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Fibrations of Groupoids

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INTRODUCTION

The notion of a covering morphism of groupoids has been developed by P. J. Higgins [4, 5] and shown to be a convenient tool in algebra, even for purely group theoretic results. That covering morphisms of groupoids model conveniently the covering maps of spaces is shown in [1].

If we weaken the conditions for a covering morphism we obtain what we shall call a *fibration* of groupoids, and our purpose is to explore this notion.

The main results are that, even if we start in the category of groups, then certain constructions lead naturally to fibrations of groupoids; that for fibrations of groupoids we can obtain a family of exact sequences of a type familiar to homotopy theorists; and that these exact sequences include not only the bottom part of the usual exact sequence of a fibration of spaces, but also the well known six term exact sequences in the non-Abelian cohomology of groups [6]. A further advantage of our procedure is that the same setup leads naturally to a definition of non-Abelian cohomology in dimensions 0 and 1 of a groupoid with coefficients in a groupoid. This cohomology (which will be dealt with elsewhere) generalises a non-Abelian cohomology of a group with coefficients in a groupoid which has been developed by A. Frohlich (unpublished) with a view to applications in Galois cohomology. Another question not touched on here is possible application of these methods to the non-Abelian H^2.

There is some overlap of this paper with techniques used by J. Gray in [3]. However, the aims of that paper are quite different from ours, and so the theory is developed here from the beginning.

1. THE CATEGORY OF GROUPOIDS

The basic theory of groupoids is covered in [1] and in [5] but in order to make this paper relatively self-contained most of the elementary notions will be recapitulated here.

A *groupoid* is a small category in which every morphism has an inverse. However such a groupoid will be regarded as an algebraic object in its own right, and so it is convenient to call the morphisms of a groupoid G *elements* of G, so that to some extent G is identified with the (disjoint) union of the sets $G(x, y)$ for all x, y in $\mathrm{Ob}(G)$.

A groupoid G is *connected* if $G(x, y)$ is nonempty for all objects x, y of G. The *components* of G are the maximal, connected subgroupoids of G. At the other extreme, a groupoid G is *discrete* if its only elements are identities; such a groupoid can be identified with its set of objects.

If x is an object of G, then under composition† the set $G(x, x)$ is a group, written $G\{x\}$, and called the *object group*, or *vertex group*, of G at x.

A *morphism* $f: G \to H$ of groupoids is simply a functor. Then $\ker f$ is the set of elements a in G such that $f(a)$ is an identity of H, and $\mathrm{Im} f$ is the set of elements $f(a)$ for a an element of G. Clearly $\ker f$ is a subgroupoid of G, but $\mathrm{Im} f$ is in general only a subgraph of H, since the composite of two elements of $\mathrm{Im} f$ may not again be an element of $\mathrm{Im} f$. For example, let $\mathscr{I}$ denote throughout this paper the groupoid with two objects 0, 1 and only two nonidentity morphisms $\iota \in \mathscr{I}(0, 1)$, $\iota^{-1} \in \mathscr{I}(0, 1)$. Then any nonconstant morphism $\mathscr{I} \to \mathbb{Z}$ (where $\mathbb{Z}$, the additive group of integers, is considered as a groupoid with one object) has image which is not a subgroup, and so not a subgroupoid, of $\mathbb{Z}$.

As is usual, a subgroupoid N of a groupoid G is called *full* if $N(x, y) = G(x, y)$ for all objects x, y of N. We say a subgroupoid N is *wide* in G if N has the same objects as G.

A subgroupoid N of G will be called *normal* if N is wide in G and for all objects x, y of G and $g \in G(x, y)$ we have

$$g^{-1}N\{y\}g = N\{x\}.$$

In such case the *quotient groupoid* is defined as follows (cf. [5]). The objects of G/N are the equivalence classes of objects of G under the relation $x \sim y$ if $N(x, y)$ is nonempty; the elements of G/N are the equivalence classes of elements of G under the relation $g \sim h$ if there are elements a, b of N such that agb is defined and equal to h; composition in G/N is induced by composition in G. The projection $p: G \to G/N$ is then a morphism of groupoids which is universal for morphisms f from G such that $\mathrm{Im} f$ is discrete. Any such universal morphism, which must be of the form p followed by an isomorphism, will be called a quotient map.

Suppose further that B is a subgroupoid of G such that B contains N. Then B/N is a subgroupoid of G/N. In particular, N/N is a subgroupoid of G/N; however this subgroupoid does not, as in the case of groups, have

† We shall write maps on the left so that the composite of $a \in G(x, y)$, $b \in G(y, z)$ is $ba \in G(x, z)$.

only one element, but instead is the discrete groupoid with one object for each component of N.

Let $f : G \to H$ be a morphism of groupoids. The *fibre* of f at an object y of H is the subgroupoid of G whose elements are mapped by f to the identity at y; this fibre is written $f^{-1}(y)$. Clearly the kernel of f, $\ker f$, is the sum (or disjoint union, as it is also called) of the fibres $f^{-1}(y)$ for all objects y of H.

Further $\ker f$ is a normal subgroupoid of G. So we have a factorization

$$\begin{tikzcd} G \arrow[dr, "\kappa(f)"] \arrow[dd, "f"'] & \\ & G/\ker f \arrow[dl, "\rho(f)"] \\ H & \end{tikzcd} \tag{1.1}$$

A vital difference now emerges between groups and groupoids. If G and H are groups and we factor f as in (1.1) through a quotient map, then $\rho(f)$ is an isomorphism onto $\operatorname{Im} f$. However, we have already pointed out that for groupoids $\operatorname{Im} f$ may not be a groupoid. A further important fact is that $\rho(f)$ may make some identifications. For example, if $f : \mathscr{I} \to \mathbb{Z}_2$ maps ι and ι^{-1} to 1, then $\kappa(f)$ is an isomorphism and $\rho(f)$ is essentially just f.

For a general morphism f, the kernel of $\rho(f)$ will consist only of identities, which we can express as $\rho(f)$ has *discrete kernel.*

Let G be a groupoid, and $x \in \operatorname{Ob}(G)$. Then $\operatorname{St}_G x$ is the union of the sets $G(x, y)$ for all $y \in \operatorname{Ob}(G)$. If $f : G \to H$ is a morphism, and $x \in \operatorname{Ob}(G)$, then $\operatorname{St}_f x$ is the restriction of f mapping $\operatorname{St}_G x \to \operatorname{St}_H f(x)$. We say f is *star injective, star surjective, star bijective* according as $\operatorname{St}_f x$ is injective, surjective, bijective for all $x \in \operatorname{Ob}(G)$.

1.2 Proposition. *A morphism $f : G \to H$ has discrete kernel if and only if f is star-injective.*

Proof. Suppose f has discrete kernel. Let $a, b \in \operatorname{St}_G x$, and suppose $f(a) = f(b)$. Then $ab^{-1} \in \ker f$, which is discrete. Hence ab^{-1} is an identity, and so $a = b$.

Conversely, let f be star injective. If $f(a)$ is an identity of H, then $f(a) = f(1_x)$ where x is the initial point of a and 1_x is the identity at x. By star injectivity, $a = 1_x$; so $\ker f$ is discrete. □

A morphism $f : G \to H$ is *faithfull* (resp *full*) if the restrictions of f mapping $G(x, y) \to H(f(x), f(y))$ are injective (resp surjective) for all objects x, y of G. Clearly star injective implies faithful, and star surjective implies full.

Going back to (1.1), the fibres of $\rho(f) : G/\ker f \to H$ are discrete, and so may be considered as sets. The construction of $\rho(f)$ shows that if $x \in \mathrm{Ob}(H)$, then the fibre of $\rho(f)$ over x is simply $\pi_0 f^{-1}(x)$, the set of components of the fibre of f over x.

The factorization (1.1) is functorial. In order to express this precisely, let $\mathscr{G}_{\mathfrak{d}}$ denote the category of groupoids and groupoid morphisms, and let $\mathscr{G}_{\mathfrak{d}}^{(1)}$ denote the category whose objects are morphisms of groupoids, and whose morphisms from $f : G \to H$ to $f' : G' \to H'$ are commutative squares of morphisms

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, "f"'] & G' \arrow[d, "f'"] \\ H \arrow[r, "\beta"'] & H' \end{tikzcd} \tag{1.3}$$

1.4 PROPOSITION. *ρ extends to a functor $\rho : \mathscr{G}_{\mathfrak{d}}^{(1)} \to \mathscr{G}_{\mathfrak{d}}^{(1)}$.*

Proof. Suppose given the commutative diagram (1.3), then $\alpha(\ker f) \subseteq \ker f'$ and so α defines $\alpha' : G/\ker f \to G'/\ker f'$, and we set $\rho(\alpha, \beta) = (\alpha', \beta)$. $\square$

We now consider homotopy notions for groupoids.

A morphism $f : G \times \mathscr{I} \to H$ of groupoids is also called a *homotopy* from f_0 to f_1, where $f_\epsilon = f(\ ,\epsilon) : G \to H$, $\epsilon = 0, 1$. Thus $f_\epsilon = f \circ i_\epsilon$, where $i_\epsilon : G \to G \times \mathscr{I}$ is the inclusion $g \mapsto (g, \epsilon)$ for $\epsilon = 0, 1$. For each object x of G, let $\theta_x = f(x, \iota) \in H$. Then for any $a \in G(x, y)$ we have a commutative square of elements of H

$$\begin{tikzcd} f_0(x) \arrow[r, "\theta_x"] \arrow[d, "f_0(a)"'] & f_1(x) \arrow[d, "f_1(a)"] \\ f_0(y) \arrow[r, "\theta_y"'] & f_1(y) \end{tikzcd} \tag{1.5}$$

so that

$$f_1(a)\, \theta_x = \theta_y f_0(a). \tag{1.6}$$

Conversely, given a collection of elements θ_x in $\mathrm{St}_H f_0(x)$ for each $x \in \mathrm{Ob}(H)$, then we can define a homotopy f from f_0 to f_1 where f_1 is determined by (1.6) (cf. [1] Section 6.5). Thus a homotopy is the same as a natural equivalence; and if G, H are groups, the relation of homotopy between morphisms $G \to H$ is just conjugation by elements of H (since in this case G has just one object).

Another way of expressing the above results is in terms of free products. Recall from [1, p. 270] that a free product $G_1 * G_2$ of groupoids G_1, G_2 is a pushout of the diagram of inclusions

$$G_1 \leftarrow \mathrm{Ob}(G_1) \cap \mathrm{Ob}(G_2) \to G_2$$

(where $\mathrm{Ob}(G_\epsilon)$ is regarded as a discrete subgroupoid of G_ϵ). Then we have: *$G \times \mathscr{I}$ is the free product*

$$(G \times 0) * (\mathrm{Ob}(G) \times \mathscr{I}),$$

because a morphism $G \times \mathscr{I} \to H$ is completely determined by morphisms $G \times 0 \to H$, $\mathrm{Ob}(G) \times \mathscr{I} \to H$ which agree on $\mathrm{Ob}(G) \times 0$.

A generalization of this will be convenient later.

1.7 Proposition. *Let A be a subgroupoid of G, and let $\underset{\sim}{Q}$ be the full subgroupoid of $G \times \mathscr{I}$ on $\mathrm{Ob}(G \times 0) \cup \mathrm{Ob}(A \times \mathscr{I})$. Then the diagram of inclusions*

$$\begin{array}{ccc} A \times 0 & \longrightarrow & A \times \mathscr{I} \\ \downarrow & & \downarrow \\ G \times 0 & \longrightarrow & \underset{\sim}{Q} \end{array}$$

is a pushout.

Proof. Suppose given a commutative square of morphisms

$$\begin{array}{ccc} A \times 0 & \xrightarrow{\subseteq} & A \times \mathscr{I} \\ {\scriptstyle\subseteq}\downarrow & & \downarrow{\scriptstyle g} \\ G \times 0 & \xrightarrow[f]{} & H \end{array}$$

Let $h : \mathrm{Ob}(G) \times \mathscr{I} \to H$ be any morphism extending $g \mid \mathrm{Ob}(A) \times \mathscr{I}$. Then f and h define a morphism $\psi : G \times \mathscr{I} \to H$, and $\varphi = \psi \mid \underset{\sim}{Q}$ extends both f and g. However, any element of $\underset{\sim}{Q}$ can be written as a product of elements from $X \times 0$ or $A \times \mathscr{I}$; so there is at most one morphism $\underset{\sim}{Q} \to H$ extending f and g. $\square$

In analogy with constructions in homotopy theory, the groupoid $\underset{\sim}{Q}$ of 1.7 will be written $G \times 0 \cup A \times \mathscr{I}$; however, note that if A is wide in G, then $G \times 0 \cup A \times \mathscr{I} = G \times \mathscr{I}$.

We now define a groupoid (GH). The objects of (GH) are the morphisms $G \to H$. The morphisms in $(GH)(f_0, f_1)$ are just the homotopies f_0 to f_1, and homotopies are composed in the obvious way [for example by composing squares such as (1.5)]. The composition of homotopies $f : f_0 \simeq f_1$, $g : f_1 \simeq f_2$ is written $g + f : f_0 \simeq f_2$.

The groupoid (GH) satisfies the exponential law:

$$((GH)K) \cong ((G \times H)K) \tag{1.8}$$

the isomorphism of (1.8) being given by the usual exponential formula.

Finally, we shall assume as known the notion of a *pullback square*

$$\begin{tikzcd} Q \arrow[r, "\bar{f}"] \arrow[d, "\bar{p}"'] & G \arrow[d, "p"] \\ X \arrow[r, "f"'] & H \end{tikzcd}$$

for which it is sometimes convenient to refer to $\bar{p}$ as the pullback of p by f.

2. Fibrations and Covering Morphisms

A *regular completion* of a square of morphisms

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & G \arrow[d, "p"] \\ Y \arrow[r, "g"'] & H \end{tikzcd}$$

is a morphism $\varphi : Y \to G$ such that

$$p\varphi = g, \qquad \varphi i = f.$$

2.1 Proposition. *Let $p : G \to H$ be a morphism of groupoids. The following conditions are equivalent.*

(i) (*The covering homotopy property*). *For any groupoid X, any commutative square*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_0"'] & G \arrow[d, "p"] \\ X \times \mathscr{I} \arrow[r, "F"'] & H \end{tikzcd} \tag{2.2}$$

has a regular completion.

(ii) (*Path-lifting property*). *Any commutative square*

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "i_0"'] & G \arrow[d, "p"] \\ \mathscr{I} \arrow[r] & H \end{tikzcd}$$

has a regular completion.

(iii) *The morphism f is star surjective.*

Proof. It is clear (by taking X to be a point groupoid) that (i) $\to$ (ii). Also (ii) $\to$ (iii) because morphisms $0 \to G$ are bijective (under evaluation on 0) with objects of G, and morphisms $\mathscr{I} \to H$ are bijective (under evaluation on ι) with elements of H.

We now prove (iii) $\to$ (i). Let f, F be as in (i). For each object $x \in X$ choose an element θ_x in $\mathrm{St}_G fx$ such that

$$p(\theta_x) = F(x, \iota);$$

this is possible by (iii). By [1, Section 6.5] the morphism f and the function $x \mapsto \theta_x$ determine a morphism $X \times \mathscr{I} \to G$; and this morphism is the required regular completion. $\square$

It may happen that the regular completion of diagram (2.2) is unique—in this case we say p has the *unique covering homotopy property*. Similarly, we have the *unique path-lifting property*.

2.2 Proposition. *For a morphism $p : G \to H$ of groupoids, the unique covering homotopy property, and unique path-lifting property, are equivalent, and are each equivalent to the condition that f is star bijective.*

The proof of this is clear.

Because of the analogies shown by 2.1, 2.2 with topological situations, we call a star surjective morphism a *fibration*, and a star-bijective morphism a *covering morphism*.

Note that a morphism p of *groups* is a fibration if and only if p is surjective and p is a covering morphism if and only if p is an isomorphism. On the other hand, the morphism $\mathscr{I} \to \mathbb{Z}_3$ of groupoids which sends $\iota \mapsto +1$, $\iota^{-1} \mapsto -1$ and the identities to 0, is surjective on elements, but is not a fibration.

2.3 Proposition. *A fibration p is a covering morphism if and only if* $\ker p$ *is discrete.*

Proof. This follows from 1.2. $\square$

On the other hand, given a fibration we can use the factorisation of Section 1 to construct a covering morphism.

2.4 Proposition. *Let $p : G \to H$ be a morphism, and let*

$$\begin{tikzcd} G \arrow[dd, "p"'] \arrow[dr, "\kappa"] & \\ & C \arrow[dl, "\rho"] \\ H & \end{tikzcd}$$

be the canonical factorization of p with $C = G/\ker p$. Then κ is a fibration, and if p is a fibration then ρ is a covering morphism.

Proof. Let $x \in \mathrm{Ob}(G)$, $y = \kappa(x)$. If $c \in \mathrm{St}_C\, y$, then $c = \kappa(b)$ for some b in $G(x', x'')$ say. Since $\kappa(x) = \kappa(x')$, there is an element d in $(\ker p)(x, x')$. Then $bd \in \mathrm{St}_G\, x$ and $\kappa(bd) = c$. This proves κ is a fibration.

Now let $y \in \mathrm{Ob}(C)$, and suppose $a \in \mathrm{St}_H\, \rho y$. Choose an $x \in \mathrm{Ob}(G)$ such that $\kappa(x) = y$, and choose, using the fibration property of p, an element b of $\mathrm{St}_G\, x$ such that $p(b) = a$. Then $c = \kappa(b)$ satisfies $c \in \mathrm{St}_C\, y$, $\rho(c) = a$. So we have proved that the restriction of ρ mapping $\mathrm{St}_C\, y \to \mathrm{St}_H\, \rho(y)$ is surjective; it is injective because ρ has discrete kernel. □

2.5 Corollary. *If N is a normal subgroupoid of G, then the quotient morphism $G \to G/N$ is a fibration.*

Proof. Apply 2.4 with p the quotient morphism $G \to G/N$ and $\kappa = p$. □

Let $\mathscr{F}\!\mathit{ib}$, $\mathscr{C}\!\mathit{ov}$ denote the full subcategories of $\mathscr{G}_{\mathfrak{d}}^{(1)}$ on the fibrations and covering morphisms respectively.

2.6 Corollary. *The functor $\rho : \mathscr{G}_{\mathfrak{d}}^{(1)} \to \mathscr{G}_{\mathfrak{d}}^{(1)}$ restricts to a functor.*

$$\rho : \mathscr{F}\!\mathit{ib} \to \mathscr{C}\!\mathit{ov}.$$

We now give some rules for deriving new fibrations from old ones.

2.7 Proposition. *Let $p : G \to H$, $q : H \to K$ be morphisms. Then*

(i) *if p, q are fibrations (resp. covering morphisms) then so also is qp;*

(ii) *if qp and p are fibrations, and* $\mathrm{Ob}(p)$ *is surjective, then q is a fibration;*

(iii) *if qp and q are covering morphisms, then so also is p.*

Proof. This amounts to considering the sequence of restrictions of p and q

$$\mathrm{St}_G\, x \to \mathrm{St}_H\, y \to \mathrm{St}_K\, q(y).$$

In (i) and (iii) we take $y = p(x)$, while in (ii) we are given y and use the fact that $\mathrm{Ob}(p)$ is surjective to find x such that $y = p(x)$. Then (i), (ii), and (iii) correspond to simple criteria for functions to be surjective or bijective. □

2.8 Proposition. *Suppose given a pull-back square*

$$\begin{array}{ccc} Q & \longrightarrow & G \\ {\scriptstyle \bar{p}}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & H \end{array}$$

If p is a fibration (resp. covering morphism) then so also is $\bar{p}$.

Proof. This is a simple consequence of the covering homotopy property of fibrations or covering morphisms, and the universal properties of pull backs. □

Our next examples involve the groupoid (XG), and will be essential for our description of the non-Abelian cohomology of groups.

2.9 PROPOSITION. *If $p : G \to H$ is a fibration (resp. covering morphism) then for any groupoid X the induced morphism*

$$(Xp) : (XG) \to (XH)$$

is a fibration (resp. covering morphism)

Proof. This is an easy consequence of the covering homotopy property and the exponential law (1.8). □

2.10 PROPOSITION. *Let G be a groupoid and $i : A \to X$ an inclusion morphism. Then the induced morphism*

$$(iG) : (XG) \to (AG)$$

is a fibration, and is a covering morphism if A is wide in X.

Proof. By the exponential law and 2.1 (ii) it is sufficient to prove that given any commutative diagram

$$\begin{tikzcd} A \times 0 \arrow[r, "\subseteq"] \arrow[d, "\subseteq"'] & A \times \mathscr{I} \arrow[d, "g"] \\ X \times 0 \arrow[r, "f"'] & G \end{tikzcd}$$

then there is a morphism $F : X \times \mathscr{I} \to G$ extending both f and g, and that such an F is unique if A is wide in X. Define a homotopy function θ on $\mathrm{Ob}(X)$ by $\theta_x = g(x, \iota)$ if $x \in \mathrm{Ob}(A)$, and θ_x is any element of $\mathrm{St}_G f(x, 0)$, if $x \notin \mathrm{Ob}(A)$. By Section 1, θ defines a homotopy $F : X \times \mathscr{I} \to G$ which extends both f and g. If A is wide in X, then F is determined by f and $\theta_x = g(x, \iota) = F(x, \iota)$, $x \in \mathrm{Ob}(X)$. Thus F is unique in this case. □

There is a proposition including both 2.9 and 2.10. We use the groupoid $X \times 0 \cup A \times \mathscr{I}$ of 1.7, which is defined when A is a subgroupoid of X.

2.11 PROPOSITION. *Let A be a supgroupoid of X and $p : G \to H$ a fibration. Then any commutative square*

$$\begin{tikzcd} X \times 0 \cup A \times \mathscr{I} \arrow[r, "h"] \arrow[d, "\subseteq"'] & G \arrow[d, "p"] \\ X \times \mathscr{I} \arrow[r, "F"'] & H \end{tikzcd}$$

has a regular completion which is unique if A is wide in X or if p is a covering morphism.

Proof. All that is needed is to lift the restriction of F mapping $\mathrm{Ob}(X) \times \mathscr{I} \to H$ to a morphism $\mathrm{Ob}(X) \times \mathscr{I} \to H$ agreeing with h on $\mathrm{Ob}(X) \times 0$ and $\mathrm{Ob}(A) \times \mathscr{I}$; this can clearly be done using the fibration property, and the lifting is unique if p is a covering morphism or if A is wide in X. $\square$

The following corollary of 2.11 is useful is classifying sections of a fibration.

2.12 Corollary. *Let $p : G \to H$ be a fibration, and $f : X \to H$ a morphism where X is connected. Let $\tilde{f} : X \to G$ be a morphism such that $p\tilde{f} = f$. Suppose $x \in \mathrm{Ob}(X)$ and z belongs to the same component of $p^{-1}f(x)$ as $\tilde{f}(x)$. Then $\tilde{f}$ is homotopic to a morphism $\tilde{f}' : X \to G$ such that $\tilde{f}'(x) = z$, and $p\tilde{f}' = f$.*

Proof. Let ω be an element from $\tilde{f}(x)$ to z lying in $p^{-1}f(x)$. We apply 2.11 with $A = \{x\}$, with h defined by

$$h(\alpha, 0) = \tilde{f}(\alpha) \qquad \alpha \in X$$

$$h(x, \iota) = \omega$$

and with F the constant homotopy of f to itself. By 2.11, F lifts to a homotopy $X \times \mathscr{I} \to G$ extending h, and the final morphism of this homotopy is the one required. $\square$

By using 2.11 and the exponential law, it is easy to prove the following generalization of both 2.9 and 2.10.

2.13 Proposition. *Let $p : G \to H$ be a fibration and $\iota : A \to X$ an inclusion. Consider the diagram*

$$\begin{tikzcd}
(XG) \arrow[dr, dashed, "k"] \arrow[drr, "(iG)"] \arrow[ddr, "(Xp)"'] & & \\
& Z \arrow[r] \arrow[d] & (AG) \arrow[d, "(Ap)"] \\
& (XH) \arrow[r, "(iH)"'] & (AH)
\end{tikzcd}$$

in which the inner square is a pullback and k is determined by (Xp) and (iG). Then k is a fibration, and is a covering morphism if A is wide in X or if p is a covering morphism.

The final proposition of this section will be used in constructing "fibrations along the fibres".

2.14 PROPOSITION. *Suppose, given a commutative diagram of morphisms*

$$\begin{array}{ccc} F & \xrightarrow{f_3} & F' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ G & \xrightarrow{f_2} & G' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ H & \xrightarrow[f_1]{} & H' \end{array}$$

in which (i) $F = p^{-1}(y)$, $F' = p'^{-1}(y')$ *are the fibres of* p, p' *over* $y \in \mathrm{Ob}(H)$ *and* $y' = f_1(y)$, *respectively*; (ii) f_1 *has discrete kernel (or at least the kernel of* $H\{y\} \to H'\{y'\}$ *is trivial). Then* f_3 *is a fibration or covering morphism if* f_2 *is a fibration or covering morphism respectively.*

Proof. Let $x \in \mathrm{Ob}(F)$, $a \in \mathrm{St}_{F'} f_3(x)$. Then also $a \in \mathrm{St}_{G'} f_2(x)$ and so a lifts to an element $b \in \mathrm{St}_G\, x$, since f_2 is a fibration. However

$$f_1 p(b) = p'f_2(b) = p'(a) = 1_{y'}.$$

By (ii), $p(b) = 1_y$, and so $b \in F$. Since i is an inclusion and $f_2(b) = a$, it follows that $f_3(b) = a$. If f_2 is a covering morphism and b, $b' \in \mathrm{St}_G\, x$ satisfy $f_3(b) = f_3(b') = a$, then $f_2(b) = f_2(b') = a$ and so $b = b'$. $\square$

We conclude this section with a more concrete example which will be useful later. This example is obtained essentially by taking fundamental groupoids of the usual fibre bundles $M \to S^1$, $\partial M \to S^1$ where M is the Möbius band, ∂M is the boundary of M and S^1 is the circle.

2.15 EXAMPLE. Let T be a free cyclic group with generator t. Let $p' : T \times \mathscr{I} \to T$ be the projection; it is easy to check that p' is a fibration. Define $p : T \times \mathscr{I} \to T$ to be the homotopy of $p_0 : t^n \to t^{2n}$ determined by the element t of T; then p is a covering morphism called the *double covering* of T. Define $g : T \times \mathscr{I} \to T \times \mathscr{I}$ to be the homotopy of $g_0 : t^n \to (t^{2n}, 0)$ determined by the element (t, ι) of $T \times \mathscr{I}$. Then $p'g = p$, so that $(g, 1)$ is a morphism $p \to p'$ of fibrations. Notice also that the (unique) fibre of p is discrete, while the fibre of p' is $\mathscr{I}$.

3. LIFTING MORPHISMS

Throughout this section, $p : G \to H$ will be a fibration of groupoids.

3.1 PROPOSITION. *Let* $x \in \mathrm{Ob}(G)$, *and let* $h_n \ldots h_1$ *be a product in* H *such that* $h_1 \in \mathrm{St}_H\, p(x)$. *Then there are elements* g_i *in* G *such that* $g_1 \in \mathrm{St}_G\, x$;

$p(g_i) = h_i$, $i = 1,\dots, n$; and the product $g_n \dots g_1$ is defined in G. These elements g_i are unique if p is a covering morphism.

Proof. The proof is easy using 2.1 (iii) and shifting in turn from the initial to the final point of $h_1, \dots, h_n$. $\square$

Let $f : X \to H$ be a morphism. If $x \in \mathrm{Ob}(X)$, then the subgroup

$$f(X\{x\})$$

of $H\{fx\}$ is called the *characteristic subgroup* of f at x. Recall that subgroups C of $H\{x'\}$, D of $H\{y'\}$ are *conjugate* if there is an element h of $H(x', y')$ such that $h^{-1}Dh = C$.

3.2 Proposition. *Let C be the characteristic group of p at x.* (i) *If D is the characteristic group of p at y, and x, y lie in the same component of G, then C and D are conjugate.* (ii) *If D is a subgroup of $H\{y'\}$ and D is conjugate to C, then D is the characteristic group of p at y for some y.*

Proof. (i) Let $g \in G(x, y)$ and let $p(g) = h$. If $c' \in C$, then c' is covered by an element c of $G\{x\}$. Clearly $p(gcg^{-1}) = hc'h^{-1}$, and so $hCh^{-1} \subseteq D$. Similarly, $h^{-1}Dh \subseteq C$, and hence $h^{-1}Dh = C$.

(ii) Suppose $h^{-1}Dh = C$. Let g be an element of $\mathrm{St}_G\, x$ covering h. Let y be the final point of g, and let D' be the characteristic group of p at y. We prove $D' = D$.

If $d' \in D$, then $d' = hc'h^{-1}$ for $c' \in C$. Then d' is covered by an element gcg^{-1} with $c \in G\{x\}$; hence $d' \in D'$. Conversely, if $d' \in D'$, then d' is covered by an element d of $G\{y\}$, whence $d' = hc'h^{-1}$ where $c' = p(g^{-1}dg)$. $\square$

The above is essentially the same as 9.3.2 of [1].

We are now interested in the following question: given a morphism $f : X \to H$, when does f lift to a morphism $\tilde{f} : X \to G$, that is, is there a morphism $\tilde{f} : X \to G$ such that $p\tilde{f} = f$? We shall see that this reduces to a group theoretic problem in the case p is a fibration; the solution of this is simple when p is a covering morphism, and in this case we shall determine the set of all liftings.

It is convenient to work first in the category of pointed groupoids. So we consider the following question: given morphisms of pointed groupoids

$$\begin{array}{ccc} & & G, x \\ & & \big\downarrow{\scriptstyle p} \\ X, z & \xrightarrow[f]{} & H, y \end{array}$$

when does f lift to a morphism $\tilde{f} : X, z \to G, x$?

3.3 PROPOSITION. *Let X be connected. Then f lifts if and only if, in the following diagram of restrictions*

$$\begin{tikzcd} & G\{x\} \arrow[d, "\bar{p}"] \\ X\{z\} \arrow[r, "\bar{f}"'] & H\{y\}, \end{tikzcd}$$

$\bar{f}$ lifts to a morphism $X\{z\} \to G\{x\}$ of groups.

Proof. Clearly, if f lifts, so also does $\bar{f}$. On the other hand suppose a lifting $\tilde{\bar{f}}'$ of $\bar{f}$ exists. Write $X = X\{z\} * T$ where T is a tree groupoid (8.1.5 of [1]). Clearly $f \mid T : T, z \to H, y$ lifts to a morphism $\tilde{f}'' : T, z \to G, x$ (since p is a fibration). The morphisms $\tilde{\bar{f}}', \tilde{f}''$ define a morphism $\tilde{f} : X, z \mapsto G, x$ which lifts f. □

3.4 COROLLARY. *Let X be connected and p a covering morphism. Then the pointed morphism f lifts if and only if the characteristic group of f (at z) is contained in the characteristic group of p (at x), and in such case the lifting is unique.*

Proof. The first part follows from 3.3 since $\bar{p} : G\{x\} \to H\{y\}$ is injective. If the lifting $\tilde{\bar{f}}'$ of $\bar{f}$ exists, it is unique. Also the lifting of $f \mid T$ (where T is as in the proof of 3.3) always exists, and is unique. □

We suppose for the rest of this section that $p : G \to H$ is a covering morphism. In order to solve the lifting problem for nonbase point morphisms, we need some notation.

Suppose K is a group, and F, C are subgroups of K. The K *normaliser of* (F, C) is the set

$$N_K(F, C) = \{k \in K : F \subseteq kCk^{-1}\}.$$

In general, this set is not a group, although it is a group if $F = C$ [in which case it is the usual normaliser $N(C)$ of C in K], or if F is normal in K. However, we do have the additional structure that C operates by right-multiplication on $N_K(F, C)$; the set of orbits is written

$$N_K(F, C)/C.$$

3.5 THEOREM. *Let $f : X \to H$ be a morphism, and let $p : G \to H$ be a covering morphism; we assume X, G, H are connected. Let*

$$F = f(X\{z\}),\ C = p(G\{x\})$$

be the characteristic groups of f, p at z, x respectively, where we assume $f(z) = p(x)$. Let $K = H\{px\}$. Then the set of liftings of f is bijective with $N_K(F, C)/C$.

Proof. Let Φ be the set of liftings of f. We define a function

$$\varphi : N_K(F, C) \to \Phi.$$

Let $k \in N_K(F, C)$, so that $F \subseteq kCk^{-1}$. Then k lifts uniquely to an element $g \in \mathrm{St}_G\, x$ with final point x' say, and C', the characteristic group of p at x', is kCk^{-1}, by 3.2. Since $F \subseteq C'$, it follows from 2.4 that f lifts uniquely to a morphism $\tilde{f} : X \to G$ such that $\tilde{f}(z) = x'$, and we set $\varphi(k) = \tilde{f}$.

If $\tilde{f} : X \to G$ is any lifting of f, then there is an element g in $G(x, f(z))$; if $k = p(g)$, we have $\tilde{f} = \varphi(k)$. This proves φ is surjective.

Now φ is not injective in general. For suppose $c' \in C$, so that $c' = p(c)$ for some $c \in G\{x\}$. If $g \in \mathrm{St}_G\, x$ lifts $k \in N_K(F, C)$, then $gc \in \mathrm{St}_G\, x$ lifts kc'; but g and gc have the same final point, so that $\varphi(kc') = \varphi(k)$.

On the other hand, suppose $\varphi(k) = \varphi(k')$, and that $g, g' \in \mathrm{St}_G\, x$ lift k, k' respectively. Then g, g' have the same final point, and so $c = g^{-1}g' \in G\{x\}$ is well defined. Clearly $k' = kp(c)$.

Thus $\varphi(k) = \varphi(k')$ if and only if $kC = k'C$, and so we have proved that φ induces a bijection $N_K(F, C)/C \to \Phi$. $\square$

In the case $f = p$, the liftings of p are called *covering transformations*, and we can make 3.5 a little more precise.

3.6 THEOREM. *Let $p : G \to H$ be a covering morphism such that G and H are connected. Let C be the characteristic group of p at a point x of* Ob(G). *Then the set of covering transformations is under composition a group anti-isomorphic to $N(C)/C$.*

Proof. Let $K = H\{px\}$. In this case, $N(C) = N_K(C, C)$ is a group containing C as normal subgroup, and $N_K(C, C)/C$ is the quotient group $N(C)/C$. Let $\varphi : N(C) \to \Phi$ be the function defined in the proof of 3.5, where Φ is now the set of lifts of p. We prove that $\varphi(k_2k_1) = \varphi(k_1)\,\varphi(k_2)$.

For $\epsilon = 1, 2$, let $k_\epsilon \in N(C)$, let $g_\epsilon \in G(x, x_\epsilon)$ lift k_ϵ and let $\varphi(k_\epsilon) = f_\epsilon$, so that $f_\epsilon(x) = x_\epsilon$. Then $f_1(g_2) \in G(x_1, f_1(x_2))$. Since $pf_1 = p$, we have $pf_1(g_2) = k_2$. Hence $f_1(g_2)\, g_1 \in \mathrm{St}_G\, x$ and lifts k_2k_1. The final point of $f_1(g_2)\, g_2$ is $\varphi(k_2k_1)(x)$. Hence

$$\varphi(k_2k_1)(x) = f_1(x_2) = f_1f_2(x).$$

It follows that $\varphi(k_2k_1) = f_1f_2 = \varphi(k_1)\,\varphi(k_2)$.

Since $\varphi : N(C) \to \Phi$ is an antihomomorphism, which is surjective and has kernel C, it follows that φ induces an antiisomorphism $N(C)/C \to \Phi$, and this implies that Φ is a group. $\square$

4. Operators and Exact Sequences

Let $p : G \to H$ be a covering morphism. If $x \in \mathrm{Ob}(H)$, then the fibre $p^{-1}(x)$ is discrete, and so we regard $p^{-1}(x)$ both as a subset of $\mathrm{Ob}(G)$ and as a subgroupoid of G.

We now define an *operation* of H on $\mathrm{Ob}(G)$.

Let $h \in H(x, y)$ and let $x' \in p^{-1}(x)$. Then h lifts to a unique element g of $\mathrm{St}_G\, x'$, and the final point of g is written $h \,.\, x'$. Clearly

$$1 \,.\, x' = x', \qquad h' \,.\, (h \,.\, x') = (h'h) \,.\, x'$$

when these are defined. However $h \,.\, x'$ is defined if and only if $h \in \mathrm{St}_H\, p(x')$. So to make the notion of an operation clear we follow Ehresmann in defining an operation of a groupoid on a set.

Let H be a groupoid. An *operation* of H is a quadruple $(H, S, w, .)$ where S is a set, $w : S \to \mathrm{Ob}(H)$ is a function and $. : (h, x') \mapsto h \,.\, x'$ is a function defined whenever $h \in \mathrm{St}_H\, w(x')$. The axioms we impose are that $w(h \,.\, x')$ shall be the final point of h, and also the usual rules $1 \,.\, x' = x'$, $h' \,.\, (h \,.\, x') = (h'h) \,.\, x'$ whenever both sides are defined. Note that if $h \in H(x, y)$, then the operation gives us a function $h_* : w^{-1}(x) \to w^{-1}(y)$, $x' \mapsto h \,.\, x'$, and we have $1_* = 1$, $(h'h)_* = h'_* h_*$. Thus an operation of H defines a functor $H \to \mathscr{S}et$, where $\mathscr{S}et$ is the category of sets.

The operations of groupoids form a category $\mathscr{O}p$ whose objects are operations $(H, S, w, .)$ and whose maps $(H, S, w, .) \to (H, 'S', w', .)$ are pairs (ψ, f) such that $\psi : H \to H'$ is a morphism of groupoids, $f : S \to S'$ is a function, and we have the axioms

(i) $w'f = \mathrm{Ob}(\psi)w$

(ii) $\psi(h) \,.\, f(x') = f(h \,.\, x')$, whenever $h \,.\, x'$ is defined.

We mention (but shall not use the fact) that the above way of constructing a functor from an operation defines an equivalence between the category $\mathscr{O}p$ and the category $\mathscr{F}\mathscr{O}p$ whose objects are functors Γ from a groupoid H to $\mathscr{S}et$, and whose morphisms $\Gamma \to \Gamma'$ are pairs (ψ, f) where $\psi : H \to H'$ is a morphism and $f : \Gamma \to \Gamma'\psi$ is a natural transformation. For our present purposes, the category $\mathscr{O}p$ is more useful that $\mathscr{F}\mathscr{O}p$, but in other situations, for example in order to have an operation of a groupoid on vector spaces, the description of an operation as a functor is perhaps more convenient.

In the third paragraph of this section we defined for each covering morphism $p : G \to H$ an operation

$$\sigma(p) = (H, \mathrm{Ob}(G), \mathrm{Ob}(p), .).$$

4.1 Proposition. *The function $p \mapsto \sigma(p)$ extends to a functor $\sigma : \mathscr{C}ov \to \mathscr{O}p$*

Proof. Suppose given a commutative diagram

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "p"'] & G' \arrow[d, "p'"] \\ H \arrow[r, "\psi"'] & H' \end{tikzcd}$$

where p, p' are covering morphisms. Let $f = \mathrm{Ob}(\varphi) : \mathrm{Ob}(G) \to \mathrm{Ob}(G')$. Then $\mathrm{Ob}(p') \,.\, f = \mathrm{Ob}(\psi) \,.\, \mathrm{Ob}(p)$, and this verifies one condition for (ψ, f) to be a map of operations.

To verify the other condition, let $h \in H(x, y)$, $x' \in p^{-1}(x)$. If $g \in G(x', y')$ lifts h, then $h \,.\, x' = y'$. Further $\varphi(g)$ in $G(fx', fy')$ lifts $\psi(h)$, and so

$$\psi(h) \,.\, f(x') = f(y') = f(h \,.\, x').$$

We have now defined σ on maps in $\mathscr{C}ov$. The verification that σ is a functor is straightforward. □

In Section 2 we defined a functor $\rho : \mathscr{F}ib \to \mathscr{C}ov$. Let τ be the composite

$$\tau = \sigma\rho : \mathscr{F}ib \to \mathscr{O}p.$$

Then τ gives the operations which are fundamental in the theory of fibrations. That τ is a functor expresses the fact that these operations are natural with respect to maps of fibrations.

Let $p : G \to H$ be a fibration. It is useful to describe the operation $\tau(p)$ explicitly. First of all,

$$\tau(p) = (H, \mathrm{Ob}(G/\ker p), \sigma(p), .),$$

so that H operates on $\mathrm{Ob}(G/\ker p)$. If $h \in H(x, y)$, then h_* is a function $\sigma(p)^{-1}(x) \to \sigma(p)^{-1}(y)$. But $\sigma(p)^{-1}(x) = \pi_0 p^{-1}(x)$, the components of the fibre over x. Thus we have

$$h_* : \pi_0 p^{-1}(x) \to \pi_0 p^{-1}(y).$$

This function can be described explicitly as follows: Let $\bar{x}'$ denote the component of x' in $\pi_0 p^{-1} p(x')$. Then $h \,.\, \bar{x}' = \bar{y}'$ where y' is the final point of some lifting of h which starts at x'. It can be verified directly and easily that $\bar{y}'$ is independent of the choice of x' in its class, and of the possible liftings of h.

However, the previous, more abstract, construction of $\tau(p)$ has the advantage of making obvious the naturality with respect to maps of fibrations.

We now use the operations to construct exact sequences.

Suppose $p : G \to H$ is a fibration, $x \in \mathrm{Ob}(G)$ and $F = p^{-1}p(x)$ is the fibre over $p(x)$. The inclusion $F \to G$ is written i; and $\bar{x}$, $\tilde{x}$ denote the components of x in $\pi_0 F$, $\pi_0 G$, respectively. Define

$$\partial : H\{px\} \to \pi_0 F$$

by

$$\partial(h) = h \,.\, \bar{x}$$

and consider the sequence

$$1 \longrightarrow F\{x\} \xrightarrow{i} G\{x\} \xrightarrow{p} H\{px\} \xrightarrow{\partial} \pi_0 F \xrightarrow{i} \pi_0 G \xrightarrow{p} \pi_0 H \qquad (4.2)$$

in which we abuse notation by writing i, p for maps induced by i, p respectively. Notice that the first four terms of the sequence are groups, and the last three terms are sets; however, we give these sets basepoints, namely $\bar{x}$ (which is, as above, the component of x in F), $\tilde{x}$ (the component of x in G) and $\overline{px}$ (the component of px in H). With this choice of base points, each function in (4.2) has a well-defined kernel, so that it makes sense to speak of the sequence being exact. However, there is also an operation of $H\{px\}$ on $\pi_0 F$; this is used in the definition of ∂, and we will see that it gives additional information about exactness.

4.3 THEOREM. *The sequence* (4.2) *of groups and based sets is exact in the usual sense. Further*:

(a) *If h, $k \in H\{px\}$, then $\partial(h) = \partial(k)$ if and only if there is a g in $G\{x\}$ such that $p(g) = k^{-1}h$.*

(b) *If $\bar{y}$, $\bar{z} \in \pi_0 F$,* then *$i(\bar{y}) = i(\bar{z})$ if and only if there is an h in $H\{px\}$ such that $h \,.\, \bar{y} = \bar{z}$.*

Proof. (i) Exactness at $G\{x\}$ is clear, since $F = p^{-1}p(x)$.

(ii) We prove exactness at $H\{px\}$.

If $g \in G\{x\}$, then g is a lift of $p(g)$, and g has final point x. Hence $\partial p(g) = \bar{x}$. So $\operatorname{Im}(p : G\{x\} \to H\{px\}) \subseteq \ker \partial$. The opposite inclusion is implied by (a), so we now prove (a).

Let $\partial(h) = \partial(k)$, so that $h \,.\, \bar{x} = k \,.\, \bar{x}$. Then $k^{-1}h \,.\, \bar{x} = \bar{x}$. This implies that $k^{-1}h$ lifts to an element $g' \in G(x, z)$, say, where $\bar{z} = \bar{x}$. So $F(z, x)$ is nonempty. Let g'' be an element of $F(z, x)$. Then $g = g''g'$ belongs to $G\{x\}$ and $p(g) = p(g') = k^{-1}h$.

On the other hand, if $k^{-1}h = p(g)$, where $g \in G\{x\}$, then $k^{-1}h \,.\, \bar{x} = \bar{x}$ and so $h \,.\, \bar{x} = k \,.\, \bar{x}$.

(iii) We prove exactness at $\pi_0 F$. This is clearly a consequence of (b).

Suppose then $\bar{y}$, $\bar{z} \in \pi_0 F$ and $i(\bar{y}) = i(\bar{z})$. Then $G(y, z)$ is nonempty; let $g \in G(y, z)$. Then $h = p(g) \in H\{px\}$, and $\bar{z} = h \,.\, \bar{y}$.

On the other hand if $h \in H\{px\}$ and $h \,.\, \bar{y} = \bar{z}$, then $h = p(g)$ for some $g \in G(y, z)$, and so y, z are in the same component of G, that is $i(\bar{y}) = i(\bar{z})$.

(iv) We prove exactness at $\pi_0 G$. Clearly $\operatorname{Im}(\pi_0 F \to \pi_0 G) \subseteq \ker(\pi_0 G \to \pi_0 H)$. Suppose then $p(\tilde{y}) = \overline{px}$. Then $H(px, py)$ is nonempty, containing an element h, say. Let $g \in \mathrm{St}_G\, y$ lift h^{-1} and let z be the final point of g. Then $p(z) = p(x)$, and $i(\bar{z}) = \tilde{y}$. $\square$

4.4 COROLLARY. *Let $p : G \to H$ be a fibration, and let $y \in \mathrm{Ob}(H)$. Then there is a bijection*

$$\eta : \bigsqcup_x H\{y\}/pG\{x\} \to \pi_0 p^{-1}(y)$$

where the disjoint union is taken over any set X of objects x of G which satisfy $p(x) = y$, and such that X is a complete set of representatives of the subset $p^{-1}(\bar{y})$ of $\pi_0 G$.

Proof. The function ∂ of 4.2 depends on x and so will here be written ∂_x. By 4.2 (a), the functions ∂_x induce η as above, and η is injective on each set of cosets $H\{y\}/pG\{x\}$. Also, if $x, x' \in X$ and $x \neq x'$, then $\tilde{x} \neq \tilde{x}'$, whence

$$\mathrm{Im}(\partial_x) \cap \mathrm{Im}(\partial_{x'}) = i^{-1}(\tilde{x}) \cap i^{-1}(\tilde{x}') = \varnothing$$

So η is injective.

Finally η is surjective, because if $\ddot{x}' \in \pi_0 p^{-1}(y)$, then $i(\ddot{x}') = i(\ddot{x})$ for some $x \in X$ and so $\ddot{x}' \in \mathrm{Im}(\partial_x)$ by exactness. $\square$

The exact sequence simplifies in case $p : G \to H$ has a section, that is, if there is a morphism $s : H \to G$ such that $ps = 1$.

4.5 PROPOSITION. *Let $p : G \to H$ be a fibration, and s a section of p. If $v \in \mathrm{Ob}(H)$ and $h \in H\{v\}$, then*

$$h \,.\, \overline{s(v)} = \overline{s(v)}.$$

Proof. Let $p = \rho\kappa$ be the canonical factorization of p. The covering morphism ρ has section κs, and $\overline{s(v)} = \kappa s(v)$. But $\kappa s(h)$ is the unique lift of h which starts at $\kappa s(v)$, and $\kappa s(h)$ finishes at $\kappa s(v)$. So $h \,.\, \kappa s(v) = \kappa s(v)$. $\square$

4.6 COROLLARY. *Let $p : G \to H$ be a fibration with fibre F over $v \in \mathrm{Ob}(H)$. Suppose* (*)*: every component of F contains an object $s(v)$ for s a section of p. Then the function $\pi_0 F \to \pi_0 G$ induced by inclusion is injective.*

Proof. This follows from 4.5 and 4.3(b) with $y = s(v)$, $z = s'(v)$ for sections s, s' of p. $\square$

Notice that it follows from 2.12 with $f : X \to H$, the identity morphism on H, that the condition (*) of 4.6 is equivalent to (**): every object of F is of the form $s(v)$ for s a section of p.

The exact sequence of 4.2 is natural in the following sense. Suppose given a commutative square of morphisms

$$\begin{array}{ccc} G & \xrightarrow{g} & G' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ H & \xrightarrow[h]{} & H' \end{array} \tag{4.7}$$

where p, p' are fibrations. Let $v \in \mathrm{Ob}(H)$, $v' = h(v)$; let $F = p^{-1}(v)$, $F' = p'^{-1}(v')$, then g induces a morphism $f : F \to F'$ of fibres. Choose $x \in \mathrm{Ob}(F)$ and let $x' = f(x)$. Then we have a diagram

$$\begin{array}{ccccccccccccc} 1 & \longrightarrow & F\{x\} & \xrightarrow{i} & G\{x\} & \xrightarrow{p} & H\{v\} & \xrightarrow{\partial} & \pi_0 F & \xrightarrow{i} & \pi_0 G & \xrightarrow{p} & \pi_0 H \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ 1 & \longrightarrow & F'\{x'\} & \xrightarrow[i']{} & G'\{x'\} & \xrightarrow[p']{} & H'\{v'\} & \xrightarrow[\partial']{} & \pi_0 F' & \xrightarrow[i]{} & \pi_0 G' & \xrightarrow[p']{} & \pi_0 H' \end{array} \tag{4.8}$$

in which $\pi_0 F$, $\pi_0 G$, $\pi_0 H$ have base points $\bar{x}$, $\tilde{x}$, $\bar{v}$; $\pi_0 F'$, $\pi_0 G'$, $\pi_0 H'$ have base points $\bar{x}'$, $\tilde{x}'$, $\bar{v}'$; and ∂, ∂' are given by operations on $\bar{x}$, $\bar{x}'$, respectively. Then 4.8 is commutative by commutativity of 4.7 and naturality of the operations.

Next we prove a 5-lemma type result, which uses the operations in a crucial way. We suppose given the situation as above.

4.9 THEOREM. (a) *If $h : H\{v\} \to H'(v')$, $g : \pi_0 G \to \pi_0 G'$ are surjective and $h : \pi_0 H \to \pi_0 H'$ is injective, then $f : \pi_0 F \to \pi_0 F'$ is surjective.*

(b) *If $h : H\{v\} \to H'\{v'\}$ is injective and $g : G\{x\} \to G'\{x'\}$ is surjective, then $f : \pi_0 F \to \pi_0 F'$ is injective on* $\ker(i : \pi_0 F \to \pi_0 G)$.

Proof. (a) Let $\bar{y}' \in \pi_0 F'$. We must construct $\bar{y} \in \pi_0 F$ such that $f(\bar{y}) = \bar{y}'$. Since $g : \pi_0 G \to \pi_0 G'$ is surjective, there is an element $\tilde{z}$ in $\pi_0 G$ such that $g(\tilde{z}) = i'(\bar{y}')$. Now

$$hp(\tilde{z}) = p'g(\tilde{z}) = p'i'(\bar{y}') = \bar{v}' = h(\bar{v}).$$

Since h is injective, it follows that $p(\tilde{z}) = \bar{v}$. Hence, by exactness, there is a $w \in \mathrm{Ob}(F)$ such that $\tilde{w} = \tilde{z}$. It follows that

$$i'f(\bar{w}) = g(\tilde{w}) = g(\tilde{z}) = i'(\bar{y}').$$

By 4.3 (b), there is an $a' \in H'\{v'\}$ such that

$$a' \,.\, f(\bar{w}) = \bar{y}.'$$

Since $h : H\{v\} \to H'\{v'\}$ is surjective, $a' = h(a)$ for some $a \in H\{v\}$ and we have

$$f(a \,.\, \bar{w}) = \bar{y}'.$$

(b) Let $y, z \in \mathrm{Ob}(F)$ satisfy $\tilde{y} = \tilde{z} = \tilde{x}, f(\bar{y}) = f(\bar{z})$. We prove that $\bar{y} = \bar{z}$. Since $\tilde{y} = \tilde{x}$, $\tilde{z} = \tilde{x}$, there are elements $c \in G(x, y)$, $d \in G(x, z)$; hence

$$p(c) \, . \, \bar{x} = \bar{y}, \qquad p(d) \, . \, \bar{x} = \bar{z}.$$

Since $f(\bar{y}) = f(\bar{z})$, we have

$$\partial' hp(c) = f(p(c) \, . \, \bar{x}) = f(p(d) \, . \, \bar{x}) = \partial' hp(d).$$

By 4.3(a) there is a $b' \in G'\{x'\}$ such that

$$p'(b') = [hp(c)]^{-1} \, . \, hp(d).$$

Since $g : G\{x\} \to G'\{x\}'$ is surjective there is a $b \in G\{x\}$ such that $b' = g(b)$. Then

$$hp(b) = p'g(b) = h(p(c)^{-1}p(d)).$$

Since h is injective

$$p(b) = p(c)^{-1}p(d)$$

and it follows that

$$\bar{y} = p(c) \, . \, \bar{x} = p(d) \, . \, \bar{x} = \bar{z}. \quad \square$$

Notice that in 4.9(b) we make no assumption on $g : \pi_0 G \to \pi_0 G'$; however, if this function is injective (as in the usual 5-lemma) and $y, z \in \mathrm{Ob}(F)$ satisfy $f(\bar{y}) = f(\bar{z})$, $i'f(\bar{y}) = \tilde{x}'$, then we can deduce $g(\tilde{y}) = g(\tilde{z}) = g(\tilde{x})$, whence $\tilde{x} = \tilde{y} = \tilde{z}$.

Another point is that the conclusion of 4.9(b) is weaker than in the ususal 5-lemma. The following example shows that the assumptions of 4.9(b) do not imply $f : \pi_0 F \to \pi_0 F'$ injective even when $g : \pi_0 G \to \pi_0 G'$ is injective.

4.10 Example. Suppose given a map of fibrations

$$\begin{array}{ccc} G & \xrightarrow{g} & G' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ H & \xrightarrow{1} & H \end{array}$$

such that G, G' are connected and the induced map $\pi_0 F \to \pi_0 F'$ of components of the fibres of p, p' over $v \in \mathrm{Ob}(H)$ is not injective. Thus p, p' could be coverings, or we could take p, p' as in Example 2.15. Then there is the following map of fibrations

$$\begin{array}{ccc} G \sqcup H & \xrightarrow{g \sqcup 1} & G' \sqcup H \\ {\scriptstyle (p,1)}\downarrow & & \downarrow{\scriptstyle (p',1)} \\ H & \xrightarrow{1} & H. \end{array}$$

The induced map of components of fibres is still not injective, although the maps $\pi_0(G \sqcup H) \to \pi_0(G' \sqcup H)$, $H\{v\} \to H\{v\}$ are injective and also $(G \sqcup H)\{v\} \to (G' \sqcup H)\{v\}$ is surjective.

If we apply the full force of 4.9 we obtain the following corollary which assumes the situation of 4.7, 4.8.

4.11 COROLLARY. *If $h : H \to H'$, $g : G \to G'$ are equivalences of groupoids, then so also is $f : F \to F'$.*

Proof. Since g, h are equivalences, the induced functions

$$h : \pi_0 H \to \pi_0 H', \qquad h : H\{v\} \to H'\{v'\}, \qquad g : G\{x\} \to G'\{x'\}$$

are bijective, the last one for all $x \in \mathrm{Ob}(F)$. Hence $f : \pi_0 F \to \pi_0 F'$ is a bijection, by 4.9. But the ordinary 5-lemma for groups applied to diagram 4.8 shows that $f : F\{x\} \to F'\{x'\}$ is bijective for all $x \in \mathrm{Ob}(F)$. It follows easily (for example, by using 6.5.13 (Corollary 1) of [1]) that $f : F \to F'$ is an equivalence. $\square$

This corollary can also be proved by using arguments for groupoids similar to those given in [2] for spaces. This sort of topic will be dealt with elsewhere by P. R. Heath.

5. NON-ABELIAN COHOMOLOGY OF GROUPS

In this section we show how the exact sequence of the previous section includes the well-known exact sequences involving H^0 and H^1 for the non-Abelian cohomology of groups.

Let G be a group operating on the left of a group A. We call A a G module. If $g \in G$, $a \in A$ we shall write ${}^g a$ for the result of operating on a by g.

We can form the split extension $G \ltimes A$ and the split exact sequence

$$1 \xrightarrow{i} A \longrightarrow G \ltimes A \xrightarrow{p} G \longrightarrow 1 \tag{5.1}$$

where the elements of $G \ltimes A$ are pairs (g, a), $g \in G$, $a \in A$; the multiplication in $G \ltimes A$ is given by

$$(g, a)(g_1, a_1) = (gg_1, a\, {}^g a_1);$$

and $i : a \mapsto (a, 1)$, $p : (g, a) \mapsto g$.

A section s of p has a *principal part* $\bar{s} : G \to A$ obtained by composing s with projection onto A. Of course $\bar{s}$, unlike s, is not a morphism but is a *crossed morphism* (or *derivation*); i.e., $\bar{s}$ satisfies

$$\bar{s}(gg_1) = \bar{s}(g) \,.\, {}^g\bar{s}(g_1) \qquad g, g_1 \in G$$

The set of such crossed morphisms is nonempty, since the constant function $G \to A$ with value 1 is a crossed morphism— this *trivial* crossed morphism is the principal part of the 1-*section* $s_1 : G \to G \tilde{\times} A, g \mapsto (g, 1)$.

Let $p_* : (G(G \tilde{\times} A)) \to (GG)$ be the morphism of groupoids induced by p; p_* is a fibration by 2.9. Let $1_G : G \to G$ be the identity morphism. Then the fibre of p_* over 1_G is written

$$Z^1(G; A).$$

Clearly the objects of $Z^1(G; A)$ are just the sections of p.

Let us abbreviate $Z^1(G; A)$ to Z^1 when this will cause no confusion.

5.2 Proposition. *Let s, t be two sections of $p : G \tilde{\times} A \to G$ with principal parts $\bar{s}$, $\bar{t}$. Then $Z^1(s, t)$ is bijective with the set of objects $a \in A$ which satisfy*

$$\bar{t}(g) = a\bar{s}(g)({}^g a)^{-1} \qquad all \quad g \in G.$$

Proof. Since G has only one object, the elements of $Z^1(s, t)$ are determined by elements (g_0, a) in $G \tilde{\times} A$ which satisfy $p(g_0, a) = 1$ and $(g_0, a)\, s(g) = t(g)(g_0, a)$ all $g \in G$. So $g_0 = 1$, and the latter equation becomes, in terms of principal parts.

$$a\bar{s}(g) = \bar{t}(g)\,{}^g a \qquad \text{all} \quad g \in G$$

from which the equation of the proposition follows. □

5.3 Corollary. *$Z^1\{s_1\}$ is isomorphic to the group A^G of elements of A fixed under G.*

Proof. This follows from 5.2 with $\bar{s}(g) = \bar{t}(g) = 1$, all $g \in G$. □

The set A^G is sometimes written $H^0(G; A)$.

Similar calculations to those of 5.3 show that $(GG)\{1_G\}$ is isomorphic to $C(G)$, the centre of G. Note that according to 4.1 $C(G)$ operates on $\pi_0 Z^1(G; A)$; but this operation is trivial by 4.5.

5.4 Definition. The 1-*dimensional cohomology set* of G with coefficients in A is

$$H^1(G; A) = \pi_0 Z^1(G; A).$$

Notice that because $C(G)$ operates trivially on $H^1(G; A)$, we can, by 4.6, identity $H^1(G; A)$ with a subset of $\pi_0(G(G \tilde{\times} A))$.

5.5 Proposition. *Let $1 \to A \xrightarrow{i} B \to C \xrightarrow{j} 1$ be an exact sequence of G modules. Then j induces a fibration*

$$j' : Z^1(G; B) \to Z^1(G; C)$$

whose fibre over $s_1 : G \to G \tilde{\times} C$ is the image of the inclusion $i' : Z^1(G; A) \to Z^1(G; B)$ induced by i.

Proof. A morphism $f: A \to A'$ of G modules induces a function $f: G \mathbin{\tilde{\times}} A \to G \mathbin{\tilde{\times}} A'$, $(g, a) \mapsto (g, f(a))$, and it is easily checked that $f: G \mathbin{\tilde{\times}} A \to G \mathbin{\tilde{\times}} A'$ is a morphism of groups which is injective or surjective according as $f: A \to A'$ is injective or surjective.

It follows that $j: B \to C$ induces a diagram

$$\begin{tikzcd}
Z^1(G; B) \arrow[r, "j'"] \arrow[d] & Z^1(G; C) \arrow[d] \\
(G(G \mathbin{\tilde{\times}} B)) \arrow[r, "j_*"] \arrow[d] & (G(G \mathbin{\tilde{\times}} C)) \arrow[d] \\
(GG) \arrow[r, "id"] & (GG)
\end{tikzcd}$$

By 2.9, j_* is a fibration, and so by 2.14 j' is a fibration.

If $s \in \mathrm{Ob}[Z^1(G; B)]$, then the condition $j'(s) = s_1 : G \to G \mathbin{\tilde{\times}} C$ is simply that $j\bar{s} = \bar{s}_1$, whence, by exactness, $\bar{s}$ factors through the inclusion $i: A \to B$. This proves that the fibre of j' is the image of $i' : Z^1(G; A) \to Z^1(G; B)$. In fact i' is injective: indeed it is obviously injective on objects; and it is injective on elements by 5.2. $\square$

5.6 Corollary. *Let* $1 \to A \to B \to C \to 1$ *be an exact sequence of* G *modules. There is a six term sequence*

$$1 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \xrightarrow{\partial} H^1(G; A) \longrightarrow H^1(G; B) \longrightarrow H^1(G; C)$$

which is exact in the sense of 4.3, *so that* C^G *operates on* $H^1(G; A)$ *and the boundary* ∂ *is defined by* $\partial(c) = c\,.\,(\mathrm{cls}\, s_1)$.

Proof. This is immediate from 5.5 and 4.2. $\square$

The sequence of 5.6 is the fundamental exact sequence of non-Abelian cohomology–see for example [6].

There is a well-known generalization of the exact sequence of 5.6 involving cohomology sets $H_\varphi{}^1(G;\ )$ determined by a morphism $\varphi: G \to G$. This comes out of the present setup as follows.

5.7 Theorem. *Let* $\varphi: G \to G$ *be a morphism and* A *a* G *module. Let* $Z_\varphi{}^1 = Z_\varphi{}^1(G; A)$ *be the fibre of* $p_* : (G(G \mathbin{\tilde{\times}} A)) \to (GG)$ *over* φ, *and let* $H_\varphi{}^1(G; A) = \pi_0 Z_\varphi{}^1(G; A)$. *Then for any exact sequence of* G *modules* $1 \to A \to B \to C \to 1$ *there is a six term sequence*

$$1 \longrightarrow A^{\varphi(G)} \longrightarrow B^{\varphi(G)} \longrightarrow C^{\varphi(G)} \xrightarrow{\partial} H_\varphi{}^1(G; A) \longrightarrow H_\varphi{}^1(G; B) \longrightarrow H_\varphi{}^1(G; C)$$

which is exact in the sense of 4.3.

Proof. The modifications in the previous proofs are that the objects of $Z_\varphi{}^1$ are morphisms $s : G \to G \mathbin{\tilde{\times}} A$ of the form $g \mapsto (\varphi(g)), \bar{s}(g)$ such that 5.2 becomes: $Z_\varphi{}^1(s, t)$ is bijective with objects $a \in A$ such that

$$\bar{t}(g) = a\bar{s}(g)({}^{\varphi(g)}a)^{-1};$$

that $Z_\varphi{}^1\{s_1\}$ is isomorphic to $A^{\varphi(G)}$, the set of elements of A fixed under the action of $\varphi(G)$; and that we obtain a fibration $Z_\varphi{}^1(G; B) \to Z_\varphi{}^1(G; C)$ with fibre $Z_\varphi{}^1(G; A)$. $\square$

Other exact sequences can be obtained by considering the fibration $Z^1(G; B) \to Z^1(G; C)$ and taking the fibre over some other object of $Z'(G; C)$ than the trivial section. The discussion of these is left to the reader.

Going back to the situation of 5.6, we can obtain some 5-lemma type results by applying 4.9.

5.8 PROPOSITION. *Suppose given a map of exact sequences of G modules*

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r] \arrow[d, "f_1"] & B \arrow[r] \arrow[d, "f_2"] & C \arrow[r] \arrow[d, "f_3"] & 1 \\
1 \arrow[r] & A_1 \arrow[r] & B_1 \arrow[r] & C_1 \arrow[r] & 1
\end{tikzcd}$$

We have (a) *If $H^1(G; C)$, $C_1{}^G$ and $H^1(G; B_1)$ consist of a single element, then $f_{1*} : H^1(G; A) \to H^1(G; A_1)$ is surjective.* (b) *If $H^1(G; B)$, C^G and $B_1{}^G$ consist of a single element, then $f_{1*} : H^1(G; A) \to H^1(G; A_1)$ is injective.*

Proof. We apply 4.9 to the map of exact sequences given by 5.6. Then 5.8(a) follows from 4.9(a), and 5.8(b) follows from 4.9(b). $\square$

We now obtain an exact sequence for the case A is a G module and H is a subgroup of G, so that A is also an H module.

5.9 PROPOSITION. *If $i : H \to G$ is an inclusion of groups, and A is a G module, then i induces a covering morphism*

$$i^* : Z^1(G; A) \to Z^1(H; A).$$

Proof. We consider the diagram

$$\begin{tikzcd}
Z^1(G; A) \arrow[r, "i_3{}^*"] \arrow[d] & W \arrow[d] & Z^1(H; A) \arrow[l, "i_3{}^*"'] \arrow[d] \\
(G(G \mathbin{\tilde{\times}} A)) \arrow[r, "i_2{}^*"] \arrow[d, "p_*"] & (H(G \mathbin{\tilde{\times}} A)) \arrow[d, "p_*"] & (H(H \mathbin{\tilde{\times}} A)) \arrow[l, "i_2{}^*"'] \arrow[d, "p_*"] \\
(GG) \arrow[r, "i_1{}^*"] & (HG) & (HH) \arrow[l, "i_1{}^*"']
\end{tikzcd}$$

in which the horizontal morphisms are induced by i; the morphisms p_* are induced by the projections of the split extension; and W is the fibre over $i : H \to G$. By 2.10, $i_1{}^*$ and $i_2{}^*$ are covering morphisms.

By 2.14, $i_3{}^* : Z^1(G; A) \to W$ also is a covering morphism.

Now i_{3*} is an inclusion. Further its image contains the image of $i_3{}^*$, essentially because a section of $G \tilde{\times} A \to G$ restricts to a section of $H \tilde{\times} A \to H$. So $i_3{}^*$ restricts to a covering morphism $i^* : Z^1(G; A) \to Z^1(H; A)$. □

(One can also give a fairly simple direct proof of 5.9).

The fibre of $i^* : Z^1(G; A) \to Z^1(H; A)$ over the trivial section s_1 is written

$$Z^1(G, H; A).$$

It is a discrete groupoid consisting of the sections s of $G \tilde{\times} A \to G$ whose principal parts $\bar{s}$ satisfy $\bar{s}(H) = \{1\}$.

5.10 COROLLARY. *There is a 5-term sequence*

$$1 \longrightarrow A^G \longrightarrow A^H \xrightarrow{\partial} Z^1(G, H; A) \longrightarrow H^1(G; A) \longrightarrow H^1(H; A)$$

in which A^H operates on $Z^1(G, H; A)$, ∂ is defined by $\partial(a) = a \,.\, s_1$, and the sequence is exact in the sense of 4.3. □

The quotient of $Z^1(G, H; A)$ by the above operation of A^H will be written $H^1(G, H; A)$. By 4.3 we obtain

5.11 COROLLARY. *There is an exact sequence of pointed sets*

$$H^1(G, H; A) \xrightarrow{j} H^1(G; A) \longrightarrow H^1(H; A)$$

in which j is injective. □

In the case H is normal in G the above relative cohomology becomes more understandable.

5.12 PROPOSITION. *If H is normal in G, then A^H is a G/H module and there is a bijection*

$$H^1(G, H; A) \to H^1(G/H; A^H)$$

Proof. That A^H is a G/H module is clear since the elements of H act trivially on A^H.

We construct a function

$$\varphi : Z^1(G, H; A) \to \mathrm{Ob}[Z^1(G/H; A^H)].$$

Let $s \in Z^1(G, H; A)$. If $h, h' \in H, g \in G$, then

$$\bar{s}(hg) = \bar{s}(h) \,.\, {}^h s(g) = {}^h s(g), \qquad \bar{s}(gh') = \bar{s}(g) \,.\, {}^g \bar{s}(h') = \bar{s}(g).$$

But given h in H, g in G, we can find h' in H such that $hg = gh'$ (as H is normal in G). We deduce that

$${}^h \bar{s}(g) = \bar{s}(hg) = \bar{s}(g).$$

Thus the image of $\bar{s}$ is contained in A^H, and $\bar{s}$ is constant on each coset gH. Hence s determines a section $\varphi(s)$ of $(G/H) \,\tilde{\times}\, A^H \to G/H$ as required.

Conversely, a section t of $(G/H) \,\tilde{\times}\, A^H \to G/H$ has principal part $\bar{t} : G/H \to A^H$, and the composite

$$G \longrightarrow G/H \xrightarrow{\bar{t}} A^H \longrightarrow A$$

is the principal part of a section $\psi(t)$ of $G \,\tilde{\times}\, A \to G$ such that $\psi(t) \in Z^1(G, H; A)$. Clearly $\psi\varphi = 1$, $\varphi\psi = 1$, and so φ is a bijection.

Two elements s, t of $Z^1(G, H; A)$ lie in the same orbit under the action of A^H if and only if they lie in the same component of $Z^1(G; A)$, and this is true if and only if $\varphi(s)$, $\varphi(t)$ are in the same component of $Z^1(G/H; A^H)$. Hence φ induces the required bijection. $\square$

5.13 Corollary. *If H is normal in G, and A is a G module, there is an exact sequence of pointed sets*

$$H^1(G/H; A^H) \xrightarrow{j} H^1(G; A) \xrightarrow{i^*} H^1(H; A)$$

with j injective. $\square$

6. Applications to Homotopy Theory

If X, Y are spaces, then πY^X will mean the track groupoid as defined in [1]; that is, the objects of πY^X are the maps $X \to Y$ and the morphisms $\pi Y^X(f, g)$ are the homotopy classes rel end maps of homotopies $f \simeq g$. If X, Y are spaces with base point, then $\pi Y^X_\cdot$ has objects the maps $X \to Y$ of spaces with base point, and the elements of $\pi Y^X_\cdot(f, g)$ are the homotopy classes rel end maps of homotopies $f \simeq g$ rel base point.

A *fibration* of spaces is a map $p : E \to B$ which has the covering homotopy property for all spaces; for spaces with base point, this will mean that all maps and homotopies are to be rel base point.

6.1 PROPOSITION. *Let $p : E \to B$ be a fibration of spaces; then for any X*

$$p_* : \pi E^X \to \pi B^X$$

is a fibration of groupoids. Further p_* is a covering morphism if, in addition, $p : E \to B$ has unique path-lifting, that is if given $e \in E$ and w a path in B starting at $p(e)$, then w is covered by a unique path in E starting at e; in particular, p_* is a covering morphism if p is a covering map.

Proof. This is almost immediate from the definition and 2.1 (iii). Indeed the necessary and sufficient condition for p_* to be a fibration is that given any map $f : X \to E$, any homotopy of pf is homotopic rel end maps to a homotopy which is covered by a homotopy of f. If p has unique path lifting, then the above covering homotopy is unique, since a homotopy F on X determines for each $x \in X$ a path $t \mapsto F(x, t)$. □

The dual result to 6.1 is:

6.2 PROPOSITION. *Let $i : A \to X$ be a cofibration. Then for any Y, the induced morphism*

$$i^* : \pi Y^X \to \pi Y^A$$

is a fibration of groupoids.

The proof here is simple.

The canonical factorization of a fibration of groupoids leads to a simple version of the first step in the Moore–Postnikov factorization of a fibration.

6.3 PROPOSITION. *Let $p : E \to B$ be a fibration, and let B be locally path-connected and semilocally simply connected. Then there is a factorization*

$$\begin{tikzcd} E \arrow[dr, "\kappa'"] \arrow[dd, "p"'] & \\ & \tilde{B} \arrow[dl, "\rho'"] \\ B & \end{tikzcd}$$

such that ρ' is a covering map and κ' is a surjective fibration with path-connected fibres.

Proof. By 2.4, the fibration of groupoids $p_* : \pi E \to \pi B$ has a factoriaztion $p_* = \rho\kappa$ where $\kappa : \pi E \to C$ is a fibration of groupoids and $\rho : C \to \pi B$ is a covering morphism. By Section 9.5 of [1], the topology of B lifts to a topology on $\tilde{B} = \mathrm{Ob}(C)$ so that if $\rho' = \mathrm{Ob}(\rho)$ then $\pi(\rho') : \pi\tilde{B} \to \pi B$ can be identified with ρ. Let $\kappa' = \mathrm{Ob}(\kappa)$; then κ' is continuous by 9.5.3 of [1], and it is easily checked that κ' is a fibration. The fibres of κ' are path-connected since the fibres of $\pi(\kappa') = \kappa$ are connected groupoids. $\square$

The results of Section 3 on lifting morphisms can be used to generalise well-known results on the group of covering transformations (for example 2.6.2 of [7]):

6.4 Proposition. *Let $p : E \to B$ be a covering map and let $f : X \to B$ be a map. We suppose X, E, B are path-connected and locally path-connected. Let $z \in X$, $x \in E$ be such that $f(z) = p(x)$. Then the set of liftings of f is bijective with*

$$N_{\pi(B,px)}(f_*\pi(X, z), p_*\pi(E, x))/p_*\pi(E, x).$$

Proof. This is immediate from 3.5 and the facts (9.5.3 of [1]) that if $f' : \pi X \to \pi E$ is any lifting of $\pi(f)$, then $\bar{f} = \mathrm{Ob}(f') : X \to E$ is continuous, $\bar{f}$ is a lifting of f, and any lifting of f is obtained in this way. $\square$

If we apply the results of Section 4 on operations to the case of fibrations of spaces, then we obtain at one blow all the usual operations, in particular all those discussed in [7], Section 7.3. First of all we prove:

6.5 Proposition. *Let $p : E \to B$ be a fibration of spaces. Then for any space X there is a functor $\Gamma : \pi B^X \to \mathcal{S}et$ which on objects sends $u \mapsto X, E \diagdown u$, where $X, E \diagdown u$ is the set of homotopy classes* rel p *of lifts of u.*

Proof. We know that $p_* : \pi E^X \to \pi B^X$ is a fibration so that by Section 4 we have a functor $\Gamma : \pi B^X \to \mathcal{S}et$ which on objects sends $u \to \pi_0 F_u$ where F_u is the fibre of p_* over u. The objects of F_u are lifts of u. If $f, g : X \to E$ are two lifts of u which are homotopic rel p, then f, g lie in the same component of F_u. Conversely if f, g lie in the same component of F_u, then there is a homotopy $G : f \simeq g$ such that pG is homotopic rel end maps to the constant homotopy of u. This homotopy of pG can be lifted to a homotopy rel end maps of G to a homotopy $G' : f \simeq g$ such that pG' is constant. $\square$

By applying the same method to the dual case given in 6.2, we can derive the operations used in Chapter 7 of [1]. As another example, recall that a pair (X, A) has *nondegenerate base point* x_0 if $x_0 \in A$ and for any map $f : (X, A) \to (Y, B)$ any homotopy of $f \mid \{x_0\}, B$ extends to a homotopy of f.

6.6 Proposition. *Let (X, A) have nondegenerate base point x_0. Then for any (Y, B) there is a functor $\pi B \to \mathcal{S}et$ which on objects sends $b_0 \mapsto [X, A, x_0; Y, B, b_0]$ this being the set of homotopy classes of maps of triples.*

Proof. Since (X, A) has nondegenerate base point, we have for any (Y, B) a fibration of groupoids (with obvious notation)

$$p : \pi(Y, B)^{(X,A)} \to \pi B$$

defined by evaluation on x_0. The required functor is thus a special case of that given by 6.5. □

The operation given in 6.5 is a special case of a more subtle operation which for a fibration $E \to B$ gives a functor from πB assigning to each $x \in B$ the fibre F_x over x, and to each class in $\pi B(x, y)$ a homotopy class of homotopy equivalences $F_x \to F_y$ (cf. [7] Theorem 2.8.12). This operation, and some generalizations, will be discussed elsewhere by P. R. Heath.

We conclude with the exact sequence of a fibration.

6.7 Proposition. *Let $p : E \to B$ be a fibration. Then for any X, map $u : X \to B$, and lift $f : X \to E$ of u there is a sequence*

$$\pi E^X\{f\} \longrightarrow \pi B^X\{u\} \xrightarrow{\partial} X, E \diagdown u \xrightarrow{i_*} [X, E] \xrightarrow{p_*} [X, B]$$

which is exact in the sense of 4.3, *where the last three sets have base points $\bar{f}, \bar{f}, \bar{u}$, respectively, $\pi B^X\{u\}$ operates on $X, E \diagdown u$, and ∂ is defined by $\partial(\alpha) = \alpha . \bar{f}$.*

Proof. This is immediate from 4.3 and 6.5. □

A similar exact sequence is valid for the case $p : E \to B$ is a fibration of spaces with base point, where πE^X, πB^X are replaced by $\pi E^X_\cdot$, $\pi B^X_\cdot$. But then we can take $f = .$, $u = .$ to be constant maps so that (modulo suitable topological assumptions, or by working in a convenient category of spaces)

$$\pi E^X_\cdot\{.\} \cong [\Sigma X, E] \cong [X, \Omega E]$$

$$\pi B^X_\cdot\{.\} \cong [\Sigma X, B] \cong [X, \Omega B]$$

where ΣX is the reduced suspension of X and Ω is the loop space of X. Further if F is the fibre of p over $.$, then $X, E \diagdown . = [X, F]$. So the exact sequence of 6.7 becomes the usual exact sequence

$$[X, \Omega E] \xrightarrow{p_*} [X, \Omega B] \longrightarrow [X, F] \longrightarrow [X, E] \longrightarrow [X, B]. \qquad (6.8)$$

Here p_* is induced by $\Omega p : \Omega E \to \Omega B$. Since Ωp is also a fibration, the sequence (6.8) can be continued indefinitely to the left.

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