QUANTUM SYMMETRIES, OPERATOR ALGEBRA AND QUANTUM GROUPOID REPRESENTATIONS: PARACRYSTALLINE SYSTEMS, TOPOLOGICAL ORDER, SUPERSYMMETRY AND GLOBAL SYMMETRY BREAKING

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ABSTRACT

Novel approaches to extended quantum symmetry, paracrystals, quasicrystals, noncrystalline solids, topological order, supersymmetry and spontaneous, global symmetry breaking are outlined in terms of quantum groupoid, quantum double groupoids and dual, quantum algebroid structures. Physical applications of such quantum groupoid and quantum algebroid representations to quasicrystalline structures and paracrystals, quantum gravity, as well as the applications of the Goldstone and Noether's theorems to: phase transitions in superconductors/superfluids, ferromagnets, antiferromagnets, mictomagnets, quasi-particle (nucleon) ultra-hot plasmas, nuclear fusion, and the integrability of quantum systems are also considered. Both conceptual developments and novel approaches to Quantum theories are here proposed starting from existing Quantum Group Algebra (QGA), Algebraic Quantum Field Theories (AQFT), standard and effective Quantum Field Theories (QFT), as well as the refined `machinery' of Non--Abelian Algebraic Topology (NAAT), Category Theory (CT) and Higher-Dimensional Algebra (HDA).The logical links between Quantum Operator Algebras and their corresponding,'dual' structure of the Quantum State Spaces are also investigated. Among the key concepts presented are: Quantum Group Algebras (QGAs)/Groupoids, Hopf and C*- algebras, Lie `algebras', Quantization and Asymptotic Morphisms, Locally Topological Groupoids, Crossed Modules of Groups or Lie Double Groupoids, Lie Algebroids, Crossed Complexes over Groupoids, Holonomy and Gauge Transformation Groupoids, Quantum Principal Bundles and Sheaves.

Keywords: *Extended quantum symmetry, topological ordering, quantum groupoid/quantum algebroid representations, quasicrystals, paracrystals, glasses and noncrystalline solids, ferromagnets, FSWR, ESR and spin waves, quantum groups and quasi-Hopf algebras, Yang-Baxter equations, R-matrix and quantum inverse scattering problem, 6-j symmetry, convolution algebroids, duality, supersymmetry, graded Lie algebroids and quantum gravity; Goldstone theorem, global SSB and NG-bosons, superuids and superconductivity, ultra-hot plasmas, QCD, EFT and nuclear fusion, approximate chiral symmetry and SSB, pions, rho and omega particles.*
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1. INTRODUCTION

In this highly-condensed review we are discussing several fundamental aspects of quantum symmetry, extended quantum symmetries, and also their related quantum groupoid and categorical representations. This is intended as an up-to-date review centered on quantum symmetry, invariance and representations. We aim at an accessible presentation, as well as a wide field of view of quantum theories, so that the hitherto `hidden' patterns of quantum relations, concepts and the underlying, extended quantum symmetries become visible to the `mathematical-ready eyes' of the theoretical physicist. To this end, we are therefore focusing here on several promising developments related to extended quantum symmetries, such as `paragroups', `quasicrystals' and quantum groupoid/quantum algebroid representations whose roots can be traced back to recent developments in solid state physics, crystallography, metal physics, nanotechnology laboratories, quantum chromodynamics theories, nuclear physics, nuclear fusion reactor engineering /designs, and so on, to other modern physics areas, including quantum gravity and supergravity theories. Then, we propose several unifying ideas such as: general representations of abstract structures and relations relevant to the treatment of extended quantum symmetry, ultra-high energy physics, and topological order theories.

Symmetry groups have been the mainstay of Euclidean structures as have been envisaged in classical dynamical systems, relativity and particle physics, etc., where single, direct transformations are usually sufficient. In contrast, as is found to be the case in sub-atomic, extreme microscopic quantum mechanical and biomolecular systems, transformations are essentially simultaneous. A transformed state-configuration can be indistinguishable from its original and at the microscopic level a precise symmetry might neglect some anomalous behaviour of the underlying physical process particularly in the excitation spectra. Thus, the role played here by classical symmetry (Lie) groups is useful but limiting. Extending the latter symmetry concepts, one looks towards the more abstract but necessary structures at this level, such as `paragroups' or `symmetry groupoids', and also encoding the ubiquitous concept of a `quantum group' into the architecture of a $C^*$-Hopf algebra. With this novel approach, a classical Lie algebra now evolves to a `Lie (bi)algebroid' as one of the means for capturing higher order and super- symmetries. The convolution algebra of the *transition groupoid* of a bounded quantum system--which is related to its spectra--is however just *matrix algebra*, when viewed in terms of its representations. The initial groupoid viewpoint-- which was initially embraced by Werner Heisenberg for a formal, quantum-theoretical treatment of spectroscopy--was thus replaced for expediency by a computable matrix formulation of Quantum Mechanics resulting from such group representations. This was happening at a time when the *groupoid* and category concepts have not yet entered the realm of either physics or the mainstream of pure mathematics. On the other hand, in Mathematics the groupoid theory and the groupoid mode of thought became established in algebraic topology [44] towards the end of the twentieth century, and now it is a fruitful, already rich in mathematical results; it is also a well-founded field of study in its own right, but also with potential, numerous applications in mathematical physics, and especially in developing *non-Abelian* physical theories.

In the beginning, the algebraic foundation of Quantum Mechanics occurred along two different lines of approach- that of John von Neumann, published in 1932, and independently, Paul A.M. Dirac's approach in 1930; these two developments followed the first analytical formulation of Quantum Mechanics of the (electron in) Hydrogen Atom by Erwin Schrödinger in 1921. The equivalence of Heisenberg's `matrix mechanics' and Schrödinger's analytical formulation is now universally accepted. On the other hand, Von Neumann's formulation in terms of operators on Hilbert spaces and W^*-algebras has proven its fundamental role and real value in providing a more general, algebraic framework for both quantum measurement theories and the mathematical treatment of a very wide range of quantum systems. To this day, however, the underlying problem of the `right' quantum logic for von Neumann's algebraic formulation of quantum theories remains to be solved, but it would seem that a modified Łukasiewicz, many-valued, LM -- *algebraic logic* is a very strong candidate [21]. Interestingly, such an LM-quantum algebraic logic is *noncommutative* by definition, and it would also have to be a *non-distributive* lattice (*loc.cit.*, and also the relevant references cited therein). A topos theory based on a concept called a 'quantum topos' was also proposed for quantum gravity by rejecting the idea of a spacetime continuum [148]. The latter concept is based however on a Heyting (intuitionistic) logic algebra which is known however to be a *commutative lattice*, instead of the (noncommutative and non-distributive, multi-valued) quantum logic expected of any quantum theory [21].

The quantum operator algebra for various quantum systems then required also the introduction of: C^*-algebras, Hopf algebras/quantum groups, Clifford algebras (of observables), graded algebras or superalgebras, Weak Hopf algebras, quantum doubles, j-symbols, Lie 2-algebras, Lie-2 groups, Lie 3-superalgebras [9], and so on. The current rapid expansion of the collection of such various types of `quantum algebras' suggests an eventual need for a *Categorical Ontology of quantum systems* which is steadily moving towards the framework of *higher-dimensional algebra (HDA)* and the related, *higher categorical, non-Abelian structures* underlying quantum field and higher gauge theories. A survey of the basic mathematical approach of HDA, also with several examples of physical applications, can be found in an extensive, recent monograph [21], complemented by a recent, introductory textbook on Quantum Algebraic Topology, Quantum Algebra and Symmetry [11]. The case of the C^*-algebras is particularly important as the von Neumann W^*- algebras can be considered as a special type of C^*-algebras. Moreover, Gelfand and Neumark [123] showed that any C^*- algebra can be given a concrete representation in a Hilbert space, that need not be separable; thus, there is an isomorphic mapping of the elements of a C^*- algebra into the set of bounded operators of a Hilbert space. Subsequently, Segal [254] completed the work begun by Gelfand and Neumark by providing a procedure for the construction of concrete Hilbert space representations of an abstract C^*-algebra, called the *`GNS construction'* [253], from the initials of Gelfand, Neumark and Segal. Segal then proceeded to an algebraic formulation of quantum mechanics based on postulates that define a Segal S-algebra structure which is more general than a C^*-algebra [254].

We also introduce the reader to a series of novel concepts that are important for numerous applications of modern physics, such as: the generalised convolution algebras of functions, convolution product of groupoids/quantum

groupoids, convolution of quantum algebroids and related crossed complexes, C^*--convolution quantum algebroids, graded quantum algebroids, the embeddings of quantum groups into *groupoid* C^**-algebras*, quantum (Lie) double groupoids with connection, the R-matrix of *the Yang-Baxter equations*, the 6 j-symbol symmetry with their representations in relation to Clebsch-Gordan theories, Extended Topological Quantum Field Theories (ETQFTs), as well as their relationship with integrable systems, solutions of the generalised Yang-Baxter equation and $6j$-symbols. Other related structures, such as Clifford algebras, Grassman algebras, R--algebroids, quantum double algebroids were discussed in detail in a recent monograph [21]. Moreover, the new establishment of the dual concepts of quantum groupoid and quantum algebroid representations in a `Hopf'-algebra/bialgebroid setting are a natural consequence of their long-accepted use in the simpler guise of finite, `quantum groups'. Many of the extended quantum symmetry concepts considered here need to be viewed in the light of fundamental theorems that already have a wide range of physical applications, such as the theorems of: Noether, Goldstone, Wigner, Stone-von Neumann and others,which we briefly recall in § 5. Further generalisations and important potential applications to theoretical physics of theorems such as the generalised Siefert-van Kampen theorem are then discussed in § 6.
Ultimately, we would like to see an *unified categorical framework* of the quantum symmetry fundamental concepts and the results based on them that are here encapsulated only as the sub-structures needed for a broader view of quantum symmetry theories than that traditionally emcountered. In this regard, our presentation also includes several novel approaches that are outlined, particulary in § 5 and § 6. The inclusion of an extensive, supporting bibliography of both experimental data and theoretical physics reports was thus required, together with an overview provided in the next subsection indicating how the cited references are grouped according to the main categories, or themes, that are discussed here.

1.1.Topic Groups and Categories

The topics covered in the references cited under our main subject of quantum symmetries and representations can be grouped, or categorized as follows:

• Extended Quantum Symmetry and Nonabelian Quantum Algebraic Topology:[21],[286, 287], [12], [14], [20], [41],[35], [57],[63, 193], [66], [121],[122], [130],[140],[148], [154], [159], [178], [179], [181], [202], [203],[205], [227], [241, 242, 243], [271], [278, 279],[284],[293], [121];

• Paracrystal Theory, Quasicrystals and Convolution Algebra: [21, 14],[143], [145, 144, 146],[162],[249], [299, 300];

• Operator algebras: Von Newmann, C^*-, W^*- algebras, Clifford algebras, Hilbert spaces, Quantum Groups and Hopf algebras: [5], [36],[115], [38], [36],[78], [95], [101], [102], [103], [109, 110], [116], [117],[111],[123], [126], [142], [163], [104],[164], [165],[185], [204], [209, 210],[213],[225], [219], [235],[239, 240], [253, 254], [257, 258], [266], [290];

• Quantum Groupoids and Weak Hopf, Quantum Doubles, Quasi-Hopf, etc. algebras and the Inverse Scattering Problem:[265], [2], [37], [30], [38], [41], [104], [167, 168],[177], [178], [186], [199, 200], [233], [262],[295],[301, 302];

• Algebraic Topology, Groupoids, Algebroids, R-algebroids, Homology/Cohomology, Double Groupoids, Algebraic Geometry and Higher-Dimensional Algebra: [30], [132], [133, 134, 135, 136], [190, 191], [34], [34], [60], [57], [63], [66], [67], [78], [95], [141], [129], [171], [177],[181], [192],[193],[199, 200], [213, 214, 215], [268], [269],[276];

• Groupoid Representations and Haar measure systems: [250, 251],[68], [71, 69, 70], [72, 73, 74, 75, 76, 77, 78] [40, 41, 42], [138, 139], [213, 214, 215], [226], [236], [253],[254],[259, 260],[284];

• Functional and Harmonic Analysis--Fourier Transforms, Generalised Fourier Transforms and Measured Groupoids: [113],[182], [213, 214, 215], [212], [228, 229],[237], [247, 248], [282, 283];

• Quantum Field theories, Local Quantum Physics and Yang-Baxter equations: [119],[274, 275], [285], [25], [98], [164], [172],[280],[187], [245], [259, 260], [267], [290],[296],[298];

• General Relativity (GR), Quantum Gravity and Supersymmetry: [234], [28], [41], [83], [131], [137], [171],[172], [198],[202], [216], [218], [231], [244], [281],[291], [275], [102];

• Quantum Topological Order and Quantum Algebraic Topology:[99], [173, 175, 174],[188], [230], [278, 279], [230], [278],[203],[11],[99], [121];

• Quantum Physics Applications related to Symmetry and Symmetry Breaking: Superconductivity [274], [238], Superfluidity [274], Quasi-crystals [227], Crystalline [1], [3], [7], [118] and Noncrystalline Solids [8], [18, 15, 19, 17, 22],[23], Mott-Anderson Transition [194, 195],[196],[197],[170], [270],[82], Many-body and Many-electron systems [278], [126],[35], Liquid Crystals, Nanotechnologies, Quantum Hall Systems [263, 264], Nuclear Fusion

[23], [274], and Astrophysics* [274], Fundamental aspects: [24],[21],[188],[274, 275], [11];
• Fundamental theorems and results:[272], [11, 21], [59], [61], [62, 176], [67],[81];
• Generalised Representation Theory and Adjunction Theorems: [21], [95];
• Galois and Generalised Galois theories: [149, 39], [58, 150], [209, 210];
• Van Kampen Theorem, its Generalisations and Potential Applications to Quantum Theories: [272, 44, 50, 57, 60, 67, 55, 59, 58];
• Category Theory and Categorical Representations: [189], [183, 184], [222], [29],[97], [95], [153], [108], [125], [124],[180], [206], [211];
• Noncommutative geometry: [273], [41], [42], [40],[188], [241, 242].
*Note that only several representative examples are given in each group or category, without any claim to either comprehensiveness or equal representation.

1.2. Paracrystal Theory and Convolution Algebra

As reported in a recent publication [21], the general theory of scattering by partially ordered, atomic or molecular, structures in terms of *paracrystals* and *lattice convolutions* was formulated by Hosemann and Bagchi in [145] using basic techniques of Fourier analysis and convolution products. A natural generalization of such molecular, partial symmetries and their corresponding analytical versions involves convolution algebras -- a functional/distribution [247, 248] based theory that we will discuss in the context of a more general and original concept of a *convolution-algebroid of an extended symmetry groupoid of a paracrystal*, of any molecular or nuclear system, or indeed, any quantum system in general; such applications also include quantum fields theories, and local quantum net configurations that are endowed with either partially disordered or `completely' ordered structures, as well as in the graded, or super-algelbroid extension of these concepts for very massive structures such as stars and black holes treated by quantum gravity theories.
A statistical analysis linked to structural symmetry and scattering theory considerations shows that a real paracrystal can be defined by a three dimensional convolution polynomial with a semi-empirically derived composition law, *, [146]. As was shown in [13, 14] - supported with computed specific examples - several systems of convolution can be expressed analytically, thus allowing the numerical computation of X -ray, or neutron, scattering by partially disordered layer lattices via complex Fourier transforms of one-dimensional structural models using fast digital computers. The range of paracrystal theory applications is however much wider than the one-dimensional lattices with disorder, thus spanning very diverse non-crystalline systems, from metallic glasses and spin glasses to superfluids, high-temperature superconductors, and extremely hot anisotropic plasmas such as those encountered in controlled nuclear fusion (for example, JET) experiments. Other applications - as previously suggested in [12] - may also include novel designs of `fuzzy' quantum machines and quantum computers with extended symmetries of quantum state spaces.

1.3. Convolution product of groupoids and the convolution algebra of functions

From a purely mathematical perspective, Alain Connes introduced the concept of a C^* -algebra of a (discrete) group (see, e.g., [91]). The underlying vector space is that of complex valued functions with finite support, and the multiplication of the algebra is the fundamental *convolution product* which it is convenient for our purposes to write slightly differently from the common formula as

$$(f * g)(z) = \sum_{xy=z} f(x)g(y),$$

and $*$ -operation

$$f(x) = \overline{f(x^{-1})}.$$

The more usual expression of these formulas has a sum over the elements of a selected group. For topological groups, where the underlying vector space consists of continuous complex valued functions, this product requires the availability of some structure of measure and of measurable functions, with the sum replaced by an integral. Notice also that this algebra has an identity, the distribution function δ_1, which has value 1 on the identity 1 of the group, and has zero value elsewhere. Given this convolution/distribution representation that combines crystalline (`perfect' or global-group, and/or group-like symmetries) with partial symmetries of paracrystals and glassy solids on the one hand, and also with non-commutative harmonic analysis [182] on the other hand, we propose that several extended quantum symmetries can be represented algebraically in terms of certain structured *groupoids*, their C^*-*convolution quantum algebroids*, paragroup/*quantized groups* and/or other more general mathematical structures that will be introduced in this report. It is already known that such extensions to groupoid and algebroid/coalgebroid symmetries require also a generalization of non-commutative harmonic analysis which involves certain Haar measures, generalized Fourier-Stieltjes transforms and certain categorical duality relationships representing very general mathematical symmetries as well. Proceeding from the abstract structures endowed with extended symmetries to numerical applications in quantum physics always involves representations through specification of concrete elements, objects and transformations. Thus, groupoid and functorial representations that generalize group representations in several, meaningful ways are key to linking abstract, quantum operator algebras and symmetry properties with actual numerical computations of quantum eigenvalues and their eigenstates, as well as a wide variety of numerical factors involved in computing quantum dynamics. The well-known connection between groupoid convolution representations and matrices [276] is only one of several numerical computations made possible via groupoid representations. A very promising approach to nonlinear (anharmonic) analysis of aperiodic quantum systems represented by rigged Hilbert space bundles may involve the computation of representation coefficients of Fourier--Stieltjes groupoid transforms.

Currently, however, there are several important aspects of quantum dynamics left out of the invariant, simplified picture provided by group symmetries and their corresponding representations of quantum operator algebras [126]. An alternative approach proposed in [140] employs differential forms to find symmetries.

Physicists deal often with such problems in terms of either spontaneous symmetry breaking or approximate symmetries that require underlying assumptions or ad-hoc dynamic restrictions that have a phenomenological basisl. A well-studied example of this kind is that of the dynamic Jahn--Teller effect and the corresponding `theorem' (Chapter 21 on pp. 807--831, as well as p. 735 of [1]) which in its simplest form stipulates that *a quantum state with electronic non-Kramers degeneracy may be unstable against small distortions of the surroundings, that would lower the symmetry of the crystal field and thus lift the degeneracy* (i.e., cause an observable splitting of the corresponding energy levels). This effect occurs in certain paramagnetic ion systems *via* dynamic distortions of the crystal field symmetries around paramagnetic or high-spin centers by moving ligands that are diamagnetic. The established physical explanation is that the Jahn--Teller coupling replaces a purely electronic degeneracy by a vibronic degeneracy (of *exactly the same* symmetry!). The dynamic, or spontaneous breaking of crystal field symmetry (for example, distortions of the octahedral or cubic symmetry) results in certain systems in the appearance of doublets of symmetry γ_3 or singlets of symmetry γ_1 or γ_2. Such dynamic systems could be locally expressed in terms of symmetry representations of a Lie algebroid, or globally in terms of a special Lie (or Lie--Weinstein) symmetry groupoid representations that can also take into account the spin exchange interactions between the Jahn--Teller centers exhibiting such quantum dynamic effects. Unlike the simple symmetries expressed by group representations, the latter can accommodate a much wider range of possible or approximate symmetries that are indeed characteristic of real, molecular systems with varying crystal field symmetry, as for example around certain transition ions dynamically bound to ligands in liquids where motional narrowing becomes very important. This well known example illustrates the importance of the interplay between symmetry and dynamics in quantum processes which is undoubtedly involved in many other instances including: *quantum chromodynamics (QCD), superfluidity, spontaneous symmetry breaking (SSB), quantum gravity and Universe dynamics* (i.e., the inflationary Universe), some of which will be discussed in further detail in § 5.

Therefore, the various interactions and interplay between the symmetries of quantum operator state space geometry and quantum dynamics at various levels leads to both algebraic and topological structures that are variable and complex, well beyond symmetry groups and well-studied group algebras (such as Lie algebras, see for example [126]). A unified treatment of quantum phenomena/dynamics and structures may thus become possible with the help of algebraic topology, non-Abelian treatments; such powerful mathematical tools are capable of revealing novel, fundamental aspects related to extended symmetries and quantum dynamics through a detailed analysis of the variable geometry of (quantum) operator algebra state spaces. At the center stage of non-Abelian algebraic topology are groupoid and algebroid structures with their internal and external symmetries [276] that allow one to treat physical spacetime structures and dynamics within an unified categorical, higher dimensional algebra framework [52]. As already suggested in our recent report [21], the interplay between extended symmetries and dynamics generates higher dimensional structures of quantized spacetimes that exhibit novel properties not found in lower dimensional representations of groups, group algebras or Abelian groupoids.

It is also our intention here to explore new links between several important but seemingly distinct mathematical approaches to extended quantum symmetries that were not considered in previous reports. An important example example is the general theory of scattering by partially ordered, atomic or molecular, structures in terms of *paracrystals* and *lattice convolutions* that was formulated in [145] using basic techniques of Fourier analysis and convolution products. Further specific applications of the paracrystal theory to X-ray scattering, based on computer algorithms, programs and explicit numerical computations, were subsequently developed by the first author [13] for one-dimensional paracrystals, partially ordered membrane lattices [14] and other biological structures with partial structural disorder [16]. Such biological structures, `quasi-crystals', and the paracrystals, in general, provide rather interesting physical examples of extended symmetries (cf. [144]). Moreover, the quantum inverse scattering problem and the treatment of nonlinear dynamics in ultra-hot plasmas of white stars and nuclear fusion reactors requires the consideration of quantum doubles, or respectively, quantum double groupoids and graded double algebroid representations [21].

1.4. Group and Groupoid Representations

Whereas group representations of quantum unitary operators are extensively employed in standard quantum mechanics, the quantum applications of *groupoid representations* are still under development. For example, a description of stochastic quantum mechanics in curved spacetime [102] involving a Hilbert bundle is possible in terms of groupoid representations which can indeed be defined on such a Hilbert bundle (X^*, H, π), but cannot be expressed as the simpler group representations on a Hilbert space H. On the other hand, as in the case of group representations, unitary groupoid representations induce associated C^*-algebra representations. In the next subsection we recall some of the basic results concerning groupoid representations and their associated groupoid *-algebra representations. For further details and recent results in the mathematical theory of groupoid representations one has also available a succint monograph [68] (and references cited therein).

Let us consider first the relationships between these mainly algebraic concepts and their extended quantum symmetries. Then we introducer several extensions of symmetry and algebraic topology in the context of local quantum physics, ETQFT, spontaneous symmetry breaking, QCD and the development of novel supersymmetry theories of quantum gravity. In this respect one can also take spacetime `inhomogeneity' as a criterion for the comparisons between physical, partial or local, symmetries: on the one hand, the example of paracrystals reveals thermodynamic disorder (entropy) within its own spacetime framework, whereas in spacetime itself, whatever the selected model, the inhomogeneity arises through (super) gravitational effects. More specifically, in the former case one has the technique of the generalized Fourier--Stieltjes transform (along with convolution and Haar measure), and in view of the latter, we may compare the resulting `broken'/paracrystal--type symmetry with that of the supersymmetry predictions for weak gravitational fields, as well as with the spontaneously broken global supersymmetry in the presence of intense gravitational fields.

Another significant extension of quantum symmetries may result from the superoperator algebra and/or algebroids of Prigogine's quantum *superoperators* which are defined only for irreversible, infinite-dimensional systems [225]. The latter extension is also incompatible with a commutative logic algebra such as the Heyting algebraic logic currently utilized to define topoi [128].

1.4.1. *Extended Quantum Groupoid and Algebroid Symmetries.*

Our intention here is to view the following scheme in terms of a weak Hopf C^*-algebroid- and/or other- extended symmetries, which we propose to do, for example, by incorporating the concepts of *rigged Hilbert spaces* and sectional functions for a small category.

Quantum groups $\rightarrow$ ***Representations*** $\rightarrow$ ***Weak Hopf algebras*** $\rightarrow$ ***Quantum groupoids and algebroids***

We note, however, that an alternative approach to quantum groupoids has already been reported [186], (perhaps also related to noncommutative geometry); this was later expressed in terms of deformation-quantization: the Hopf algebroid deformation of the universal enveloping algebras of Lie algebroids [295] as the classical limit of a quantum `groupoid'; this also parallels the introduction of quantum `groups' as the deformation-quantization of Lie bialgebras. Furthermore, such a Hopf algebroid approach [177] leads to categories of Hopf algebroid modules [295] which are monoidal, whereas the links between Hopf algebroids and monoidal bicategories were investigated by Day and Street [95].

As defined under the following heading on groupoids, let (G_{lc}, τ) be a *locally compact groupoid* endowed with a (left) Haar system, and let $A = C^*(\mathsf{G}_{lc}, \tau)$ be the convolution C^*-algebra (we append A with $\mathbf{1}$ if necessary, so that A is unital). Then consider such a *groupoid representation* $\Lambda : (\mathsf{G}_{lc}, \tau) \rightarrow \{\mathsf{H}_x, \sigma_x\}_{x \in X}$ that respects a compatible measure σ_x on H_x (cf. [68]). On taking a state ρ on A, we assume a parametrization

$$(\mathsf{H}_x, \sigma_x) := (\mathsf{H}_\rho, \sigma)_{x \in X} \; . \tag{1}$$

Furthermore, each H_x is considered as a *rigged Hilbert space*[38], that is, one also has the following nested inclusions:

$$\Phi_x \subseteq (\mathsf{H}_x, \sigma_x) \subseteq \Phi_x^{\times}, \tag{2}$$

in the usual manner, where Φ_x is a dense subspace of H_x with the appropriate locally convex topology, and $\Phi_x^{\times}$ is the space of continuous antilinear functionals of Φ. For each $x \in X$, we require Φ_x to be invariant under Λ and $\mathrm{I}m\,\Lambda \,|\, \Phi_x$ is a continuous representation of G_{lc} on Φ_x. With these conditions, representations of (proper) quantum groupoids that are derived for weak C^*-Hopf algebras (or algebroids) modeled on rigged Hilbert spaces could be suitable generalizations in the framework of a Hamiltonian generated semigroup of time evolution of a quantum system via integration of Schrödinger's equation $i\hbar \frac{\partial \psi}{\partial t} = H\psi$ as studied in ıthe case of Lie groups [284]. The adoption of the rigged Hilbert spaces is also based on how the latter are recognized as reconciling the Dirac and von Neumann approaches to quantum theories [38].

Next, let G be a *locally compact Hausdorff groupoid* and X a locally compact Hausdorff space. (G will be called a *locally compact groupoid, or lc- groupoid* for short). In order to achieve a small C^*-category we follow a suggestion of A. Seda (private communication) by using a general principle in the context of Banach bundles [250, 251]. Let $q = (q_1, q_2) : \mathsf{G} \to X \times X$ be a continuous, open and surjective map. For each $z = (x, y) \in X \times X$, consider the fibre $\mathsf{G}_z = \mathsf{G}(x, y) = q^{-1}(z)$, and set $\alpha_z = C_0(\mathsf{G}_z) = C_0(\mathsf{G}(x, y))$ equipped with a uniform norm $\| \|_z$. Then we set $\alpha = \bigcup_z \alpha_z$. We form a Banach bundle $p : \alpha \to X \times X$ as follows. Firstly, the projection is defined via the typical fibre $p^{-1}(z) = \alpha_z = \alpha_{(x,y)}$. Let $C_c(\mathsf{G})$ denote the continuous complex valued functions on G with compact support. We obtain a sectional function $\tilde{\psi} : X \times X \to \alpha$ defined via restriction as $\tilde{\psi}(z) = \psi \,|\, \mathsf{G}_z = \psi \,|\, \mathsf{G}(x, y)$. Commencing from the vector space $\gamma = \{\tilde{\psi} : \psi \in C_c(\mathsf{G})\}$, the set $\{\tilde{\psi}(z) : \tilde{\psi} \in \gamma\}$ is dense in α_z. For each $\tilde{\psi} \in \gamma$, the function $\| \tilde{\psi}(z) \|_z$ is continuous on X, and each $\tilde{\psi}$ is a continuous section of $p : \alpha \to X \times X$. These facts follow from Theorem 1 in [251]. Furthermore, under the convolution product $f * g$ *the space* $C_c(\mathsf{G})$ *forms an associative algebra over* $\mathbb{C}$ (cf. Theorem 3 in [251]).

We refer readers to [105] for the description and properties of Banach bundles.

1.4.2. *Groupoids*

Recall that a groupoid G is, loosely speaking, a small category with inverses over its set of objects $X = \mathrm{O}b(\mathsf{G})$. One often writes G_x^y for the set of morphisms in G from x to $\boldsymbol{y}$. A *topological groupoid* consists of a space G, a distinguished subspace $\mathsf{G}^{(0)} = \mathrm{O}b(\mathsf{G}) \subseteq \mathsf{G}$, called *the space of objects* of G, together with maps

$$\boldsymbol{r}, \boldsymbol{s} : \mathsf{G} \underset{s}{\overset{r}{\rightrightarrows}} \mathsf{G}^{(0)} \tag{3}$$

called the *range* and *source maps* respectively, together with a law of composition

$$\circ : \mathsf{G}^{(2)} := \mathsf{G} \times_{\mathsf{G}^{(0)}} \mathsf{G} = \{ (\gamma_1, \gamma_2) \in \mathsf{G} \times \mathsf{G} : s(\gamma_1) = r(\gamma_2) \} \to \mathsf{G}, \tag{4}$$

such that the following hold :

1. $s(\gamma_1 \circ \gamma_2) = r(\gamma_2)\,, r(\gamma_1 \circ \gamma_2) = r(\gamma_1)$, for all $(\gamma_1, \gamma_2) \in \mathsf{G}^{(2)}$.
2. $s(x) = r(x) = x$, for all $x \in \mathsf{G}^{(0)}$.
3. $\gamma \circ s(\gamma) = \gamma\,, r(\gamma) \circ \gamma = \gamma$, for all $\gamma \in \mathsf{G}$.
4. $(\gamma_1 \circ \gamma_2) \circ \gamma_3 = \gamma_1 \circ (\gamma_2 \circ \gamma_3)$.

5. Each γ has a two--sided inverse γ^{-1} with $\gamma\gamma^{-1} = r(\gamma)$, $\gamma^{-1}\gamma = s(\gamma)$.

Furthermore, only for topological groupoids the inverse map needs be continuous. It is usual to call $\mathsf{G}^{(0)} = Ob(\mathsf{G})$ *the set of objects* of G . For $u \in Ob(\mathsf{G})$, the set of arrows $u \to u$ forms a group G_u, called the *isotropy group of* G *at* u.

Thus, as is well kown, a topological groupoid is just a groupoid internal to the category of topological spaces and continuous maps. The notion of internal groupoid has proved significant in a number of fields, since groupoids generalize bundles of groups, group actions, and equivalence relations. For a further study of groupoids we refer the reader to [48].

Several examples of groupoids are:

- (a) locally compact groups, transformation groups, and any group in general (e.g. [59]);
- (b) equivalence relations;
- (c) tangent bundles;
- (d) the tangent groupoid (e.g. [4]);
- (e) holonomy groupoids for foliations (e.g. [4]);
- (f) Poisson groupoids (e.g. [81]);
- (g) graph groupoids (e.g. [47, 64]).

As a simple example of a groupoid, consider (b) above. Thus, let *R* be an *equivalence relation* on a set X. Then *R* is a groupoid under the following operations: $(x, y)(y, z) = (x, z), (x, y)^{-1} = (y, x)$. Here, $\mathsf{G}^0 = X$, (the diagonal of $X \times X$) and $r((x, y)) = x, s((x, y)) = y$.

So R^2 = $\{((x, y),(y, z)) : (x, y),(y, z) \in R\}$. When $R = X \times X$, *R* is called a *trivial* groupoid. A special case of a trivial groupoid is $R = R_n = \{1,2,...,n\}$ $\times$ $\{1,2,...,n\}$. (So every *i* is equivalent to every *j*). Identify $(i, j) \in R_n$ with the matrix unit e_{ij}. Then the groupoid R_n is just matrix multiplication except that we only multiply e_{ij}, e_{kl} when $k = j$, and $(e_{ij})^{-1} = e_{ji}$. We do not really lose anything by restricting the multiplication, since the pairs e_{ij}, e_{kl} excluded from groupoid multiplication just give the 0 product in normal algebra anyway.

Definition 1.1.

For a groupoid G_{lc} to be a *locally compact groupoid*, G_{lc} is required to be *a (second countable) locally compact Hausdorff space*, and the product and also inversion maps are required to be continuous. Each G_{lc}^u as well as the unit space G_{lc}^0 is closed in G_{lc}.

Remark 1.1:

What replaces the left Haar measure on G_{lc} is a system of measures λ^u ($u \in \mathsf{G}_{lc}^0$), where λ^u is a positive regular Borel measure on G_{lc}^u with dense support. In addition, the λ^u 's are required to vary continuously (when integrated against $f \in C_c(\mathsf{G}_{lc})$) and to form an invariant family in the sense that for each x, the map $y \mapsto xy$ is a measure preserving homeomorphism from $\mathsf{G}_{lc}^s(x)$ onto $\mathsf{G}_{lc}^r(x)$. Such a system $\{\lambda^u\}$ is called a *left Haar system* for the locally compact groupoid G_{lc}. This is defined more precisely in the next subsection.

1.4.3. Haar systems for locally compact topological groupoids

Let

$$\mathsf{G} \underset{s}{\overset{r}{\rightrightarrows}} \mathsf{G}^{(0)} = X \tag{5}$$

be a locally compact, locally trivial topological groupoid with its transposition into transitive (connected) components. Recall that for $x \in X$, the *costar of* x denoted $\mathrm{CO}^*(x)$ is defined as the closed set

$\bigcup\{\mathsf{G}(y,x) : y \in \mathsf{G}\}$, whereby

$$\mathsf{G}(x_0, y_0) \longrightarrow \mathrm{CO}^*(x) \to X \ , \tag{6}$$

is a principal $\mathsf{G}(x_0, y_0)$-bundle relative to fixed base points (x_0, y_0). Assuming all relevant sets are locally compact, then following [250], a *(left) Haar system on* G denoted (G, τ) (for later purposes), is defined to comprise of i) a measure κ on G, ii) a measure μ on X and iii) a measure μ_x on $\mathrm{CO}^*(x)$ such that for every Baire set E of G, the following hold on setting $E_x = E \cap \mathrm{CO}^*(x)$:

- $x \mapsto \mu_x(E_x)$ is measurable
- $\kappa(E) = \int_x \mu_x(E_x)\, d\mu_x$
- $\mu_z(tE_x) = \mu_x(E_x)$, for all $t \in \mathsf{G}(x,z)$ and $x, z \in \mathsf{G}$.

The presence of a left Haar system on G_{lc} has important topological implications: it requires that the range map $r : \mathsf{G}_{lc} \to \mathsf{G}_{lc}^0$ is open. For such a G_{lc} with a left Haar system, the vector space $C_c(\mathsf{G}_{lc})$ is a *convolution* *--*algebra*, where for $f, g \in C_c(\mathsf{G}_{lc})$:

$$f * g(x) = \int f(t) g(t^{-1}x) d\lambda^{r(x)}(t),$$

with

$$f * (x) = \overline{f(x^{-1})}.$$

One has $C^*(\mathsf{G}_{lc})$ to be the *enveloping* C^**-algebra* of $C_c(\mathsf{G}_{lc})$ (and also representations are required to be continuous in the inductive limit topology). Equivalently, it is the completion of $\pi_{univ}(C_c(\mathsf{G}_{lc}))$ where π_{univ} is the *universal representation* of G_{lc}. For example, if $\mathsf{G}_{lc} = R_n$, then $C^*(\mathsf{G}_{lc})$ is just the finite dimensional algebra $C_c(\mathsf{G}_{lc}) = M_n$, the span of the e'_{ij} s.

There exists (cf. [68]) a *measurable Hilbert bundle* $(\mathsf{G}_{lc}^0, H, \mu)$ with

$$H = \left\{ H^u_{u \in \mathsf{G}_{lc}^0} \right\}$$

and a G-representation L on H. Then, for every pair ξ, η of square integrable sections of H, it is required that the function $x \mapsto (L(x)\xi(s(x)), \eta(r(x)))$ be ν --measurable. The representation Φ of $C_c(\mathsf{G}_{lc})$ is then given by $\langle \Phi(f)\xi |, \eta \rangle = \int f(x)(L(x)\xi(s(x)), \eta(r(x))) d\nu_0(x)$. The triple (μ, H, L) is called a *measurable* G_{lc}*-Hilbert bundle*.

1.5. Groupoid C^* --convolution Algebras and Their Representations

Jean Renault introduced in ref. [235] the C^**-algebra of a locally compact groupoid* G as follows: the space of continuous functions with compact support on a groupoid G is made into a *-algebra whose multiplication is the *convolution*, and that is also endowed with the smallest C^*-norm which makes its representations continuous, as shown in ref.[76]. Furthermore, for this convolution to be defined, one needs also to have a *Haar system* associated to the locally compact groupoid G that are then called *measured groupoids* because they are endowed with an associated Haar system which involves the concept of measure, as introduced in ref. [138] by P. Hahn.

With these concepts one can now sum up the definition (or construction) of the *groupoid* C^**-convolution algebra*, or the *groupoid* C^**-algebra* [79] as follows.

Definition 1.2: A *groupoid* C^* *-convolution algebra*, G_{CA}, is defined for *measured groupoids* as * -algebra, with "*" being defined by convolution so that it has a smallest C* -norm which makes its representations continuous. representations continuous.

Remark 1.2: One can also produce a functorial construction of G_{CA} that has additional interesting properties. Next we recall a result due to P. Hahn [139] which shows how groupoid representations relate to induced *-algebra representations and also how-under certain conditions- the former can be derived from the appropriate *-algebra representations.

Theorem 1. *(Source: ref. [139]). Any representation of a groupoid* (G, C) *with Haar measure* (ν, μ) *in a separable Hilbert space* H *induces a *-algebra representation* $f \mapsto X_f$ *of the associated groupoid algebra* $\Pi(\mathsf{G}, \nu)$ *in* $L^2(U_{\mathsf{G}}, \mu, H)$ *with the following properties:*

(1) For any $l, m \in \Gamma$, one has that:

$$\left| \langle X_f (u \mapsto l), (\mathrm{u} \mapsto \mathrm{m}) \rangle \right| \leq \| f_l \| \, \| l \| \, \| m \|$$

and

(2) $M_r(\alpha) X_f = X_{f\alpha \circ r}$, where $M_r : L^\infty(U_{\mathsf{G}}, \mu) \to L[L^2(U_{\mathsf{G}}, \mu)]$, with $M_r(\alpha) j = \alpha \cdot j$.

Conversely, any *- algebra representation with the above two properties induces a groupoid representation, X, as follows:

$$\langle X_{f,j,k} \rangle = \int f(x)[X(x) j(d(x)), k(r(x)) d\nu(x)], \tag{8}$$

(viz. p. 50 of ref. [139]).

Furthermore, according to Seda (ref. [252] and also *personal communication from A. Seda*), the continuity of a Haar system is equivalent to the continuity of the convolution product $f * g$ for any pair f, g of continuous functions with compact support. One may thus conjecture that similar results could be obtained for functions with *locally compact* support in dealing with convolution products of either locally compact groupoids or quantum groupoids. Seda's result also implies that the convolution algebra $C_c(\mathsf{G})$ of a groupoid G is closed with respect to convolution if and only if the fixed Haar system associated with the measured groupoid G_m is *continuous* (see ref. [68]).

Thus, in the case of groupoid algebras of transitive groupoids, it was shown in [68] that any representation of a measured groupoid $(\mathsf{G}, [\int \nu^u d\tilde{\lambda}(u)] = [\lambda])$ on a separable Hilbert space H induces a *non-degenerate* *-representation $f \mapsto X_f$ of the associated groupoid algebra $\Pi(\mathsf{G}, \nu, \tilde{\lambda})$ with properties formally similar to (1) and (2) above in **Theorem 1**. Moreover, as in the case of groups, *there is a correspondence between the unitary representations of a groupoid and its associated* C^* *-convolution algebra representations* (p. 182 of [68]), the latter involving however *fiber bundles of Hilbert spaces*, instead of single Hilbert spaces.

2. SYMMETRIES OF VON NEUMANN ALGEBRAS. EXTENDED SYMMETRIES. HOPF AND WEAK HOPF ALGEBRAS

2.1. Symmetries and Representations

The key for symmetry applications to physical problems and numerical computations lies in utilizing *representations* of abstract structures such as groups, double groups, groupoids and categories. Thus, an abstract structure has an infinite number of equivalent representations such as matrices of various dimensions that follow the same multiplication operations as for example those of an abstract group; such representations are therefore called group representations. Among the important representations in physics are:

- Representations of Lie algebras and Lie groups
- Representations of the symmetry groups

- $U(1), SU(2)$ and $SU(3)$ symmetry group representations
- ***6j*** - symmetry representations
- Quantum Group, Hopf and Weak Hopf algebra representations
- Representations of the Poincaré group
- Representations of the Lorentz group of transformations
- Double-group representations.

Lie groups and Lie algebras are representative examples of a very well developed and elegant theory of continuous symmetry of mathematical objects and structures that are also indispensible tools in modern theoretical physics; they provide a natural framework for the analysis of continuous symmetries related to differential equations in a Differential Galois theory somewhat similar to the use of premutation groups in the Galois theory for analysing the discrete symmetries of algebraic equations. Sophus Lie's principal motivation for developing the theory was the extension of the original Galois theory to the case of continuous symmetry groups. We shall consider in § 6 further extensions of the Galois theory, well beyond that of Lie's theory.

A widely employed type of symmetry in many quantum computations for solid crystals is the point-group symmetry of various kinds, and the representations of the point-groups are matrices of lower dimensions in some cases, but typically infinite matrices as in Heisenberg's formulation of Quantum Mechanics. In finite dimensions, a representation of the Abelian, local symmetry group $U(1)$ is related to electrical charges and it is gauged to yield Quantum Electrodynamics (QED). Moreover, in quantum mechanics there are several quite useful and widely employed, lower-dimensional representations of symmetry groups, such as the Pauli (spin) matrix representations of the group $SU(2)$ and the three dimensional matrix representations of $SU(3)$ in QCD for the strong interactions *via* gluons. Thus, there are two types of $SU(3)$ symmetry: the exact gauge symmetry mediated by gluons, which symmetry acts on the different colors of quarks, and there is the distinct flavor $SU(3)$ symmetry which is only an approximate (not a fundamental) symmetry of the vacuum in QCD. Moreover the vacuum is symmetric under $SU(2)$ isospin rotations between up and down orientations, but it is less symmetric under the strange or full flavor group $SU(3)$; such approximate flavor symmetries still have associated gauge bosons, that are actually observed particles such as the ρ and the ω, but they are not masssles and behave very differently from gluons. In an approximate QCD version with n_f flavors of massless quarks one would have an approximate global, chiral symmetry group for flavors, $SU_L(n_f)\times SU_R(n_f)\times U_B(1)\times U_A(1)$, whose symmetry is spontaneously broken by the QCD vacuum with the formation of a chiral condensate. The axial symmetry $U_A(1)$ is exact classically, but broken in the quantum theory; it is sometimes called an `anomaly'. On the other hand, the $U_B(1)$ vector symmetry is an *exact* symmetry in the quantum theory and relates to the baryon quark number [11].

The representations, or realizations, of quantum groupoids and quantum categories are however much more complex, especially if numerical computations are desired based on such representations. Both quantum groupoids and quantum categories can be defined in several ways depending on the type of quantum system envisaged, e.g. finite boundary problems or quantum fields.

2.1. The QCD Lagrangian and formal Cross-relations with Disordered Magnetic Systems in Solids

Quark and gluon dynamics are governed by a QCD Lagrangian of the form:

$$L_{QCD} = \overline{\psi}_i(i\gamma^{\mu}\partial_{\mu} - m)\psi_i - gG_{\mu}{}^{a}\overline{\psi}_i\gamma^{\mu}T_{ij}{}^{a}\psi_j - (1/4)G_{\mu\nu}{}^{a}G_{a}{}^{\mu\nu}, \qquad (9)$$

where $\overline{\psi}_i(x)$ is a dynamic function of spacetime called the quark field in the fundamental representation of the gauge group $SU(3)$, which has indces $i, j,...$, and $G_{\mu}{}^{a}(x)$ are the gluon fields, also dynamic functions of spacetime x, but in the *adjoint representation* of the $SU(3)$ gauge group indexed by $a,b,...$; the Lagrangian L_{QCD} also includes the Dirac matrices γ^{μ} which connect the spinor representation to the vector representation of the Lorentz group of transformations. $T_{ij}{}^{a}$'s are called the generators which connect fundamental, anti-fundamental and adjoint representations of the $SU(3)$ gauge group; Gell-Mann matrices provide one such representation, thus playing a central role in QCD. $G_{\mu\nu}$ is the gauge invariant, gluonic field strength tensor, somewhat analogous in form to the electromagnetic field strength tensor of QED; this gluon field tensor includes also the *structure constants* of $SU(3)$. The gluon color field is represented by a SU(3)-Lie algebra-valued `curvature' 2-form: $G = d\widetilde{G} - g\widetilde{G} \wedge \widetilde{G}$, where $\widetilde{G}$ is a `vector-potential' 1-form corresponding to $\widetilde{G}$. The constants m and g in eq. (2.1) represent, respectively, the quark mass and the coupling constants of the QCD theory expressed by this Lagrangian, and are subject to renormalisation in the complete quantum theory. Then, the gluon terms represent the non-Abelian character of the symmetry group $SU(3)$.

Hinting at an underlying quantum symmetry, there is the gauge invariance that gives rise to a formal similarity between Mattis spin glasses in certain disordered magnetic solid systems and the coupling degrees of freedom $J_{i,k}$ in QCD which correspond to gluons; in such magnetic solids there are fixed ``random'' couplings $J_{i,k} = \varepsilon_i J_0 \varepsilon_k$, as a result of quenching or ``freezing'', whereas in QCD such coupling degrees of freedom ``fluctuate''. When $J_0$ is positive the Mattis spin glass corresponds to a ferromagnet because such systems are not subject to any ``frustration''. This notion of ``frustration'' in a spin glass corresponds to the Wilson loop quantity of QCD, but in the latter case where the symmetry is given by matrix representations of the $SU(3)$ group, the coupling degrees of freedom `` fluctuate''. This formal cross-correlation between disorderd magnetic systems (including spin glasses and mictomagnets) was considered in some detail in [120].

2.2. Quantum Theories and Symmetry

Following earlier attempts by Segal to formulate postulates [253] for quantum mechanics (and also to identify irreducible representations of operator algebras [254]), quantum theories adopted a new lease of life post 1955 when von Neumann beautifully re-formulated Quantum Mechanics (QM) in the mathematically rigorous context of Hilbert spaces and operator algebras. From a current physics perspective, von Neumann's approach to quantum mechanics has done however much more: it has not only paved the way to expanding the role of symmetry in physics, as for example with the Wigner-Eckhart theorem and its applications, but also revealed the fundamental importance in quantum physics of the state space geometry of (quantum) operator algebras.

Subsequent developments of these latter algebras were aimed at identifying more general quantum symmetries than those defined for example by symmetry groups, groups of unitary operators and Lie groups, thus leading to the development of theories based on various quantum groups [101]. The basic definitions of von Neumann and Hopf algebras (see for example [185]), quasi-Hopf algebra, quasi-triangular Hopf algebra, as well as the topological groupoid definition, are recalled in the Appendix to maintain a self-contained presentation. Several, related quantum algebraic concepts were also fruitfully developed, such as: the Ocneanu *paragroups*-later found to be represented by Kac--Moody algebras, quantum groups represented either as Hopf algebras or locally compact groups endowed with Haar measure, `quantum' groupoids represented as weak Hopf algebras, and so on.

2.3. Ocneanu Paragroups, Quantum Groupoids and Extended Quantum Symmetries

The Ocneanu paragroup case is particularly interesting as it can be considered as an extension through quantization of certain finite group symmetries to infinitely-dimensional von Neumann type $\mathbf{II_1}$ algebras [112], and are, in effect, *quantized groups* that can be nicely constructed as Kac algebras; in fact, it was recently shown that a paragroup can be constructed from a crossed product by an outer action of a Kac-Moody algebra. This suggests a relation to categorical aspects of paragroups (rigid monoidal tensor categories [271, 298]). The strict symmetry of the group of (quantum) unitary operators is thus naturally extended through paragroups to the symmetry of the latter structure's unitary representations; furthermore, if a subfactor of the von Neumann algebra arises as a crossed product by a finite group action, the paragroup for this subfactor contains a very similar group structure to that of the original finite group, and also has a unitary representation theory similar to that of the original finite group. Last-but-not least, a paragroup yields a *complete invariant* for irreducible inclusions of AFD von Neumannn II_1 factors with finite index and finite depth (Theorem 2.6. of [245]). This can be considered as a kind of internal, 'deeper' quantum symmetry of von Neumann algebras.

On the other hand, unlike paragroups, quantum locally compact groups are not readily constructed as either Kac or Hopf C^*-algebras. In recent years the techniques of Hopf symmetry and those of weak Hopf C^*-algebras, sometimes called *`quantum groupoids'* (cf. Böhm et al. [38]), provide important tools-in addition to the paragroups-for studying the broader relationships of the Wigner fusion rules algebra, $6j$-symmetry [233], as well as the study of the noncommutative symmetries of subfactors within the Jones tower constructed from finite index depth 2 inclusion of factors, also recently considered from the viewpoint of related Galois correspondences [204].

2.4. Quantum groupoids, Lie Algebroids and Quantum Symmetry Breaking

The concept of a *quantum groupoid* may be succinctly presented as that of a *weak* C^* *- Hopf algebra* which admits a faithful *-representation on a Hilbert space (see **Appendix** § 0.4 and [24, 21]). On the other hand, one can argue that locally compact groupoids equipped with a Haar measure (after quantization) come even closer to defining quantum groupoids. Nevertheless, there are sufficiently many examples in quantum theories that justify introducing weak C*--Hopf algebras and hence quantum groupoids as the essentially the same concept. Further importance is attached to the fact that notions such as (proper) *weak* C^* *-algebroids* provide a significant framework for symmetry breaking and quantum gravity. Related notions are the quasi-group symmetries constructed by means of special transformations of a coordinate space M . These transformations along with M define certain Lie groupoids, and also their infinitesimal version - the Lie algebroids $\mathbf{A}$. Lifting the algebroid action from M to the principal homogeneous space R over the cotangent bundle $T^*M \to M$, one obtains a physically significant algebroid structure. The latter was called the Hamiltonian algebroid, A^H, related to the Lie algebroid, $\mathbf{A}$. The Hamiltonian algebroid is an analog of the Lie algebra of symplectic vector fields with respect to the canonical symplectic structure on R or T^*M . In this example, the Hamiltonian algebroid, A^H over R, was defined over the phase space of W_N -gravity, with the anchor map to Hamiltonians of canonical transformations [171]. Hamiltonian algebroids thus generalize Lie algebras of canonical transformations; canonical transformations of the Poisson sigma model phase space define a *Hamiltonian algebroid* with the Lie brackets related to such a Poisson structure on the target space. The Hamiltonian algebroid approach was utilized to analyze the symmetries of generalized deformations of complex structures on Riemann surfaces $\sum_{g,n}$ of genus g with n marked points. One recalls that the Ricci flow equation introduced by Richard Hamilton is the dynamic evolution equation for a Riemannian metric $g_{ij}(t)$. It was then shown that Ricci flows ``cannot quickly turn an almost Euclidean region into a very curved one, no matter what happens far away" [218], whereas a Ricci flow may be interpreted as an entropy for a canonical ensemble. However, the implicit algebraic connections of the Hamiltonian algebroids to von Neumann *--algebras and/or *weak* C^* *--algebroid representations* have not yet been investigated. This example suggests that algebroid (quantum) symmetries are implicated in the foundation of relativistic quantum gravity theories and of supergravity.

The fundamental interconnections between quantum symmetries, supersymmetry, graded Lie algebroids/their duals and quantum groupoid representations are summarized in Figure 2.1. Several physical systems that exhibit such extended quantum symmetries, and in which spontaneous symmetry breaking does occur, are also indicated in *Figure 2.1*. The example of quasicrystals is then further discussed in the following section.

3. QUASICRYSTALS. SYMMETRY GROUPOIDS.NONCOMMUTATIVE STRUCTURES

3.1. Quasicrystals

Penrose [216] considered the problem of coverings of the whole plane by shifts of a finite number of non-overlapping polygons without gaps. These tilings, though being non-periodic, are *quasi-periodic* in the sense that any portion of the tiling sequence, displayed as a non-periodic lattice, appears infinitely often and with extra symmetry (there are more general examples in 3-dimensions). In such tiling patterns there is a requirement for matching rules if the structure is to be interpreted as scheme of an energy ground state [227]. Remarkably, further examples arise from icosahedral symmetries as first observed in solid state physics by [249] who described the creation of alloys $\mathrm{Al_6Mn}$ with unusual icosahedral, 10-fold symmetries *forbidden* by the crystallographic rules for Bravais lattices. These very unsual symmetries were discovered in the electron diffraction patterns of the latter solids which consisted of sharp Bragg peaks (true δ -functions) that are typical of all crystalline structures that are highly ordered, and are thus in marked contrast to those of metallic glasses and other noncrystalline solids which exhibit only broad scattering bands instead of discrete Bragg diffraction peaks. Such unusual lattices were coined *quasicrystals* because they contain relatively small amounts of structural disorder in such lattices of 10-fold symmetry, formed by closely packed icosahedra. Further investigation of 10 - and higher- fold symmetries has suggested the use of noncommutative geometry to characterize the underlying electron distributions in such quasicrystals, as outlined for example in [31, 91] in the setting of C^* -algebras and K-theory on a variety of non-Hausdorff spaces, and also attempting to relate this theory to the quantum Hall effect.

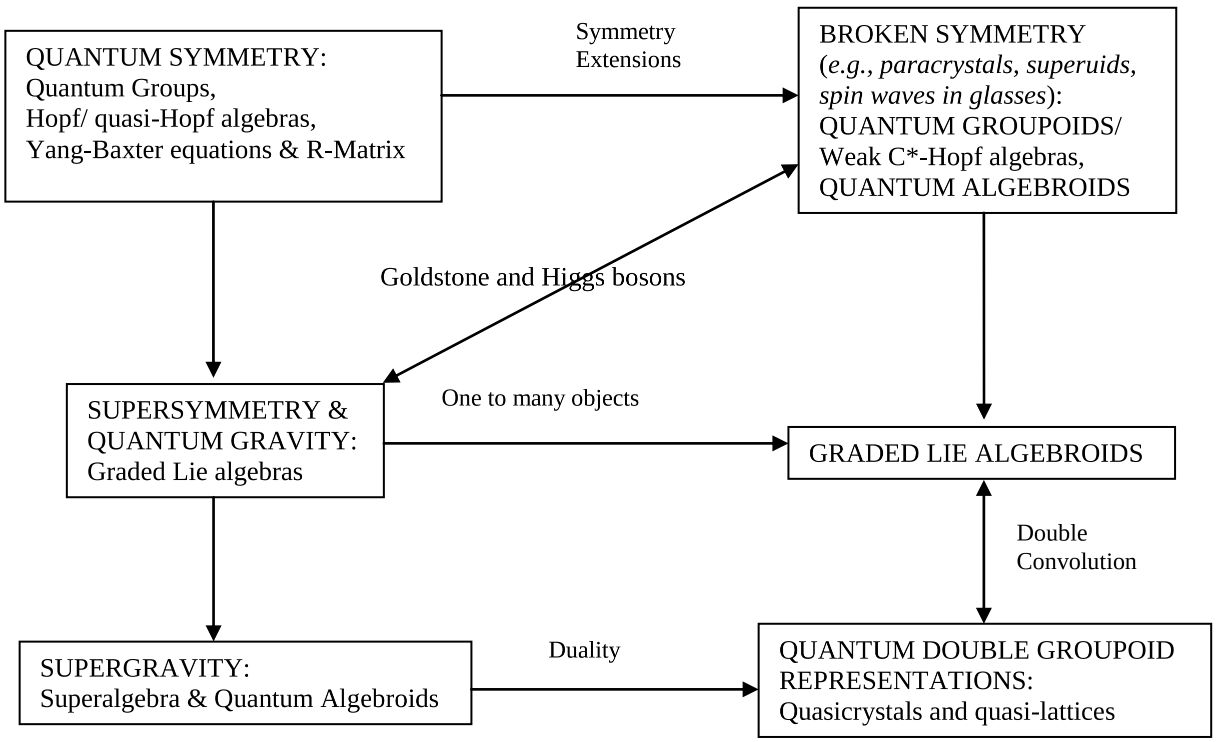


Figure 2.1. Extensions of quantum symmetry concepts in Quantum Algebra, Supersymmetry, Quantum Gravity, Superfluid and Paracrystal quantum theories.

More specifically, as explained in [299, 300] there is an apparent lack of direct correlation between the symmetry of the diffraction patterns and the expected periodicity in the quasi-crystalline lattice; hence, there is an absence of a group lattice action. Furthermore, there are no distinct Brillouin zones present in such quasicrystals. Here is where groupoids enter the picture by replacing the single group symmetry of crystalline lattices with many distinct symmetries of the quasi-lattice, and noncommutative C^*-algebras replacing the Brillouin zones of the crystalline lattices. The quasicrystal can also be modeled by a tiling T and its hull Ω_T, regarded as the space of all tilings can be equipped with a suitable metric d_{Ω_T}, so that Ω_T is the metric space completion of $(\{T + x : x \in \mathrm{R}^d\}, d_{\Omega_T})$ thus giving a structure more general than the space of Penrose tilings; moreover, Ω_T / $\boldsymbol{R^d}$ is, in general, a non-Hausdorff space. This leads to a groupoid $\Omega_T \rtimes \boldsymbol{R^d}$, and from the space of continuous functions with compact support $\boldsymbol{C}_\mathrm{c}(\Omega_T \rtimes \boldsymbol{R^d})$, a completion in the supremum norm provides a noncommutative C^*-algebra, $\boldsymbol{C}_\mathrm{c}^{*}(\Omega_T \rtimes \boldsymbol{R^d})$, which can be interpreted as a 'noncommutative Brillouin zone' [31, 299]. This procedure related to an overall noncommutativity thus characterizes a transition from a periodic state structure to one that is either non-periodic or aperiodic. From another perspective, [162] has considered exactly solvable (integrable) systems in quasicrystals constructing an 8-vertex model for the Penrose non-periodic tilings of the plane equivalent to a pair of interacting Ising spin models. Further, it is shown that the 8-vertex model is solvable, and indeed that any solution of the Yang-Baxter equations can be used for constructing an unique, integrable model of a quasicrystal.

4. YANG-BAXTER EQUATIONS

4.1. Parameter-dependent Yang--Baxter equation

Consider A to be an unital associative algebra. Then *the parameter--dependent Yang--Baxter equation* below is an equation for $R(u)$, the parameter--dependent invertible element of the tensor product $A \otimes A$ and $R = R(u)$ is usually referred to as the (quantum) R-matrix (see **Appendix** § 0.3). Here, u is the parameter, which usually ranges over all real numbers in the case of an additive parameter, or over all positive real numbers in the case of a multiplicative parameter. For the dynamic Yang--Baxter equation see also ref. [111]. The Yang--Baxter equation is usually stated (e.g., [259, 260]) as:

$$R_{12}(u)\ R_{13}(u+v)\ R_{23}(v) = R_{23}(v)\ R_{13}(u+v) R_{12}(u), \tag{10}$$

for all values of u and v, in the case of an additive parameter, and

$$R_{12}(u)\ R_{13}(uv)\ R_{23}(v) = R_{23}(v)\ R_{13}(uv) R_{12}(u), \tag{11}$$

for all values of u and v, in the case of a multiplicative parameter, where

$$\begin{aligned} R_{12}(w) &= \phi_{12}(R(w)) \\ R_{13}(w) &= \phi_{13}(R(w)) \\ R_{23}(w) &= \phi_{23}(R(w)) \end{aligned} \tag{12}$$

for all values of the parameter w, and

$$\begin{aligned} \phi_{12} &: H \otimes H \to H \otimes H \otimes H \\ \phi_{13} &: H \otimes H \to H \otimes H \otimes H \\ \phi_{23} &: H \otimes H \to H \otimes H \otimes H \end{aligned} \tag{13}$$

are algebra morphisms determined by the following (strict) conditions:

$$\begin{aligned} \phi_{12}(a \otimes b) &= a \otimes b \otimes 1 \\ \phi_{13}(a \otimes b) &= a \otimes 1 \otimes b \\ \phi_{23}(a \otimes b) &= 1 \otimes a \otimes b \end{aligned} \tag{14}$$

The importance of the equation (and Yang-Baxter algebras) is that they are ubiquitous in (integrable) quantum systems such as [88]:

- 1-dimensional quantum chains such as the Toda lattice and the Hesienberg chain.
- Factorizable scattering in $(1+1)$-dimensions.
- 2-dimensional statistical lattice/vertex models.
- Braid groups.

The quantum R-matrix itself also appears in many guises, such as a correspondent to 2-pt Schlesinger transformations in the theory of isomonodromic deformations of the torus [187].

4.2. The Parameter-independent Yang--Baxter equation

Let A be a unital associative algebra. The parameter--independent Yang--Baxter equation is an equation for R, an invertible element of the tensor product $A \otimes A$. The Yang--Baxter equation is:

$$\begin{aligned} R_{12}\ R_{13}\ R_{23} &= R_{23}\ R_{13}\ R_{12}, \text{ where } R_{12} = \phi_{12}(R) \\ R_{13} &= \phi_{13}(R), \text{ and } R_{23} = \phi_{23}(R). \end{aligned} \tag{15}$$

Let V be a module over A. Let $T : V \otimes V \to V \otimes V$ be the linear map satisfying $T(x \otimes y) = y \otimes x$ for all $x, y \in V$. Then a representation of the braid group B_n, can be constructed on $V^{\otimes n}$ by $\sigma_i = 1^{\otimes i-1} \otimes \breve{R} \otimes 1^{\otimes n-i-1}$ for $i = 1, \ldots, n-1$, where $\breve{R} = T \circ R$ on $V \otimes V$. This representation may thus be used to determine quasi--invariants of braids, knots and links.

4.3. Generalization of the Quantum Yang--Baxter Equation

The quantum Yang--Baxter equation was generalized in [164] to:

$$R = qb(\sum_{i=1}^{n} e_{ii} \otimes e_{ii}) + b(\sum_{i>j} e_{ii} \otimes e_{jj}) + c(\sum_{i<j} e_{ii} \otimes e_{jj}) + (qb - q^{-1}c)(\sum_{i>j} e_{ij} \otimes e_{ji}), \tag{16}$$

for $b, c \neq 0$. A solution of the quantum Yang--Baxter equation has the form $R : M \otimes M \to M \otimes M$, with M being a finite dimensional vector space over a field k. Most of the solutions are stated for a given ground field but in many cases a commutative ring with unity may instead be sufficient.

4.4. $6j$ -Symmetry, Extended Topology Quantum Field Theories, and $6j$ -Symmetry Representations $6j$ -Symbols and $6j$ -Symmetry Representations

An important development linking classical with quantum group symmetries occurs in the Clebsch-Gordan theory involving the recoupling formulation for representations of classical and quantum $U(sl(2))$ groups *via* the spin networks of Penrose [217] and Kauffman [157, 158]. In such formulations the finite dimensional irreducible representations are expressed in spaces of homogeneous polynomials V^j in two variables of degree $2j = r + s$, where $j \in \{0,1/2,3/2,...\}$. $V^{1/2}$ is called the *fundamental representation*.

For the quantum $sl(2)$ case the variables *`commute up to a factor of* q *'*, that is

$$yx = qxy,$$

and when the parameter is a root of unity one only decomposes representations modulo those with trace 0. In general, however, the tensor product of two representations is decomposed as a direct sum of irreducible ones. Let us consider first the set of (2 by 2) matrices of determinant 1 over the field of complex numbers which form the `special' group $SL(2)$. There is a **Well Known Theorem** for representations on V^j: *The representations of the classical group* $SL(2)$ *on* V^j *are irreducible*, [84].

Then, the classical group $U(sl(2))$ constructed from the algebra generated by three symbols E, F and H subject to a few algebraic relations, has the same finite dimensional representations as the group $SL(2)$. As an example, when A is a primitive $4r$ -th root of unity, one has the relations: $E^* = F^* = 0$ and $K^{4r} = 1$, and the quantum group $U_q(sl(2))$ has the structure of a modular ribbon Hopf algebra as defined by Reshetikhin-Turaev [232]. In general, E, F and H are subject to the following algebraic relations:

$$EF - FE = 2H, HE - EH = E,$$

and

$$HF - FH = -F,$$

analogous to the Lie bracket in the Lie algebra $sl(2)$. The $sl(2)$ Lie algebra is related to the Lie group $SL(2)$ via the exponential function , $exp : sl(2) \to SL(2)$ defined by the power series:

$$exp(Q) = \sum_{j=0}^{\infty}(Q^j/j!),$$

for $Q \in sl(2)$. This exponentiation function exp maps a trace 0 matrix to a matrix with determinant 1.

A *representation* of either $sl(2)$ or $U(sl(2))$ is determined by assigning to E, F and H corresponding operators on a vector space V that are also subject to the above relations, and the enveloping algebra acts by composition: $E^2 v = E(Ev)$, where v is a vector in the representation V.

Moreover, the tensor product of such representations can be naturally decomposed in two distinct ways that are compared in the so-called *recoupling theory* or formulation with recoupling coefficients that are called `$6j$ -symbols'.

4.5.1. Extended Topological Quantum Field Theories (ETQFT)

A useful geometric visualisation of $6j$ -symbols is also available as the corners of regular tetrahedra, but in fact the $6j$ -symbols satisfy two fundamental identities--the Elliott-Biedenharn and the orthogonality identities--that can be interpreted in terms of the decomposition of the union of two regular tetrahedra; in the case of the Elliott-Biedenharn identity the two tetrahedra are glued only along one face, and then recomposed as the union of three tetrahedra glued along an edge. In the case of the orthogonality identity the tetrahedra are glued along two faces but the recomposition is no longer simplicial. This peculiar symmetry of the $6j$ -symbols and their relationship to tetrahedra was explained when Turaev and Viro [271] were able to construct 3-manifold invariants based on a roughly analogous theory for the quantum $sl(2)$ group case, and interestingly, the identities satisfied by the $6j$ -symbols are exactly the same in the quantum case as in the classical one [84]. The Turaev-Viro invariants were

derived using the results of Kirilov and Reshetikhin for quantum group representations [160] and are a good example of a Topological Quantum Field Theory (TQFT), defined as a functor $\mathsf{F}: Cob \to Hilb$, from the category Cob of smooth manifold cobordisms to the category of Hilbert spaces, $Hilb$. An extension to higher dimensional Homotopy QFTs (HHQFTs) has also been reported, but this novel approach [223] is distinct from the previous work in ETQFTs. A related, formal approach to HQFTs in terms of formal maps and crossed C--algebras was also recently reported [224]. Potential physical applications of the latter HHQFT developments are in the area of topological higher gauge theory [127, 9].
The solutions to the tetrahedral analogue of the quantum Yang-Baxter equation lead to a 4-algebra, and therefore a search is on for the higher-dimensional extensions of such equations, and their related ETQFT invariants. Significant efforts are currently being made to generalise such theories in higher dimensions and one such formulation of an Extended TQFT is due to Lawrence [169] in terms of the structure associated to a 3-manifold called a `3-algebra'. Note, however, that the latter should be distinguished from the cubical structure approaches mentioned in § 6 that could lead to a Cubical Homotopy QFT (CHQFT) instead of the 3-algebras of Lawrence's ETQFT. In the latter case of CHQFTs, as it will be further detailed in § 6 and § 7, the generalised van Kampen theorem might play a key role for filtered spaces. Such recent developments in higher-dimensional ETQTs point towards `` *deep connections to theoretical physics that require much further study from the mathematical, theoretical and experimental sides*" [84].

4.6. κ -Poincaré symmetries

In keeping with our theme of quantization of classical (Poisson-Lie) structures into Hopf algebras, we consider the case as treated in [202] of how the usual Poincaré symmetry groups of (anti) de-Sitter spaces can be deformed into certain Hopf algebras with a bicrossproduct structure and depend on a parameter κ. Given that Hopf algebras arise in the quantization of a $(2+1)$-dimensional Chern-Simons quantum gravity, one may consider this theory as workable. However, as pointed out in [202] the Hopf algebras familiar in the $(2+1)$-gravity are not κ-symmetric, but are deformations of the isometry groups of the latter, namely, the Drinfeld doubles in relation to respectively *zero, positive* and *negative* cosmological constant: $D(\mathrm{U}(su(1,1))), D(\mathrm{U}_q(su(1,1)))$, $q \in \mathsf{R}$ and

$D(\mathrm{S}U_q(su(1,1)))$, $q \in \mathrm{U}(1)$. .

Now suppose we have take the quantization of a classical Poisson Lie group G into a Hopf algebra. In the case of the quasi-triangular Hopf algebras (see *R-matrix* in the **Appendix** § 0.3) such as the Drinfeld doubles and the κ-Poincaré structures, the Lie bialgebras are definable on taking an additional structure for the corresponding Lie algebra $\mathfrak{g}$. As shown in [202], this turns out to be an element $r = r^{\alpha\beta} X_\alpha \otimes X_\beta \in \mathfrak{g} \otimes \mathfrak{g}$ which satisfies the classical Yang-Baxter equations:

$$[[r,r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \tag{17}$$

where $r_{12} = r^{\alpha\beta} X_\alpha \otimes X_\beta \otimes \mathbf{1}, r_{13} = r^{\alpha\beta} X_\alpha \otimes \mathbf{1} \otimes X_\beta, r_{23} = r^{\alpha\beta} X_\alpha \otimes \mathbf{1} \otimes X_\alpha \otimes X_\beta$, and $\{X_\alpha\}$, $1 \leqslant \alpha \leqslant \dim \mathfrak{g}$, is a basis for $\mathfrak{g}$. The ensuing relations between the Hopf algebras, Poisson Lie groups, Lie bialgebras and classical r-matrices is given explicitly in [202]. From another point of view, works such as [257, 258] demonstrate that the C^*-algebra structure of a compact quantum group, such as $\mathrm{S}U_q(n)$, can be studied in terms of the groupoid C^*-algebra into which the former can be embedded. These embeddings thus describe the structure of the C^*-algebras of such groups and that of various related homogeneous spaces, such as $\mathrm{S}U_q(n+1)/\mathrm{S}U_q(n)$ (a `quantum sphere').

4.7. Towards a Quantum Category

We remark that the Drinfeld construction of the quantum doubles of (finite dimensional) Hopf algebras can be extended to various bilalgebras [262]. The bialgebra (algebroid) treatment leads into a more categorical framework (cf. *-autonomous bicategories), namely to that of a *quantum category* [96] where a quantum groupoid is realized via the antipode structure (cf. *weak Hopf algebra* in **Appendix**). If $\boldsymbol{M}_{\mathrm{BA}}$ denotes a braided monoidal category with coreflexive equalizer, one considers the right autonomous monoidal bicategory ***Comod***$(\boldsymbol{M}_{\mathrm{BA}})$ and the quantum category (in) consists of `basic data' in ***Comod***$(\boldsymbol{M}_{\mathrm{BA}})$ in [96, 262]. By adding an invertible antipode to an associated weak Hopf algebra (see definition in the **Appendix**) and on transferring Hopf-basic data into $\boldsymbol{Comod}(\boldsymbol{M}_{\mathrm{BA}})^{\boldsymbol{co}}$ we obtain a specialised form of quantum groupoid (cf. [21, 24]). However, we point out that for such constructions at least one *Haar measure* should be attached in order to allow for groupoid representations that are associated with observables and their operators, and that also correspond to certain extended quantum symmetries that are much less restrictive than those exhibited by quantum groups and Hopf algebras. Note also that this concept of quantum category may not encounter the problems faced by the 'quantum topos' concept in its applications to quantum physics [21].

5. THEOREMS AND RESULTS

In this section we recall some of the important results relevant to extended quantum symmetries and their corresponding representations. This leads us to consider wider classes of representations than the group representations usually associated with symmetry; they are the more general representations for groupoids, arbitrary categories and functors.

5.1. General Definition of Extended Symmetries *via* Representations

We aim here to define extended quantum symmetries as general representations of mathematical structures that have as many as possible physical realizations, i.e. *via* unified quantum theories. In order to be able to extend this approach to very large ensembles of composite or complex quantum systems one requires general procedures for quantum `coupling' of component quantum systems; several relevant examples will be given in the next sections.
Because a group G can be viewed as a category with a single object, whose morphisms are just the elements of G, a *general representation* of G in an arbitrary category $\mathbf{C}$ is a functor R_G from G to $\mathbf{C}$ that selects an object X in $\mathbf{C}$ and a group homomorphism from γ to $\mathrm{A}ut(X)$, the automorphism group of X. Let us also define an *adjoint representation* by the functor $R_{\mathbf{C}}^{*}:\mathbf{C}\to G$. If $\mathbf{C}$ is chosen as the category $\mathbf{Top}$ of topological spaces and homeomorphisms, then *representations* of G in $\mathbf{Top}$ are homomorphisms from G to the homeomorphism group of a topological space X. Similarly, a *general representation of a groupoid* G (considered as a category of invertible morphisms) in an arbitrary category $\mathbf{C}$ is a functor R_{G} from G to $\mathbf{C}$, defined as above simply by substituting G for G. In the special case of Hilbert spaces, this categorical definition is consistent with the `standard' representation of the groupoid on a bundle of Hilbert spaces.

Remark 5.1. Unless one is operating in supercategories, such as 2-categories and higher dimensional categories, one needs to distinguish between the *representations of an (algebraic) object* -- as defined above -- and the *representation of a functor* S (from $\mathbf{C}$ to the category of sets, $\mathbf{Set}$) by an object in an arbitrary category $\mathbf{C}$ as defined next. Thus, in the latter case, a *functor representation* will be defined by a certain *natural equivalence between functors*. Furthermore, one needs also consider the following sequence of functors:

$$R_G : G \to \mathbf{C}, \qquad R_{\mathbf{C}}^{*} : \mathbf{C} \to G, \qquad \mathsf{S} : G \to \mathbf{Set},$$

where R_G and $R_{\mathbf{C}}^{*}$ are adjoint representations as defined above, and S is the forgetful functor which `forgets' the group structure; the latter also has a right adjoint S^{*}. With these notations one obtains the following commutative diagram of adjoint representations and adjoint functors that can be expanded to a square diagram to include either $\mathbf{Top}$ -- the category of topological spaces and homeomorphisms, or $\mathbf{TGrpd}$, and/or $\mathbf{C}_{\mathsf{G}} = \mathbf{CM}$ (respectively, the category of topological groupoids, and/or the category of categorical groups and homomorphisms) :

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, shift right, "S^{*}"'] & \mathbf{G} \arrow[l, shift right, "S"'] \arrow[d, shift left, "\mathbf{R}_{C}^{*}"] \\
 & \mathbf{C} \arrow[u, shift left, "\mathbf{R}_{G}"] \arrow[ul, "{F,\ F^{*}}"]
\end{tikzcd}$$

5.2. Representable Functors and Their Representations

The key notion of *representable functor* was first reported by Grothendieck (also with Dieudonné) during 1960--

1962 [136, 133, 134], (see also the earlier publication by Grothendieck [132]). This is a functor $\mathsf{S}:\mathbf{C}\rightarrow\mathbf{Set}$, from an arbitrary category $\mathbf{C}$ to the category of sets, $\mathbf{Set}$, if it admits a (functor) *representation* defined as follows. A *functor representation* of S is a pair, (R,ϕ), which consists of an object R of $\mathbf{C}$ and a family ϕ of equivalences $\phi(\mathbf{C})$: $\mathrm{Hom}_{\mathbf{C}}(R,\mathbf{C}) \cong \mathsf{S}(\mathbf{C})$ which is natural in $\mathbf{C}$. When the functor S has such a representation, it is also said to be *represented by the object* R of $\mathbf{C}$. For each object R of $\mathbf{C}$ one writes $h_R:\mathbf{C}\rightarrow\mathbf{Set}$ for the covariant *Hom*-functor $h_R(\mathbf{C}) \cong \mathrm{Hom}_{\mathbf{C}}(R,\mathbf{C})$. A *representation* (R,ϕ) of S is therefore *a natural equivalence of functors* $\phi: h_R \cong \mathsf{S}$.

The equivalence classes of such functor representations (defined as natural equivalences) obviously determine an *algebraic groupoid* structure. As a simple example of an *algebraic* functor representation, let us also consider (cf. [183]) the functor $N:\mathbf{Gr}\rightarrow\mathbf{Set}$ which assigns to each group G its underlying set, and to each group homomorphism f the same morphism but regarded just as a function on the underlying sets; such a functor N is called a *forgetful* functor because it ``forgets" the group structure. N is a representable functor as it is represented by the additive group Z of integers and one has the well-known bijection $\mathrm{Hom}_G(Z,\ G) \cong \mathsf{S}(G)$ which assigns to each homomorphism $f:\mathsf{Z}\rightarrow G$ the image $f(1)$ of the generator 1 of Z.

In the case of groupoids there are also two natural forgetful functors F: $\mathbf{Grpd}\rightarrow\mathbf{Set}$ and E: $\mathbf{Grpd}\rightarrow\mathbf{DirectedGraphs}$; the left adjoint of E is the free groupoid on a directed graph, i.e. the groupoid of all paths in the graph. One can therefore ask the question:

Is F representable, and if so, what is the object that represents F ?

Similarly to the group case, a functor F can be defined by assigning to each groupoid $\mathbf{G_X}$ its underlying set of arrows, $\mathbf{G_X}^{(1)}$, but `forgetting' the structure of $\mathbf{G_X}^{(1)}$. In this case, F is representable by the indiscrete groupoid $\mathbf{I}(S)$ on a set S since the morphisms of $\mathbf{G_X}^{(1)}$ are determined by the morphisms from $\mathbf{I}(S)$ to $\mathbf{G_X}^{(1)}$. One can also describe (viz. [183]) representable functors in terms of certain universal elements called *universal points*. Thus, consider $\mathsf{S}:\mathbf{C}\rightarrow\mathbf{Set}$ and let $\mathbf{C}_{s*}$ be the category whose objects are those pairs (A,x) for which $x\in\mathsf{S}(A)$ and with morphisms $f:(A,x)\rightarrow(B,y)$ specified as those morphisms $f:A\rightarrow B$ of $\mathbf{C}$ such that $\mathsf{S}(f)x=y$; this category $\mathbf{C}_{s*}$ will be called the *category of* S*-pointed objects* of $\mathbf{C}$. Then one defines a *universal point* for a functor $\mathsf{S}:\mathbf{C}\rightarrow\mathbf{Set}$ to be an initial object (R,u) in the category $\mathbf{C}_{s*}$. At this point, a general connection between representable functors/functor representations and universal properties is established by the following, fundamental functor representation theorem [183].

Theorem 5.2. Functorial Representation Theorem 7.1 of MacLane [183]: *For each functor* $\mathsf{S}:\mathbf{C}\rightarrow\mathbf{Set}$, *the formulas* $u=(\phi R)1_R$ *, and* $(\phi c)h=(\mathsf{S}h)u$ *, (with the latter holding for any morphism* $h:R\rightarrow C$*), establish a one-to-one correspondence between the functor representations* (R,ϕ) *of* S *and the universal points* (R,u) *for* S.

5.3. Physical Invariance under Symmetry Transformations, Generalised Representation and Quantum Algebraic Topology Theorems

A statement of Noether's theorem as *a conservation law* is a s follows.

Theorem 5.3. Noether's Theorem and Generalisations [207]:
Any differentiable symmetry of the action of a physical system has a corresponding conservation law; to every differentiable symmetry generated by local actions, there corresponds a conserved current. Thus, if a system has a continuous symmetry property, then there are corresponding physical quantities that are invariant (conserved) in time.

5.3.1. Consequences of Noether's theorem, extensions and application examples:

(a) The angular momentum and the energy of a system must be conserved;

(b) There are also Conservation Laws for tensor fields ϕ which are described by partial differential equations, and the conserved physical quantity is called in this case a ``Noether charge''; the flow carrying the Noether charge is called a ``Noether current''; for example, the electric charge is conserved, and Noether's theorem states that there are N conserved current densities when the action is invariant under N transformations of the spacetime coordinates and fields; an application of the Noether charge to stationary black holes allows the calculation of the black hole's entropy;

(c) A *quantum* version of Noether's first theorem is known as the *Ward-Takahashi identity*. Note that symmetry here is expressed as a covariance of the form that a physical law has with respect to the one-dimensional Lie group of transformations (with an uniquely associated Lie algebra);

(d) In the case of the Klein-Gordon(relativistic) equation for spin-0 particles, Noether's theorem provides an exact expression for the conserved current, which multiplied by the charge equals the electrical current density, the physical system being invariant under the transformations of the field ϕ and its complex conjugate ϕ^* that leave $|\phi|^2$ unchanged; such transformations were first noted by Hermann Weyl and they are the *fundamental gauge symmetries* of contemporary physics;

(e) Interestingly, the relativistic version of Noether's theorem holds rigorously true for the conservation of 4-momentum and the zero covariant divergence of the stress-energy tensor in GR, even though the conservation laws for momentum and energy are only valid up to an approximation;

(f) Noether's theorem can be extended to conformal transformations, and also Lie algebras or certain superalgebras, such as graded Lie algebras.

In terms of the *invariance* of a physical system, Noether's theorem can be expressed for:

- spatial translation--the law of conservation of linear momentum;
- time translation-- the law of energy conservation;
- rotation--the law of conservation of angular momentum.
- change in the phase factor of a quantum field and associated gauge of the electric potential--the law of conservation of electric charge (the Ward-Takahashi identity);

Theorem 5.4. Goldstone's Theorem: *Let us consider the case of a physical system in which a (global) continuous symmetry is spontaneously broken. In this case, both the action and measure are initially invariant under a continuous symmetry. Subsequent to a global, spontaneous symmetry breaking, the spectrum of physical particles of the system must contain one particle of zero rest mass and spin for each broken symmetry; such particles are called **Goldstone bosons** or **Nambu-Goldstone** (NG) **bosons**, [274].* An alternate formulation in terms of the energy spectrum runs as follows:

Theorem 5.4.1 [43]: *The spontaneous breaking of a continuous, global internal symmetry requires the existence of a mode in the spectrum with the property :*

$$\lim_{k \to 0} E_k = 0. \tag{18}$$

A Corollary of the Goldstone theorem can be then stated as follows:

``If there are two different Green functions of a quantum system which are connected by a symmetry transformation then there must exist a Goldstone mode in the spectrum of such a system'', [43].

One assumes that in the limit of zero gauge couplings, the effective quantum field theory is invariant under a certain group G of global symmetries, which is spontaneously broken to a subgroup H of G. Then we `turn on' the gauge couplings and the gauge group $\mathsf{G}_g \subseteq G$, for G being the group of all symmetries of the effective QFT or an effective field theory (EFT); moreover, when G is spontaneously broken to H, the gauge subgroup G_g must be sponatneously broken to a subgroup H_g, equal to the intersection of G_g with H. Furthermore, the generators T_α of the gauge group G_g can be expressed as a linear combination of the generators T_A of the full (global) symmetry group G [274]. One can also simply define a *spontaneously broken symmetry* (SBS) as a global symmetry whose ground state is not an eigenstate of its generator T_0 [43]. Because the charge operator commutes with the Hamiltonian of the system, a finite symmetry transformation generated by the charge operator also commutes with the Hamiltonian. Then, the ground state will be transformed into another state of the same energy. If the symmetry group is continuous there will be infinitely many degenerate ground states, all connected by symmetry transformations; therefore, all such ground states must be physically equivalent, and any of these ground states can

be excited to yield a full spectrum of excited states.
The proof of the theorem begins with a consequence of Noether's theorem which requires that any continuous symmetry of the action leads to the existence of a conserved Noether current J^{μ}, with a charge Q that induces the associated symmetry transformation $Q = \int (d^3 x J^0 (r,0))$, and the proof proceeds by calculating the vacuum expectation value of the commutator of the current and field. Note that the integral charge Q is conserved/time independent, and also that the presence of a density of non-Abelian charge implies the presence of a certain type of Glodstone bosons (also called the *Nambu-Golstone*, ***NG type-II bosons***). The Goldstone theorem does not apply when the spontaneously broken symmetry is a *local* rather than a *global* one, and *no massless Goldstone bosons are generated in this case as a result of the local symmetry breaking*. However, when the broken symmetry is *local*, the Goldstone degrees of freedom appear in helicity zero states of the vector particles associated with the broken local symmetries, thereby acquiring mass, a process called the *Higgs mechanism*-which is considered to be an important extension of the Standard Model of current physics; the vector particles are therefore called *Higgs bosons*, and a real `hunt' is now ongoing at the latest built accelerators operating at ultra-high energies for the observation of such massive particles. Similar considerations played a key role in developing the electroweak theory, as well as in the formulation of unified quantum theories of electromagnetic, electroweak and strong interactions summarised in the *Standard Model* [274]. Here is, at last, a chance for the experimental high-energy physics to catch up with the theoretical physics, and test its predictions; so far, there has been no report of Higgs bosons up to 175 GeV, above the Higgs bosons' mass estimates of about 170GeV from noncommutative geometry-based theories. On the other hand, in the case of spontaneously broken approximate symmetries (SBAS), low-mass, spin-0 particles, called *pseudo-Goldstone bosons*, are generated, instead of the massless Goldstone bosons. This case is important in the theory of strong, nuclear interactions, as well as in superconductivity. There is also an approximate symmetry of strong interactions known as *chiral symmetry,* $SU(2) \times SU(2)$, which arises because there are two quark fields, u and d of relatively small masses. This approximate symmetry is spontaneously broken leading to the isospin subgroup $SU(2)$ of $SU(2) \times SU(2)$. Because the u and d quarks do not have zero rest mass the chiral symmetry is not exact. Therefore, the breaking of this approximate symmetry entails the existence of approximately massless pseudo-Goldstone bosons of spin-0 and with the same quantum numbers as the symmetry broken generator X ; thus, such pseudo-Goldstone bosons should have zero spin, negative parity, unit isospin, zero baryon number and zero strangeness. The experimental fact is that the lightest observed of all hadrons is the *pion* which has precisely these quantum numbers; therefore, one identifies the pion with the pseudo-Goldstone boson associated with the spontaneous breaking of the approximate chiral symmetry.

Another interesting situation occurs when by lowering the temperature in a certain quantum system, this is brought very close to a *second-order phase transition* which goes smoothly from unbroken to broken global symmetry. Then, according to [274], on the side of the transition where the global symmetry is broken there will be massless Goldstone bosons present together with other massive excitations that do not form complete multiplets which would yield *linear representations* of the broken symmetry group. On the other side of the transition-- where the global symmetry of the system is not broken-- there are, of course, complete linear multiplets, but they are, in general, massive, not massless as in the case of Godstone bosons. If the transition is second-order, that is continuous, then very near the phase transition, the Goldstone bosons must also be part of a complete linear multiplet of excitations that are almost massless; such a multiplet would then form only *one irreducible* representation of the broken symmetry group. The irreducible multiplet of fields that become massless only at the second-order phase transition then defines the *order parameter* of the system, which is *time independent*. The calculation of the order parameter can be approached on an experimental basis by introducing an *effective* field selected with the transformation properties under the observed symmetry that create the Goldstone bosons whose ground state expectation value determines the order parameter. This approach is also considered in conjunction with either the Higgs boson mechanism in the Standard Model (SUSY) or the Ginzburg-Landau theory of phase transitions. As a specific example, for ferromagnets it is the expectation value of the spontaneous magnetization which determines the order parameter. However, in certain unifed field theories this is no longer straightforward because the order parameter is associated with transformation properties corresponding to higher dimensional representations of the symmetry group of such a grand unifiication theory (p. 619 of [43]).
It is possible to construct a quantum description of SBS by employing a symmetric ground state; then the spontaneous symmetry breaking is manifested as *long-range correlations* in the system rather than as nonzero vacuum expectation values [297]. Thus, the Nambu-Goldstone mode can be considered in this case to be *a long-wavelength fluctuation of the corresponding order parameter*. Similar explanations hold for *coupled magnons in long-range spin wave excitations* of certain ferromagnets-- even in the presence of *long-range* structural disorder

[23]-- leading to nonlinear magnon dispersion curves that are the result of two-magnon, and higher groups, of magnon excitations in such noncrystalline, or *glassy* solids; the latter ferromagnetic metallic glasses were sometimes called `mictomagnets'. A magnon is a propagating, `magnetic' perturbation caused by flipping one of the electron spins, and can also be considered as a spin wave that carries a $[-1]$ unbroken charge, being the projection of the electron spin along the direction of the total magnetization of the ferromagnet.
The application of the Goldstone theorem to this case leads to the result that for each broken symmetry generator there is a state in the spectrum that couples to the corresponding Noether current. Calculations for the amplitudes corresponding to particle states $\boldsymbol{k} := |k\rangle$ can be carried out either in the Schrödinger or the Heisenberg representation, and provide the following important result for the energy eigenvalues:

$$E_{\mathbf{k}} = \boldsymbol{k}^2/2m \tag{19}$$

where the state $|k\rangle$ represents a Goldstone, or NG, boson of momentum $\boldsymbol{k}$. Therefore, the magnon dispersion curve is often quadratic in ferromagnets, and the coupled magnon pair provides an example of a type II NG-boson; this is a single NG-boson coupled to two broken symmetry generators [43]. On the other hand, in antiferromagnets there are two distinct Goldstone modes--which are still magnons or spin-waves, but the dspersion relation at low momentum is linear. In both the ferromagnet and antiferromagnet case the $SU(2)$ group symmetry is spontaneously broken by spin alignments (that are respectively parallel or anti-parallel) to its $U(1)$ subgroup symmetry of spin rotations along the direction of the total magnetic moment. In a crystalline ferromagnet all spins sitting on the crystall lattice are alligned in the same direction and the ferromagnet possesses in general an strongly anisotropic total magnetization associated with the crystal symmetry of the ferromagnet. In a glassy ferromagnet the spontaneous magnetization plays the role of the order parameter even if the system may manifest a significant, residual magnetic anisotropy [22]. Although the magnetization could in principle take any direction even a weak external magnetic field is sufficient to align the total sample magnetization along such a (classical) magnetic field. The ferromagnet's ground state is then determined only by the perturbation, and this is an example of vacuum alignnment. Moreover, the ferromagnetic ground state has nonzero net spin density, whereas the antiferromagnet ground state has zero net spin density. The full spectrum of such SBS systems has soft modes--the Goldstone bosons. In the case of glassy ferromagnets, the transitions to excited states induced by microwaves in the presence of a static, external magnetic field can be observed at resonance as a *spin-wave excitation spectrum* [23]. The quenched-in *magnetic anisotropy* of the ferromagnetic glass does change measurably the observed resonance frequencies for different sample orientations with respect to the external, static magnetic field, and of course, the large total magnetization always shifts considerably the observed microwave resonance frequency in *Ferromagnetic Resonance* (FMR) and *Ferromagnetic Spin Wave Resonance* (FSWR) spectra from that of the free electron spin measured for paramagnetic systems by Electron Spin Resonance (ESR). On the other hand, for an isotropic ferromagnet one can utilize either the simple Hamiltonian of a *Heisenberg ferromagnet model*:

$$H = -(1/2)\sum_{ij} J_{ij} s_i\, s_j, \tag{20}$$

In the case of a ferromagnetic glass, however, other more realistic Hamiltonians need be employed that also include anisotropic exchange couplings, coupled local domains and localised ferromagnetic clusters of various (local) approximate symmetries [23] that can be, and often are, larger than 10 nm in size.
As required by the isotropic condition, the Hamiltonian expression (20) is invariant under simultaneous rotation of all spins of the ferromagnet model, and thus forms the $SU(2)$ symmetry group; if all J_{ij} spin couplings are positive, as it would be the case for any ferromagnet, the ground state of the Heisenberg ferromagnet model has all spins parallel, thus resulting in a considerable, total magnetization value. The calculated ground state energy of the Heisenberg ferromagnet is then $E_0 = -(1/8)\sum_{ij} J_{ij}$. With simplifying assumptions about the one-particle Hamiltonian and plane $|k\rangle$ waves one also obtains all of the Heisenberg ferromagnet energy eigenvalues for the excited states:

$$E_k = (1/2)(J_0 - J_k), \tag{21}$$

where

$$J_k = \sum_i J(x_i)\exp(-ik\; x_i), \tag{22}$$

(see ref. [43]). As already discussed, one obtains with the above equations the result that the dispersion relation for the Heisenberg ferromagnet is quadratic at low-momenta, and also that the NG- bosons are of type-II [43]. The Hamiltonian for the Heisenberg ferromagnet model in eq. (5.3) is a significant simplification in the isotropic case

because in this model a magnon, or a spin wave, propagates in the homogeneous magnetic field background of all the other randomly aligned spins. Moreover, the Larmor precession of a spin wave can only occur clockwise, for example, because its sense of rotation is uniquely determined by the magnetic moment of the electron spin, and the axis for the Laromor precession is determined by the total sample magnetization.
In a three-component Fermi gas the global SU(3) x U(1) symmetry is spontaneously broken by the formation of Cooper pairs of fermions, but still leaving unbroken a "residual SU(2) x U(1) symmetry".
In a system with three spin polarizations, such as a Bose-Einstein condensate of an alkali gas, the global symmetry is instead that of the SO(3) x U(1) group, which corresponds to rotational invariance and conservation of particle number [43].
One of the key features of SSB is that the symmetry, in this case, is not realised by unitary operators on a Hilbert space, and thus, it does not generate multiplets in the spectrum. Another main feature is the presence of the order parameter{expressed as a nonzero expectation value of an operator that transforms under the symmetry group; the ground states are then degenerate and form a continuum, with each degenerate state being labeled by different values of the order parameter; such states also form a basis of a distinct Hilbert space. These degenerate ground states are unitarily equivalent representations of the broken symmetry and are therefore called the `Nambu-Goldstone realisation' of symmetry. For an introduction to SSB and additional pertinent examples see also [43].
On the other hand, the above considerations about Goldstone bosons and linear multiplets in systems exhibiting a second-order phase transition are key to understanding, for example, superconductivity phenomena at both low and higher temperatures, in both type I and type II superconductors. The applications extend however to spin-one color superconductors, that is a theory of cold dense quark matter at moderate densities. However, symmetry, Goldstone bosons and SSB are just as important in understanding quantum chromodynamics in general, thus including ultra-hot dense quark plasmas, and nuclear fusion in particular. Thus, similar SSB behavior to that of solid ferromagnets can be observed in nuclear matter, as well as several colour superconducting phases made of dense quark matter. As a further example, Kapitsa [156] in his Nobel lecture address, pointed out that the symmetry of the configuration in a controlled nuclear fusion reactor is very important, and also that in view of the theoretical, major computational problems encountered wit dense and ultra-hot plasmas, in systems with toroidal symmetry, such nuclear fusion reactors are not optimal for the nuclear fusion confinement and control, and therefore are unmanageable for optimizing their output generation efficieny. To date, even though one of the largest, existing nuclear fusion reactors (NFRs), **JET** in UK, generated significant amounts of energy, the input required to the toroidal geometry/toroidal confinement field, unoptimised NFR is much greater than the NF energy output of the NFR (as for example in the JET or the future ITER NFRs). Therefore, this makes the NF energy breakeven point unattainable in the short term (i.e., < 10 years), which is obviously required for any practical use of the tokamak NFRs. Because any NFR system operates nonlinearly with ultra-hot plasmas in which the deuterium (D^+) ion oscillations are strongly coupled to the accelerated electron beams [147], the groupoid C*-algebra representation treatments discussed above in Section 4 can be applied, in the case of simple symmetry configurations, to determine the corresponding extended quantum symmetries of such (D^+; e^-) processes for optimising the NFRs' energy output and their energy generation efficiency. At least in principle, if not in practice, such symmetry-based simplification of the NF computational problems may provide clues for significant increases in the energy efficiency of such novel NFRs, beyond their breakeven point, and therefore relevant for near future, practical applications. This was precisely Kapitsa's major point also in his Nobel lecture about the importance of selecting the more advantageous NFR configurations, except for the fact that, at the time, there were available only semi-empirical approaches, based mostly on physical intuition and brief experimental trial runs in very small size, low-cost NFRs [156]. On the other hand, in white, as well as red stars, their global spherical configuration is stable in the presence of nuclear fusion reactions that continue to burn for extremely long times on the order of many billion years, as one would expect from general symmetry considerations related to quantum groups such as sl(2).
An extension of the Goldstone theorem to the case when translational invariance is not completely broken and long-range interactions are absent is known as the Nielsen-Chadha theorem; it relates the number of Goldstone bosons generated to their dispersion relations [43].

Theorem 5.5. Wigner's Theorem [288]:
Any symmetry acts as a unitary or anti-unitary transformation in Hilbert space: there is a surjective map $T: H \to H$ on a complex Hilbert space H, which satisfies the condition:

$$|\langle Tx, Ty\rangle| = |\langle x, y\rangle|$$

for all x, y in H, has the form $Tx = \varphi(x)\, Ux$ for all x in H, where $\varphi : H \to C$ has modulus one and $U : H \to H$ is either unitary or antiunitary.

Theorem 5.6. Peter-Weyl Theorem:
I. The matrix coefficients of a compact topological group G are dense in the space *C(G)* of continuous complex-valued functions on G, and thus also in the space $L^2(G)$ of square-integrable functions;
II. The unitary representations of G are completely reducible representations, and there is a decomposition of a unitary representation of G into finite-dimensional representations;
III. There is a decomposition of the regular representation of G on $L^2(G)$ as the direct sum of all irreducible unitary representations. Moreover, the matrix coefficients of the irreducible unitary representations form an orthonormal basis of $L^2(G)$. A *matrix coefficient* of the group G is a complex-valued function φ, on G given as the composition

$$\varphi = L \circ \pi, \tag{23}$$

where $\pi : G \to GL(V)$ is a finite-dimensional (continuous) group representation of G, and L is a linear functional on the vector space of endomorphisms of V (that is, the *trace*), which contains $GL(V)$ as an open subset. Matrix coefficients are continuous because by their definition representations are continuous, and moreover, linear functionals on finite-dimensional spaces are also continuous.

Theorem 5.7. Stone-von Neumann theorem and its Generalisation:
The cannonical commutation relations between the position and momentum quantum operators are unique.
More precisely, **Stone's theorem** states that:

There is a one-to-one correspondence between self-adjoint operators and the strongly continuous, one-parameter unitary groups. In a form using representations it can be rephrased as follows: *For any given quantization value h every strongly continuous unitary representation is unitarily equivalent to the standard representation as position and momentum.*

Theorem 5.8. Generalisation
*Let H_n be a general Heisenberg group for n a positive integer. The representation of the center of the Heisenberg group is determined by a scale value, called the `quantization value' (i.e., Planck's constant, $\hbar$). Let us also define the Lie algebra of H_n whose corresponding Lie group is represented by $(n+2)\times(n+2)$ square matrices $M(a,b,c)$ realized by the quantum operators P,Q. Then, for each non-zero real number **h** there is an irreducible representation U_h of H_n acting on the Hilbert space $L^2(R^n)$ by*

$$[U_h(M(a,b,c))]\psi(x) = e^{i(b\cdot x+hc)}\psi(x+ha) \tag{24}$$

All such ρ_i representations are then unitarily inequivalent; moreover, any irreducible representation which is not trivial on the center of H_n is unitarily equivalent to exactly one of the ρ_i representations. (The center of H_n is represented by the matrices $M(0,0,c)$ in this notation). For a locally compact group G and its Pontryagin dual G^o the theorem can be also stated by using the Fourier-Plancherel transform and also considering the group C^* --algebra of G, $C^*(G)$. It turns out that the spectrum of $C^*(G)$ is precisely G^o, the Pontryagin dual of G; one obtains Stone's theorem for the one-parameter unitary groups when the elements of G are real numbers, with the usual number multiplication.

Another fundamental theorem is **Mitchell's theorem** for *compact Lie groups* acting smoothly on a real manifold, which constrains the possible stationary points of group-invariant potentials [43]. An interesting question is if Michel's theorem could be extended to symmetry *Lie groupoids* acting `smoothly' on a manifold; such a generalised Michel theorem might be derived from the fundamental holonomy theorem for groupoids. As a particular example, for inframanifolds one has the Anosov theorem involving odd-order Abelian holonomy groups, but in the groupoid case non-Abelian extensions of the theorem are to be expected. More generally, in the loop space formulation of **(3+1)** canonical quantum gravity the physical information is contained within the holonomy loop functionals, and a ***generalisation of the Reconstruction Theorem for groupoids*** (*GRT*) was reported involving principal fiber bundles [292], obtained by extension to a base path space instead of a loop space; thus, an abstract *Lie groupoid* was constructed by employing a *holonomy groupoid map and a path connection*. Unlike the holonomy group reconstruction theorem-- which is applicable only to connected manifolds-- the generalised groupoid reconstruction theorem is valid for both connected and nonconnected base manifolds. Therefore, *GRT* provides an alternative approach to the conventional Wilson loop theory of quantum gravity.

6. EXTENDED SYMMETRY, GENERALISED GALOIS AND GENERALISED REPRESENTATION THEORY

In this section we shall present without proof the following theorems and results:

(a) **The Equivalence Theorem** of Brown and Mosa (1986)

(b) **An Univalence Theorem**-- *Proposition 9.1* in [21] and the **Adjointness Lemma**

(c**) A Hilbert-functor Representation Theorem** and **Rigged-Hilbert Space Corollary.**

In two related papers Janelidze [149, 150] outlined a categorical approach to a generalised, or extended Galois theory. Subsequently, Brown and Janelidze [58] reported a *homotopy double groupoid* construction of a surjective fibration of Kan simplicial sets based on a generalized, *categorical Galois (GCG) theory* which under certain, well-defined conditions gives a *Galois groupoid* from a pair of adjoint functors. As an example, the standard fundamental group arises in GCG from an adjoint pair between topological spaces and sets. Such a homotopy double groupoid (HDG, explicitly given in diagram 1 of [58]) was also shown to contain the 2-groupoid associated to a map defined by Kamps and Porter [155]; this HDG includes therefore the 2-groupoid of a pair defined by Moerdijk and Svenson [192], the cat^1-group of a fibration defined by Loday [176], and also the classical fundamental crossed module of a pair of pointed spaces introduced by J.H.C. Whitehead. Related aspects concerning homotopical excision, Hurewicz theorems for n-cubes of spaces and van Kampen theorems [272] for diagrams of spaces were subsequently developed in [61, 62].

Two major advantages of this generalized Galois theory construction of Higher Dimensional Groupoids (HDGs) that were already reported are:

- the construction includes information on the map $q : M \to B$ of topological spaces, and
- one obtains different results if the topology of M is varied to a finer topology.

Another advantage of such a categorical construction is the possibility of investigating the global relationships among the category of simplicial sets, $\mathbf{C}_S = \mathbf{Set}^{\Delta^{op}}$, the category of topological spaces, **Top**, and the category of groupoids, **Grpd**. Let I be the fundamental groupoid functor $I = \pi_1 : \mathbf{C}_S \to X$ from the category $\mathbf{C}_S$ to the category $X = Grpd$ of (small) groupoids.

Let us introduce first the notations needed to present the general representation theorems and related results. Consider next diagram 11 on page 67 of Brown and Janelidze [58]:

$$
\begin{tikzcd}
\mathbf{Top} \arrow[r, shift right, "S"'] & \mathbf{Set}^{\Delta^{\mathrm{op}}} \arrow[l, shift right, "R"'] \arrow[r, shift left, "I"] & \mathbf{Grpd} \arrow[l, shift left, "H"] \\
& \Delta \arrow[ul, "r"] \arrow[u, "y"'] \arrow[ur, "i"'] &
\end{tikzcd} ,
\qquad (25)
$$

where:

- Top is the category of topological spaces, S is the singular complex functor and R is its left-adjoint, called the geometric realisation functor;
- $I \vdash H$ is the adjoint pair introduced in Borceux and Janelidze [39], with I being the fundamental groupoid functor, and H being its unique right-adjoint *nerve functor*; and
- y is the Yoneda embedding, with r and i being, respectively, the restrictions of R and I respectively along y; thus, r is the singular simplex functor and i carries finite ordinals to codiscrete groupoids on the same sets of objects.

The adjoint functors in the top row of the above diagram are uniquely determined by r and i - up to isomorphisms as a result of the universal property of y, the Yoneda embedding construction. Furthermore, one notes that there is a natural completion to a square, commutative diagram of the double triangle diagram (25) reproduced above by three adjoint functors of the corresponding forgetful functors related to the Yoneda embedding. This natural diagram completion, that may appear trivial at first, leads however to the following Lemma and several related theorems.

6.1. Generalised Representation Theorems and Results from Higher-Dimensional Algebra

In this subsection we recall several recent generalised representation theorems [21] and pertinent, previous results involving higher-dimensional algebra (HDA).

6.1.1. Adjointness Lemma [21]-- **Theorem 6.1**: *Diagram (26) is commutative, and there exist canonical natural equivalences between the compositions of the adjoint functor pairs and their corresponding identity functors of the four categories present in diagram (26):*

$$\begin{tikzcd}
\mathbf{Top} \arrow[r, shift left, "R"] \arrow[d, shift right, "f"'] & \mathbf{Set}^{\Delta^{op}} \arrow[l, shift left, "S"] \arrow[d, shift left, "I"] \\
\mathbf{Set} \arrow[r, shift left, "G"] \arrow[u, shift right, "g"'] & \mathbf{Grpd} \arrow[l, shift left, "F"] \arrow[u, shift left, "H"]
\end{tikzcd} \tag{26}$$

The forgetful functors $f:\mathbf{Top}\to\mathbf{Set}$, $F:\mathbf{Grpd}\to\mathbf{Set}$ *and* $\Phi:\mathbf{Set}^{\Delta^{op}}\to\mathbf{Set}$ *complete this commutative diagram of adjoint functor pairs. The right adjoint of* Φ *is denoted by* Φ^**, and the adjunction pair* $[\Phi,\Phi^*]$ *has a mirror-like pair of adjoint functors between* $\mathbf{Top}$ *and* $\mathbf{Grpd}$ *when the latter is restricted to its subcategory* $\mathbf{TGrpd}$ *of topological groupoids, and also when* $\phi:\mathbf{TGrpd}\to\mathbf{Top}$ *is a functor that forgets the algebraic structure -- but not the underlying topological structure of topological groupoids, which is fully and faithfully carried over to* $\mathbf{Top}$ *by* ϕ.

Theorem 6.2. Univalence Theorem (Proposition 9.4 on p. 55 in [21]):
If $T:\mathbf{C}\to\mathbf{Grpd}$ *is any groupoid valued functor then* T *is naturally equivalent to a functor*
$\Theta:\mathbf{C}\to\mathbf{Grpd}$ which is univalent with respect to objects in $\mathbf{C}$.

This recent theorem for groupoid valued functors is a natural extension of the corresponding theorem for the T' group univalued functors (**Proposition 10.4** of Mitchell, on p.63 in [189]).

Theorem 6.3. The Equivalence Theorem of Brown and Mosa [63].
The category of crossed modules of R *- algebroids is equivalent to the category of double* R *-algebroids with thin structure.* **Remark.** An interesting application of this theorem is the novel representation of certain cross-modules, such as the Yetter--Drinfeld modules for crossed structures [303], in terms of double R -algebroid representations following the construction scheme recently employed for double groupoid representations [80]; this is also potentially important for quantum algebroid representations [21].

6.2. Functorial representations of topological groupoids

A representable functor $S:\mathbf{C}\to\mathbf{Set}$ as defined above is also determined by the equivalent condition that there exists an object X in $\mathbf{C}$ so that S is isomorphic to the *Hom*-functor h^X. In the dual, categorical representation, the *Hom*--functor h^X is simply replaced by h_X . As an immediate consequence of the Yoneda--Grothendieck lemma the set of natural equivalences between S and h^X (or alternatively h_X) -- which has in fact a groupoid structure -- is isomorphic with the object $S(X)$. Thus, one may say that if S is a *representable functor* then $S(X)$ is its (isomorphic) *representation object*, which is also unique up to an isomorphism [189, p.99]. As an especially relevant example we consider here the *topological groupoid representation* as a functor $\gamma:\mathbf{TGrpd}\to\mathbf{Set}$, and related to it, the more restrictive definition of $\gamma:\mathbf{TGrpd}\to\mathbf{BHilb}$, where $\mathbf{BHilb}$ can be selected either as the category of Hilbert bundles or as the category of rigged Hilbert spaces generated through a *GNS* construction:

$$\begin{tikzcd} \mathbf{Top} \arrow[r, shift right, "M"'] & \mathbf{BHilb} \arrow[l, shift right, "L"'] \arrow[r, shift left, "J"] & \mathbf{TGrpd} \arrow[l, shift left, "K"] \\ & \mathbf{Set} \arrow[ul, "g"] \arrow[u, "m"] \arrow[ur, "n"'] & \end{tikzcd} \tag{27}$$

Considering the forgetful functors f and F as defined above, one has their respective adjoint functors defined by g and n in diagram (27); this construction also leads to a diagram of adjoint functor pairs similar to the ones shown in diagram (26). The functor and natural equivalence properties stated in *the Adjointness Lemma* (**Theorem 6.1**) also apply to diagram (27) with the exception of those related to the adjoint pair $[\Phi, \Phi^*]$ that are replaced by an adjoint pair $[\Psi, \Psi^*]$, with $\Psi : \mathbf{BHilb} \to \mathbf{Set}$ being the forgetful functor and Ψ^* its left adjoint functor. With this construction one obtains the following proposition as a specific realization of **Theorem 6.2** adapted to topological groupoids and rigged Hilbert spaces.

Theorem 6.4. Hilbert-functor Representation Theorem

If $R_o : \mathbf{BHilb} \to \mathbf{TGrpd}$ is any topological groupoid valued functor then R_o is naturally equivalent to a functor $\rho : \mathbf{BHilb} \to \mathbf{TGrpd}$ which is univalent with respect to objects.

Remark: R_o and ρ can be considered, respectively, as adjoint Hilbert-functor representations to groupoid, and respectively, topological groupoid functor representations R_o^* and ρ^* in the category $\mathbf{BHilb}$ of rigged Hilbert spaces.

The connections of the latter result for groupoid representations on rigged Hilbert spaces to the weak C^*-Hopf symmetry associated with quantum groupoids and to the generalised categorical Galois theory warrant further investigation in relation to quantum systems with extended symmetry. Thus, the following **Corollary 6.4** and the previous **Theorem 6.4** suggest several possible applications of GCG theory to extended quantum symmetries *via* Galois groupoid representations in the category of rigged Hilbert families of quantum spaces that involve interesting adjoint situations and also natural equivalences between such *functor* representations. Then, considering the definition of quantum groupoids as *locally compact* (topological) groupoids with certain extended (quantum) symmetries, their functor representations also have the unique properties specified in **Theorem 6.4** and **Corollary 6.4**, as well as the unique adjointness and natural properties illustrated in diagrams (26) and (27).

Corollary 6.4 - ***Rigged Hilbert Space Duality:***

The composite functor $\Psi \circ R_o : \mathbf{TGrpd} \to \mathbf{BHilb} \to \mathbf{Set}$, *with* $\Psi : \mathbf{TGrpd} \to \mathbf{BHilb}$ *and* $R_o : \mathbf{BHilb} \to \mathbf{Set}$, has the left adjoint **n** which completes naturally diagram (30), with both $\Psi : \mathbf{BHilb} \to \mathbf{Set}$ and $\Psi \circ R_o$ being forgetful functors. Ψ also has a left adjoint Ψ^*, and R_o has a defined inverse, or duality functor $\mathrm{I}m$ which assigns in an univalent manner a topological groupoid to a family of rigged Hilbert spaces in $\mathbf{BHilb}$ that are specified *via* the GNS construction.

6.3. Groups, Groupoids and Higher Groupoids in Algebraic Topology

An area of mathematics in which nonabelian structures have proved important is algebraic topology, where the fundamental group $\pi_1(X, a)$ of a space X at a base point a goes back to Poincaré [221]. The intuitive idea behind this is the notion of paths in a space X with a standard composition.

An old problem was to compute the fundamental group and the appropriate theorem of this type is known as the Siefert--van Kampen Theorem, recognising work of Seifert [255] and van Kampen [272]. Later important work was done by Crowell in [94], formulating the theorem in modern categorical language and giving a clear proof.

Theorem 6.5. The Seifert-van Kampen theorem for groups.

For fundamental groups this may be stated as follows [272]:

Let X be a topological space which is the union of the interiors of two path connected subspaces X_1, X_2.

Suppose $X_0 := X_1 \cap X_2$, X_1 *and* X_2 *are path connected and* $* \in X_0$. *Let* $i_k : \pi_1(X_0,*) \to \pi_1(X_k,*)$, $j_k \pi_1(X_k,*) \to \pi_1(X,*)$ *be induced by the inclusions for* $k = 1,2$. *Then* X *is path connected and the natural morphism*

$$\pi_1(X_1,*) \star_{\pi_1(X_0,*)} \pi_1(X_2,*) \to \pi_1(X,*) \,, \tag{28}$$

from the free product of the fundamental groups of X_1 and X_2 with amalgamation of $\pi_1(X_0,*)$ to the fundamental group of X is an isomorphism, or, equivalently, , the following diagram

$$\begin{tikzcd}
\pi_1(X_0,a) \arrow[r, "\pi_1(i_1)"] \arrow[d, "\pi_1(i_2)"'] & \pi_1(X_1,a) \arrow[d, "\pi_1(j_1)"] \\
\pi_1(X_2,a) \arrow[r, "\pi_1(j_2)"'] & \pi_1(X,a)
\end{tikzcd} \tag{29}$$

is a pushout of groups.

Usually the morphisms induced by inclusion in this theorem are not themselves injective, so that the more precise version of the theorem is in terms of pushouts of groups. However this theorem did not calculate the fundamental group of the circle, or more generally of a union of two spaces with non connected intersection. Since the circle is a basic example in topology, this deficiency is clearly an anomaly, even if the calculation can be made by other methods, usually in terms of covering spaces.

Theorem 6.6. Seifert--van Kampen Theorem for Fundamental Groupoids [44, 48] .

The anomaly mentioned above was remedied with the use of the fundamental groupoid $\pi_1(X, A)$ on a set of base points, introduced in [44]; its elements are homotopy classes rel end points of paths in X with end points in $A \cap X$, and the composition is the usual one.

Because the underlying geometry of a groupoid is that of a directed graph, whereas that of a group is a set with base point, the fundamental groupoid is able to model more geometry than the fundamental group, and this has proved crucial in many applications. In the non connected case, the set A could be chosen to have at least one point in each component of the intersection. If X is a contractible space, and A consists of two distinct points of X , then $\pi_1(X, A)$ is easily seen to be isomorphic to the groupoid often written $\mathbf{I}$ with two vertices and exactly one morphism between any two vertices. This groupoid plays a role in the theory of groupoids analogous to that of the group of integers in the theory of groups. Again, if the space X is acted on by a group, than the set A should be chosen to be a union of orbits of the action; in particular, it could consist of all fixed points.

The notion of *pushout* in the category $Grpd$ of groupoids allows for a version of the theorem for the non path connected case, using the fundamental groupoid $\pi_1(X, A)$ on a set A of base points, [48]. This groupoid consists of homotopy classes rel end points of paths in X joining points of $A \cap X$.

Theorem 6.6.1. Fundamental Groupoid S--vKT

Let the topological space X *be covered by the interiors of two subspaces* X_1, X_2 *and let* A *be a set which meets each path component of* X_1, X_2 *and of* $X_0 := X_1 \cap X_2$. *Then* A *meets each path component of* X *and the following diagram of morphisms of groupoids induced by inclusion:*

$$\begin{array}{ccc} \pi_1(X_0,A) & \xrightarrow{\pi_1(i_1)} & \pi_1(X_1,A) \\ \big\downarrow{\scriptstyle \pi_1(i_2)} & & \big\downarrow{\scriptstyle \pi_1(j_1)} \\ \pi_1(X_2,A) & \xrightarrow[\pi_1(j_2)]{} & \pi_1(X,A) \end{array} \tag{30}$$

is a pushout diagram in the category $Grpd$ of groupoids . The use of this theorem for explicit calculation involves the development of a certain amount of *combinatorial groupoid theory* which is often implicit in the frequent use of directed graphs in combinatorial group theory.
The most general theorem of this type is, however, as follows:

Theorem 6.7. Generalised Theorem of Seifert-van Kampen, [65]: *Suppose* X *is covered by the union of the interiors of a family* $\{U_\lambda : \lambda \in \Lambda\}$ *of subsets. If* A *meets each path component of all 1,2,3-fold intersections of the sets* U_λ*, then* A *meets all path components of* X *and the diagram*

$$\coprod_{(\lambda,\mu)\in\Lambda^2} \pi_1(U_\lambda \cap U_\mu, A) \ \underset{b}{\overset{a}{\rightrightarrows}} \ \coprod_{\lambda\in\Lambda} \pi_1(U_\lambda, A) \xrightarrow{\ c\ } \pi_1(X,A)$$

(*coequaliser-π_1*)

of morphisms induced by inclusions is a *coequaliser* in the category $Grpd$ of groupoids. Here the morphisms a,b,c are induced respectively by the inclusions:

$$a_{\lambda\mu} : U_\lambda \cap U_\mu \to U_\lambda, \quad b_{\lambda\mu} : U_\lambda \cap U_\mu \to U_\mu, \quad c_\lambda : U_\lambda \to X. \tag{31}$$

Note that the above coequaliser diagram is an algebraic model of the diagram

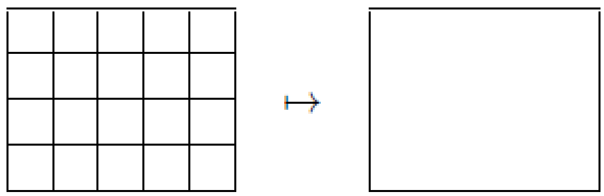
$$\coprod_{(\lambda,\mu)\in\Lambda^2} U_\lambda \cap U_\mu \ \underset{b}{\overset{a}{\rightrightarrows}} \ \coprod_{\lambda\in\Lambda} U_\lambda \xrightarrow{\ c\ } X$$

(*coequaliser-π_2*)

which intuitively says that X is obtained from copies of U_λ by gluing along their intersections.
The remarkable fact about these theorems is that even though *the input information involves two dimensions, namely 0 and 1* they enable, through a variety of further combinatorial techniques, the explicit computation of a nonabelian invariant, the fundamental group $\pi_1(X,a)$ at some base point a. In algebraic topology, the use of such information in two neighbouring dimensions usually involves exact sequences, sometimes with sets with base points, and does *not* give complete information. The success of this groupoid generalisation seems to stem from the fact that groupoids have structure in dimensions 0 and 1, and this enables us to compute groupoids, which are models of homotopy 1-types. In homotopy theory, identifications in low dimensions have profound implications on homotopy invariants in high dimensions, and it seems that in order to model this by gluing information we require algebraic invariants which have structure in a range of dimensions, and which completely model aspects of the homotopy type. Also the input is information not just about the spaces but spaces with structure, in this case a set of base points.
The suggestion is then that other situations involving the analysis of the behaviour of complex hierarchical systems

might be able to be analogously modelled, and that this modelling might necessitate a careful choice of the algebraic system. Thus there are many algebraic models of various kinds of homotopy types, but not all of them might fit into this scheme of being able directly to use gluing information.

The successful use of groupoids in 1-dimensional homotopy theory suggested the desirability of investigating the use of groupoids in higher homotopy theory. One aspect was to find a mathematics which allowed higher dimensional *`algebraic inverses to subdivision'*, in the sense that it could represent multiple compositions as in the following diagram:



(multi-composition)

in a manner analogous to the use of

$$(a_1, a_2, \ldots, a_n) \mapsto a_1 a_2 \ldots a_n$$

in both abstract categories and groupoids, but in dimension 2. Note that going from right to left in the diagram is subdivision, a standard technique in mathematics.

Another crucial aspect of the proof of the Seifert-van Kampen Theorem for groupoids is the use of commutative squares in a groupoid. Even in ordinary category theory we need the 2-dimensional notion of commutative square:

$$\begin{tikzcd} \cdot \arrow[r, "c"] \arrow[d, "a"'] & \cdot \arrow[d, "d"] \\ \cdot \arrow[r, "b"'] & \cdot \end{tikzcd} \qquad ab = cd \quad (a = cdb^{-1} \text{ in the groupoid case})$$

An easy result is that any composition of commutative squares is commutative. For example, in ordinary equations:

ab=cd, ef=bg implies *aef= cdg*.

The commutative squares in a category form a *double category*, and this fits with the above (multi-composition) diagram.

There is an obstacle to an analogous construction in the next dimension, and the solution involves a new idea of *double categories or double groupoids with connections*, which does not need to be explained here in detail here as it would take far too much space. What we can say is that in groupoid theory, we can stay still, `move forward, or turn around and go back'. In double groupoid theory, we need in addition to be able `to turn left or right'! This leads to an entirely new world of 2-dimensional algebra, which is explained for example in [47],[64, 57].

A further subtle point is that to exploit these algebraic ideas in homotopy theory in dimension 2 we find we need not just spaces but spaces X with subspaces $C \subseteq A \subseteq X$ where C is thought of as a set of base points. In higher dimensions it turns out that we need to deal with a *filtered space*, which is a space X and a whole increasing sequence of subspaces,

$$X_* := \quad X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X,$$

which in dimension 0 is often a set of base points. With such a structure it is possible to generalise the Seifert-van Kampen Theorem to all dimensions, yielding new results including nonabelian results in dimension 2, which are independent of and not seemingly obtainable by traditional methods, such as homology. Indeed this method gives a new foundation for algebraic topology, see [57], of which a central feature is a cubical homotopy groupoid $\rho(X_*)$ for any filtered space X_* and a Higher Homotopy Seifert--van Kampen Theorem analogous to the coequaliser diagram (coequaliser-π_1) but in which the term $\pi_1(X, A)$ is replaced by $\rho(X_*)$ and analogously for the other terms. It is this theorem which replaces and strengthens some of the foundations of homology theory.

One of the points of this development is that in geometry spaces often, even usually, arise with some kind of structure, and a filtration is quite common. Therefore it is quite natural to consider gluing of spaces with structure rather than just gluing of general spaces. What is clear is that the use of some forms of strict higher homotopy groupoids can be made to work in this context, and that this links well with a number of classical results, such as the absolute and relative Hurewicz theorems.

A further generalisation of this work involves not filtered spaces but n-cubes of spaces. The related algebraic structures are known as *cat^n-groups*, introduced in [176], and the equivalent structure of *crossed n-cubes of groups*, of [107]. This work is surveyed in [46]. All these structures should be seen as forms of n-fold groupoids; the step from 1 to $n>1$ gives extraordinarily rich algebraic structures which have had their riches only lightly explored. Again one has a Seifert--van Kampen type theorem, [61], with other surprising consequences, [62, 107]. Thus one sees these methods in terms of `higher dimensional groupoid theory', developed in the spirit of group theory, so that, in view of the wide importance of group theory in mathematics and science, one seeks for analogies and applications in wider fields than algebraic topology.

In particular, since the main origin of group theory was in symmetry, one seeks for *higher order notions of symmetry* or *extended symmetry*. A set can be regarded as an algebraic model of a homotopy **0**-type. The symmetries of a set form a group, which is an algebraic model of a pointed homotopy 1-type. The symmetries of a group G should be seen as forming a crossed module, $\chi : G \to \mathrm{A}ut(G)$, given by the inner automorphism map, and crossed modules form an algebraic model of homotopy **2**-types: for a recent account of this, see [57]. The situation now gets more complicated, and studies of this are in [208] and [51]: one gets a structure called a *crossed square*, which is an algebraic model of homotopy **3**-types. Crossed squares are homotopy invariants of a square of pointed spaces, which is a special case of an n-cube of spaces, for which again a Seifert--van Kampen type theorem is available, as said above.

Since representation theory is a crucial aspect of group theory and its applications, this raises the question of *what should be the representation theories for double and higher groupoids*. A recent preprint [80] is a first step in this direction by providing a formal definition of *double groupoid representations*. Again, groupoids are heavily involved in noncommutative geometry and other related aspects of physics, but it is unknown how to extend these methods to the intrinsically `more nonabelian' higher groupoids. Both of these problems may be hard: it took 9 years of experimentation to move successfully from the fundamental groupoid on a set of base points to the fundamental double groupoid of a based pair. There is an extensive literature on applications of higher forms of groupoids, particularly in areas of high energy physics, and even of special cases such as what are called sometimes called 2-groups. A recent report following many of the ideas of `algebraic inverse to subdivision' as above, but in a smooth manifold context, with many relations to physical concepts, was presented in ref.[114].

A general point about the algebraic structures used is that they have *partial operations which are defined under geometric conditions*: this is the generalisation of the notion of composition of morphisms, or, more ordinarily, of journeys, where the end point of one has to be the start of the next. The study of such structures may be taken as a general definition of *higher dimensional algebra*, and it is not too surprising intuitively that such structures can be relevant to gluing problems or to the local-to-global problems which are fundamental in many branches of mathematics and science.

As shown in [50] crossed complexes and higher homotopy groupoids provide useful noncommutative tools for higher-dimensional, local-to-global problems. Such *local-to-global problems* also occur in modern quantum physics, as for example in Extended Topological Quantum Field Theories (ETQFTs) and in Local Quantum Physics (AQFT) or Axiomatic QFTs. Therefore, one would expect crossed complexes and higher homotopy groupoids, as well as the generalised, higher homotopy SvKT theorem, to have potential applications in quantum theories, and especially in quantum gravity where the structure of quantised spacetimes is expected to be *non-Abelian*, as for example it is assumed from the outset in the gravitational theories based on Noncommutative Geometry [91].

There is also a published extension of the above Seifert--van Kampen Theorem (*S-vKT*) in terms of double groupoids [50, 60] for Hausdorff spaces rather than triples as above. The Seifert--van Kampen theorem for *double groupoids with connections* has also indirect consequences *via* quantum R-algebroids [63], for example, in theories relying on *2-Lie algebraic structures with connections*; thus, it has been recently suggested-- albeit indirectly- that such fundamental HDA results would have higher-dimensional applications in mathematical physics, as in the case of *higher gauge theory*, representations of the Lorentz group on **4**-dimensional Minkowski spacetimes, parallel transport for higher-dimensional extended 'objects', Lie **3**-superalgebras and **11**-dimensional supergravity [10, 246, 9].

6.4. Potential Applications of Novel Algebraic Topology methods to the problems of Quantum Spacetime and Extended Topological Quantum Field Theories

Traditional algebraic topology works by several methods, but all involve going from a space to some form of combinatorial or algebraic structure. The earliest of these methods was `triangulation': a space was supposed to be represented as a simplicial complex, i.e. was subdivided into simplices of various dimensions glued together along faces, and an algebraic structure such as a chain complex was built out of this simplicial complex, once assigned an orientation, or, as found convenient later, a total order on the vertices. Then, in the 1940s a convenient form of singular theory was found, which assigned to any space X a `singular simplicial set' SX, using continuous mappings from $\Delta^n \to X$, where Δ^n is the standard n-simplex. From this simplicial set, the whole of the weak homotopy type could in principle be determined. Further, the geometric realisation $|SX|$ is naturally a filtered space and so the methods above apply.

An alternative approach was found by Čech, using open covers $\cup$ of X to determine a simplicial set $N\cup$, and then refining the covers to get better `approximations' to X. It was this method which Grothendieck discovered could be extended, especially combined with new methods of homological algebra, and the theory of sheaves, to give new applications of algebraic topology to algebraic geometry, via his theory of schemes. The 600-page manuscript, *`Pursuing Stacks'* conceived by Alexander Grothendieck in 1983 was aimed at a *non-Abelian homological algebra*; it did not achieve this goal but has been very influential in the development of weak n-categories and other *higher categorical structures*.

Now, if new quantum theories were to reject the notion of a *continuum*, then it must also reject the notion of the real line and the notion of a path. How then is one to construct a homotopy theory?
One possibility is to take the route signalled by Čech, and which later developed in the hands of Borsuk into a `Shape Theory' . Thus, a quite general space, or spacetime in relativistic physical theories might be studied by means of its *approximation by open covers*.

With the advent of Quantum Groupoids, Quantum Algebra and perhaps Quantum Algebraic Topology, several fundamental concepts and new theorems of Algebraic Topology may also acquire an enhanced importance through their potential applications to current problems in theoretical and mathematical physics [21]. Such potential applications were briefly outlined, based upon algebraic topology concepts, fundamental theorems and HDA constructions. Moreover, the higher homotopy van Kampen theorem might be utilzed for certain types of such quantum spacetimes and Extended TQFTs to derive invariants beyond those covered by the current generalisations of Noether's theorem in General Relativity if such quantised spacetimes could be represented, or approximated in the algebraic topology sense, either in terms of open covers or as filtered spaces. If such approximations were valid then one would also be able to define a quantum fundamental groupoid of the quantised spacetime and derive consequences through the applications of GS-vKT, possibly extending this theorem to higher dimensions.

7. CONCLUSIONS AND DISCUSSION

The mathematical and physical symmetry background relevant to this review may be summarized in terms of a comparison between the Lie group `classical' symmetries with the following schematic representations of the extended groupoid and algebroid symmetries that we discussed in this paper :

Standard Classical and Quantum Group/Algebra Symmetries:

Lie Groups $\Rightarrow$ **Lie Algebras** $\Rightarrow$ **Universal Enveloping Algebra** $\Rightarrow$ **Quantization** $\to$ **Quantum Group Symmetry** (or Noncommutative (quantum) Geometry).

Extended, Quantum Groupoid and Algebroid Symmetries:

Quantum Groupoids/Quantum Algebroids $\leftarrow$ **Weak Hopf Algebras** $\Leftarrow$ **Representations** $\leftarrow$ **Quantum Groups**

Supported by a very wide array of examples from: solid state physics, spectroscopy, SBS, QCD, nuclear fusion reactors/ulltra-hot stars, EFT, ETQFT, HQFT, Einstein-Bose condensates, SUSY with the Higgs boson mechanism, Quantum Gravity and HDA-- as the generous provision of references reveals-- we have surveyed and applied several of these mathematical representations related to extended symmetry for the study of quantum systems (paracrystalline/quasicrystal structures, superfluids, superconductors, spin waves and magnon dispersion in ferromagnets, gluon coupled nucleons, nuclear fusion reactions, etc.) as specifically encapsulated within the framework of (nonabelian) Hopf symmetries, nonabelian algebraic topology so leading to a categorical formalism underlying a schemata that is apt for describing supersymmetric invariants of quantum space-time geometries. We propose that the need for investigation of (quantum) groupoid and algebroid representations is the natural consequence of the existence of local quantum symmetries, symmetry breaking, topological order, and other extended quantum symmetries in which transitional states are realized, for example, as noncommutative (operator) C*-algebras. Moreover, such representations-- when framed in their respective categories (of representation spaces)-- may be viewed in relation to several functorial relations that have been established between the categories $\mathbf{BHilb}, \mathbf{TGrpd}$ and $\mathbf{Set}$ as described in § 5 and § 6. We view these novel symmetry-related concepts as being essential ingredients for the formulation of a categorical ontology of quantum symmetries in the universal setting of the higher dimensional algebra and higher quantum homotopy/HHQFT of spacetimes.

APPENDIX: Hopf algebras- the basic definitions

Firstly, a unital associative algebra consists of a linear space A together with two linear maps

$$\begin{aligned} m \quad &: A \otimes A \to A, \text{(multiplication)} \\ \eta \quad &: \mathbb{C} \to A, \text{(unity)} \end{aligned} \tag{32}$$

satisfying the conditions

$$\begin{aligned} m(m \otimes \mathbf{1}) \quad &= m(\mathbf{1} \otimes m) \\ m(\mathbf{1} \otimes \eta) \quad &= m(\eta \otimes \mathbf{1}) = \mathrm{id}. \end{aligned} \tag{33}$$

This first condition can be seen in terms of a commuting diagram:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes m"'] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd} \tag{34}$$

Next let us consider `reversing the arrows', and take an algebra A equipped with a linear homorphisms $\Delta : A \to A \otimes A$, satisfying, for $a, b \in A$:

$$\begin{aligned} \Delta(ab) \quad &= \Delta(a)\Delta(b) \\ (\Delta \otimes \mathrm{id})\Delta \quad &= (\mathrm{id} \otimes \Delta)\Delta. \end{aligned} \tag{35}$$

We call Δ a *comultiplication*, which is said to be *coassociative* in so far that the following diagram commutes

$$\begin{tikzcd} A \otimes A \otimes A & A \otimes A \arrow[l, "\Delta\otimes\mathrm{id}"'] \\ A \otimes A \arrow[u, "\mathrm{id}\otimes\Delta"] & A \arrow[l, "\Delta"'] \arrow[u, "\Delta"'] \end{tikzcd} \tag{36}$$

There is also a counterpart to η, the *counity* map $\varepsilon : A \to \mathbb{C}$ satisfying

$$(\mathrm{id} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}. \tag{37}$$

A *bialgebra* $(A, m, \Delta, \eta, \varepsilon)$ is a linear space A with maps $m, \Delta, \eta, \varepsilon$ satisfying the above properties.

Now to recover anything resembling a group structure, we must append such a bialgebra with an antihomomorphism $S : A \to A$, satisfying $S(ab) = S(b)S(a)$, for $a, b \in A$. This map is defined implicitly via the property:

$$m(S \otimes \mathrm{id}) \circ \Delta = m(\mathrm{id} \otimes S) \circ \Delta = \eta \circ \varepsilon. \tag{38}$$

We call S the *antipode map*. A *Hopf algebra* is then a bialgebra $(A, m, \eta, \Delta, \varepsilon)$ equipped with an antipode map S.

Commutative and non--commutative Hopf algebras form the backbone of quantum groups [86, 185] and are essential to the generalizations of symmetry. Indeed, in many respects a quantum group is closely related to a Hopf algebra. When such algebras are actually associated with proper groups of matrices there is considerable scope for their representations on both finite and infinite dimensional Hilbert spaces.

0.1 Example: the $SL_q(2)$ Hopf algebra

This algebra is defined by the generators a, b, c, d and the following relations:

$$ba = qab\,, db = qbd, ca = qac, dc = qcd\ , bc = cb, \tag{39}$$

together with

$$adda = (q^{-1} - q)bc, adq^{-1}bc = 1, \tag{40}$$

and

$$\Delta \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix}\ ,\ \varepsilon \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ ,\ S \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -qb \\ -q^{-1}c & a \end{bmatrix} \tag{41}$$

0.2 Quasi--Hopf algebra

A quasi-Hopf algebra is an extension of a Hopf algebra. Thus, a quasi-Hopf algebra is a *quasi-bialgebra* $\mathsf{B}_{\mathsf{H}} = (\mathsf{H}, \Delta, \varepsilon, \Phi)$ for which there exist $\alpha, \beta \in \mathsf{H}$ and a bijective antihomomorphism S (the `antipode') of H such that $\sum_i S(b_i)\alpha c_i = \varepsilon(a)\alpha, \sum_i b_i \beta S(c_i) = \varepsilon(a)\beta$ for all $a \in \mathsf{H}$, with $\Delta(a) = \sum_i b_i \otimes c_i$, and the relationships

$$\sum_i X_i \beta S(Y_i) \alpha Z_i = \mathbf{I},\ \sum_j S(P_j) \alpha Q_j \beta S(R_j) = \mathbf{I}, \tag{42}$$

where the expansions for the quantities Φ and Φ^{-1} are given by

$$\Phi = \sum_i X_i \otimes Y_i \otimes Z_i,\ \Phi^{-1} = \sum_j P_j \otimes Q_j \otimes R_j. \tag{43}$$

As in the general case of a quasi-bialgebra, the property of being quasi-Hopf is unchanged by ``twisting''. Thus, twisting the comultiplication of a coalgebra

$$\mathsf{C} = (C, \Delta, \varepsilon) \tag{44}$$

over a field k produces another coalgebra C^{cop}; because the latter is considered as a vector space over the field k, the new comultiplication of C^{cop} (obtained by ``twisting'') is defined by

$$\Delta^{cop}(c) = \sum c_{(2)} \otimes c_{(1)}, \tag{45}$$

with $c \in \mathsf{C}$ and

$$\Delta(c) = \sum c_{(1)} \otimes c_{(2)}. \tag{46}$$

Note also that the linear dual C^* of C is an algebra with unit ε and the multiplication being defined by

$$\langle c^* * d^*, c \rangle = \sum \langle c^*, c_{(1)} \rangle \langle d^*, c_{(2)} \rangle, \tag{47}$$

for $c^*, d^* \in \mathsf{C}^*$ and $c \in \mathsf{C}$ (see [164]).

Quasi-Hopf algebras emerged from studies of Drinfeld twists and also from F-matrices associated with finite-dimensional irreducible representations of a quantum affine algebra. Thus, F-matrices were employed to factorize the corresponding R-matrix. In turn, this leads to several important applications in Statistical Quantum Mechanics, in the form of quantum *affine* algebras; their representations give rise to solutions of the quantum Yang-Baxter equation. This provides solvability conditions for various quantum statistics models, allowing characteristics of such models to be derived from their corresponding quantum affine algebras. The study of F-matrices has been applied to models such as the so-called Heisenberg `XYZ model', in the framework of the algebraic *Bethe ansatz*. Thus, F-matrices and quantum groups together with quantum affine algebras provide an effective framework for solving two-dimensional integrable models by using the Quantum Inverse Scattering method as suggested by Drinfeld and other authors.

0.3 Quasi--triangular Hopf algebra and the R-matrix

We begin by defining the quasi--triangular Hopf algebra, and then discuss its usefulness for computing the *R*-matrix of a quantum system.

Definition: A Hopf algebra, H, is called *quasi--triangular* if there is an invertible element R, of $H \otimes H$ such that:

(1) $R\,\Delta(x) = (T \circ \Delta)(x)\,R$ for all $x \in H$, where Δ is the coproduct on H, and the linear map $T : H \otimes H \to H \otimes H$ is given by

$$T(x \otimes y) = y \otimes x, \tag{48}$$

(2) $(\Delta \otimes 1)(R) = R_{13}\,R_{23}$,

(3) $(\mathbf{1} \otimes \Delta)(R) = R_{13}\,R_{12}$, where $R_{12} = \phi_{12}(R)$,

(4) $R_{13} = \phi_{13}(R)$, and $R_{23} = \phi_{23}(R)$, where $\phi_{12} : H \otimes H \to H \otimes H \otimes H$,

(5) $\phi_{13} : H \otimes H \to H \otimes H \otimes H$, and $\phi_{23} : H \otimes H \to H \otimes H \otimes H$, are algebra morphisms determined by

$$\begin{aligned} \phi_{12}(a \otimes b) &= a \otimes b \otimes 1, \\ \phi_{13}(a \otimes b) &= a \otimes 1 \otimes b, \\ \phi_{23}(a \otimes b) &= 1 \otimes a \otimes b. \end{aligned} \tag{49}$$

R is called the *R-matrix*.

An important part of the above algebra can be summarized in the following commutative diagrams involving the algebra morphisms, the coproduct on H and the identity map, id:

$$\begin{tikzcd} H \otimes H \otimes H & H \otimes H \arrow[l, "\phi_{12},\ \phi_{13}"'] \\ H \otimes H \otimes H \arrow[u, "\mathrm{id}\otimes\mathrm{id}\otimes\Delta"] & H \otimes H \arrow[l, "\phi_{23},\ \mathrm{id}\otimes\Delta"'] \arrow[u, "\Delta"'] \end{tikzcd} \tag{50}$$

and

$$\begin{tikzcd} H \otimes H \otimes H & H \otimes H \arrow[l, "\Delta\otimes\mathrm{id}"'] \\ H \otimes H \arrow[u, "\mathrm{id}\otimes\Delta"] & H \arrow[l, "\Delta"'] \arrow[u, "\Delta"'] \end{tikzcd} \tag{51}$$

Because of this property of quasi--triangularity, the R-matrix, R, becomes a solution of *the Yang-Baxter equation*. Thus, a module M of H can be used to determine quasi--invariants of links, braids, knots and higher dimensional structures with similar quantum symmetries. Furthermore, as a consequence of the property of quasi--triangularity, one obtains:

$$(\varepsilon \otimes 1)R = (1 \otimes \varepsilon)R = 1 \in H. \tag{52}$$

Finally, one also has:

$$R^{-1} = (S \otimes 1)(R),\ R = (1 \otimes S)(R^{-1}) \text{ and } (S \otimes S)(R) = R. \tag{53}$$

One can also prove that the antipode S is a linear isomorphism, and therefore S^2 is an automorphism: S^2 is obtained by conjugating by an invertible element, $S(x) = uxu^{-1}$, with

$$u = m(S \otimes 1)R^{21}. \tag{54}$$

By employing Drinfel'd's quantum double construction one can assemble a quasi--triangular Hopf algebra from a Hopf algebra and its dual.

0.4 The weak Hopf algebra

In order to define a *weak Hopf algebra*, one can relax certain axioms of a Hopf algebra as follows :

- The comultiplication is not necessarily unit--preserving.
- The counit ε is not necessarily a homomorphism of algebras.
- The axioms for the antipode map $S : A \to A$ with respect to the counit are as follows.

For all $h \in H$:

$$\begin{aligned} m(id \otimes S)\Delta(h) &= (\varepsilon \otimes \mathrm{id})(\Delta(1)(h \otimes 1)) \\ m(S \otimes \mathrm{id})\Delta(h) &= (\mathrm{id} \otimes \varepsilon)((1 \otimes h)\Delta(1)) \\ S(h) &= S(h_{(1)})Sh_{(2)}S(h_{(3)}) . \end{aligned} \tag{55}$$

These axioms may be appended by the following commutative diagrams :

$$\begin{tikzcd} A \otimes A \arrow[r, "S\otimes\mathrm{id}"] & A \otimes A \arrow[d, "m"] \\ A \arrow[u, "\Delta"] \arrow[r, "u\circ\varepsilon"] & A \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \arrow[r, "\mathrm{id}\otimes S"] & A \otimes A \arrow[d, "m"] \\ A \arrow[u, "\Delta"] \arrow[r, "u\circ\varepsilon"] & A \end{tikzcd} \tag{56}$$

along with the counit axiom:

$$\begin{tikzcd} A \otimes A \arrow[d, "\varepsilon\otimes 1"'] & A \arrow[l, "\Delta"'] \arrow[d, "\Delta"] \arrow[dl, "\mathrm{id}_A"'] \\ A & A \otimes A \arrow[l, "1\otimes\varepsilon"] \end{tikzcd} \tag{57}$$

Often the term *quantum groupoid* is used for a weak C*-Hopf algebra. Although this algebra in itself is not a proper groupoid, it may have a component *group* algebra as in, say, the example of the quantum double discussed previously. See [21, 52] and references cited therein for further examples.

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