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The classifying space of a crossed complex

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Introduction

The aim of this paper is to show one more facet of the role of crossed complexes as generalisations of both groups (or groupoids) and of chain complexes. We do this by defining and establishing the main properties of a classifying space functor $B:\mathscr{C}rs \to \mathscr{T}op$ from the category of crossed complexes to the category of spaces.

The basic example of a crossed complex is the fundamental crossed complex $\pi\mathbf{X}$ of a filtered space $\mathbf{X}$. Here $\pi_1\mathbf{X}$ is the fundamental groupoid $\pi_1(X_1, X_0)$ and $\pi_n\mathbf{X}$ for $n \geqslant 2$ is the family of relative homotopy groups $\pi_n(X_n, X_{n-1}, p)$ for all $p \in X_0$. These come equipped with the standard operations of $\pi_1\mathbf{X}$ on $\pi_n\mathbf{X}$ and boundary maps δ: $\pi_n\mathbf{X} \to \pi_{n-1}\mathbf{X}$. The axioms for crossed complexes are those universally satisfied for this example. Every crossed complex is of this form for a suitable filtered space $\mathbf{X}$ ([**17**], corollary 9·3, see also Section 2 below).

Thus a crossed complex C is like a chain complex of modules with a groupoid G as operators, but with non-Abelian features in dimensions 1 and 2, in the sense that the part $C_2 \to C_1$ is a crossed module with cokernel G. So crossed complexes have the virtues of chain complexes, in the sense of having a familiar homological algebra, at the same time as being able to carry non-Abelian information, such as that involved in a presentation of a group G. Their convenience is also shown by results of [**2**], which give them as the first level or linear approximation to combinatorial homotopy theory. Further details of the history of their use are given in [**18**].

Our main result is the following classification theorem.

THEOREM A. *If X is a CW-complex with skeletal filtration $\mathbf{X}$ then there is a natural bijection of homotopy classes*

$$[X, BC] \to [\pi\mathbf{X}, C].$$

This generalises a classical result on maps to an Eilenberg-Mac Lane space, and includes also the case of local coefficients (see Proposition 4·9 and Section 7). The special case when C is a reduced crossed module, and with a different definition of BC, is proved by simplicial group methods in [**23**]. However, these methods do not allow for the determination given below of the weak homotopy type of the function space TOP (X, BC).

A Corollary of Theorem A is that if X is a reduced CW-complex then there is map $f: X \to B\pi\mathbf{X}$ (where $\mathbf{X}$ is the skeletal filtration of X) such that if $\pi_i(X) = 0$ for

$1 < i < n$ then f induces an isomorphism of $\pi_i(X)$ for $1 \leqslant i \leqslant n$. Thus, under these assumptions, $B\pi\mathbf{X}$ models the n-type of X. In the case $n = 2$, this in effect recovers the result of [38] that crossed modules are algebraic models of 2-types (called '3-types' in [38]). In view of the Van Kampen Theorem for the fundamental crossed complex proved in [17], this result also allows for the computation of the n-types of certain colimits, a result difficult to express in the traditional cohomological language of Postnikov invariants.

Another property of the classifying space is that if $p: E \to D$ is a fibration of crossed complexes ([31] and Section 6) then the induced map $Bp: BE \to BD$ is a (Serre) fibration of spaces. This is convenient for relating BC with Postnikov invariants.

For the proof of these properties we use in an essential way the Van Kampen Theorem for the fundamental crossed complex of a CW-complex which is the union of subcomplexes ([17], corollary 5·2). We also use the monoidal closed structure on the category $\mathscr{C}rs$ of crossed complexes, with tensor product - $\otimes$ - and internal hom CRS (-, -), established in [20]. This enables us to enrich the category of filtered spaces over the category of crossed complexes.

The closed structure on $\mathscr{C}rs$ can also be used to generalise a result of Thom ([47], see also [29]) on the function space TOP $(X, K(G, n))$, where TOP (-, -) is the internal hom in the (convenient) category of compactly generated spaces. We prove that if C is a crossed complex and $\mathbf{X}$ is the skeletal filtered space of a CW-complex X, then there is a weak equivalence

$$B(\mathrm{CRS}\,(\pi\mathbf{X}, C)) \to \mathrm{TOP}\,(X, BC)$$

whose homotopy class is natural.

A key role in this work is played by an Eilenberg-Zilber type theorem, namely an associative and commutative natural transformation

$$\theta: \pi\mathbf{X} \otimes \pi\mathbf{Y} \to \pi(\mathbf{X} \otimes \mathbf{Y})$$

for all filtered spaces $\mathbf{X}$ and $\mathbf{Y}$. Further, θ is an isomorphism if $\mathbf{X}$ and $\mathbf{Y}$ are the skeletal filtrations of CW-complexes. The proof of the existence of θ is given in the final Section 8, since the proof requires the explanation of some results on ω-groupoids. The further result that θ is an isomorphism if $\mathbf{X}$ and $\mathbf{Y}$ are cofibred connected filtered spaces is proved in [5].

The structure of the paper is as follows. Section 1 sets out our conventions for the closed monoidal category of filtered spaces, and relates this to the category of simplicial sets. Section 2 recalls basic facts on the fundamental crossed complex functor π from filtered spaces to crossed complexes. In order to obtain the enrichment of $\mathscr{F}\mathscr{T}op$ over $\mathscr{C}rs$ we make a minor change to the conventions of [17]. It is in this section that we define the classifying space BC of a crossed complex C. Section 3 uses the tensor product of crossed complexes defined in [20] and proves the above mentioned enrichment. This gives sufficient information on homotopies to prove Theorem A. In Section 4 we give applications. In Section 5 we prove that the tensor product of crossed complexes of relative free type is of relative free type. The proof uses crucially the monoidal closed structure to verify that the functor $A \otimes$ - preserves colimits. In Section 6 we give applications to fibrations of crossed complexes, in particular proving that a fibration $E \to D$ of crossed complexes induces a fibration $BE \to BD$ of classifying spaces. In Section 7 we show that crossed complexes give a convenient expression for local systems. Section 8 proves Theorem 3·1.

The homotopy classification of Theorem A was announced in [**10**]. The paper [**21**] explains the relation of these results to work of Whitehead [**49**] on chain complexes with operators.

For R. Brown, this paper returns to the spirit of work in his thesis, written under the direction of Michael Barratt, and which used techniques of chain complexes and simplicial abelian groups to study Postnikov invariants of function spaces [**8**, **9**].

1. *The monoidal closed category $\mathscr{F}\mathscr{T}op$ of filtered spaces*

By a space is meant a compactly generated topological space X, i.e. one which has the final topology with respect to all continuous functions $C \to X$ for all compact Hausdorff spaces C. The category of spaces and continuous maps will be written $\mathscr{T}op$. This category is well-known to be cartesian closed, and the space of continuous maps $Y \to Z$ will be written $\mathrm{TOP}(Y, Z)$. (See for example [**12**].)

A *filtered space* $\mathbf{X}$ is a space X and a sequence $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots$ of subspaces of X. A *filtered map* $f: \mathbf{X} \to \mathbf{Y}$ of filtered spaces is a map $f: X \to Y$ such that $f(X_n) \subseteq Y_n$ for all n. So we have a category $\mathscr{F}\mathscr{T}op$ of filtered spaces and filtered maps.

If $\mathbf{X}, \mathbf{Y}$ are filtered spaces, their *tensor product* $\mathbf{X} \otimes \mathbf{Y}$ is the filtered space consisting of $X \times Y$ and the family of subspaces

$$(X \otimes Y)_n = \bigcup_{p+q=n} X_p \times Y_q$$

where the union is simply the union of subspaces of $X \times Y$.

If $\mathbf{Y}, \mathbf{Z}$ are filtered spaces, we denote by $\mathrm{FTOP}(\mathbf{Y}, \mathbf{Z})$ the filtered space with total space $\mathrm{TOP}(Y, Z)$ and family of subspaces

$$\mathrm{FTOP}(\mathbf{Y}, \mathbf{Z})_n = \{f \in \mathscr{T}op(Y, Z) : f(Y_q) \subseteq Z_{n+q} \text{ for all } q \geqslant 0\}.$$

Proposition 1·1. *Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be filtered spaces. Then there is a natural isomorphism of filtered spaces*

$$e: \mathrm{FTOP}(\mathbf{X} \otimes \mathbf{Y}, \mathbf{Z}) \to \mathrm{FTOP}(\mathbf{X}, \mathrm{FTOP}(\mathbf{Y}, \mathbf{Z}))$$

given by the usual exponential rule

$$e(f)(x)(y) = f(x, y), \quad f: X \times Y \to Z, \quad x \in X, \quad y \in Y. \quad \blacksquare$$

Thus we conclude that $\otimes$ and FTOP give the category $\mathscr{F}\mathscr{T}op$ the structure of symmetric, monoidal, closed category.

We denote the standard n-simplex by Δ^n, and the same space with its skeletal filtration by $\boldsymbol{\Delta}^n$. In particular, we write I for Δ^1, and $\mathbf{I}$ for the corresponding filtered space. We write $\mathbf{I}^n$ for the n-fold tensor product of $\mathbf{I}$ with itself – then $\mathbf{I}^n$ is the CW-filtered space of the standard n-cube.

The category $\mathscr{F}\mathscr{T}op$ is a natural home for the realization functor $|\;|$ from the category of simplicial sets to $\mathscr{T}op$. That is, any simplicial set K has a filtration by skeleta, and this induces the skeletal filtration of the realization $|K|$. This filtered space is written $|\boldsymbol{K}|$.

For a filtered space $\mathbf{X}$ there is a filtered singular complex $R\mathbf{X}$ which in dimension

n consists of the filtered maps $\Delta^n \to \mathbf{X}$. This $R\mathbf{X}$ has the structure of simplical set, and it is easily verified that if $\mathcal{S}imp$ is the category of simplicial sets, then

$$R : \mathcal{F}\mathcal{T}op \to \mathcal{S}imp$$

is right adjoint to $\qquad | \; | : \mathcal{S}imp \to \mathcal{F}\mathcal{T}op.$

The category $\mathcal{S}imp$ is also cartesian closed. However, for simplicial sets K and L the standard homeomorphism

$$|K \times L| \cong |K| \times |L|$$

is not cellular. We will return to this later.

The internal hom functor in the category $\mathcal{S}imp$ is written SIMP (-, -).

We now discuss homotopies in $\mathcal{F}\mathcal{T}op$. If $f_0, f_1 : \mathbf{X} \to \mathbf{Y}$ are filtered maps, a *homotopy* $f_0 \simeq f_1$ is a homotopy f_t with $f_t(X_n) \subseteq Y_{n+1}$ for all n; that is, a homotopy is a map $\mathbf{I} \otimes \mathbf{X} \to \mathbf{Y}$, which may be viewed as an element h of $R_1(\text{FTOP}(\mathbf{X}, \mathbf{Y}))$ with $\partial^0 h = f_0, \partial^1 h = f_1$. Similarly, the elements of $R_n(\text{FTOP}(\mathbf{X}, \mathbf{Y}))$ may also be viewed as maps $\Delta^n \otimes \mathbf{X} \to \mathbf{Y}$.

There are four possible definitions of n-fold homotopy in $\mathcal{F}\mathcal{T}op$, corresponding to four enrichments of this category, two of which will be used in later sections.

Firstly, let $\mathbf{E}^n$ be the filtered space of the skeletal filtration of the n-ball so that $\mathbf{E}^1 = \mathbf{I} = \{0, 1\} \cup e^1$, and for $n \geqslant 2$

$$\mathbf{E}^n = \{1\} \cup e^{n-1} \cup e^n.$$

Then an n-fold homotopy could be defined as a map $\mathbf{E}^n \otimes \mathbf{X} \to \mathbf{Y}$. This definition is relevant to the enrichment of the category $\mathcal{F}\mathcal{T}op$ over the category $\mathcal{C}rs$ of crossed complexes, and it is this enrichment which will be our primary concern in this paper. However, in establishing this enrichment we will use at one point of the proof the equivalence of the categories of crossed complexes and ω-groupoids established in [**16**], together with the monoidal closed structures established in [**20**].

Secondly, one could take an n-fold homotopy to be a map $\mathbf{I}^n \otimes \mathbf{X} \to \mathbf{Y}$. This is relevant to the enrichment of the category $\mathcal{F}\mathcal{T}op$ over the category of ω-groupoids. Since the latter objects are also Kan cubical sets, this enrichment allows the abstract homotopy theory of Kamps [**33**] to be applied.

The other two possibilities are to use homotopies $\Delta^n \otimes \mathbf{X} \to \mathbf{Y}$ or $\mathbf{G}^n \otimes \mathbf{X} \to \mathbf{Y}$, where $\mathbf{G}^n$ is the unit ball with its equatorial cell decomposition. These homotopies are relevant to enrichments of $\mathcal{F}\mathcal{T}op$ over the categories of simplicial T-complexes [**1**], and of ∞-groupoids [**19**], respectively. They will not concern us here.

2. *The nerve and classifying space of a crossed complex*

We assume as known the definition of the category $\mathcal{C}rs$ of crossed complexes (see for example [**16**], [**20**], [**21**]). In the Introduction, we explained the fundamental crossed complex functor

$$\pi : \mathcal{F}\mathcal{T}op \to \mathcal{C}rs.$$

Remark 2·1. The functor π is defined in [**17**] on J_0 filtered spaces, and the convention there makes $\pi_0 \mathbf{X} = \pi_0 X_0$. However, the modified definition above with $\pi_0 \mathbf{X} = X_0$ not only concurs with standard usage, but also is essential in the present context. We wish to form the crossed complex $\pi(\text{FTOP}(\mathbf{Y}, \mathbf{Z}))$ – but FTOP $(\mathbf{Y}, \mathbf{Z})$ is rarely a J_0 filtered space. In any case, we do want $\pi\mathrm{FTOP}(\mathbf{Y},\mathbf{Z})$ to be in dimension 0 simply the set $\mathscr{FT}op(\mathbf{Y},\mathbf{Z})$ of filtered maps $\mathbf{Y}\to\mathbf{Z}$.

The functor π composes with the realisation functor $|\,|:\mathscr{S}imp\to\mathscr{FT}op$ to give a functor, also written $\pi:\mathscr{S}imp\to\mathscr{C}rs$.

PROPOSITION 2·2. *The fundamental crossed complex πK of a simplicial set K is given as a coend*

$$\pi K = \int^n K_n \times \pi\Delta^n.$$

Proof. This follows from the Van Kampen Theorem for the fundamental crossed complex of a CW-complex covered by a family of subcomplexes (Corollary 5·2 of [**17**]). ∎

Definition 2·3. The *nerve* functor $N:\mathscr{C}rs\to\mathscr{S}imp$ is defined by

$$(NC)_n = \mathscr{C}rs(\pi\boldsymbol{\Delta}^n, C)$$

where $\boldsymbol{\Delta}^n$, as in Section 1, is the standard n-simplex with its standard cell structure and cellular filtration, and NC has the simplicial structure induced by the usual maps of $\boldsymbol{\Delta}^n, n\geqslant 0$.

The crossed complex $\pi\boldsymbol{\Delta}^n$ is a free crossed complex on the cells of Δ^n, and the boundaries are determined by the universal example, namely $\delta:\pi_r\boldsymbol{\Delta}^r\to\pi_{r-1}\boldsymbol{\Delta}^r$, which is itself given by the homotopy addition lemma ([**48**], p. 175).

For $n\geqslant 2$, the crossed complex $\pi\boldsymbol{\Delta}^n$ involves non-Abelian groups in dimension 2, and a groupoid in dimension 1 which acts on the crossed complex. The homotopy addition lemma (which says, intuitively, that the boundary of a simplex is the sum of its faces) therefore needs to be stated with care.

If σ is an r-simplex with $r\geqslant 4$, then, analogously to the purely additive theory of homology, we have the formula

$$\delta\sigma = (\partial_0\sigma)^{-a} + \sum_{i=1}^{r}(-)^i\partial_i\sigma,$$

where $a = \partial_2\partial_3\ldots\partial_r\sigma$. Here the action of $-a$ transports the base point of $\partial_0\sigma$ to the common basepoint of the other faces so as to make addition possible. For a 3-simplex σ, we have the non-Abelian formula

$$\delta\sigma = (\partial_0\sigma)^{-a} + \partial_2\sigma - \partial_1\sigma - \partial_3\sigma$$

and for a 2-simplex σ, we have the groupoid formula

$$\delta\sigma = \partial_2\sigma + \partial_0\sigma - \partial_1\sigma.$$

One easily verifies that $\delta\delta\sigma = 0$ for an r-simplex σ with $r\geqslant 3$ and that $\delta_0\delta\sigma = \delta_1\delta\sigma$ for a 2-simplex σ. Cubical versions of such formulae are also well known (see, for example, [**16**]).

A morphism $f:\pi\boldsymbol{\Delta}^n\to C$ for a crossed complex C can now be described as given by a family of elements in varying dimensions of C, indexed by the cells of Δ^n, and related by the homotopy addition lemma. Blakers in [7] uses such families to construct for a reduced crossed complex C (there called a group system) a simplicial set associated to C which is essentially our nerve of C, and this construction is also used in [**1**].

One of our main results is the following. The proof involves standard manipulations with ends and coends.

THEOREM 2·4. *The functor* $\pi : \mathscr{S}imp \to \mathscr{C}rs$ *is left adjoint to the nerve functor* N: $\mathscr{C}rs \to \mathscr{S}imp$.

Proof. Let K be a simplicial set and let C be a crossed complex. Then we have natural bijections

$$\begin{aligned}\mathscr{S}imp(K, NC) &\cong \int_n \mathscr{S}et(K_n, \mathscr{C}rs(\pi\Delta^n, C))\\ &\cong \int_n \mathscr{C}rs(K_n \times \pi\Delta^n, C)\\ &\cong \mathscr{C}rs\left(\int^n (K_n \times \pi\Delta^n), C\right)\\ &\cong \mathscr{C}rs(\pi K, C). \quad \blacksquare\end{aligned}$$

Remark 2·5. The fact that the functor $\pi : \mathscr{S}imp \to \mathscr{C}rs$ is a left adjoint implies that it preserves all colimits. However, the generalised Van Kampen Theorem of [**17**] (Theorem C) is not an immediate consequence of this fact since it is a theorem about the functor π from filtered spaces to crossed complexes, and one of the conditions for $\pi \operatorname{colim}_\lambda \mathbf{U}^\lambda \cong \operatorname{colim}_\lambda \pi\mathbf{U}^\lambda$ is that each filtered space $\mathbf{U}^\lambda$ should be homotopy full, in the sense of [**17**]. (A better term, following the usage of [**22**], would be 'connected'.) It would be interesting to know whether this generalised Van Kampen Theorem can be deduced from the fact that $\pi : \mathscr{S}imp \to \mathscr{C}rs$ preserves all colimits.

It is useful to know that NC is a Kan complex, so that homotopy of simplicial maps $K \to NC$ is an equivalence relation. This follows from a more general result in Section 6 on Kan fibrations. We mention here that NC has the structure of Kan complex in a strong way. Define an element $f : \pi\Delta^n \to C$ of $(NC)_n$ to be *thin* if f maps the top dimensional element of $\pi\Delta^n$ to zero in C. The thin elements satisfy Dakin's axioms: degenerate elements are thin; any horn has a unique thin filler; if all faces but one of a thin element are thin, then so also is the last face. Thus NC has the structure of a T-complex [**24**], and in fact N yields an equivalence between crossed complexes and simplicial T-complexes ([**1**], reproved in [**41**, **42**]).

We define the classifying space BC of the crossed complex C by

$$BC = |NC|,$$

the geometric realization of the nerve of C. This defines the classifying space functor

$$B : \mathscr{C}rs \to \mathscr{T}op.$$

Note that a crossed complex C is filtered by its skeleta $C^{[n]}$, $n \geqslant 0$, where $C^{[n]}$ agrees with C in dimensions $\leqslant n$, and is trivial in dimensions $> n$. This filtered crossed complex is written $\mathbf{C}$. Then $B\mathbf{C}$ is a filtered space. It is proven in [**1**] (following the analogous cubical result from [**17**]) that there is a natural isomorphism

$$\pi B\mathbf{C} \cong C.$$

In order to describe the homotopy groups of BC, we recall some facts on crossed complexes ([**16**], p. 258). Let C be a crossed complex. Then $\pi_0 C$ is the set of components of C. The *fundamental groupoid* $\pi_1 C$ of C is the quotient of the groupoid C_1 by the normal, totally intransitive subgroupoid $\delta(C_2)$. For $n \geqslant 2$, and for $p \in C_0$, the *homology group* $H_n(C, p)$ is the quotient group

$$\operatorname{Ker}[\delta : C_n(p) \to C_{n-1}(p)]/\operatorname{Im}[\delta : C_{n+1}(p) \to C_n(p)].$$

This group is Abelian, and the action of C_1 on C_n induces an action of $\pi_1 C$ on the family $H_n C$. In other words, $H_n C$ is naturally a module over the groupoid $\pi_1 C$. Clearly, $\pi_0 C$, $\pi_1 C$ and $H_n C$ are functors of C.

Proposition 2·6. *For any crossed complex C, there are natural isomorphisms*

$$\pi_0 BC \cong \pi_0 C, \quad \pi_1(BC, C_0) \cong \pi_1 C, \quad \pi_n(BC, p) \cong H_n(C, p),$$

and the two latter isomorphisms preserve the actions.

Proof. These isomorphisms follow from Remark 2 on p. 258 of [**16**], which is easily extended to include the facts about the actions. ∎

This result shows that the spaces BC should be regarded as generalising the classical Eilenberg-Mac Lane spaces. If C is a crossed complex which is trivial except in dimension n, where it is a group G (Abelian if $n \geqslant 2$), then NC is isomorphic to the classical $K(G, n)$.

3. *Tensor products and homotopies*

In order to obtain the homotopy classification theorem from Theorem 2·4, we need to use tensor products and homotopies of crossed complexes.

The tensor product $A \otimes B$ of crossed complexes A and B is defined in [**20**], and shown in proposition 3·10 of that paper to be generated by elements $a \otimes b$ in dimension $m+n$, where $a \in A_m$, $b \in B_n$ with a number of defining relations determined by those in the universal examples $\pi\mathbf{E}^m \otimes \pi\mathbf{E}^n$. We omit further details. Some calculations of $A \otimes B$ are given in section 6 of [**20**], and in particular it is proved in corollary 6·6 that the canonical morphism $A \otimes B_0 \to A \otimes B$ is an injection.

It follows from the definition of $A \otimes B$ that

$$\pi_0(A \otimes B) \cong \pi_0 A \times \pi_0 B, \quad \pi_1(A \otimes B) \cong \pi_1 A \times \pi_1 B.$$

In order to construct the enrichment of $\mathscr{F}\mathscr{T}\mathit{op}$ over $\mathscr{C}\mathit{rs}$ we need the first part of the following basic result.

Theorem 3·1. *If* $\mathbf{X}$ *and* $\mathbf{Y}$ *are filtered spaces, then there is a natural morphism*

$$\theta : \pi\mathbf{X} \otimes \pi\mathbf{Y} \to \pi(\mathbf{X} \otimes \mathbf{Y})$$

such that:

(i) *θ is associative;*

(ii) *if $*$ denotes a singleton space or crossed complex, then the following diagrams are commutative*

$$\begin{tikzcd} \pi\mathbf{X} \arrow[dr, "\cong"'] & (\pi\mathbf{X}) \otimes * \arrow[l, "\cong"'] \arrow[d, "\theta"] \\ & \pi(\mathbf{X} \otimes *) \end{tikzcd} \qquad \begin{tikzcd} * \otimes \pi\mathbf{X} \arrow[r, "\cong"] \arrow[d, "\theta"'] & \pi\mathbf{X} \arrow[dl, "\cong"] \\ \pi(* \otimes \mathbf{X}) & \end{tikzcd}$$

(iii) *θ is commutative in the sense that if $T_c: C \otimes D \to D \otimes C$ is the natural isomorphism of crossed complexes described in* [**20**], *and $T_t: \mathbf{X} \otimes \mathbf{Y} \to \mathbf{Y} \otimes \mathbf{X}$ is the twisting map, then the following diagram is commutative*

$$\begin{tikzcd} \pi\mathbf{X} \otimes \pi\mathbf{Y} \arrow[r] \arrow[d, "T_c"'] & \pi(\mathbf{X} \otimes \mathbf{Y}) \arrow[d, "\pi(T_t)"] \\ \pi\mathbf{Y} \otimes \pi\mathbf{X} \arrow[r] & \pi(\mathbf{Y} \otimes \mathbf{X}); \end{tikzcd}$$

(iv) *if $\mathbf{X}, \mathbf{Y}$ are the skeletal filtrations of CW-complexes, then θ is an isomorphism.*

The proof is deferred to Section 8 since it uses the techniques of ω-groupoids. Note that the construction of the natural transformation θ could in principle be proved directly, but this would be technically difficult because of the complications of the relations for the tensor product of crossed complexes.

It is shown in [**20**] that $- \otimes B$ has right adjoint an internal hom functor $\mathrm{CRS}(B, -)$, so that for any crossed complexes A, B, C there is a natural isomorphism

$$\mathscr{C}rs\,(A \otimes B, C) \cong \mathscr{C}rs\,(A, \mathrm{CRS}\,(B, C)).$$

An element of $\mathrm{CRS}\,(B, C)$ in dimension 0 is simply a morphism $B \to C$. Let $\mathscr{I}$ be the crossed complex $\pi\mathbf{I}$. An element of $\mathrm{CRS}\,(B, C)$ in dimension 1 may be identified with a morphism $\mathscr{I} \to \mathrm{CRS}\,(B, C)$ and so with a morphism $\mathscr{I} \otimes B \to C$. Such morphisms are called (left) homotopies $B \to C$. They may be identified with pairs (h, f) where $f: B \to C$ is a morphism and $h: B_m \to C_{m+1}, m \geqslant 0$, is a family of functions satisfying conditions which are given in (3·1) of [**20**].

THEOREM 3·2. *The category $\mathscr{F}\mathscr{T}op$ may be enriched over the monoidal closed category $\mathscr{C}rs$ of crossed complexes.*

Proof. If $\mathbf{Y}, \mathbf{Z}$ are filtered spaces then we form the crossed complex $\pi\mathrm{FTOP}\,(\mathbf{Y}, \mathbf{Z})$. This gives the factorisation required for an enrichment

$$\begin{tikzcd} \mathscr{F}\mathscr{T}op^{op} \times \mathscr{F}\mathscr{T}op \arrow[rr] \arrow[dr] & & \mathscr{S}et \\ & \mathscr{C}rs \arrow[ur, "u"'] & \end{tikzcd}$$

where $u(C) = C_0$. The required composition functor $c_{\mathscr{C}rs}$ is defined by the following commutative diagram

$$\begin{tikzcd} \pi\mathrm{FTOP}(\mathbf{Y}, \mathbf{Z}) \otimes \pi\mathrm{FTOP}(\mathbf{X}, \mathbf{Y}) \arrow[rr, "c_{\mathscr{C}rs}"] \arrow[dr, "\theta"'] & & \pi\mathrm{FTOP}(\mathbf{X}, \mathbf{Z}) \\ & \pi\mathrm{FTOP}(\mathbf{Y}, \mathbf{Z}) \otimes \pi(\mathrm{FTOP}(\mathbf{X}, \mathbf{Y})) \arrow[ur, "\pi(c_{\mathscr{F}\mathscr{T}op})"'] & \end{tikzcd}$$

where $c_{\mathscr{F}\mathscr{T}op}$ is the composition in the monoidal closed category $\mathscr{F}\mathscr{T}op$. The properties required for an enrichment, as given in [**34**], are clearly satisfied. ∎

In a similar spirit, we now prove that the functor $\pi : \mathscr{FT}op \to \mathscr{C}rs$ is a homotopy functor.

PROPOSITION 3·3. *There is a natural morphism of crossed complexes*

$$\psi : \pi(\mathrm{FTOP}\,(\mathbf{Y},\mathbf{Z})) \to \mathrm{CRS}\,(\pi\mathbf{Y}, \pi\mathbf{Z})$$

which is the identity in dimension 0. *In particular, a homotopy* $f_t : f_0 \simeq f_1 : \mathbf{Y} \to \mathbf{Z}$ *in* $\mathscr{FT}op$ *induces a (left) homotopy* $\pi f_t : \pi f_0 \simeq \pi f_1 : \pi\mathbf{Y} \to \pi\mathbf{Z}$ *in* $\mathscr{C}rs$.

Proof. It is sufficient to construct the morphism $\hat{\psi}$ as the composition in the following commutative diagram

$$\begin{tikzcd}
\pi(\mathrm{FTOP}(\mathbf{Y},\mathbf{Z})) \otimes \pi\mathbf{Y} \arrow[rr, "\hat{\psi}"] \arrow[dr, "\theta"'] & & \pi\mathbf{Z} \\
& \pi(\mathrm{FTOP}(\mathbf{Y},\mathbf{Z}) \otimes \mathbf{Y}) \arrow[ur, "\pi(\epsilon)"'] &
\end{tikzcd}$$

where $\epsilon : \mathrm{FTOP}\,(\mathbf{Y},\mathbf{Z}) \otimes Y \to \mathbf{Z}$ is the evaluation morphism, i.e. the adjoint to the identity on $\mathrm{FTOP}\,(\mathbf{Y},\mathbf{Z})$.

The last statement of the proposition is the information given by ψ in dimension 1. ∎

Similar statements hold for right homotopies of crossed complexes. A right homotopy $C \to D$ is a morphism $C \otimes \mathscr{I} \to D$, or, equivalently, a morphism $C \to \mathrm{CRS}\,(\mathscr{I}, D)$. We may also define a right homotopy in $\mathscr{FT}op$ to be a map $\mathbf{Y} \otimes \mathbf{I} \to \mathbf{Z}$. By Theorem 3·1, such a map gives rise to a right homotopy $\pi\mathbf{Y} \otimes \mathscr{I} \to \pi\mathbf{Z}$.

We can now show that the classifying space functor $B : \mathscr{C}rs \to \mathscr{T}op$ is also a homotopy functor. For this we need the crossed complex version of the Eilenberg-Zilber Theorem.

PROPOSITION 3·4. *Let K and L be simplicial sets. Then there is a natural homotopy equivalence of crossed complexes*

$$\pi K \otimes \pi L \to \pi(K \times L)$$

extending the inclusions $\pi K \otimes L_0 \to \pi(K \times L)$, $K_0 \otimes \pi L \to \pi(K \times L)$.

Proof. This follows by the methods of acyclic models. ∎

COROLLARY 3·5. *The nerve functor* $N : \mathscr{C}rs \to \mathscr{S}imp$ *and the classifying space functor* $B : \mathscr{C}rs \to \mathscr{T}op$ *are homotopy functors. More precisely, a homotopy* $f_0 \simeq f_1 : C \to D$ *in* $\mathscr{C}rs$ *induces homotopies* $Nf_0 \simeq Nf_1$ *in* $\mathscr{S}imp$ *and* $Bf_0 \simeq Bf_1 : BC \to BD$ *in* $\mathscr{T}op$.

Proof. Let K and L be simplicial sets. Then we have natural maps

$$\begin{aligned}
\mathscr{S}imp\,(K, \mathrm{SIMP}\,(L, NC)) &\cong \mathscr{S}imp\,(K \times L, NC) \\
&\cong \mathscr{C}rs\,(\pi(K \times L), C) \\
&\leftrightarrows \mathscr{C}rs\,(\pi K \otimes \pi L, C) \\
&\cong \mathscr{C}rs\,(\pi K, \mathrm{CRS}\,(\pi L, C)) \\
&\cong \mathscr{S}imp\,(K, N(\mathrm{CRS}\,(\pi L, C))).
\end{aligned}$$

We deduce that the adjunction of Theorem 2·4 extends to simplicial maps

$$\mathrm{SIMP}\,(L, NC) \leftrightarrows N(\mathrm{CRS}\,(\pi L, C)).$$

Now let $f_0 \simeq f_1 : C \to D$. Because the tensor product of crossed complexes is symmetric ([20]), we may view such a 1-fold homotopy either as a left homotopy $\mathscr{I} \otimes C \to D$ or as a right homotopy $C \otimes \mathscr{I} \to D$. Choosing the latter form, we obtain a morphism $C \to \mathrm{CRS}(\mathscr{I}, D)$ and hence simplicial maps $NC \to N(\mathrm{CRS}(\mathscr{I}, D)) \to \mathrm{SIMP}(\mathbf{I}, ND)$. This gives a homotopy $(NC) \times \mathbf{I} \to ND$ with end maps Nf_0, Nf_1, and so a homotopy $Bf_0 \simeq Bf_1 : BC \to BD$. ∎

Remark. This last result is one place where cubical methods are more efficient, since for cubical sets K and L we have a cellular isomorphism $|K \otimes L| \cong |K| \times |L|$.

We are now able to prove our main result, giving the weak homotopy type of the function space $\mathrm{TOP}(X, BC)$. In the case BC is an Eilenberg-Mac Lane space $K(G, n)$ for Abelian G, this result is essentially a theorem of Thom [47] (see also [9, 29]). It also includes a result of Gottlieb [28] for the case BC is a space $K(G, 1)$ for any group G (see Proposition 4·7).

THEOREM A. *If X is a CW-complex, and C is a crossed complex, then there is a weak homotopy equivalence*

$$\eta : B(\mathrm{CRS}(\pi\mathbf{X}, C)) \to \mathrm{TOP}(X, BC),$$

and a bijection of sets of homotopy classes

$$[X, BC] \cong [\pi\mathbf{X}, C]$$

which is natural with respect to morphisms of C and cellular maps of X.

Proof. By the previous corollary, it is sufficient to assume that X is the realisation $|L|$ of a simplicial set L. The argument of Corollary 3·5 yields a homotopy equivalence of simplicial sets

$$N(\mathrm{CRS}(\pi L, C)) \to \mathrm{SIMP}(L, NC).$$

The result now follows from the well-known weak equivalence

$$|\mathrm{SIMP}(L, M)| \to \mathrm{TOP}(|L|, |M|)$$

for any simplicial sets L and M.

Since

$$\pi_0 \mathrm{TOP}(X, BC) = [X, BC], \quad \text{and} \quad \pi_0 N(\mathrm{CRS}(\pi\mathbf{X}, C)) = [\pi\mathbf{X}, C],$$

this gives the bijection of homotopy classes, whose naturality follows from the fact that the homotopy class of η is natural. ∎

We denote by $[X, Y]_*$ the set of pointed homotopy classes of pointed maps $X \to Y$ of pointed spaces X, Y. Similarly, for pointed crossed complexes D, C, we denote by $[D, C]_*$ the set of pointed homotopy classes of pointed morphisms $D \to C$. The definition is given in detail in section 5 of [20]. If C is a pointed crossed complex, then BC is naturally a pointed space.

THEOREM A_*. *If X is a pointed CW-complex and C is a pointed crossed complex, there is a bijection of sets of pointed homotopy classes which is natural with respect to pointed morphisms of C and pointed, cellular maps of X, and which fits into a commutative diagram*

$$\begin{tikzcd}[X, BC]_* \arrow[rr, "\cong"] \arrow[dr] & & {[\pi\mathbf{X}, C]_*} \arrow[dl] \\ & \mathrm{Hom}(\pi_1(X, *), \pi_1(C, *)), & \end{tikzcd}$$

in which we identify $\pi_1(BC, *)$ with $\pi_1(C, *)$, $\pi_1(X, *)$ with $\pi_1(\mathbf{X}, *)$.

Proof. The proof of the existence of the isomorphism of sets of pointed homotopy classes is similar to the proof of Theorem A, but using the pointed constructions $\otimes_*$ and CRS_* described in [**20**]. We omit further details.

The slanting maps are induced by the functor $\pi_1(-,*)$ and the given identifications. To prove commutativity, it is sufficient to assume that $X = |L|$ for some Kan simplicial set L. One then has to check that maps transformed by the following arrows induce the same map of fundamental groups:

$$\mathcal{T}\!op(|L|, BC) \leftarrow \mathcal{S}imp(L, NC) \rightarrow \mathcal{C}rs(\pi L, C).$$

But this is clear on checking the values of these maps on 1-dimensional elements. ∎

4. *Applications*

We first apply Theorem A to show how to realise some homotopy n-types as BC for some crossed complex C.

THEOREM 4·1. *Let $n \geqslant 1$, and let X be a reduced CW-complex with $\pi_i X = 0$, $1 < i < n$. (This condition is vacuous if $n = 1, 2$.) Then there is a crossed complex C with $C_i = 0, i > n$, together with a map $f: X \to BC$ inducing an isomorphism of homotopy groups $\pi_i X \to \pi_i BC$ for $1 \leqslant i \leqslant n$.*

Proof. Let $\mathbf{X}$ be the CW-filtration of X, and let p be the vertex of X. Let $D = \pi\mathbf{X}$ and let C be the crossed complex such that

$$C_i = \begin{cases} D_i & \text{if } \ 0 \leqslant i < n, \\ D_n/\delta D_{n+1} & \text{if } \ i = n, \\ 0 & \text{if } \ i > n. \end{cases}$$

Then there is a unique morphism $g: D \to C$ which is the identity in dimensions $<n$ and is the quotient morphism in dimension n. This morphism g induces an isomorphism of fundamental groupoids, and of homology groups $H_i(D,p) \to H_i(C,p)$ for $2 \leqslant i \leqslant n$.

By Theorem A_* there is a pointed morphism $f: X \to BC$ whose homotopy class corresponds to $g: \pi\mathbf{X} \to C$. Without loss of generality we may assume f is cellular. Then for all $i \geqslant 1$, the following diagram is commutative, where $S^i = e^0 \cup e^i$ is the i-sphere:

$$\begin{tikzcd}
{[S^i, X]_*} \arrow[r, "f_*"] \arrow[d, "\cong"'] & {[S^i, BC]_*} \arrow[d, "\cong"] \arrow[dr, "\cong"] & \\
{[\pi\mathbf{S}^i, \pi\mathbf{X}]_*} \arrow[r, "(\pi f)_*"] \arrow[d, "\cong"'] & {[\pi\mathbf{S}^i, \pi\mathbf{BC}]_*} \arrow[r, "\cong"'] & {[\pi\mathbf{S}^i, C]_*} \arrow[d, "\cong"] \\
H_i(\pi\mathbf{X}, p) \arrow[rr, "\cong", "H_i g"'] & & H_i(C, p)
\end{tikzcd}$$

The assumptions on X imply that the map $[S^i, X]_* \to [\pi \mathbf{S}^i, \pi \mathbf{X}]_*$ is bijective for $1 \leqslant i \leqslant n$. So the result on π_i follows. ∎

Remarks 4·2. (i) This Corollary shows that if $\pi_i X = 0$, $1 < i < n$, then the n-type of X is described completely by a crossed complex. For $n = 1$, this is well known, and for $n = 2$ it is essentially due to Mac Lane and Whitehead [**38**]. Indeed, they prove that the 2-type (for which they use the term 3-type) of a reduced CW-complex X is described by the crossed module $\pi_2(X, X^1) \to \pi_1 X^1$, which is the same crossed module as arises for $n = 2$ in the proof of Theorem 4·1.

(ii) The crossed complex C constructed in the proof of Theorem 4·1 has the property that $H_i(C) = 0$, $1 < i < n$; $H_j(C) \cong \pi_j(X)$, $j = 1, n$; $C_i = 0$, $i > n$. It is known that $H^{n+1}(\pi_1 X, \pi_n X)$ can be represented by equivalence classes of such complexes ([**30**], [**32**], [**36**], [**37**]). In particular, the equivalence class of C can be regarded as giving the first k-invariant of X.

The following result is sometimes useful for giving an explicit presentation of a crossed module representing the 2-type of a space. It is due to Loday [**35**], with a different proof.

PROPOSITION 4·3. *Let X be a reduced CW-complex and let P be a group such that there is a map $f: BP \to X$ which is surjective on fundamental groups. Let $F(f)$ be the homotopy fibre of f and let $M = \pi_1 F(f)$, so that we have a crossed module $M \to P$. Then there is a map $X \to B(M \to P)$ inducing an isomorphism of π_1 and π_2.*

Proof. Let $f: BP \to X$ be a cellular map which is surjective on fundamental groups. Let Y be the reduced mapping cylinder $M(f)$ of f, and let $j: BP \to Y$ be the inclusion. Then the crossed module $\pi_2(Y, BP) \to \pi_1 BP$ is isomorphic to $\mu: M \to P$. Also j is surjective on fundamental groups, and it follows that the inclusion $X^1 \to Y$ is deformable by a homotopy to a map g', say, with image in BP. This homotopy extends to a homotopy of the inclusion $X \to Y$ to a map $g: X \to Y$ extending g'. Let $\mathbf{Y}$ be the filtered space in which Y_0 is the base point of Y, $Y_1 = BP$, $Y_i = Y$ for $i \geqslant 2$. Then $C = \pi \mathbf{Y}$ is the trivial extension by zeros of the crossed module $M \to P$. The map $g: X \to Y$ induces a morphism $g_*: \pi \mathbf{X} \to \pi \mathbf{Y}$ which is realised by a map $X \to B(M \to P)$ inducing an isomorphism of π_1 and π_2. ∎

Example 4·4. Here is an application of the last proposition which uses the Generalised Van Kampen Theorem for crossed modules. Let X be a CW-complex which is the union of connected subcomplexes Y and Z such that $W = Y \cap Z$ is a $K(P, 1)$, i.e. is a space BP. Suppose that the inclusions of W into Y and Z induce isomorphisms of fundamental groups. Then, as in Proposition 4·3, the 2-types of Y and Z may be described by crossed modules $M \to P$ and $N \to P$ respectively, say. By results of [**17**] which are applied in [**11**] to this situation, the crossed module describing the 2-type of X is the coproduct $M \circ N \to P$ of the crossed P-modules M and N. In fact $M \circ N$ is a quotient of the semidirect product $M \rtimes N$, where N acts on M via P. Note also that if M and N are finite, so also is $M \circ N$. For more information on this construction, see [**26**]. Thus crossed module methods do yield explicit computations of 2-types.

We now give another application to the homotopy classification of maps, using the following result which is essentially the remark on p. 37 of [**17**].

PROPOSITION 4·5. *If Y is a CW-complex with skeletal filtration $\mathbf{Y}$, then there is a* *homotopy fibration $F \to Y \to B\pi\mathbf{Y}$ whose homotopy exact sequence at a base point y is isomorphic to Whitehead's exact sequence*

$$\ldots \to \Gamma_n(Y,y) \to \pi_n(Y,y) \to H_n(\tilde{Y}_y) \to \ldots$$

where $\tilde{Y}_y$ denotes the universal cover of Y based at y. Further, if $\pi_i(Y,y) = 0$ for $1 < i < n$, then the fibre F is n-connected.

Proof. Results of [1] (compare the cubical results in [17]) give a Kan fibration $R\mathbf{Y} \to N\pi\mathbf{Y}$ (since in the terminology of [1], $N\pi\mathbf{Y}$ is the underlying simplicial set of the simplicial T-complex $\rho\mathbf{Y}$). One shows that for a CW-complex $\mathbf{Y}$ the inclusion of $R\mathbf{Y}$ into the singular complex of Y is a homotopy equivalence. The identification of the homotopy groups of the fibres with Whitehead's Γ-groups is carried out in Theorem 8·5 of [17] for the cubical case, and the simplicial case is analogous. ∎

COROLLARY 4·6. *If Y is a connected CW-complex such that $\pi_i Y = 0$ for $1 < i < n$, and X is a CW-complex with* $\dim X \leqslant n$, *then there is a natural bijection of homotopy classes*

$$[X, Y] \cong [\pi\mathbf{X}, \pi\mathbf{Y}].$$

Proof. The assumptions imply that the fibration $Y \to B\pi\mathbf{Y}$ induces a bijection $[X, Y] \to [X, B\pi\mathbf{Y}]$. The bijection $[X, B\pi\mathbf{Y}] \to [\pi\mathbf{X}, \pi\mathbf{Y}]$ follows from Theorem A. ∎

This Corollary may also be obtained as a concatenation of results proved in [**49**]. It is also proved in general circumstances in [**2**].

There is one particular case when we can identify CRS (C, A). For a groupoid H let $X(H, 1)$ denote the crossed complex which consists of H in dimension 1 and which is trivial elsewhere. If G and H are groupoids, then GPD (G, H) denotes the internal hom object in the category of groupoids, and $[G, H]$ denotes the set π_0GPD (G, H) of components of GPD (G, H). If G is connected, $x \in G_0$, and $f\colon G \to H$ is a morphism, then the vertex group GPD $(G, H)(f)$ is isomorphic to the centraliser of $f(G(x))$ in $H(fx)$. So the following result with Theorem A yields a result of Gottlieb [**28**] on the fundamental group of spaces of maps into an Eilenberg-Mac Lane space $K(H, 1)$.

PROPOSITION 4·7. *If B is a crossed complex and H is a groupoid, then there is a homotopy equivalence of crossed complexes*

$$\mathrm{CRS}\,(B, X(H, 1)) \simeq X(\mathrm{GPD}\,(\pi_1 B, H), 1).$$

Proof. Let Z be a crossed complex. Then there are natural bijections

$$\begin{aligned}
[Z, \mathrm{CRS}\,(B, X(H, 1))] &\cong [Z \otimes B, X(H, 1)] \\
&\cong [\pi_1(Z \otimes B), H] \\
&\cong [\pi_1 Z \times \pi_1 B, H] \\
&\cong [\pi_1 Z, \mathrm{GPD}\,(\pi_1 B, H)] \\
&\cong [Z, X(\mathrm{GPD}\,(\pi_1 B, H), 1)].
\end{aligned}$$

The result follows. ∎

In the pointed case we get an even simpler result.

PROPOSITION 4·8. *If B is a pointed, connected crossed complex and H is a pointed* *groupoid, then the crossed complex* $\mathrm{CRS}_*(B, X(H,1))$ *has set of components bijective with the set* $\mathscr{G}p(\pi_1(B,*), H(*))$ *of morphisms of groups* $\pi_1(B,*) \to H(*)$, *and all components of* $\mathrm{CRS}_*(B, X(H,1))$ *have trivial* π_1 *and* H_i *for* $i \geqslant 2$.

Proof. An argument similar to that in the proof of the previous proposition yields

$$[*, \mathrm{CRS}_*(B, X(H,1))] \cong [*, \mathrm{GPD}_*(\pi_1 B, H)],$$

which gives the first result. The second result follows since for any pointed crossed complex Z and morphism $f: B \to X(H,1)$ we have

$$\begin{aligned}[(Z,*), (\mathrm{CRS}_*(B, X(H,1), f)] &\cong [Z \otimes B, X(H,1) \mid 1 \otimes f : * \otimes B \to X(H,1), \\ &\qquad *: Z \otimes * \to X(H,1)] \\ &\cong [\pi_1 Z \times \pi_1 B, H \mid 1 \times f : * \times \pi_1 B \to H, * : \pi_1 Z \times * \to H] \\ &\cong [\pi_1 B, H \mid f] \cong *. \quad \blacksquare\end{aligned}$$

There is another interesting special case of the homotopy classification. Let $n \geqslant 2$. Let M be an abelian group and let $\mathrm{Aut}M$ be the group of automorphisms of M. Let $\chi(M,n)$ be the pointed crossed complex which is: $\mathrm{Aut}M$ in dimension 1; M in dimension n; which has the given action of $\mathrm{Aut}M$ on M; and has trivial boundaries. Let C be a crossed complex; in useful cases, C will be of free type. We suppose C reduced and pointed. Let $\alpha: \pi_1(C,*) \to \mathrm{Aut}M$ be a morphism. The set of pointed homotopy classes of morphisms $C \to \chi(M,n)$ which induce α on fundamental groups is written $[C, \chi(M,n)]^{\alpha}_*$. This set is easily seen to have an Abelian group structure, induced by the addition on operator morphisms $C_n \to M$ over α. So we obtain the homotopy classification:

PROPOSITION 4·9. *If* X *is a pointed reduced CW-complex, and* $\alpha: \pi_1(X,*) \to \mathrm{Aut}M$, *then there is a natural bijection*

$$[X, B\chi(M,n)]^{\alpha}_* \cong [\pi \mathbf{X}, \chi(M,n)]^{\alpha}_*,$$

where the former set of homotopy classes denotes the set of pointed homotopy classes of maps inducing α *on fundamental groups.* $\blacksquare$

The proof is immediate from Theorem A_*.

This result is related in Section 7 to the case of local coefficients, so that Proposition 4·9 is essentially a result of [**27**]. (See also [**40**, **45**].)

We remark that another application of the classifying space of a crossed complex is given in [**5**], where it is shown that for a natural filtration on the James construction JBC of the classifying space of a crossed complex C, there is an isomorphism $\pi JBC \cong JC$, where JC is the James construction (i.e. free monoid with respect to the tensor product) of the pointed crossed complex C.

5. *Tensor products and free crossed complexes*

The results of this section will be used in discussing fibrations of crossed complexes in Section 6.

The category $\mathscr{C}rs$ is complete and cocomplete. So we may specify a crossed complex by a presentation, that is, by giving a set of generators in each dimension and a set of defining relations of the form $u = v$, where u, v are well-formed formulae of the same dimension made from the generators and the additions, negatives, actions, and boundary maps.

Write $\mathbb{C}(n)$ for the crossed complex freely generated by one generator c_n in dimension n. So $\mathbb{C}(0)$ is $\{1\}$; $\mathbb{C}(1)$ is essentially the groupoid $\mathscr{I}$ which has two objects $0, 1$ and non-identity elements $c_1 : 0 \to 1$ and $c_1^{-1} : 1 \to 0$; and for $n \geqslant 2$ $\mathbb{C}(n)$ is in dimensions n and $n-1$ an infinite cyclic group with generators c_n and δc_n respectively, and is otherwise trivial. Let $\mathbb{S}(n-1)$ be the subcomplex of $\mathbb{C}(n)$ which agrees with $\mathbb{C}(n)$ up to dimension $n-1$ and is trivial otherwise. If $\mathbf{E}^n$ and $\mathbf{S}^{n-1}$ denote the skeletal filtrations of the standard n-ball and $(n-1)$-sphere, where $E^0 = \{0\}$, $S^{-1} = \varnothing$, $E^1 = I = \{0, 1\} \cup e^1$, $S^0 = \{0, 1\}$, and for $n \geqslant 2$, $E^n = \{1\} \cup e^{n-1} \cup e^n$, $S^{n-1} = \{1\} \cup e^{n-1}$, then it is clear that for all $n \geqslant 0$, $\mathbb{C}(n) \cong \pi \mathbf{E}^n$ and $\mathbb{S}(n-1) \cong \pi \mathbf{S}^{n-1}$.

We now follow [**13**] and define a particular kind of morphism $j : A \to C$ called a crossed complex morphism of relative free type. Let A be any crossed complex. A sequence of morphisms $j_n : C^{n-1} \to C^n$ may be defined with $C^0 = A$ by choosing any family of morphisms $f_n^\lambda : \mathbb{S}(m_\lambda - 1) \to C^{n-1}$ for $\lambda \in \Lambda_n$ and any m_λ, and forming the pushout

$$\begin{tikzcd} \coprod_{\lambda \in \Lambda_n} \mathbb{S}(m_\lambda - 1) \arrow[r, "(f_n^\lambda)"] \arrow[d] & C^{n-1} \arrow[d, "j_n"] \\ \coprod_{\lambda \in \Lambda_n} \mathbb{C}(m_\lambda) \arrow[r] & C^n. \end{tikzcd}$$

Let $C = \operatorname{colim}_n C^n$, and let $j : A \to C$ be the canonical morphism. We call $j : A \to C$ a *crossed complex morphism of relative free type*. The images x^{m_λ} of the elements c_{m_λ} in C are called basis elements of C relative to A. We can conveniently write

$$C = A \cup \{x^{m_\lambda}\}_{\lambda \in \Lambda_n,\, n \geqslant 0},$$

and may abbreviate this in some cases, for example to $C = A \cup x^n \cup x^m$, analogously to standard notation for CW-complexes. Also we may without loss of generality assume that the basis elements are added in order of increasing dimension, so that in forming C^n from C^{n-1} as above, all the m_λ are n. In this case C has the following structure.

C_0 is the disjoint union of A_0 and Λ_0;

C_1 is the coproduct of C_0-groupoids A_1^* and $F(\Lambda_1)$, where A_1^* is the groupoid obtained from A_1 by adjoining the objects of C not already in A, and $F(\Lambda_1)$ is the free groupoid on Λ_1 considered as a graph over C_0 via the maps f_1^λ;

C_2 is the coproduct of crossed C_1-modules A_2^* and $F(\Lambda_2)$, where A_2^* is the C_1-crossed module induced from the A_1-crossed module A_2 by the morphism of groupoids $A_1 \to C_1$, and $F(\Lambda_2)$ is the free crossed C_1-module on Λ_2 via the maps f_2^λ;

C_n, for $n \geqslant 3$, is the direct sum of $(C_1/\delta C_2)$-modules A_n^* and $F(\Lambda_n)$, where A_n^* is the module induced from the $(A_1/\delta A_2)$-module A_n by the morphism of groups $(A_1/\delta A_2) \to (C_1/\delta C_2)$, and $F(\Lambda_n)$ is the free $(C_1/\delta C_2)$-module on Λ_n.

The boundary maps are in all cases induced by the boundary maps in A and by the maps f_n^λ.

We remark that for $A = \varnothing$ we get by this construction the crossed complexes of free type which were considered in [18] under the name 'free crossed complexes'. If $\mathbf{X}$ is the skeletal filtration of a CW-complex, then the crossed complex $\pi\mathbf{X}$ is of free type; if $\mathbf{Y}$ is a subcomplex of $\mathbf{X}$ then the induced morphism $\pi\mathbf{Y} \to \pi\mathbf{X}$ is of relative free type. (The reason for avoiding the term free crossed complex is that such crossed complexes do not seem to arise from adjoints of a forgetful functor.) Reduced crossed complexes of free type are called homotopy systems in [49], and free crossed chain complexes in [2], [3].

The exponential law in $\mathscr{C}\!\mathit{rs}$ and the symmetry of the tensor product have the following consequence of which special cases are dealt with in [13] and are used in [4] (lemma III·9·2).

PROPOSITION 5·1. *If $A \to C$ and $U \to W$ are morphisms of relative free type then so also is $A \otimes W \cup C \otimes U \to C \otimes W$, where $A \otimes W \cup C \otimes U$ denotes the pushout of the pair of morphisms*

$$A \otimes W \leftarrow A \otimes U \to C \otimes U.$$

Proof. Since the tensor product $-\otimes-$ is symmetric and $-\otimes B$ has a right adjoint, the functors $-\otimes B$ and $A \otimes -$ preserve colimits. Using this fact and standard properties of pushouts, one easily proves the following four statements:

(i) If in a pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

the morphism $A \to C$ is of relative free type, so is the morphism $B \to D$.

(ii) If in a sequence of morphisms of crossed complexes

$$A^0 \to A^1 \to \ldots \to A^n \to \ldots$$

each morphism is of relative free type, so are the composites $A^0 \to A^n$ and the induced morphism $A^0 \to \operatorname{colim}_n A^n$.

(iii) If in a commutative diagram

$$\begin{tikzcd} A^0 \arrow[r] \arrow[d] & A^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & A^n \arrow[r] \arrow[d] & \cdots \\ C^0 \arrow[r] & C^1 \arrow[r] & \cdots \arrow[r] & C^n \arrow[r] & \cdots \end{tikzcd}$$

each vertical morphism is of relative free type, so is the induced morphism $\operatorname{colim}_n A^n \to \operatorname{colim}_n C^n$.

(iv) If the following squares are pushouts

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ W \arrow[r] & X \end{tikzcd}$$

then so is the induced square

$$\begin{tikzcd} A \otimes W \cup C \otimes U \arrow[r] \arrow[d] & B \otimes X \cup D \otimes V \arrow[d] \\ C \otimes W \arrow[r] & D \otimes X. \end{tikzcd}$$

These four facts are the basis for an inductive proof. One starts by verifying the proposition when $A \to C$, $U \to W$ are of the type $\mathbb{S}(n-1) \to \mathbb{C}(n)$ and $\mathbb{S}(m-1) \to \mathbb{C}(m)$. Using the fact that $Y \otimes -$ and $- \otimes Y$ preserve coproducts, one deduces the result in the case when $A \to C$, $U \to W$ are of the type $\coprod_\lambda \mathbb{S}(n-1) \to \coprod_\lambda \mathbb{C}(n)$ and $\coprod_\lambda \mathbb{S}(m-1) \to \coprod_\lambda \mathbb{C}(m)$. Putting morphisms of this type in (iv), and using (i) one finds that the proposition is true for morphisms of simple relative free type, that is for morphisms $B \to D$, $V \to X$ obtained as pushouts

$$\begin{tikzcd} \amalg_\lambda \mathbb{S}(n-1) \arrow[r] \arrow[d] & B \arrow[d] \\ \amalg_\lambda \mathbb{C}(n) \arrow[r] & D \end{tikzcd} \qquad \begin{tikzcd} \amalg_\mu \mathbb{S}(m-1) \arrow[r] \arrow[d] & V \arrow[d] \\ \amalg_\mu \mathbb{C}(m) \arrow[r] & X. \end{tikzcd}$$

Next, using (i), (ii), (iv) one proves the result for composites of morphisms of simple relative free type. A general morphism of relative free type is a colimit of simple ones, as in (ii), and the full result now follows from (ii) and (iii). ∎

COROLLARY 5·2. *If $A \to C$ is a morphism of relative free type and W is a crossed complex of free type, then $A \otimes W \to C \otimes W$ is of relative free type.* ∎

6. *Fibrations of crossed complexes*

We recall a definition due to Howie in [**31**].

Definition 6·1. A morphism $p: E \to D$ of crossed complexes is a *fibration* if

(i) the morphism $p_1: E_1 \to D_1$ is a fibration of groupoids;

(ii) for each $n \geq 2$ and $x \in E_0$, the morphism of groups $p_n: E_n(x) \to D_n(px)$ is surjective.

The morphism p is a *trivial fibration* if it is a fibration, and also a weak equivalence, by which is meant that p induces a bijection on π_0 and isomorphisms $\pi_1(E, x) \to \pi_1(D, px)$, $H_n(E, x) \to H_n(D, px)$ for all $x \in E_0$ and $n \geq 2$.

Howie shows in [**31**] that a fibration of crossed complexes leads to a family of exact sequences involving the H_n, π_1 and π_0.

We now consider cofibrations, following methods of [**43**] which were developed for crossed complexes in [**13**].

Consider the following diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\ C \arrow[r] \arrow[ur, dashed] & D. \end{tikzcd}$$

If given i the dotted completion exists for all morphisms p in a class $\mathscr{F}$, then we say that i has the left lifting property (LLP) with respect to $\mathscr{F}$. We say a morphism $i:A\to C$ is a *cofibration* if it has the LLP with respect to all trivial fibrations. We say a crossed complex C is *cofibrant* if the inclusion $\varnothing\to C$ is a cofibration. We shall also need the definition that p has the right lifting property (RLP) with respect to a class $\mathscr{C}$ if in the above diagram, given p, then the dotted completion exists for all i in the class $\mathscr{C}$.

PROPOSITION 6·2. *Let $p:E\to D$ be a morphism of crossed complexes. Then the following conditions are equivalent*:

(i) *p is a fibration*;

(ii) (covering homotopy property) *p has the* RLP *with respect to the inclusion $C\otimes 1\to C\otimes \mathbb{C}(m)$ for all cofibrant crossed complexes C and $m\geqslant 1$*;

(ii)′ *the covering property (ii) holds for $m=1$*;

(iii) *if C is a cofibrant crossed complex then the induced morphism $p_*:\mathrm{CRS}(C,E)\to \mathrm{CRS}(C,D)$ is a fibration*;

(iv) *the induced map of nerves $Np:NE\to ND$ is a Kan fibration.*

Proof. (i) ⇒ (ii) We follow the method of [**43**]. We verify the covering property by constructing a lifting in the left hand of the following diagrams, where $1\to\mathbb{C}(m)$ is the inclusion. Let p' in the right hand

$$\begin{tikzcd} C\otimes 1 \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ C\otimes\mathbb{C}(m) \arrow[r] & D \end{tikzcd} \qquad \begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \mathrm{CRS}(\mathbb{C}(m),E) \arrow[d, "p'"] \\ C \arrow[r] & \mathrm{CRS}(1,E)\times_D \mathrm{CRS}(\mathbb{C}(m),D) \end{tikzcd}$$

diagram be induced by p and the inclusion $1\to\mathbb{C}(m)$. Then a lifting in the left hand diagram is equivalent to a lifting in the right hand diagram. Since C is cofibrant, such a lifting exists if p' is a trivial fibration. But by the exponential law, for this it is sufficient to show that p has the RLP with respect to the inclusion

$$\mathbb{S}(n)\otimes\mathbb{C}(m)\cup\mathbb{C}(n+1)\otimes 1\to\mathbb{C}(n+1)\otimes\mathbb{C}(m).$$

For $n=-1$, this corresponds precisely to the fibration property of p. In general, a lifting of the image of the top basis element of $\mathbb{C}(n+1)\otimes\mathbb{C}(m)$ is chosen, and the value of the lifting on the remaining basis element of $\mathbb{C}(n+1)\otimes\mathbb{C}(m)$, namely $c_{n+1}\otimes\delta c_m$ if $m\geqslant 2$, $c_n\otimes 0$ if $m=1$, is determined by the boundary formula for $c_n\otimes c_m$ and the values on $\delta c_{n+1}\otimes c_m$ if $n\geqslant 1$ and $0\otimes c_m$ and $1\otimes c_m$ if $n=0$.

(ii)′ ⇒ (i) This is easily proved on taking C to be the crossed complex of free type on one generator of dimension n.

(ii) ⇒ (iii) This is clear.

(iii) ⇒ (ii) Again, one takes C as in (ii)′ ⇒ (i).

(i) ⇔ (iv) Let Λ^n_i for $i=1,\dots,n$ be the simplicial subset of Δ^{n+1} generated by all faces $\partial_j c^{n+1}$ except for $j=i$. To say that $Np:NE\to ND$ is a Kan fibration is equivalent [**39**] to saying that any diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & NE \arrow[d, "Np"] \\ \Delta^{n+1} \arrow[r] \arrow[ur, dashed] & ND \end{tikzcd} \tag{**}$$

has a regular completion given by the dotted arrow. By adjointness, this is equivalent to the existence of a regular completion in $\mathscr{C}rs$ of the following diagram:

$$\begin{tikzcd} \pi(\Lambda_i^n) \arrow[r, "k"] \arrow[d, "j"'] & E \arrow[d, "p"] \\ \pi(\Delta^{n+1}) \arrow[r, "k'"] \arrow[ur, dashed, "g"] & D \end{tikzcd}$$

If $n = 0$, this last condition is equivalent to $E_1 \to D_1$ being a fibration of groupoids. If $n \geqslant 1$, this last condition is equivalent to each $E_{n+1}(x) \to D_{n+1}(px)$ being surjective. To see this, note that if these maps are surjective, and v is the usual base point of Δ^{n+1}, then we can choose $a \in E_{n+1}(kv)$ such that $pa = k'c^{n+1}$. If we now define $g(c^{n+1}) = a$ and $g(x) = k(x)$ for each non-degenerate element x of Λ_1^n, then there is a unique value for $g(\partial_i c^{n+1})$, determined by the homotopy addition lemma, which defines a morphism $g:\pi(\Delta^{n+1}) \to E$. This g is a regular completion of (**).

On the other hand, suppose each diagram (**) has a regular completion. Let $b \in D_{n+1}(px)$. Define $k:\pi(\Lambda_1^n) \to E$ to be the trivial morphism with value 0_x. Define $k':\pi(\Delta^{n+1}) \to D$ by $k'(c^{n+1}) = b$, $k'(\Lambda_1^n) = 0_{px}$ and $k'(\partial_1 c^{n+1}) = \delta b$. Then $pk = k'j$. Let g be a regular completion. Then $pg(c^{n+1}) = b$. ∎

COROLLARY 6·3. *Let $p:E \to D$ be a fibration of crossed complexes and let $x \in D_0$. Let $F = p^{-1}(x)$. Then the sequence of classifying spaces $BF \to BE \to BD$ is a fibration sequence.*

Proof. By a fibration of spaces we will mean a map which has the covering homotopy property with respect to all maps of compactly generated spaces. It is known that the realisation of a simplicial Kan fibration is a Serre fibration [**44**], and in fact has the covering homotopy property with respect to maps of all compactly generated spaces [**46**]. Thus all we have to check is that the fibre of $NE \to ND$ over x is precisely NF. This follows from the formula $(NF)_n = \mathscr{C}rs(\pi\Delta^n, F)$ given in Section 2. ∎

We now give some applications of BC to the case where the crossed complex C is essentially a crossed module. Similar results are proved in [**35**] using a classifying space of a crossed module defined using the equivalence of crossed modules and 1-cat-groups.

We use the same notation $\mu:M \to P$ for a crossed module and for the crossed complex obtained from it by trivial extension. Hence the above crossed module has a classifying space which we write $B(M \to P)$. We will use the fact that the identity crossed module $M \to M$ has contractible classifying space $B(M \to M)$. This can be proved either by noting that it has all homotopy groups zero, or by realising a contracting homotopy of the crossed complex extending $M \to M$.

Example 6·4. Let $\mu:M \to P$ be an inclusion of a normal subgroup. Then we have an exact sequence of crossed modules

$$\begin{tikzcd} M \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\mu"] & 1 \arrow[d] \\ M \arrow[r] & P \arrow[r] & P/M \end{tikzcd}$$

It follows from the fibration sequence of classifying spaces that the induced map $B(M \to P) \to B(P/M)$ is a homotopy equivalence.

Example 6·5. Let $\mu : M \to P$ be a crossed module. Then we have a short exact sequence of crossed modules

$$\begin{tikzcd} \operatorname{Ker}\mu \arrow[r] \arrow[d] & M \arrow[r, "\mu'"] \arrow[d, "\mu"] & \operatorname{Im}\mu \arrow[d] \\ 1 \arrow[r] & P \arrow[r] & P \end{tikzcd}$$

in which μ' is the restriction of μ and the unlabelled maps are inclusions. This exact sequence yields a fibration sequence which up to homotopy is

$$\mathrm{K}(\operatorname{Ker}\mu, 2) \to B(M \to P) \to B(\operatorname{Coker}\mu).$$

Example 6·6. [35] Again let $\mu : M \to P$ be a crossed module. Then we have short exact sequences of crossed modules

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & M \arrow[r, "1"] \arrow[d, "i"] & M \arrow[d, "\mu"] \\ M \arrow[r, "j"'] & M \rtimes P \arrow[r, "\tau"'] & P \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "1"] \arrow[d, "1"'] & M \arrow[r] \arrow[d, "i"] & 1 \arrow[d] \\ M \arrow[r, "i"'] & M \rtimes P \arrow[r, "\sigma"'] & P \end{tikzcd}$$

in which $i : m \mapsto (m, 1), j : m \mapsto (m^{-1}, (\mu m)\, p), \sigma : (m, p) \mapsto p, \tau : (m, p) \mapsto (\mu m)\, p$. It follows that we have a homotopy fibration

$$B(M) \to B(P) \to B(M \to P).$$

This shows as in [35] that a morphism $\mu : M \to P$ of groups can arise from a homotopy fibration $B(M) \to B(P) \to X$ if and only if μ can be given the structure of a crossed module.

Example 6·7. Let A be a crossed complex, and let $n \geqslant 1$. We write $A^{(n)}$ for the crossed complex D where

$$D_i = \begin{cases} A_i & \text{if } i \leqslant n, \\ \delta(A_{n+1}) & \text{if } i = n+1, \\ 0 & \text{if } i > n+1, \end{cases}$$

with the boundary $D_{n+1} \to D_n$ being inclusion, and other structures induced by that of A. The natural map $A \to A^{(n)}$ is a fibration and induces isomorphisms of π_0 and π_1 and of H_i for $i \leqslant n$. Further $H_i(A^{(n)}) = 0$ for $i > n$. Thus the induced map $BA \to BA^{(n)}$ is a Postnikov fibration. Let $A^{[n]} = \operatorname{Ker}(A \to A^{(n)})$. Suppose A is reduced. Then $A^{[n]}$ may be regarded as a chain complex. It is clear that $NA^{[n]}$ is a simplicial Abelian group and so $BA^{[n]}$ may be given the structure of a topological Abelian group. If $n = 1$, then $BA^{(n)}$ is a $K(\pi, 1)$ space. Thus, as first pointed out by Loday (private communication), the homotopy types of the form BA for A a crossed complex are restricted. In particular, all Whitehead products in BA are zero. One should think of these homotopy types as giving a first approximation to homotopy theory. This idea is developed in the tower of homotopy theories due to H. J. Baues [2], in which crossed complexes (reduced, of free type) give the first level of the tower, and in a sense represent the linear approximation to homotopy theory.

However, the use of classifying spaces in conjunction with crossed complexes A with some kind of algebra structure $A \otimes A \to A$ does allow some deeper levels of homotopy information to be obtained (see [**5**, **6**]). Such objects could be called crossed differential graded algebras.

7. *Local systems*

The homotopy classification result of Theorem A suggests that if A is a crossed complex, and X is a space, with singular complex SX, then the set $[\pi|SX|, A]$ may be thought of as singular cohomology of X with coefficients in A, and written $H^0(X, A)$, [**18**]. With our present machinery, it is easy to see that this cohomology is a homotopy functor of both X and of A. We show in this section that $H^0(X, A)$ is (non-naturally) a union of Abelian groups, each of which is a kind of cohomology with coefficients in a generalised local system.

Let C and A be crossed complexes. In examples, C is to be thought of as $\pi\mathbf{X}$ for some CW-complex X. By a local system $\mathscr{A}$ of type A on C we mean the crossed complex A together with a morphism of groupoids $\mathscr{A}: C_1 \to A_1$ such that $\mathscr{A}(\delta C_2) \subseteq \delta A_2$. This last condition ensures that $\mathscr{A}$ induces a morphism of groupoids $\pi_1 C \to \pi_1 A$. It is also a necessary condition for there to exist a morphism $C \to A$ extending $\mathscr{A}$. The morphism $\mathscr{A}$ induces an operation of C_1 on all the groupoids A_n for $n \geqslant 2$. By a cocycle of C with coefficients in $\mathscr{A}$ we mean a morphism $f: C \to A$ of crossed complexes such that $f_1 = \mathscr{A}$. By a homology of such cocycles f, g we mean a homotopy $(h, g): f \simeq g$ of morphisms of crossed complexes such that $h_0 x$ is a zero for all $x \in C_0$, and $\delta h_1 = 0$. The set of homology classes of cocycles of C with coefficients in $\mathscr{A}$ is written $[C, A]_{\mathscr{A}}$.

PROPOSITION 7·1. *Let C, A be crossed complexes and let $\mathscr{A}$ be a local system of type A on C. Let $H = \pi_1 A$ and let A' be the crossed complex which is H in dimension 1 and agrees with $A^{[1]}$ in higher dimensions, with H as groupoid of operators and with trivial boundary from dimension 2 to dimension 1. Let $\mathscr{A}'$ be the composite*

$$C_1 \xrightarrow{\mathscr{A}} A_1 \to H.$$

Then a choice of cocycle f with coefficients in $\mathscr{A}$ determines a bijection

$$[C, A]_{\mathscr{A}} \to [C, A']_{\mathscr{A}'},$$

and hence an abelian group structure on $[C, A]_{\mathscr{A}}$.

Proof. We are given f extending $\mathscr{A}$. Let g be another morphism $C \to A$ extending $\mathscr{A}$. Then $g_1 = f_1$ and $\delta g_2 = \delta f_2$. For such a g we define $rg_n = g_n - f_n$, $n \geqslant 2$. Clearly rg_n is a morphism of abelian groups for $n \geqslant 3$. We prove that it is also a morphism for $n = 2$. Let $c, d \in C_2$. Then

$$\begin{aligned}
rg_2(c+d) &= g_2(c+d) - f_2(c+d) \\
&= g_2 c + g_2 d - f_2 d - f_2 c \\
&= g_2 c - (f_2 c)^{-\delta(g_2 d - f_2 d)} + g_2 d - f_2 d \\
&= g_2 c - f_2 c + g_2 d - f_2 d \qquad \text{since } \delta g_2 = \delta f_2 \\
&= rg_2 c + rg_2 d.
\end{aligned}$$

Clearly rg_n is a C_1-operator morphism where C_1 acts on $A^{[1]}$ via $\mathscr{A}$. Also $\delta r g_2 = 0$ and for $n \geqslant 3$, $\delta r g_n = r\delta g_{n-1}$. So we may regard rg as a morphism $C \to A'$ extending $\mathscr{A}'$. It is clear that r defines a bijection between the morphisms $g: C \to A$ extending $\mathscr{A}$ and the morphisms $g': C \to A'$ extending $\mathscr{A}'$.

Next suppose that (h, g) is a homology $\bar{g} \simeq g$ as defined above. Then $\delta h_1 = 0$. Hence h_1 defines uniquely $k_1: C_1 \to \operatorname{Ker} \delta_2$. Further we have for $n \geqslant 2$

$$\bar{g}_n = g_n + h_{n-1}\delta_n + \delta_{n+1} h_n.$$

For $n \geqslant 2$ let $k_n = h_n$. Then k_n is a C_1-operator morphism. Now for $x \in C_n$ and $n \geqslant 3$, we have $h_{n-1}\delta_n x + \delta_{n+1} h_n x$ lies in an abelian group, while for $n = 2$ it lies in the centre of A_2 and so commutes with $f_2 x$. It follows that (k, rg) is a homology $r\bar{g} \simeq rg$. Conversely, a homology $r\bar{g} \simeq rg$ of $\mathscr{A}'$-cocycles determines uniquely a homology $\bar{g} \simeq g$ of $\mathscr{A}$-cocycles. It follows that r defines a bijection $[C, A]_{\mathscr{A}} \to [C, A']_{\mathscr{A}'}$ as required.

Notice also that the set $[C, A']_{\mathscr{A}'}$ obtains an Abelian group structure, by addition of values, and with the class of rf as zero. ∎

Let C be a reduced cofibrant crossed complex, and let A be a reduced crossed complex. We are interested in analysing the fibres of the function

$$\eta : [C, A]_* \to \operatorname{Hom}(\pi_1 C, \pi_1 A).$$

We write $[C, A]_*^\alpha$ for $\eta^{-1}(\alpha)$. This set may be empty. We will elsewhere analyse the first obstruction to an element α lying in the image of η. Here our aim is to show that if $f: C \to A$ is a morphism realising $\alpha: \pi_1 C \to \pi_1 A$ then f determines an Abelian group structure on $\eta^{-1}(\alpha)$.

We recall from [**21**] the relations between crossed complexes and chain complexes with operators.

There is a category $\mathscr{C}\mathit{hn}$ of chain complexes with groupoids as operators and two functors

$$\Delta : \mathscr{C}\mathit{rs} \leftrightarrows \mathscr{C}\mathit{hn} : \Theta$$

such that Δ is left adjoint to Θ. Hence if D is a chain complex with a trivial group of operators, then

$$\begin{aligned}(N\Theta D)_r &= \mathscr{C}\mathit{rs}(\pi\Delta^r, \Theta D)\\ &= \mathscr{C}\mathit{hn}(\Delta\pi\Delta^r, D).\end{aligned}$$

In this last formula, $\Delta\pi\Delta^r$ consists of the chain complex $C_* \tilde{\Delta}^r$ of cellular chains of the universal covers of Δ^r based at the vertices of Δ^r, with the action of the groupoid $\pi_1 \Delta^r$. Since D has trivial group acting, it follows that

$$\mathscr{C}\mathit{hn}(\Delta\pi\Delta^r, D) = \mathscr{C}\mathit{hn}(C_* \Delta^r, D),$$

where $C_* \Delta^r$ is the usual chain complex of cellular chains of Δ^r. This shows that ND coincides with the simplicial Abelian group of the Dold-Kan Theorem [**25**].

Let $H = \pi_1 A$. In the last example we defined $A^{[1]}$. We consider the pair $(A^{[1]}, H)$ to be a chain complex with H as groupoid of operators.

PROPOSITION 7·2. *Let C, A be reduced crossed complexes such that C is cofibrant. Let $f: C \to A$ be a morphism inducing $\alpha: \pi_1 C \to \pi_1 A$ on fundamental groups. Then f determines a bijection*

$$[C, A]_*^\alpha \cong [\Delta C, (A^{[1]}, H)]^\alpha,$$

where the latter term is the set of pointed homotopy classes of morphisms which are morphisms of chain complexes with operators and which induce α on operator groups.

Proof. Let $p: A \to A^{(1)}$ denote the fibration of the last example, so that $A^{[1]}$ is the kernel of p. Let $G = \pi_1 C$. Recall that $X(H, 1)$ denotes the crossed complex which is H in dimension 1 and is zero elsewhere. The projection $A^{(1)} \to X(H, 1)$ is a trivial fibration. Since C is cofibrant, the induced morphism $\mathrm{CRS}_*(C, A^{(1)}) \to \mathrm{CRS}_*(C, X(H, 1))$ is also a trivial fibration. It follows from Proposition 4·8 that $\mathrm{CRS}_*(C, A^{(1)})$ has component set $\mathrm{Hom}\,(G, H)$ and has trivial fundamental and homology groups.

Suppose that $f: C \to A$ induces $\alpha: G \to H$ on fundamental groups. Let $F(f)$ be the fibre of $\mathrm{CRS}_*(C, A) \to \mathrm{CRS}_*(C, A^{(1)})$ over pf. Then the exact sequence of this fibration yields an exact sequence

$$1 \to \pi_0 F(f) \rightarrowtail [C, A]_* \xrightarrow{\eta} \mathrm{Hom}\,(G, H)$$

such that the first map is an inclusion with image $\eta^{-1}(\alpha)$.

Let $\mathscr{A} = f_1$. Then $\pi_0 F(f) = [C, A]_{\mathscr{A}}$. So by Proposition 7·1, $\pi_0 F(f)$ is bijective with $[C, A']_{\mathscr{A}'}$. But in terms of the functors relating crossed complexes and chain complexes with operators given in [**21**] recalled above, we have $A' = \Theta(A^{[1]}, H)$. The proposition follows immediately from the adjointness of Δ and Θ. ∎

COROLLARY 7·3. *If $\alpha: \pi_1 C \to \pi_1 A$ is realisable by a morphism $f: C \to A$, then a choice of such morphism determines an Abelian group structure on $[C, A]^{\alpha}_*$.* ∎

This result gives the expected Abelian group structure on this generalised cohomology with local coefficients.

8. *Proof of Theorem* 3·1

In view of the equivalence $\gamma: \omega\text{-}\mathscr{G}pd \to \mathscr{C}rs$ of symmetric, monoidal closed categories, established in [**20**], it is sufficient to prove similar results for ω-groupoids. For this we need the fundamental ω-groupoid $\rho\mathbf{X}$ of a filtered space $\mathbf{X}$. For current purposes, it is convenient to depart from the conventions of [**17**] as follows, and so to give a definition which makes ρ a functor on $\mathscr{F}\mathscr{T}op$ itself.

We write $R^{\square}\mathbf{X}$ for the relative cubical singular complex of the filtered space $\mathbf{X}$. Thus $(R^{\square}\mathbf{X})_n$ consists of the filtered maps $\mathbf{I}^n \to \mathbf{X}$. We now factor this cubical complex to give a quotient map of cubical sets

$$p: R^{\square}\mathbf{X} \to \rho\mathbf{X}.$$

We recall the definitions.

Let $\mathbf{X}, \mathbf{Y}$ be filtered spaces. A *filter-homotopy* $f_t: f_0 \simeq f_1$ of filtered maps f_0, f_1: $\mathbf{Y} \to \mathbf{X}$ is a homotopy f_t such that each $f_t: \mathbf{Y} \to \mathbf{X}$ is a filtered map.

We define $\rho_n \mathbf{X}$ to be the quotient of $(R^{\square}\mathbf{X})_n$, the set of filtered maps $\mathbf{I}^n \to \mathbf{X}$, by the relation of filter-homotopy rel vertices (the condition 'rel vertices' was not used in [**17**], and this is the change in definition). The family $\{\rho_n \mathbf{X}\}_{n \geqslant 0}$ clearly inherits from $R^{\square}\mathbf{X}$ the structure of cubical complex. We also recall from [**14, 16**] that $R^{\square}\mathbf{X}$ has an extra structure of connections $\Gamma_i: (R^{\square}\mathbf{X})_n \to (R^{\square}\mathbf{X})_{n+1}$, $1 \leqslant i \leqslant n$, induced by the maps

$$I^{n+1} \to I^n, \quad (t_1, \ldots, t_{n+1}) \mapsto (t_1, \ldots, \max\{t_i, t_{i+1}\}, \ldots, t_n).$$

Also $R^{\square}\mathbf{X}$ has a structure of compositions, written additively as $+_i$, partially defined on $(R^{\square}\mathbf{X})_n$ for $1 \leqslant i \leqslant n$, and given by the usual gluing of singular cubes in direction i. It is the use of this simple and intuitive composition structure which gives cubical methods a strong advantage over simplicial methods. A list of laws for the Γ_i, $+_i$ is given in [**14**, **16**], but the details will not concern us much in this paper.

It is clear that the structure of connections on the cubical complex $R\mathbf{X}$ is inherited by $\rho\mathbf{X}$.

THEOREM 8·1. *The compositions on $R^{\square}\mathbf{X}$ are inherited by $\rho\mathbf{X}$ to give $\rho\mathbf{X}$ the structure of ω-groupoid.* ∎

Remark 8·2. Let us write $\rho'_n\mathbf{X}$ for the quotient of $R_n\mathbf{X}$ by the relation of filter-homotopy, so that $\rho'_n\mathbf{X}$ is also a quotient of $\rho_n\mathbf{X}$. Theorem A of [**17**] states that if $\mathbf{X}$ is J_0, then the compositions $+_i$ on $(R^{\square}\mathbf{X})_n$ are inherited by $\rho'_n\mathbf{X}$, so that $\rho'\mathbf{X}$ becomes an ω-groupoid. The J_0 condition is that each loop in X_0 is contractible in X_1, and this condition was used in the proof of corollary 1·2 in [**17**] to ensure that any map $\{v\} \times \dot{I}^2 \to X$ which is a filter double homotopy (i.e. has image in X_0) extends to a filter double homotopy $\{v\} \times I^2 \to X$ (i.e. with image in X_1). This corollary 1·2, or a deduction from it, is applied in all later theorems of [**17**] which use the J_0 condition. In our new definition of $\rho\mathbf{X}$, any required filter double homotopy $\{v\} \times \dot{I}^2 \to X_0$ is a constant map, and so automatically extends to a constant map $\{v\} \times I^2 \to X_0$. This gives theorem 2·1, and also the results of section 3 of [**17**], without the J_0 condition, and applied to the new $\rho\mathbf{X}$. In a similar spirit we have the following analogue of theorem 5·1 of [**17**], for which we recall that $\gamma : \omega\text{-}\mathcal{G}pd \to \mathcal{C}rs$ is an equivalence of categories.

THEOREM 8·3. *If $\mathbf{X}$ is a filtered space, then $\gamma\rho\mathbf{X}$ is naturally isomorphic to the fundamental crossed complex $\pi\mathbf{X}$ of $\mathbf{X}$.* ∎

We remark that the Van Kampen type theorems for $\rho\mathbf{X}$ and $\pi\mathbf{X}$ (theorems B and C of [**17**]) are true with our new definition, again without the J_0 assumptions which only tend to confuse the issue. The proofs need only minor changes in the proof of lemma 4·2 and lemma 4·5 of [**17**] to ensure that constant homotopies on elements of X_0 are not altered during the deformations used there.

Proof of Theorem 3·1

In view of the above results, it is sufficient to prove a similar result to Theorem 3·1 for ω-groupoids. That is, we construct a natural morphism

$$\theta' : \rho\mathbf{X} \otimes \rho\mathbf{Y} \to \rho(\mathbf{X} \otimes \mathbf{Y}).$$

For this, it is sufficient to construct a bimorphism of ω-groupoids ([**20**], p. 9)

$$\theta'' : (\rho\mathbf{X}, \rho\mathbf{Y}) \to \rho(\mathbf{X} \otimes \mathbf{Y}).$$

Let $f : \mathbf{I}^p \to \mathbf{X}$, $g : \mathbf{I}^q \to \mathbf{Y}$ be representatives of elements of $\rho_p\mathbf{X}$, $\rho_q\mathbf{Y}$ respectively. We define $\theta''([f],[g])$ to be the class of the composite

$$\mathbf{I}^{p+q} \xrightarrow{\cong} \mathbf{I}^p \otimes \mathbf{I}^q \xrightarrow{f \otimes g} \mathbf{X} \otimes \mathbf{Y}.$$

It is easy to check that $\theta''([f],[g])$ is independent of the choice of representatives. The conditions that θ'' be a bimorphism are essentially a translation of conditions (2·1)(i)–(v) of [20]. Their validity is almost automatic.

The proofs of associativity of θ' and of the relations corresponding to (ii) are clear.

The proof of symmetry follows from the description in [20], p. 24 of the isomorphism $G \otimes H \to H \otimes G$ of ω-groupoids as given by $x \otimes y \mapsto (y^* \otimes x^*)^*$ where, in the geometric case $G = \rho\mathbf{X}$, $x \mapsto x^*$ is induced by the map $(t_1, \ldots, t_p) \mapsto (t_p, \ldots, t_1)$ of the unit cube.

To prove (iv), recall that $\mathbf{X} \otimes \mathbf{Y}$ is a CW-filtration, and so the crossed complex $\pi(\mathbf{X} \otimes \mathbf{Y})$ is of free type, with basis the characteristic maps of the product cells $e^p \times e^q$ of $\mathbf{X} \otimes \mathbf{Y}$. So the theorem follows from Corollary 5·2. (It is clear that the results of Section 5 do not in any way use Theorem 3·1.) ∎

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