + [\alpha] - 1_y + (1_y - [\alpha]) + (-1_s + [\lambda]) - [\lambda] + 1_s + 1_x \quad \text{by 2.3}$$

$$= 0.$$

It follows that $\bar{\phi}_2 : \pi_2(K^2, K^1) \to St_{\pi X} e$

induces a morphism of crossed modules

$$\begin{array}{ccccccccccc}
0 & \to & \pi_2(K) & \to & \pi_2(K,K^1) & \to & \pi_1(K^1) & \to & \pi_1(K) & \to & 0 \\
 & & \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} & & \\
0 & \to & \pi(X,e) & \to & St_{\pi X} e & \to & X & \to & \pi_0 X & \to & 0
\end{array}$$

Now $\phi_0 : \pi_1(K) \to \pi_0 X$ is an isomorphism. It is also true that $\phi_3 : \pi_2(K) \to \pi(X, e)$ is an isomorphism since it is easily checked that it is induced by the standard map $|SX| \to X$ (under the identification $\pi_2(B_{SX}) \simeq \pi_1(SX)$) and this is known to be a weak homotopy equivalence.

It follows easily that if ϕ_0, ϕ_3 are regarded as identifications, then the k-invariants of the two crossed sequences coincide. This proves Theorem 5.2.

We have not been able to find an example of a Lie group X for which this k-invariant is non-zero. The next proposition gives a family of groups for which the k-invariant is easily shown to be zero.

Proposition 5.4 The first k-invariant of B_X is zero if X is any quotient of O(n) by a normal subgroup.

Proof We examine the construction of the k-invariant, as described in [11], p.43, for the crossed sequence

$$0 \longrightarrow Z_\alpha \longrightarrow St_{\pi O(n)}\ I \longrightarrow O(n) \xrightarrow{\nu} Z_2 \longrightarrow 0$$

where ν is the determinant map and $Z_\alpha = Z$ for $n=2$, $Z_\alpha = Z_2$, for $n > 2$. As a section f for ν we can take

$$f(0) = I,\ f(1) = \begin{bmatrix} -1 & & & \\ & 1 & & 0 \\ & & \ddots & \\ & 0 & & 1 \end{bmatrix}$$

whence $f(q)f(q')(f(q\,q'))^{-1} = I$ for all $q, q' \in Z_2$. It follows easily that 0 is a representative cocycle for the k-invariant, which proves the result for $X = O(n)$.

If $X = O(n)/F$ where F is a normal subgroup of $O(n)$, then either X is connected (in which case the k-invariant is 0) or $\pi_0 X = Z_2$. In the latter case, a section of $X \to \pi_0 X$ can be chosen to be $f' = pf$ where f is as above and $p: O(n) \to X$ is the projection. It follows as before that the k-invariant is 0.

Let Y be a connected finite simplicial complex with non-trivial k-invariant in $H^3(\pi_1(Y); \pi_2(Y))$. Let $G(Y)$ be Milnor's topological group model of the loop space of Y [13]. Then $B_{G(Y)}$ is of the homotopy type of Y, and so $\pi G(Y)$ will be a G-groupoid with non-trivial k-invariant.

Note that there is a general problem here. If G is a topological group, and G_o is its identity component, then we have an exact sequence

$$1 \to G_o \to G \to F \to 1$$

so that F is discrete. This extension is described by an element $k^2 \in H^2(F; G_o)$ (since F is discrete).

<u>Problem</u> <u>Relate k^2 and the obstruction class $k^3 \in H^2(F; \pi_1(G_o, e))$.</u>

PART II : DOUBLE GROUPOIDS

§6. Double groupoids

In this section we introduce a category DG of double groupoids and consider the relationship of DG to C and G.

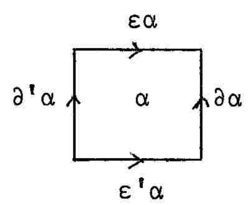
Double categories have been considered by Ehresmann [6] and Wyler [15] as a class with two commuting category structures and of course this idea specialises to that of double groupoid. However such a definition is too general for the present purposes, and we adopt a different one based on semi-cubical theory. This seems to be a useful definition for some, but definitely not all, of the applications of double groupoids.

Our definition could also be generalised to n-tuple groupoids, and even to ω-tuple groupoids. So there are arguments for using a notation for double groupoids which extends easily to n-tuple groupoids. We shall not adopt this course as this would only further obscure some of the formulae.

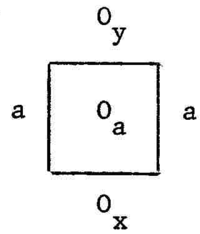
A double groupoid G consists in the first place of the part in dimensions ⩽ 2 of a semi-cubical complex. Thus G consists of sets G_o, G_1, G_2 and certain face operators $G_1 \to G_o$, $G_2 \to G_1$ and degeneracy operators $G_o \to G_1$, $G_1 \to G_2$, satisfying the usual relations. In order to fix the notation the faces of a in G_1 are given by

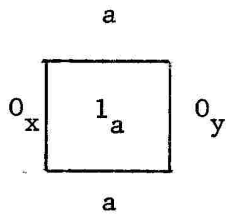
$$\partial' a \xrightarrow{\quad a \quad} \partial a$$

while if α is in G_2 its faces are



The degeneracy operation $G_o \to G_1$ is written $x \mapsto 0_x$, and the two degeneracy operations $G_1 \to G_2$ are written $a \mapsto 0_a$, $a \mapsto 1_a$ as specified by the diagrams





where $\partial' a = x$, $\partial a = y$.

The further structures we introduce are

(a) a groupoid structure written + with G_1 as elements, G_o as objects, and ∂', ∂, $x \mapsto 0_x$ the initial, final and unit maps respectively;

(b) two groupoid structures written + and $\circ$ with G_2 as elements, G_1 as objects and

∂', ∂, $a \mapsto 0_a$ the initial, final and unit maps for +

ε', ε, $a \mapsto 1_a$ the initial, final and unit maps for $\circ$.

These structures are related by the rules

(i) $\varepsilon, \varepsilon' : (G_2,+) \to (G_1,+)$ are morphisms,

(ii) $\partial, \partial' : (G_2,\circ) \to (G_1,+)$ are morphisms,

(iii) if $b + a$ is defined in G_1, then $0_{b+a} = 0_b \circ 0_a$, $1_{b+a} = 1_b + 1_a$

(iv) The <u>interchange lemma</u> in G_2:

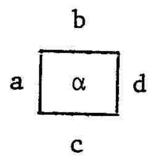
$$(\beta' + \alpha') \circ (\beta + \alpha) = (\beta' \circ \beta) + (\alpha' \circ \alpha)$$

whenever both sides are defined.

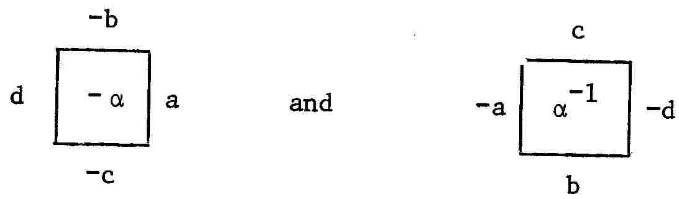
It is convenient to represent the structures $+$, $\circ$ on G by composition of squares as follows:

where of course $\beta + \alpha$ is defined if and only if $\partial'\beta = \partial\alpha$, and $\alpha' \circ \alpha$ is defined if and only if $\varepsilon'\alpha' = \varepsilon\alpha$. The interchange lemma can then be expressed by a diagram in the same way as for G-groupoids.

The inverse for $+$ on G_2 or G_1 is written $\alpha \mapsto -\alpha$; the inverse for $\circ$ is written $\alpha \mapsto \alpha^{-1}$. So if α in G_2 has faces given by



then those of $-\alpha$ and α^{-1} are given by



It is a consequence of the interchange lemma that if $\alpha \in G_2$ then

$$-(\alpha^{-1}) = (-\alpha)^{-1}.$$

Also if $x \in G_0$, then

$$1_{o_x} = 0_{o_x}$$

and so both are written 0_x or ambiguously 0.

Morphisms $f : G \to H$ of double groupoids are defined in the obvious way as given by functions $f : G_i \to H_i$, $i = 0, 1, 2$ which preserve all the structure. So we have a category $\underset{\sim}{DG}$ of double groupoids.

Let $\underset{\sim}{DG}^*$ be the category of double groupoids H with base point $e \in H_o$, and morphisms preserving base point. Then we may describe two functors

$$\omega_1, \omega_2 : \underset{\sim}{DG}^* \to \underset{\sim}{G}$$

where if H is a double groupoid with base point, then $\omega_1(H)$, $\omega_2(H)$ both have object set $H_1\{e\}$, while if $a,b \in H_1\{e\}$, then

$$\omega_1(H)(a,b) = \{\alpha \in H_2 : \varepsilon\alpha = b,\ \varepsilon'\alpha = a,\ \partial'\alpha = \partial\alpha = 0_e\}$$

$$\omega_2(H)(a,b) = \{\alpha \in H_2 : \varepsilon\alpha = b,\ \partial'\alpha = a,\ \varepsilon'\alpha = \varepsilon\alpha = 0_e\}$$

(6.1)

The composition and addition in $\omega_1(H)$ are defined by restriction of those of H; while $\omega_2(H)$ is a groupoid under + with a group structure given by $\circ$!

There are also two functors $\omega_1^!, \omega_2^! : \underset{\sim}{G} \to \underset{\sim}{DG}^*$. If G is a G-groupoid then for $i = 1,2$, $\omega_i^!(G)_o$ consists of a single element, e, say, $\omega_i'(G)_1 = Ob(G)$ and $\omega_i^!(G)_2 = G$. If $\alpha \in G$, then diagrams (5.1) define ∂', ∂, ε', ε for $\omega_1^!(G), \omega_2^!(G)$ respectively, where $0_e = 0$, the zero in $Ob(G)$.

In $\omega^!_1(G)_2$ the composition $\circ$ and addition + are determined by $\circ$ and + in G as in §2. In $\omega^!_2(G)_2$ the composition and addition are determined by the addition and composition in G respectively. Notice that $\omega_1\omega^!_1(G)$ is isomorphic to G, while $\omega_2\omega^!_1(G)$ is simply the group G{0} with its two equal and abelian group structures.

It is easy to check that $\omega^!_1 : \underset{\sim}{G} \to \underset{\sim}{DG}^*$ is a left adjoint to ω_1. Our object now is to put extra structure on double groupoids so as to make ω an equivalence of categories. This structure will introduce also an extra symmetry into double groupoids ensuring for example that the G-groupoids $\omega_1(H)$ and $\omega_2(H)$ are isomorphic.

First of all we define a functor $\rho : \underset{\sim}{C} \to \underset{\sim}{DG}^*$. Consider the crossed sequence

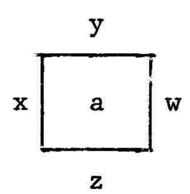
$$\mathcal{C} : 0 \to B \to \Lambda \xrightarrow{\partial} L \to Q \to 1 .$$

Then a double groupoid $\rho(C)$ is defined by: $\rho(C)_o$ consists of a single point e, say; $\rho(C)_1 = L$ with its structure of group under addition; $\rho(C)_2$ consists of quintuples $\alpha = (a; x, y, z, w)$ such that $a \in \Lambda$, $x, y, z, w \in L$ and

$$\partial a = y + x - z - w .$$

The boundaries ∂, ∂', ε, ε' are specified by the diagram for α

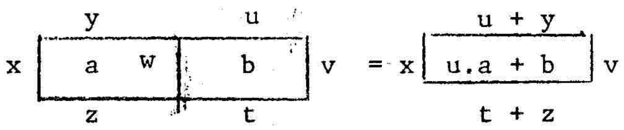
(6.2)



α

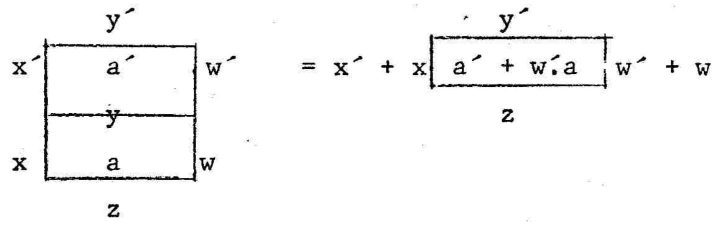
while addition and composition on $\rho(C)_2$ are defined by the diagrams

(6.3) (i)



$\beta + \alpha$

(ii)



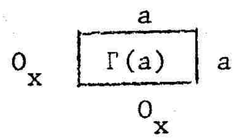
$\alpha' \circ \alpha$

If $x \in L$, then $0_x = (0;\ x,\ 0,\ x,\ 0)$ and $1_x = (0;\ 0,\ x,\ 0,\ x)$ provide the zero and identity for x with respect to $+$ and $\circ$ in $\rho\ (C)_2$. Associativity for $\circ$ and $+$, and the Interchange Lemma, are straightforward to verify, as are the existence of inverses for $\circ$ and $+$.

So $\rho : \underset{\sim}{C} \to \underset{\sim}{DG^*}$ is defined on objects as above, and ρ is extended to morphismsm in the obvious way.

Next we impose conditions on a double groupoid H with base point so that there is an isomorphism $\eta\ :\ \rho\psi\omega(H)\ \to\ H$.

6.4 Definition Let H be a double groupoid. A connection for H is a function $\Gamma : H_1 \to H_2$ such that if $a : x \to y$ in H_1 then $\Gamma(a)$ has boundaries given by the diagram

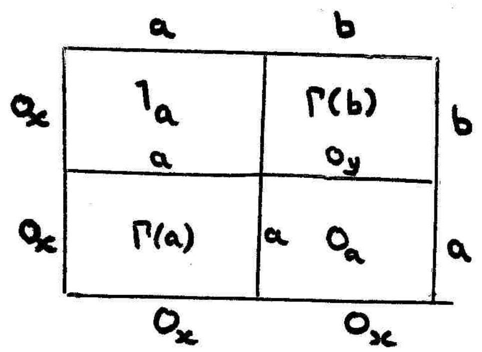


Further if $b : y \to z$ in H_1, then

(6.5) $\Gamma(b + a) = (\Gamma(b) \circ 0_a) + \Gamma(a)$

Condition (6.5) can be expressed as: $\Gamma(b+a)$ is given by the diagram

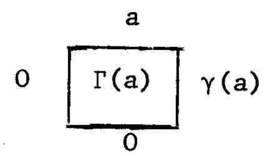
(6.6)



and so the interchange lemma gives us that

$$\Gamma(b+a) = (\Gamma(b) + 1_a) \circ \Gamma(a)$$

<u>Remark</u> The word connection is used because of a relationship with the connections of differential geometry; this relationship will be discussed elsewhere. However for the connections of differential geometry a more general notion of double groupoid has to be used, and the face $\Gamma(a)$ is represented by a diagram

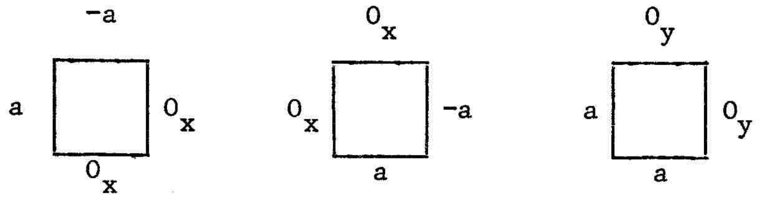


still satisfying the condition $\Gamma(b+a) = (\Gamma(b) + 1_a) \circ \Gamma(a)$, but allowing for the additional structure of a morphism $a \mapsto \gamma(a)$ of groupoids, called the <u>holonomy</u> of the connection. Continuing the analogy with differential geometry, we describe (6.5) as the <u>transport</u> property of the connection.

By transport we have for $x \in G_0$ (remembering that $1_{0_x} = 0_{0_x}$ is abbreviated to 0)

$$\Gamma(0_x) = \Gamma(0_x + 0_x) = \Gamma(0_x) \circ 0 + \Gamma(0_x)$$

so that $\Gamma(0_x) = 0$. Then applying transport to $\Gamma(-a+a)$ we may obtain various identities relating $\Gamma(-a)$ and $\Gamma(a)^{-1}$ (which for convenience is written $\Gamma^{-1}(a)$) for example $\Gamma(-a) = \Gamma(a)^{-1} - 1_a = -(1_a \circ \Gamma(a))$. The following are other useful faces provided by Γ:



$-\Gamma(a)$ $\Gamma(a)^{-1}$ $-\Gamma(-a)^{-1}$

Not all double groupoids admit connections. For example if G is a G-groupoid, then $\omega_1'(G)$ clearly does not admit a connection. However we have:

6.7 <u>Proposition</u> *If C is a crossed sequence, then the double groupoid $\rho(C)$ has a connection.*

<u>Proof</u> Let $C : 0 \to B \to \Lambda \to L \to Q \to 1$. For $a \in L$, set $\Gamma(a) = (0;\ 0,\ a,\ 0,\ a)$. Then (6.5) is easily verified.

Suppose now G is a double groupoid with connection Γ and $f : G \to H$ is a morphism of double groupoids which is isomorphic on $G_1 \to H_1$. Then the composite $H_1 \xrightarrow{f^{-1}} G_1 \xrightarrow{\Gamma} G_2 \xrightarrow{f} H_2$

is clearly a connection for H. In particular, if H is isomorphic to $\rho\ \psi\ \omega(H)$, then H admits a connection.

Our next objective is to prove a converse to this.

6.8 Theorem Let H be a double groupoid with connection Γ. Suppose that H_o has exactly one element e, say. Then there is an isomorphism $\eta: \rho\ \psi\ \omega(H) \to H$.

Proof Let $K = \rho\ \psi\ \omega(H)$.

Define η to be the identity on K_o and K_1, and on K_2 by

$$\eta(a\ ;\ x,\ y,\ z,\ w) = \Gamma(w) - 1_w + a + 1_{w+z} - \Gamma(x)$$

	$-x$	$w+z$	$y+x-z-w$	$-w$	w	
x	$-\Gamma(x)$	1_{w+z}	a	-1_w	$\Gamma(w)$	w
	1_e	$w+z$	1_e	$-w$	1_e	

Clearly η is a bijection $K_2 \to H_2$, so that it suffices to prove that η is a morphism for $+$, $\circ$ on K_2.

For + we have

$$\eta((b\ ;\ w,\ u,\ t,\ v) + (a\ ;\ x,\ y,\ z,\ w)) = \eta(u.a + b;\ x,\ u+y,\ t+z, v)$$

$$= \Gamma(v) - 1_v + u.a + b + 1_{v+t+z} - \Gamma(x) \qquad (*)$$

On the other hand

$$\eta(b\ ;\ w,u,t,v) + \eta(a\ ;\ x,y,z,w)$$

$$= \Gamma(v) - 1_v + b + 1_{v+t} - \Gamma(w) + \Gamma(w) - 1_w + a + 1_{w+z} - \Gamma(x) \qquad (**)$$

But $\partial b = u + w - t - v$, so that $u.a + b = b + (v + t - w).a$.

The equality of (*) and (**) follow easily.

For $\circ$ we have

$$\eta((a' \,;\, x',\, y',\, y,\, w') \circ (a \,;\, x,\, y,\, z,\, w))$$
$$= \eta\,(a' + w'\cdot a;\; x' + x,\, y',\, z,\, w' + w)$$
$$= \Gamma(w' + w) - 1_{w'+w} + a' + w'\cdot a + 1_{w'+w+z} - \Gamma(x' + x) \qquad (!)$$

On the other hand

$$\eta(a' \,;\, x',\, y',\, y,\, w') \circ \eta(a \,;\, x,\, y,\, z,\, w)$$
$$= (\Gamma(w') - 1_{w'} + a' + 1_{w'+y} - \Gamma(x')) \circ (\Gamma(w) - 1_{w} + a + 1_{w+z} - \Gamma(x)) \qquad (!!)$$

The following diagram, with the interchange lemma, exhibits the equality of (!) and (!!):

$-\Gamma(x')$	-1_x	1_z	1_w	$1_{y+x-z-w}$	$1_{w'}$	a'	$-1_{w'}$	-1_w	1_w	$\Gamma(w')$
0_x	$-\Gamma(x)$	1_z	1_w	a	$1_{w'}$	1_0	$-1_{w'}$	-1_w	$\Gamma(w)$	0_w

6.9 <u>Definition</u> Let $DG^{!}_{\times}$ be the category whose objects are pairs (G, Γ) consisting of a double groupoid G and a connection Γ on G with the further condition that G_0 is a singleton. The morphisms $f : (G, \Gamma) \to (H, \Delta)$ of $DG^{!}_{\times}$ are morphisms $f : G \to H$ of double groupoids such that the following diagram commutes

$$\begin{array}{ccc} G_2 & \xrightarrow{\;f\;} & H_2 \\ \Gamma \uparrow & & \uparrow \Delta \\ G_1 & \xrightarrow[\;f\;]{} & H_1 \end{array}$$

<u>Theorem</u> B <u>The categories $\mathbf{G}$ and $\mathbf{DG}^{!}$ are equivalent.</u>

<u>Proof</u> We define $\lambda : \mathbf{DG}^{!} \to \mathbf{G}$ to be the composite of the functors $\mathbf{DG}^{!} \longrightarrow \mathbf{DG} \xrightarrow{\omega} \mathbf{G}$, where the first functor is $(G, \Gamma) \mapsto G$. We define $\mu : \mathbf{G} \to \mathbf{DG}^{!}$ to be $\rho\ \psi$ where $\rho\ \psi(G)$ is equipped with the connection given by Proposition 6.7.

Clearly $\lambda\ \mu = \theta\ \psi : \mathbf{G} \to \mathbf{G}$, which is naturally equivalent to the identity by §3.

On the other hand, the isomorphism $\eta : \mu\omega(H) \to H$ of Theorem 6.8 is easily seen to preserve the connection, in the sense that

$$\eta(0; 0, x, 0, x) = \Gamma(x).$$

This completes the proof.

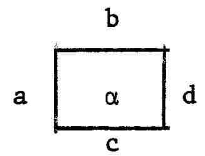
According to Theorem B a double groupoid with connection and exactly one vertex is isomorphic to a double groupoid obtained from a G-groupoid or equivalently from a crossed sequence. We now show how to model in terms of crossed sequences any connected double groupoid with connection. This will facilitate the proof of general results on double groupoids with connection.

6.10 <u>Definition</u> A double groupoid G is <u>connected</u> if the groupoid G_1 is connected. Let $\mathbf{DG}^{\wedge}$ be the category whose objects are quadruples (G, e, Γ, v) where G is a connected double groupoid, e is a preferred base point, Γ is a connection and $v : G_o \to St_{G_1} e$ is a function such that $v(e) = 0_e$ and $v(x) \in G_1(0, x)$ for all x in G_o. (Thus the choice of G_o is equivalent to the choice of tree subgroupoid of G_1.) A morphism $f : (G, e, \Gamma, v) \longrightarrow (H, e', \Delta, u)$ in $\mathbf{DG}^{\wedge}$ is a morphism $f : G \to H$ of double groupoids such that $f(e) = e'$, $f\Gamma = \Delta f$, $fv = uf$.

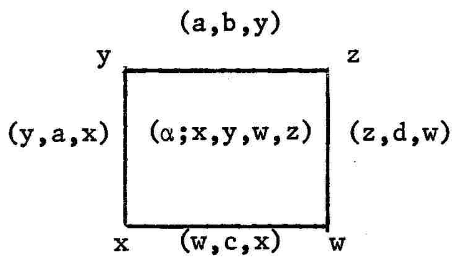
Let Ens* be the category of sets with base point.

<u>Theorem C</u> <u>The categories $C \times$ Ens* and $DG^{\wedge}$ are equivalent.</u>

<u>Proof</u> By Theorems A and B it is sufficient to prove the categories $DG^{!\cdot} \times$ Ens* and $DG^{\wedge}$ equivalent. Given an object (G,Γ) of $DG^{!\cdot}$ and a set S with base point e, we define $\sigma(G,\Gamma,S)$ to be the quadruple (H, e,Δ,u) where $H_0 = S$, $H_1(x,y) = \{(y,a,x) : a \in G_1\}$ and $H_2 = \{(\alpha;x,y,z,w) : \alpha \in G_2\}$. If $\alpha \in G_2$ has edges given by



then the edges of $(\alpha;x,y,w,z)$ are given by



The double groupoid operations + and $\circ$ are inherited in the obvious way from G; thus if $\beta \circ \alpha$ is defined in G_2 then

$$(\beta;y,p,z,q) \circ (\alpha;x,y,w,z) = (\beta \circ \alpha;\ x,p,w,q)$$

and if $\alpha' + \alpha$ is defined in G_2 then

$$(\alpha';w,z,r,s) + (\alpha;x,y,w,z) = (\alpha' + \alpha;\ x,y,r,s).$$

The connection Δ is defined by

(1) $$\Delta(y,a,x) = (\Gamma(a);x,x,x,y), \qquad x,y \in S,\ a \in G_1$$

and the tree u by

(2) $$u(x) = (x,0_e,e), \qquad x \in S$$

The extension of σ to a functor $\sigma : DG^{!\cdot} \times \text{Ens*} \to DG^{\wedge}$ is obvious.

We define $\delta : DG^{\wedge} \to DG^{!\cdot} \times$ Ens* to be the functor

$$(G,e,\Gamma,v) \mapsto (G\{e\},\ \Gamma|G_1\{e\},\ G_0)$$

Where G {e} is the largest sub-double groupoid of G with { e} as object set. Clearly $\delta\sigma$ is naturally equivalent to the identity so that is remains to show $\sigma\delta$ to be naturally equivalent to the identity.

Now $\sigma\delta(G, e, \Gamma, v) = (H, e, \Delta, u)$, where $H_0 = G_0$,

$$H_1 = \{(y, a, x) : x, y \in G_0, a \in G_1\{e\}\},$$

$$H_2 = \{(\alpha; x,y,w,z) : x,y,w,z \in G_0, \alpha \in G_2\{e\}\},$$

$\Delta(y, a, x) = (\Gamma(a); x, x, x, y)$ and $u(x) = (x, 0_e, e)$.

A morphism $\gamma : H \longrightarrow G$ is defined on H_0 by the identity, on H_1 by $\gamma(y, a, x) = v(y) + a - v(x)$ and $\gamma(\alpha; x, y, w, z)$ is defined to be the face

$-\Gamma v(y)$	1_b	$\Gamma v(z)$
0_a	α	0_d
$-\Gamma v(x)^{-1}$	1_c	$\Gamma v(w)^{-1}$

It is straightforward to check using the transport property of Γ that γ is an isomorphism of double groupoids with base point. Furthermore,

$\gamma\Delta(y, a, x)$ =

$-\Gamma v(x)$	1_a	$\Gamma v(y)$
$0_{s(x)}$	$\Gamma(a)$	0_a
$-\Gamma v(x)^{-1}$	$0_{s(x)}$	$\Gamma v(x)^{-1}$

=

<table>
<tr><td colspan="2">$1_{-v(x)}$</td><td>$\Gamma v(y)$</td></tr>
<tr><td>$1_{-v(x)}$</td><td>$\Gamma(a)$</td><td rowspan="2">0_a</td></tr>
<tr><td>$\Gamma(-v(x))$</td><td>$0_{-v(x)}$</td></tr>
</table>

$= \Gamma(v(y) + (a - v(x)))$, by transport,

$= \Gamma\gamma(y, a, x)$,

and $\gamma u(x) = \gamma(x, 0_e, e) = v(x) + 0_e - v(x) = v(x) = v\gamma(x)$; so γ preserves the connection and tree and thus γ determines a natural isomorphism $\gamma : \sigma\delta \longrightarrow 1$.

In a double groupoid with connection it is possible to define an operation which interchanges horizontal and vertical compositions. This fact is of interest in itself and is also useful in our discussion of homotopy in the next section.

6.11 Definition Let G be a double groupoid with connection Γ. The rotation τ associated with Γ is the function $\tau : G_2 \to G_2$ such that if $\alpha \in G_2$ then the edges of α and $\tau(\alpha)$ are related by

and $\tau(\alpha)$ is defined by

1_a	$\Gamma(b)^{-1}$	0_{-b}
$\Gamma(a)$	α	$-\Gamma(-d)^{-1}$
0_{-c}	$-\Gamma(-c)$	1_d

6.12 Theorem Let G be a double groupoid with connection. Then the associated rotation τ satisfies

(i) $\tau(\beta+\alpha) = \tau(\alpha) \circ \tau(\beta)$ whenever $\beta+\alpha$ is defined

(ii) $\tau(\alpha' \circ \alpha) = \tau(\alpha') + \tau(\alpha)$ whenever $\alpha' \circ \alpha$ is defined

(iii) $\tau^2(\alpha) = -\alpha^{-1}$

(iv) $\tau^4 = 1$

(v) τ is a bijection

Proof Clearly (iv) is a consequence of (iii), which also implies that τ^2 is a bijection; (v) follows easily.

For the proofs of (i), (ii) and (iii) we use Theorem C, which implies that it is sufficient to prove the theorem for the case of a double groupoid G with connection Γ arising from a crossed sequence $C : 0 \to B \to \Lambda \overset{\partial}{\to} L \to Q \to 0$, and set S, so that $G_1 = \{(y,a,x) : x,y \in G_0, a \in L\}$, $G_2 = \{(\alpha;a,b,c,d); x,y,z,w)$ where $x,y,z,w \in G_0$, $a,b,c,d \in L$, $\alpha \in \Lambda$ and $\partial\alpha = b + a - c - d\}$ with connection Γ given by $\Gamma(y,a,x) = ((0;0,a,0,a); x,x,x,y)$. The formula for τ then gives by a direct computation

$$\tau((\alpha;a,b,c,d) : x,y,z,w) = (((-b).a;-c.a.d.;b);w,x,z,y).$$

Properties (i), (ii) and (iii) follow easily from this.

We have found for (i) and (ii) of Theorem 6.12 direct proofs not involving the use of Theorem C. We have not found such proofs for (iii) or (iv).

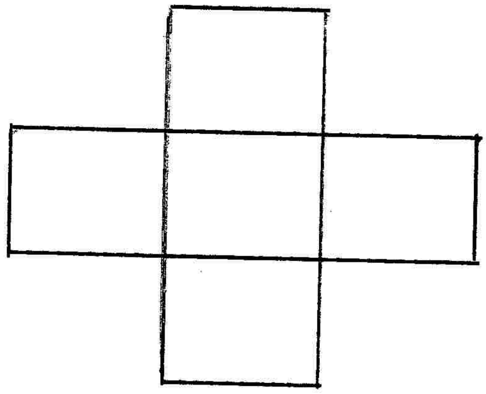
7. Homotopies for double groupoids

There are various possible definitions for homotopy $f \simeq g$ of morphisms of double groupoids $f, g : G \to H$. For our purposes we wish to impose the following conditions on such a definition.

1. The definition could depend on having a connection for H, but the homotopy classes should not depend on the choice of connection.
2. It should be rotation invariant: that is, if $\tau : H \to H$ is the rotation of (Definition 6.11) determined by a connection on H, then $f \simeq g$ should imply $\tau f \simeq \tau g$.
3. If H is a connected double groupoid, then H should have as strong deformation retract a double groupoid with only one object.
4. The Whitehead Theorem should hold, namely that if $f : G \to H$ is a morphism of connected double groupoids, then f is a homotopy equivalence if and only if f induces isomorphisms of π_1 and π_2.

Remarks 1. Condition 1 is reminiscent of the use of the Kan condition for defining homotopies in css-theory. In fact the existence of a connection implies the Kan condition (in its semi-cubical sense and in the dimensions 0, 1,2 we have available).
2. Condition 3 is reminiscent of the existence of minimal complexes.
3. Condition 4 will be expected to require some freeness assumptions, e.g. G_1 and H_1 are free groupoids.

The following definition is motivated by the idea that a homotopy of a square should in some way be determined by a cube. However for double groupoids we have no cubes! So we fold flat the surface of a cube less one face to give the figure



and then use the connection to fill in the corners so as to make a square.

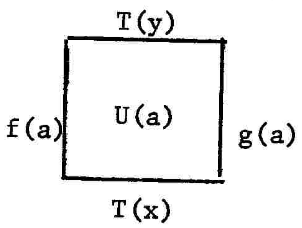
7.1 Definition Let $f, g : G \to H$ be a morphism of double groupoids and suppose H has a connection. A homotopy $f \simeq g$ is a triple (T, U, V) such that

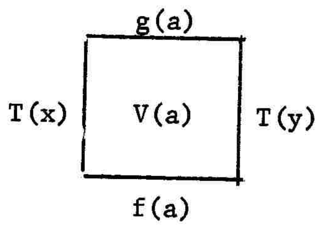
$$T : G_o \to H \ ; \quad U, V : G_1 \to H_2$$

with the following properties:

(i) if $x \in G_o$, then $T(x) : f(x) \to g(x)$ in H_1,

(ii) if $a : x \to y$ in G_1 then $U(a)$, $V(a)$ have edges given by



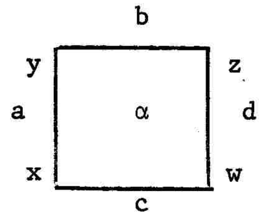


(iii) Linearity If $b, a \in G$ and $b + a$ is defined then

$$U(b+a) = U(b) \circ U(a)$$

$$V(b+a) = V(b) + V(a)$$

(iv) For each $\alpha \in G_1$ whose boundaries are given by



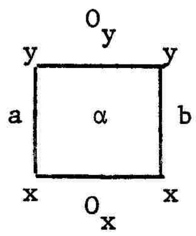
(*)

the following element of H_2 is $f(\alpha)$:

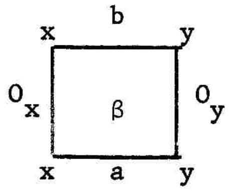
$\Gamma^{-1}T(y)$	$V(b)^{-1}$	$-\Gamma^{-1}T(z)$
$U(a)$	$g(a)$	$-U(d)$
$\Gamma T(x)$	$V(c)$	$-\Gamma T(w)$

(**)

Immediate consequences of (iii) are $U(0_x)=1_{T(x)}$, $V(0_x) = 0_{T(x)}$. Also it can be deduced from (iii) and (iv) that for faces of the form



we have $g(\alpha) + U(a) = U(b) + f(\alpha)$. Similarly, for a face



we have $g(\beta) \circ V(a) = V(b) \circ f(\beta)$.

We now show that if G, H are double groupoids and H has a connection Γ, then the homotopies of morphisms $G \to H$ form a groupoid HOM(GH). The object set of HOM(G,H) is the set Hom(G,H) of morphisms $G \to H$; if $f,g : G \to H$ are morphisms, then the arrows $f \to g$ of HOM(G,H) are to be the homotopies $f \simeq g$. If $\phi = (T,U,V) : f \simeq g$, $\psi = (T',U',V') : g \simeq h$ are homotopies, then the composite $\psi + \phi$ of the homotopies is defined to be $(T' + T,\ U' + U,\ V' \circ V)$; the following diagram may be simplified using the transport property for Γ to show that $\psi + \phi$ is a homotopy $f \simeq h$.

$\Gamma^{-1}T(y)$	$0_{-T(y)}$	$V(b)^{-1}$	$0_{-T(z)}$	$-\Gamma^{-1}T(z)$
$1_{T(y)}$	$\Gamma^{-1}T'(y)$	$V'(b)^{-1}$	$-\Gamma^{-1}T'(z)$	$1_{-T(z)}$
$U(a)$	$U'(a)$	$h(\alpha)$	$-U'(d)$	$-U(d)$
$1_{T(x)}$	$\Gamma T'(x)$	$V'(c)$	$-\Gamma T'(w)$	$1_{-T(w)}$
$\Gamma T(x)$	$0_{T(x)}$	$V(c)$	$0_{T(w)}$	$-\Gamma T(w)$

It is easily verified that with this composition HOM(G,H) is a groupoid. This structure depends on the connection Γ on H, and when we wish to emphasise this we write this groupoid as $\mathrm{HOM}^{\Gamma}(G,H)$.

We shall also need the following result :

7.2 Proposition Suppose given morphisms of double groupoids $f : F \to G$, $g, g' : G \to H$, $h : H \to K$ such that H and K have connections; then $g \simeq g'$ implies $hgf \simeq hg'f$.

Notice that 7.2 would be automatic if homotopy defined in 7.1 gave a 2-category structure for double groupoids with connection.

What certainly exists is a weak 2-category structure which is sufficient to imply 7.2.

7.3 Proposition Let $f : F \to G$, $h : H \to K$ be morphisms of double groupoids and let H,K have connections Γ,Δ respectively. Then there are induced morphisms

$$f^* : HOM^{\Gamma}(G,H) \to HOM^{\Gamma}(F,H)$$
$$h_* : HOM^{\Gamma}(G,H) \to HOM^{\Delta}(G,K)$$

satsifying the functorial rules

$$(kh)_* = k_*h_*, \quad (fe)^* = e^*f^*,$$
$$h_*f^* = f^*h_*, \quad 1_* = 1, \quad 1^* = 1.$$

Proof The definition of f^* is easy: if $\phi = (T,U,V) : g \simeq g'$, then $f^*(\phi) = (Tf,Uf,Vf)$ is easily seen to be a homotopy $gf \simeq g'f$, and the verification that f^* is a morphism is easy. To define h_*, again let $\phi = (T,U,V) : g \simeq g'$. Then $h_*(\phi) = (hT,U',hV)$ where if $a : x \to y$ in G,

$U'(a) = \Delta hT(y) \circ h\Gamma^{-1}T(y) \circ hU(a) \circ h\Gamma T(x) \circ \Delta^{-1}hT(x)$. (Notice that if h preserves the connections Γ, Δ, then $U' = hU$.) A direct computation shows that $h_*(\alpha)$ is a homotopy $hg \simeq hg'$, and that h_* is a morphism of groupoids. The verification of the functorial rule is straightforward.

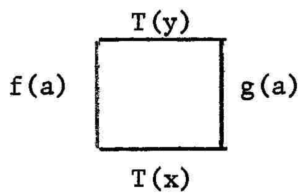
An easy consequence is that if $h = 1 : H \to H$, then we have an isomorphism of groupoids $HOM^\Gamma(G,H) \to HOM^\Delta(G,H)$ which is the identity on $Hom(G,H)$. Hence the set $[G,H]$ of homotopy classes of morphisms $G \to H$ is independent of the connection on H.

In the case of double groupoids G,H both with connection and morphisms preserving connection, a different expression for homotopies can be obtained.

7.4 Theorem Let $f,g : (G,\Delta) \to (H,\Gamma)$ be connection preserving morphisms of double groupoids. Then a triple (T,U,V) where $T : G_0 \to H_1$, $U,V : G_1 \to H_2$ is a homotopy $f \simeq g$ if and only if the following conditions hold:

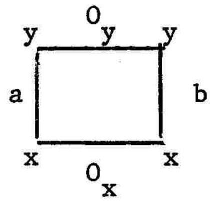
(i) if $x \in G_0$, then $T(x) : f(x) \to g(x)$ in H_1,

(ii) if $a : x \to y$ in G_1, then $U(a)$ has edges given by



(iii) $U(b+a) = U(b) \circ U(a)$ whenever $b+a$ is defined in G_1,

(iv) for eash $\alpha \in G_2$ with edges given by

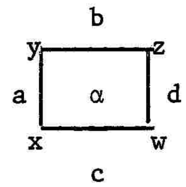


we have $g(\alpha) + U(a) = U(b) + f(\alpha)$,

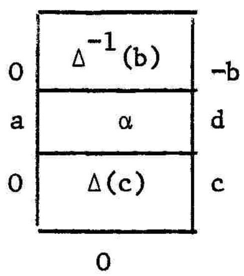
(v) if $a : x \to y$ in G_1, and τ is the rotation in H determined by Γ, then $V(a) = (\tau U(a))^{-1}$.

Proof Suppose first of all that (T,U,V) is a homotopy $f \simeq g$. All the conditions (i) - (iv) above follow from 7.1 (i) - (iv) directly, while the condition $V(a) = (\tau U(a))^{-1}$ comes from substituting $\alpha = \Delta(\alpha)$ in 7.1 (iv) and making a straightforward reduction using $f\Delta(a) = \Gamma f(a)$, $g\Delta(a) = \Gamma g(a)$, and the transport condition.

Now suppose U satisfies (i) - (iv) of the Theorem, and that V is given by (v). Let $\alpha \in G_2$ have faces given by



Then we may apply naturality for the face



to obtain

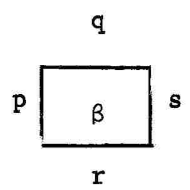
$$g(\Delta^{-1}(b)\circ\alpha\circ\Delta(c)) + U(a) = U(-b+d+c) + f(\Delta^{-1}(b)\circ\alpha\circ\Delta(c))$$

from which we deduce using linearity of U and the morphism properties of f,g that $f(\alpha)$ is the face

	$\Gamma f(b)$	
$1_{T(y)}$	$\Gamma^{-1}g(b)$	$-U(b)^{-1}$
$U(a)$	$g(\alpha)$	$-U(d)$
$1_{T(x)}$	$\Gamma g(a)$	$-U(c)$
	$\Gamma^{-1}f(c)$	

(1)

Now for any face



in H_2 a consequence of the definition of τ and transport is

	$\Gamma(q)$	
1_p	$\Gamma^{-1}(r)$	$\tau(\beta)$

=

$\Gamma(-p)$	β	$-\Gamma^{-1}(-s)$

Thus since $-U(b)^{-1} = \tau^2 U(b)$ (by 6.12), replacing β by $\tau U(b)$ we have

	$\Gamma f(b)$	
$1_{T(y)}$	$\Gamma^{-1}g(b)$	$-U(b)^{-1}$

=

$\Gamma^{-1}T(y)$	$\tau U(b)$	$-\Gamma^{-1}T(z)$

(2)

Similarly

$1_{T(x)}$	$\Gamma g(a)$	$-U(c)$
	$\Gamma^{-1}f(c)$	

=

$\Gamma T(x)$	$(\tau U(c))^{-1}$	$-\Gamma T(w)$

Finally combining (1), (2) and (3) we have $f(\alpha)$ is the face (*) and this completes the proof.

7.5 Remarks (i) The notion of homotopy as given by a pair (T,U) satisfying conditions on U (i) - (iv) of 7.4 seems to be essentially the same as that given by Gray [8] p. 281 -284 for a 2-natural transformation of 2-functors of 2-categories. These conditions do not depend on a connection for G or H, and so can be used to define a notion of homotopy

for morphisms of any double groupoids. There is also a similar notion of homotopy using the other coordinate.

(ii) We have not yet found a suitable abstract homotopy theory to include the notion of homotopy given by Definition 7.1. In particular, it seems unlikely that homotopy can be defined as a morphism $M(G) \to H$ from a cylinder object $M(G)$ in DG, basically because such a morphism would define a morphism $M(G)_1 \to H_1$ of groupoids, suggesting that homotopic morphisms in DG would have to restrict to homotopic morphisms of the 1-skeletons, which are groupoids. Such a notion would be too restrictive for our purposes. The contrast here is similar to that between the loose and strict double functor categories of Wyler [15] §16.

We now wish to relate homotopies for double groupoids, G-groupoids and crossed sequences. Recall that $\omega_1 : \underset{\sim}{DG}^* \to \underset{\sim}{G}$ is a functor from double groupoids with base point to G-groupoids. We shall require a homotopy (T,U,V) in $\underset{\sim}{DG}^*$ to be rel base point, that is to satisfy $T(*) = 0$.

7.6 Proposition. Let G,H be double groupoids with base point, let H have a connection, and let $(T,U,V): f \simeq g$ be a homotopy rel base point of morphisms $G \to H$ in $\underset{\sim}{DG}^*$. Then V is a homotopy $\omega_1(f) \simeq \omega_1(g)$ of morphisms $\omega_1(G) \to \omega_1(H)$ of G-groupoids.

The proof is trivial.

The previous proposition is not expressed in terms of 2-categories because of a difficulty in obtaining such a structure on double groupoids with the present definition of homotopy. (Such a structure may be possible using homotopy classes rel end maps of homotopies, but we do not pursue this line.)

7.7 REMARK Let $\underset{\sim}{DG}'$ be the category of double groupoids with connection and morphisms preserving the connection. In order to obtain a 2-category structure using $HOM^{\Gamma}(G,H)$ we require a pairing

$$HOM^{\Gamma}(G,H) \times HOM^{\Delta}(H,K) \to HOM^{\Delta}(G,K)$$

which is a morphism of groupoids; so if $\phi : f \simeq g : G \to H$ and $\phi' : f' \simeq g' : H \to K$ are homotopies, we require the interchange lemma

$$g^*(\phi') + f'_*(\phi) = g'_*(\phi) + f^*(\phi') : f'f \simeq g'g \quad (*)$$

If $\phi = (T,U,V)$, $\phi' = (T',U',V')$, then(*) is equivalent to the equations

$$T'g + f'T = g'T + T'f \qquad (1)$$

$$U'g + f'U = g'U + U'f \qquad (2)$$

$$V'g \circ f'V = g'V \circ V'f \qquad (3)$$

We have been unable to verify these equations in $\underset{\sim}{DG}'$ although in this category either side of (*) does give a homotopy $f'f \simeq g'g$. However our category $\underset{\sim}{DG}_!'$ of double groupoids with connection and exactly one object, and with morphisms preserving the connections, can be extended to a 2-category $\underset{\sim}{DG}^{!\bullet}$ by requiring the homotopies to be rel base point, i.e. of the form (0,U,V). Then (1) is trivially valid;

(2) follows from naturality of U' for the face

$$\begin{array}{ccc} & 0 & \\ fa & \boxed{Ua} & ga \\ & 0 & \end{array}$$

and (3) follows similarly.

7.8 Theorem The functor $\mu: G \to DG'$ extends to an equivalence $G \to DG'$ of 2-categories.

Proof Let $S : f \simeq g$ be a G-homotopy of G-groupoid morphisms $G \to H$. Let e denote the unique vertex of $\mu(G)$ and set $T(e) = 0 \in H_1$. We define for each $a \in \mu(G)_1 = Ob(G)$

$$U(a) = (1_{f(a)} - S(a) ; f(a), 0, 0, g(a)).$$

Then 7.4 (i) and (ii) certainly hold. To check 7.4 (iii), let $a,b \in Ob(G)$. Then

$$U(b+a) = (1_{f(b)} - S(b+a); \ f(b) + f(a), 0, 0, g(b) + g(a))$$

while

$$U(b) \circ U(a) = (1_{f(b)} - S(b) + 1_{g(b)} - S(a) - 1_{g(b)}; \ f(b)+f(a), \\ 0, 0, g(b) + g(a)).$$

But $-S(b) + 1_{g(b)}$ and $1_{f(a)} - S(a)$ commute (Corollary 2.3). Hence $U(b+a) = U(b) \circ U(a)$

In order to verify 7.4 (iv) we must prove in $\mu(H)$ the identity

$$\mu g(\alpha; a,0,0,b) + U(a) = U(b) + \mu f(\alpha; a, 0, 0, b),$$

for $\alpha \in St_G 0$, $a,b \in Ob(G)$ and $\partial\alpha = a - b$. According to the definition of + in $\mu(H)$, it will be enough to show

$$1_{f(a)} - S(a) + g(\alpha) = f(\alpha) + 1_{f(b)} - S(b) \qquad (*)$$

Now by linearity and naturality of S, using (2.2), we have

$g(\alpha) = f(\alpha) + 1_{f(b)} - 1_{f(a)} + S(a) - S(b)$. Thus the left hand side of (*) becomes $(1_{f(a)} - S(a)) + (f(\alpha) + 1_{f(b)} - 1_{f(a)}) + S(a) - S(b)$ which, because the terms in parentheses commute (being elements of $St_H O$ and $Cost_H O$ respectively) is equal to $f(\alpha) + 1_{f(b)} - S(b)$.

This verifies 7.4 (i) - (iv) and so proves that, with $V(a) = (\tau U(a))^{-1}$, the triple (T,U,V) is a homotopy $\mu(f) \simeq \mu(g)$.

It is clear that the correspondence $\phi : S \mapsto U$ defined above is a bijection. So to complete the proof that μ is an equivalence of 2-categories it is sufficient to prove ϕ preserves composition of homotopies.

Let $S_1 : f \simeq g$, $S \ : g \simeq h$ be G-homotopies of morphisms of G-groupoids. Then

$$(\phi(S_2) + \ (S_1))(a) = (1_{f(a)} - S_1(a) + 1_{g(a)} - S_2(a);\ f(a),0,0,h(a))$$

$$\phi(S_2 \circ S_1)(a) = (1_{f(a)} - S_2(a) \circ S_1(a);\ \ f(a),\ 0,0,h(a))$$

which are equal by 2.3.

8. The Whitehead Theorem

The Whitehead Theorem requires a number of preliminary results. First we define homotopy groups.

8.1 Definition Let G be a double groupoid and let $e \in G_o$. The first and second homotopy groups of G at e, which are written $\pi_1(G,e)$, $\pi_2(G,e)$, are respectively the last and first terms of the crossed sequence $\psi\omega_1(G,e)$ - that is, we have the crossed sequence

$$0 \to \pi_2(G,e) \to St_{\omega_1 G}e \to Ob(\omega_1 G) \to \pi_1(G,e) \to 0.$$

8.2 Theorem Let $f : G \to H$ be a homotopy equivalence of double groupoids with connection. Then f induces isomorphisms

$$f_* : \pi_i(G,e) \to \pi_i(H,f(e)) \qquad i = 1,2.$$

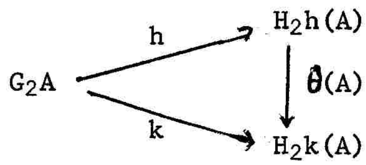
The case i = 2 of this theorem requires some lemmas.

First given a double groupoid G and elements a, b, c, d of G , let A be the 'matrix'

$$\begin{pmatrix} & b & \\ a & & d \\ & c & \end{pmatrix}$$

and write G_2A for the set of α in G_2 whose edges are given by A. Note that a morphism $h : G \to H$ induces $h_* : G_2A \to H_2h(A)$.

8.3 Lemma Given a homotopy $\theta : h \simeq k : G \to H$, where H has a connection, there is a bijection $\theta(A)$ making the following diagram commute



Proof Let $\theta = (T,U,V)$. Let β' be the element obtained from the second diagram in 7.1 (iv) by replacing $g(\alpha)$ by $\beta \in H_2h(A)$. Clearly $\theta(A) : \beta \mapsto \beta'$ is a bijection as required.

8.4 Lemma Let G have a connection, and let $h \simeq 1 : G \to G$. Then $h : G_2A \to G_2h(A)$ is a bijection.

Proof This follows from 8.3 on taking $H = G$, $k = 1$.

8.5 Lemma Let G,H have connections and let $f : G \to H$ be a homotopy equivalence. Then $f : G_2A \to H_2f(A)$ is a bijection.

Proof Consider the composites gf, fg in the sequence

$$G_2A \xrightarrow{f} H_2f(A) \xrightarrow{g} G_2gf(A) \xrightarrow{f} H_2fgf(A)$$

By 8.4, these are bijections. Hence f is a bijection.

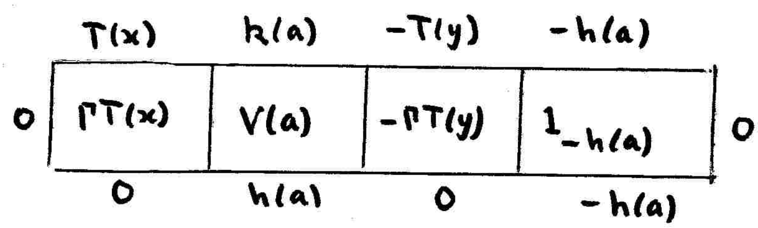
The case $i = 2$ of Theorem 8.2 is of course the special case of 8.5 when $a = b = c = d = 0_e$.

The case $i = 1$ of Theorem 8.2 is non-trivial only because it is not assumed that f is a homotopy equivalence rel base point. To express the lemma we need it is helpful to use the fundamental groupoid πH of the double groupoid H, defined to be the quotient groupoid H_1/N where N is the totally disconnected normal subgroupoid of H_1 such that at $x \in H_o$, $N\{x\} = \partial(St_{\omega_1(H,x)}x)$. Clearly π is a functor from double groupoids to groupoids.

8.6 <u>Lemma</u> <u>If $h \simeq k : H \to K$, where K is a double groupoid with connection, then h_*, $k_* : \pi H \to \pi K$ are homotopic morphisms of groupoids.</u>

<u>Proof</u> Let (T,U,V) be a homotopy $h \simeq k$. For each $x \in H_0$, let $\phi(x)$ be the class in $\pi K(h(x),k(x))$ of $T(x)$.

For $a : x \to y$ in H_1 we have the following element of $St_{\omega,K}\, h(x)$



showing that the classes in K of $T(y) + h(a)$, $k(a) + T(x)$ agree.

The proof of the case $i = 1$ of Theorem 8.2 now follows from standard results on groupoids.

The Whitehead Theorem we want is of course a converse to Theorem 8.2. We first prove a special case which allows us to reduce the general case to that of double groupoids with only one object.

Let H be a sub-double groupoid of the double groupoid G. We say H is full in G if H_1 is a full subgroupoid of G_1, and $H_2A = G_2A$ for every matrix A of elements of H_1. We say H is representative in G if the groupoid H_1 is representative in G_1. Now let $f,g : G \to G'$ be morphisms where G' has a connection, and let $\theta = (T,U,V)$ be a homotopy $f \simeq g$. We say $\theta : f \simeq g$ rel H if $x \in H_0$ implies $T(x) = f(0_x)$, and $a \in H_1$ implies $U(a) = 0_{f(a)}$, $V(a) = 1_{f(a)}$. We say H is a deformation retract of G if there is a morphism $r : G \to H$ such that $ri = 1_H$, $ir \simeq 1_G$ rel H, where $i : H \to G$ is the inclusion.

8.7 Theorem Let H be a representative, full sub-double groupoid of the double groupoid with connection G. Then H is a deformation retract of G.

Proof Let $i : H \to G$ be the inclusion. By 6.5.13 of [1] there is a morphism of groupoids $r : G_1 \to H_1$ such that $ri = 1_{H_1}$, and there is a function $T : G_0 \to G_1$ with $T(x) : r(x) \to x$ for x in G_0 and $T(y) = 0_{r(y)}$ if $y \in H_0$.

We now extend r over G_2.

First of all for $a : x \to y$ in G_1 we set

$$U(a) = (-\Gamma(-T(y))) \circ 0_a \circ (-\Gamma^{-1}(-T(x)))$$

$$V(a) = \Gamma^{-1}(-T(y)) + 1_a - \Gamma^{-1}(-T(x))$$

It is easy to verify that U, V are linear. Next if α is the face of G_2 given by (*) of 7.1 (iv) then $r(\alpha)$ is defined to be the face (**) of 7.1 (iv) with $g(\alpha)$ replaced by α. That r is a morphism of double groupoids and in fact a deformation retraction $G \to H$ follows from:

8.8 *Lemma* *Let $g : G \to H$ be a morphism of double groupoids such that H has a connection. Let $f : G_1 \to H_1$ be a morphism of groupoids and let T, U, V be functions satisfying 7.1 (i), (ii), (iii). Then 7.1 (iv) defines an extension of f to a morphism $f : G \to H$ of double groupoids such that (T, U, V) is a homotopy $f \simeq g$.*

The proof is straightforward.

8.9 *Corollary* *A connected double groupoid with connection is homotopy equivalent to a double groupoid with exactly one vertex.*

Theorem D (The Whitehead Theorem) *Let $f : G \to H$ be a morphism of connected double groupoids with connection such that G_1, H_1 are free groupoids and for some x in G_0 $f_* : \pi_i(G,x) \to \pi_i(H,f(x))$ is isomorphic for $i = 1,2$. Then f is a homotopy equivalence.*

Proof Let us first assume G, H have each exactly one vertex. We apply a Whitehead Theorem for crossed sequences (Theorem 3 of [3]) to obtain that $\psi\omega_1(f) : \psi\omega_1 G \to \psi\omega_1 H$ is a homotopy equivalence of crossed sequences. By Theorem 7.7 $f' = \rho\psi\omega_1(f) : \rho\psi\omega_1(G) \to \rho\psi\omega_1(H)$ is a homotopy equivalence of double groupoids.

Consider the diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ \eta_G \uparrow & & \uparrow \eta_H \\ \rho\psi\omega_1(G) & \xrightarrow[f']{} & \rho\psi\omega_1(H) \end{array}$$

where η_G, η_H are determined by connections Γ for G, Δ for H (Theorem 6.8), and let $g = \eta_H \; f' \; \eta_G^{-1}$ Then there is a homotopy $(T,U,V) : f \simeq g$ where $T(x) = 0_{f(x)}$, $U(a) = \Delta f(a) - f\Gamma(a)$, $V(a) = 1_{f(a)}$, $a \in G_1$. By Proposition 7.2 the composite of homotopy equivalences is a homotopy equivalence; hence g is a homotopy equivalence. Since $f \simeq g$ it follows again from Proposition 7.2 that f is a homotopy equivalence.

The general case follows from the special case, Corollary 8.9 and again Proposition 7.2.

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