Analogy, concepts and methodology in mathematics *

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Introduction

Motivation for this article is a question put to Ronnie Brown in June 2004 by a young woman, Bree, at a party while he was visiting the Center for Computational Biology in Bozeman, Montana, to give seminars. Bree said that she was studying maths and 'found proofs neat'. Also she had asked a lecturer why she should study mathematics, and he had said: 'Because it has lots of applications'. But, Bree remarked, really she wanted to know *why* it has lots of applications.

We believe it is important for anyone teaching or studying mathematics to reflect on this sort of question and to have formulated some kind of answer. We will say below what answer was then given.

If indeed mathematics is important primarily for its applications, then the reactions of Governments and the public is likely to be: 'Fine: we will fund mathematics as and when it is applied.' This is largely what happens. On the other hand, if there is some fundamental reason internal to mathematics as to why it has lots of applications, then this itself should be strongly nurtured, even if the aim is principally that the golden eggs of applications should continue. Most of us would argue that this 'fundamental reason' should be strongly supported for its own sake.

To examine these questions we should, in part, look at history, and see what mathematics has contributed to culture, to technology, to science and art, and to the individuals who have given their lives to its study.

The natural mathematician is an asker of questions. A motivation for studying mathematics is the desire to understand, to see what is true and why it is true. Our thesis, and the kind of answer given to Bree, is that to this end, *mathematics has over the centuries developed a language, or even a set of evolving and interacting languages, for expression, description, deduction, verification and calculation.* These languages involve a myriad of concepts and their interrelations.

As E. Wigner wrote in a famous article [9]:

"Mathematics is the science of skilful operations with concepts and rules invented just for this purpose." [this purpose being the skilful operation ....]

"The principal emphasis is on the invention of concepts."

"The depth of thought which goes into the formation of mathematical concepts is later justified by the skill with which these concepts are used."

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As one example, the mathematics of error correcting codes is necessary not only for CDRoms and hard disks, but also for telecommunications, and the interpretation of the messages from the Voyager space crafts. Some of this mathematics involves quite highbrow concepts and tools of algebraic geometry, which were developed for entirely geometric reasons.

It is this vast language of concepts, which is developed for description, deduction, verification, calculation, as suggested above, which makes mathematics necessary for high technology. For more on "conceptualism", see the book [4].

That also raises the question: To what extent should mathematics students be aware of, and be trained in, the *formation*, as well as the *skilled use*, of mathematical concepts? How would one teach this? How would one assess it? Who would teach it?

1 Analogies, Abstraction and Concepts

This article is *about*, rather than *on*, mathematics. It discusses some aspects of the nature of this peculiar subject, mathematics, and its methodology.

The methodology of mathematics is little discussed in teaching or in research. Yet any human activity benefits from a level of knowledge at a so-to-speak meta-level. At a humdrum level, if you decide to go on holiday, you do not rush to the station to buy tickets – there is usually some kind of analysis, e.g. where are you planning to go! At a higher level, we expect a director of a play to be able to know and to express in words what she or he is trying to achieve; a singer at a high level will still go to an experienced singing teacher, whose experience and knowledge will help the singer to reach an even higher level of expression.

For mathematics, we made a start in discussing methodology in the article [2]. Here we will signal some additional features, not emphasised in that article, but which are discussed at length in [3]. These are centred around 'analogy'.

This is a word little used in mathematics. After a lecture on knots by one of us to schools in Leicester in the 1980s, where the analogy was made between addition of knots and multiplication of numbers, leading to the notion of prime knot, a teacher came up and said that was the first time in his mathematical career that anyone had used the word analogy in relation to mathematics. After another similar lecture, a teenage boy had clearly got the message, since he asked if there were infinitely many prime knots! He had also realised that in mathematics the interest is often not in 'What is the answer?' but in 'What is the question?'.

A crucial feature of an analogy in mathematics is that it is largely not between objects but between relations between objects. So in the example mentioned above there is no *direct* analogy between knots and numbers, but you can add knots and multiply numbers. The laws for adding knots (commutativity, associativity, zero, prime, ...) have analogies to the laws for multiplying numbers (with zero replaced by 1).

An advantage of this notion of analogy is that it can be easily appreciated by non mathematicians. The rules $2+3=3+2$, $2\times 3=3\times 2$ are clearly analogous. Thus one can convey that the process of abstraction is actually the process of recognising and then exploring patterns and analogies. It is a fundamental method in mathematics and indeed in science and thought.

The advantages of abstraction are at least three fold:

- Covering many examples by one theory;
- Developing a theory which can be applied to new examples as they arise;
- Simplifying proofs.

The last advantage may be surprising to you. It arises because, in trying to see why something is true, one seeks out the essential, and tries to discard the inessential. This often involves the process of abstraction and generalisation, and allows one to see more clearly what is going on.

Let us take a simple example. A prime number is defined to be a (positive) integer which is not expressible as a product of two other positive integers neither of which is 1, and so each is smaller than the first. The classic argument that any positive whole number has a factorisation as a product of prime numbers is then something as follows: if n is given to us, it is either prime or it is not. If it is prime, we stop and relax; if not then it can be written as $n = ab$ where a, b are less than n, and we repeat, cascading down to get the result. The process stops because eventually the factors get too small.

Thinking about this, we can use the analogy of knots and numbers hinted at above to put forward a definition of a 'prime knot'. A knot is prime if it cannot be expressed as a sum of two non-trivial knots. The analogy suggests that any knot should decompose as a sum of prime knots. All seems to work well but how do we know the 'cascade' terminates? Trying a similar proof in a different situation clarifies and emphasises the things that made the proof for numbers work. What you need is some invariant number, $i(K)$, defined for knots so that i of the unknot is 0, and conversely, and if $K = L + M$, non-trivially, then $i(L), i(M) < i(K)$. Such invariants do exist[1], so a moment's thought shows the cascade must stop eventually and the decomposition theorem works.

The two proofs are clearly 'the same'. The important point to note in any abstraction from these is the role of the measure of something that might be called 'size' either 'as is' in the number case or measured by $i(K)$. We could go further and abstract the essential structure, giving lots of additional examples of its application. We will mention the case of polynomials where 'prime' polynomials are called 'irreducible' and the size is given by 'degree'.

The process of twigging a particular proof, and then seeing how that insight might be used in other situations, is one way on which mathematics progresses. Sometimes one has a proof in search of a theorem! That is, the proof would work if certain gadgets existed, but they are not available. Trying to construct gadgets to give existence to a proof has advantages, since the aim is clear, even if the theorems which would result, and their conditions, are not. The construction of such gadgets can open new worlds of mathematical structures. An important phrase in the progress of mathematics is: 'What if?'

The original idea for a proof may be applied elsewhere through analogy. This illustrates that mathematical progress is not a mystery, open to a few geniuses: mathematicians apply many basic and common methods of discovery, but to their own material.

Of course, analogy is also at the heart of the modelling process as used in 'Applied Mathematics'. Even in the origins of Newton's work on motion, there was analogy, as to model the position of a particle by three real functions of time is exactly an analogy between a 'real-life' situation and a mathematical one. Analogies are rarely exact and exploring the limits of them is another source of ideas for pushing mathematics into new territory.

What are 'concepts', as referred to by Wigner? An example of a concept is distance, say between towns. We then see that concepts are not things, but describe for us relations between things, in a convenient way. The progress of mathematics is marked by the finding of important concepts and their properties. Some concepts familiar to the public are:

> length, area, volume, addition, zero, time, speed, velocity, mass, force, random, function, ...

[1] You can use genus or bridge number as suitable invariants here.

Students of mathematics will know many more!

Now we can see again that analogies are usually not between things themselves, but between the relations among things. Knots are not analogous to numbers, but the relation between three knots given by addition has analogies to the relation between three numbers given by multiplication. We see a similarity of pattern.

Curiously, the notions of concept, analogy, pattern, themselves have some relationship. What is it, precisely? Could there be a mathematics of this?

2 Training Professional Mathematics?

Answers to the questions of what is mathematics and what are the reasons for its success should influence the teaching of mathematics. This is analogous to the fact that a musician is expected to be taught not only technique but also musicianship: both are needed. It is therefore interesting to look at aims in other areas of study.

Here are the aims which have been given for a course in design:

1. To teach students the principles of good design;
2. To encourage independence and creativity;
3. To give students a range of practical skills so that they can apply the principles of good design in an employment situation.

Is there something here from which mathematics degree courses can learn? Is it reasonable aims for a mathematics course to replace in the above the word "design" by the word "mathematics"? If not, why not?

The report [7] gives employers' views that mathematics graduates are not good at problem formulation, planning, work evaluation, communication. The report [5] analyses skills that mathematics degrees do teach.

A common statement is that there is no agreement on what constitutes 'good mathematics'. This is part of the point! We do not want to follow the argument, sometimes put, that 'top mathematics is what is done by top mathematicians', since that is not only circular but also a cop-out. We do want to encourage individual judgement and creativity. We do want students to be able to evaluate, in various modes, what they have learned.

For some, the main fascination in mathematics is the challenge of problems. Certainly, the solution of a famous problem will give the solver fame, in at least the mathematical world. Yet problems can be stated only at a given level of conceptualisation; so others see the progress of mathematics as strongly involving the development of a rigorous language, involving interlinked concepts which also *allow the formulation of new problems.* It can be argued that it is the development of such a language which has been the main contribution of mathematics to culture, science and technology over the centuries. Stanislaw Ulam told Ronnie Brown in 1964 that taking up the challenge of famous problems may in fact distract young people from developing the mathematics which is most appropriate to them. It is interesting that someone as good as Ulam should make this point.

Mathematics is not only about doing difficult things, but also providing the framework to make difficult things easy (thus giving new opportunities for difficult tasks!). As Grothendieck wrote to Ronnie Brown ([6], 5/5/1982): 'The introduction of the cipher 0 or the group concept was general nonsense too, and mathematics was more or less stagnating for thousands of years because nobody was around to take such childish steps ...'

In this direction of developing language, we can usefully quote G.-C. Rota [8, p.48]:

> “What can you prove with exterior algebra that you cannot prove without it?” Whenever you hear this question raised about some new piece of mathematics, be assured that you are likely to be in the presence of something important. In my time, I have heard it repeated for random variables, Laurent Schwartz’ theory of distributions, ideles and Grothendieck’s schemes, to mention only a few. A proper retort might be: “You are right. There is nothing in yesterday’s mathematics that could not also be proved without it. Exterior algebra is not meant to prove old facts, it is meant to disclose a new world. Disclosing new worlds is as worthwhile a mathematical enterprise as proving old conjectures.

One problem is that there is no one or easy answer to the question “What is good mathematics?” Few students are given opportunity or language for any answer. But for someone who loves, or teaches, the subject, it is a key question.

Finally we give a small extract from another letter of Alexander Grothendieck to Ronnie Brown:

> The question you raise “how can such a formulation lead to computations” doesn’t bother me in the least! Throughout my whole life as a mathematician, the possibility of making explicit, elegant computations has always come out by itself, as a byproduct of a thorough conceptual understanding of what was going on. Thus I never bothered about whether what would come out would be suitable for this or that, but just tried to understand – and it always turned out that understanding was all that mattered. ([6],12/04/1983)

It is clear that we are raising more questions than we are answering!

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