TENSOR PRODUCTS AND HOMOTOPIES FOR ω-GROUPOIDS AND CROSSED COMPLEXES*

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Abstract

Crossed complexes have longstanding uses, explicit and implicit, in homotopy theory and the cohomology of groups. It is here shown that the category of crossed complexes over groupoids has a symmetric monoidal closed structure in which the internal Hom functor is built from morphisms of crossed complexes, nonabelian chain homotopies between them and similar higher homotopies. The tensor product involves non-abelian constructions related to the commutator calculus and the homotopy addition lemma. This monoidal closed structure is derived from that on the equivalent category of ω-groupoids where the underlying cubical structure gives geometrically natural definitions of tensor products and homotopies.

Introduction

The definition of a *crossed complex* is motivated by the principal example, the *fundamental crossed complex* ΠX_* of a filtered space

$$X_* : X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X.$$

Here $(\Pi X_*)_1$ is the fundamental groupoid $\pi_1(X_1, X_0)$ and for $n \geqslant 2$, $(\Pi X_*)_n$ is the family of relative homotopy groups $\pi_n(X_n, X_{n-1}, p), p \in X_0$, together with the standard boundary operators $\delta : (\Pi X_*)_n \to (\Pi X_*)_{n-1}$ and the actions of $(\Pi X_*)_1$ on $(\Pi X_*)_n$, $n \geqslant 2$. The axioms for a crossed complex are those universally satisfied by this standard example.

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We have shown earlier [6, 7] that the fundamental crossed complex satisfies a Higher Homotopy van Kampen type theorem[1] (i.e., it preserves certain colimits), which is related to and generalises many basic facts in homotopy theory, such as the relative Hurewicz theorem, and which leads to new results [4]. Crossed complexes have also been used by a number of writers [19, 20, 24, 25] to interpret the cohomology groups $H^n(G; A)$ of a group G with coefficients in a G-module A. These results suggest that the category of crossed complexes is both a convenient first approximation to the homotopy theory of CW-complexes and a suitable context for the development of non-abelian homological algebra (see also [8] and the surveys [5, 6]).

Our aim here is to give additional arguments for this view by endowing the category Crs of crossed complexes with appropriate notions of tensor product $A \otimes B$ and internal hom-functor $\mathsf{CRS}(B, C)$, thereby *giving* Crs *the structure of symmetric monoidal closed category*. The crossed complex $\mathsf{CRS}(B, C)$ is in dimension 0 the set of all morphisms $B \to C$. In dimension $m \geqslant 1$, it consists of m*-fold homotopies* $h : B \to C$ over morphisms $f : B \to C$, that is, maps of degree m from B to C in which the component $B_1 \to C_{m+1}$ is a derivation and the components $B_n \to C_{m+n}, n \geqslant 2$, are operator morphisms, all over the morphism $f : B_1 \to C_1$ of groupoids. the defining formulae are given in (3.1) below, and Proposition 3.14 gives a complete description of the crossed complex structure on $\mathsf{CRS}(B, C)$. In dimension 1, the elements of $\mathsf{CRS}(B, C)$ determine homotopies between morphisms $B \to C$, and these are in essence the same as the homotopies defined by Whitehead in [26] for particular kinds of crossed complexes. It should be noted that, just as the category of groups has no internal hom-functor, while the category of groupoids does have one, so also, to obtain a monoidal closed structure on the category Crs it is essential to use crossed complexes over groupoids, not over groups.

The tensor product $A \otimes B$ of crossed complexes A and B is generated as crossed complex by elements $a \otimes b$ in dimension $m + n$ for all $a \in A_m$, $b \in B_n$, $m, n \geqslant 0$, with defining relations given in (3.11). This tensor product is associative, symmetric (Section 4) and satisfies the adjointness condition required for a monoidal closed structure:

$$\mathsf{Crs}(A \otimes B, C) \cong \mathsf{Crs}(A, \mathsf{CRS}(B, C)).$$

Given the formulae (3.1), (3.11) and (3.14), it is possible , in principle, to verify all the above facts within the category of crossed complexes, although the computations, with their numerous special cases, would be long. We prefer to prove these facts using the equivalent category ω-Gpd of ω-groupoids where the formulae are simpler and have clearer geometric content.

The relationships developed between crossed complexes and the underlying cubical structure of ω-groupoids will in any case be required in other contexts (in particular for the construction of classifying spaces [10]). Although the usual simplicial notions are more firmly embedded in the literature than the cubical ones, the latter are more convenient for handling higher homotopies. An analysis of the symmetric, monoidal closed structure on simplicial T-complexes, which form the simplicial analogue of ω-groupoids (Ashley [1]), has not yet been attempted.

The results of this paper will form the basis of further papers [9, 10] extending work of Whitehead and Eilenberg-Mac Lane on abstract homotopy theory and the cohomology of groups. In particular, in [10], we define the classifying space BC of a crossed complex C and show that, for any CW-complex X, there is a natural bijection of homotopy classes $[X, BC] \cong [\Pi X_*, C]$. We also show that

[1]It has recently (2007) been suggested by Jim Stasheff that this term should replace the previous 'Generalised van Kampen Theorem'.

$\Pi(X_* \otimes Y_*) \cong \Pi X_* \otimes \Pi Y_*$ for CW-complexes X and Y. The corresponding results for cubical sets are proved in Section 3 below.

The structure of the paper is as follows. Tensor products and hom-functors are described for cubical sets in Section 1, for ω-groupoids in Section 2 and for crossed complexes in Section 3. The transition from ω-groupoids to crossed complexes uses the equivalences

$$\gamma : \omega\text{−}\mathsf{Gpd} \rightleftarrows \mathsf{Crs} : \lambda$$

established in [6]. In Section 4 we establish the symmetry of the tensor product which, by contrast with the other results, is easier to prove for crossed complexes than for ω-groupoids. (It is interesting to note that the tensor product of cubical sets is not symmetric; the extra structure of ω-groupoids is needed to define the symmetry map $G \otimes H \to H \otimes G$.) In Section 5 we give a brief account of the case of crossed complexes with base-point. Finally, in Section 6, we calculate some special tensor products; in particular, the tensor product of two groupoids involves an interesting new construction akin to the Cartesian subgroup of a free product of groups.

1 Cubical sets

We start by examining the tensor products and internal hom functor in the category of cubical sets. These well-known constructions are related by exponential laws which we shall exploit later when we study similar but more difficult constructions for ω-groupoids and crossed complexes. We recall the notations and main results.

A cubical set $K = \{K_n, \partial_i^\alpha, \varepsilon_i\}$ consists of a family of sets K_n $(n \geqslant 0)$ and functions $\partial_i^\alpha : K_n \to K_{n-1}$, $\varepsilon_i : K_{n-1} \to K_n$ $(i = 1, 2, \dots, n;\ \alpha = 0, 1)$ satisfying the usual cubical laws (see, for example, [6]). A *cubical map* $f : K \to L$ is a family of functions $f_n : K_n \to L_n$ $(n \geqslant 0)$ preserving the ∂_i^α and ε_i. These form the category Cub of cubical sets.

If H,K are cubical sets, their *tensor product* $H \otimes K$ has

$$(H \otimes K)_n = \left(\bigsqcup_{p+q=n} H_p \times K_q \right) / \sim$$

where $\sim$ is the equivalence relation generated by $(\varepsilon_{r+1}x, y) \sim (x, \varepsilon_1 y)$ for $x \in H_r, y \in K_s$ $(r+s = n-1)$. We write $x \otimes y$ for the equivalence class of (x, y). The maps $\partial_i^\alpha, \varepsilon_i$ defined for $x \in H_p, y \in K_q$ by

$$\partial_i^\alpha(x \otimes y) = \begin{cases} (\partial_i^\alpha x) \otimes y & \text{if } 1 \leqslant i \leqslant p, \\ x \otimes (\partial_{i-p}^\alpha y) & \text{if } p + 1 \leqslant i \leqslant p + q, \end{cases}$$

$$\varepsilon_i(x \otimes y) = \begin{cases} (\varepsilon_i x) \otimes y & \text{if } 1 \leqslant i \leqslant p + 1, \\ x \otimes (\varepsilon_{i-p} y) & \text{if } p + 1 \leqslant i \leqslant p + q + 1 \end{cases}$$

make $H \otimes K$ a cubical set. We note, in particular, that

$$(\varepsilon_{p+1}x) \otimes y = x \otimes (\varepsilon_1 y) \text{ when } x \in H_p.$$

The universal property possessed by this tensor product is the following. Any cubical map $f : H \otimes K \to L$ defines a family of functions $F_{pq} : H_p \times K_q \to L_{p+q}$ (given by $F_{pq}(x, y) = f_{p+q}(x \otimes y)$) satisfying

1.1

$$\partial_i^\alpha F_{pq}(x \otimes y) = \begin{cases} F_{p-1,q}(\partial_i^\alpha x, y) & \textit{if } 1 \leqslant i \leqslant p \\ F_{p,q-1}(x, \partial_{i-p}^\alpha y) & \textit{if } p+1 \leqslant i \leqslant p+q, \end{cases}$$

$$\varepsilon_i F_{pq}(x \otimes y) = \begin{cases} F_{p+1,q}(\varepsilon_i x, y) & \textit{if } 1 \leqslant i \leqslant p+1 \\ F_{p,q+1}(x, \varepsilon_{i-p} y) & \textit{if } p+1 \leqslant i \leqslant p+q+1. \end{cases}$$

Such a family $F = \{F_{p,q}\}$ will be called a *bicubical map* from (H, K) to L. Conversely, given a a bicubical map $F : (H, K) \to L$, there is a unique cubical map $f : H \otimes K \to L$ such that $F_{pq}(x, y) = f_{p+q}(x \otimes y)$. This is because the defining equations (1.1) for a bicubical map imply that, for $x \in H_r$ and $y \in K_s$

$$F_{r+1,s}(\varepsilon_{r+1} x, y) = \varepsilon_{r+1} F_{rs}(x, y) = F_{r,s+1}(x, \varepsilon_1 y),$$

that is, the maps F_{pq} respect the equivalence $\sim$ used in defining $H \otimes K$. The resulting map $H \otimes K \to L$ is cubical by (1.1).

We denote by $\mathbb{I}^n$ the cubical set freely generated by one element c_n in dimension n. It is free in the sense that, for any cubical set K and any $x \in K_n$, there is a unique cubical map $\hat{x} : \mathbb{I}^n \to K$ such that $\hat{x}(c_n) = x$. The solution of the word problem for $\mathbb{I}^n$ is well known. The non-degenerate elements are uniquely of the form $\partial_{i_1}^{\alpha_1} \partial_{i_2}^{\alpha_2} \cdots \partial_{i_r}^{\alpha_r} c_n$ where $\alpha_j = 0, 1$ and $1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n$. An arbitrary element is then (as in all cubical sets) uniquely of the form $\varepsilon_{j_1} \varepsilon_{j_2} \cdots \varepsilon_{j_s} x$, where x is non-degenerate of dimension t, say, and $s + t \geqslant j_1 > j_2 > \cdots > j_s$. We shall show below that $\mathbb{I}^m \otimes \mathbb{I}^n \cong \mathbb{I}^{m+n}$. In what follows, the cubical set $\mathbb{I} = \mathbb{I}^1$ plays the role of the unit interval in homotopy theory. We denote its vertices by $0 = \partial_1^0 c_1$ and $1 = \partial_1^1 c_1$.

We now look at the internal hom functor $\mathsf{CUB}(K, L)$ for cubical sets K, L. This construction embraces the notions of homotopies and higher homotopies for cubical maps (cf. Kan [21]). However we need to distinguish between left and right homotopies because tensor products of cubical sets are not symmetric (we return to this point later).

We first define, for any cubical set L, the *left path complex* PL which is the cubical set with

$$(PL)_r = L_{r+1}$$

and cubical operations $\partial_2^\alpha, \partial_3^\alpha, \dots, \partial_{r+1}^\alpha : (PL)_r \to (PL)_{r+1}$, $\varepsilon_2, \varepsilon_3, \dots, \varepsilon_{r+1} : (PL)_{r-1} \to (PL)_r$ (that is, we ignore the first operations $\partial_1^0, \partial_1^1, \varepsilon_1$ in each dimension.) The bicubical maps from $(\mathbb{I}, K)$ to L are in natural one-one correspondence with (a) the cubical maps $f : \mathbb{I} \otimes K \to L$, and with (b) the cubical maps $\bar{f} : K \to PL$. Here corresponding maps $f, \tilde{f}$ are related by $\tilde{f}(x) = f(c_1 \otimes x)$ and either of them is termed a *left homotopy* from f_0 to f_1, where $f_\alpha : K \to L$ is given by

$$f_\alpha x = f(\alpha \otimes x) = \partial_1^\alpha \tilde{f} x \qquad (\alpha = 0, 1).$$

We note that the functor $\mathbb{I} \otimes -$ is left adjoint to $P : \mathsf{Cub} \to \mathsf{Cub}$ and we generalise this adjointness as follows.

First we define the n-fold left path complex $P^n L$ inductively by $P^n L = P(P^{n-1} L)$, so that

$$(P^n L)_r = L_{n+r}$$

with cubical operations $\partial_{n+1}^\alpha, \partial_{n+2}^\alpha, \dots, \partial_{n+r}^\alpha : (P^n L)_r \to (P^n L)_{r-1}$ and $\varepsilon_{n+1}, \varepsilon_{n+2}, \dots, \varepsilon_{n+r} : (P^n L)_{r-1} \to (P^n L)_r$. The omitted operations $\partial_1^\alpha, \dots, \partial_n^\alpha$ in each dimension induce morphisms of cubical sets

$\partial_1^\alpha, \dots, \partial_n^\alpha : P^n L \longrightarrow P^n L$, and similarly we have morphisms of cubical sets $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n : P^n L \longrightarrow P^n L$. These morphisms satisfy the cubical laws, so the family

$$P^* L = \{P^n L\}_{n \geqslant 0}$$

is an internal cubical set in Cub.

We now define

$$\mathsf{CUB}_n(K, L) = \mathsf{Cub}(K, P^n L)$$

and observe that, because of the internal cubical structure on P^*L, the family $\mathsf{CUB}(K, L)$ of sets $\mathsf{CUB}_n(K, L)$ for $n \geqslant 0$ inherits a cubical structure. Its cubical operations $\mathsf{CUB}_n(K, L) \rightleftarrows \mathsf{CUB}_{n-1}(K, L)$ are induced by the operations $\partial_1^\alpha, \partial_2^\alpha, \dots, \partial_n^\alpha, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ of L. Thus a typical $f \in \mathsf{CUB}_n(K, L)$ is a family of maps $f_r : K_r \to L_{n+r}$ satisfying

$$f_{r-1}\partial_i^\alpha = \partial_{n+i}^\alpha f_r, \qquad f_r \varepsilon_j = \varepsilon_{n+j} f_{r-1} \quad (i, j = 1, 2, \dots, r)$$

and its faces are given by

$$(\partial_i^\alpha f)_r = \partial_i^\alpha f_r \qquad (i = 1, 2, \dots, n, \alpha = 0, 1).$$

In geometric terms, the elements of $\mathsf{CUB}_0(K, L)$ are the cubical maps $K \to L$, the elements of $\mathsf{CUB}_1(K, L)$ are the (left) homotopies between such maps, the elements of $\mathsf{CUB}_2(K, L)$ are homotopies between homotopies, etc.

Proposition 1.2 (i) *The functor* $\mathsf{CUB}(K, -) : \mathsf{Cub} \to \mathsf{Cub}$ *is right adjoint to* $- \otimes K$.

(ii) *For cubical sets* H, K, L *there are natural isomorphisms of cubical sets*

$$(H \otimes K) \otimes L \cong H \otimes (K \otimes L),$$
$$\mathsf{CUB}(H \otimes K, L) \cong \mathsf{CUB}(H, \mathsf{CUB}(K, L)),$$

giving Cub *the structure of a monoidal closed category.* □

Corollary 1.3 (i) $- \otimes \mathbb{I}^n$ *is left adjoint to* $\mathsf{CUB}(\mathbb{I}^n, -) : \mathsf{Cub} \to \mathsf{Cub}$.

(ii) $\mathbb{I}^n \otimes -$ *is left adjoint to* $P^n : \mathsf{Cub} \to \mathsf{Cub}$.

(iii) *There are natural (and coherent) isomorphisms of cubical sets*

$$\mathbb{I}^m \otimes \mathbb{I}^n \cong \mathbb{I}^{m+n}.$$

Proof (i) is a special case of (1.2)(i).

(ii) follows from the natural bijections

$$\begin{aligned} \mathsf{Cub}(\mathbb{I}^n \otimes K, L) &\cong \mathsf{Cub}(\mathbb{I}^n, \mathsf{CUB}(K, L)) \\ &\cong \mathsf{CUB}_n(K, L) = \mathsf{Cub}(K, P^n L). \end{aligned}$$

(iii) follows from (ii) since $P^m \circ P^n = P^{m+n}$. □

This corollary serves to remind the reader that for cubical sets the tensor product is not symmetric since $(x, y) \mapsto y \otimes x$ is not a bicubical map. However, there is a 'transposition' functor $T : \mathsf{Cub} \to \mathsf{Cub}$, where TK has the same elements as K in each dimension but has its face and degeneracy operations numbered in reverse order, that is, the cubical operations $d_i^\alpha : (TK)_n \to (TK)_{n-1}$ and $e_i : (TK)_{n-1} \to (TK)_n$ are defined by $d_i^\alpha = \partial_{n+1-i}^\alpha, e_i = \varepsilon_{n+1-i}$. Clearly T^2K is naturally isomorphic to K and $T(K \otimes L)$ is naturally isomorphic to $T(L) \otimes T(K)$. Instead of the expected isomorphism of $\mathsf{CUB}(\mathbb{I}^n, L)$ with $P^n L$, we have:

Corollary 1.4 *There is a natural isomorphism of cubical sets*

$$\mathsf{CUB}(\mathbb{I}^n, L) \cong TP^n TL.$$

Proof By (1.3), P^n is right adjoint to $\mathbb{I}^n \otimes -$, so $P^n T$ is right adjoint to $T(\mathbb{I}^n \otimes -)$. Hence TP^nT is right adjoint to $T(\mathbb{I}^n \otimes T-) \cong (- \otimes T\mathbb{I}^n)$. But there is an obvious cubical isomorphism $T\mathbb{I}^n \cong \mathbb{I}^n$ and this induces a natural isomorphism $(- \otimes T\mathbb{I}^n) \cong (- \otimes \mathbb{I}^n)$. Hence TP^nT is naturally isomorphic to the right adjoint $\mathsf{CUB}(\mathbb{I}^n, -)$ of $- \otimes \mathbb{I}^n$. $\square$

Note. A simpler argument shows that the functor $A \otimes - : \mathsf{Cub} \to \mathsf{Cub}$ has a right adjoint T $\mathsf{CUB}(TA, T-)$ and hence that the monoidal closed category Cub is *biclosed*, in the sense of Kelly [22], even though it is not symmetric.

2 ω-groupoids

An ω-groupoid is a cubical set with extra structure, namely, (i) connections (which are extra degeneracies) and (ii) n groupoid structures in dimension n (one composition along each of the n directions). The precise definition is in [6]. The prime example is the fundamental ω-groupoid $\rho(X)$ of a filtered space X, which is the quotient of the (cubical) singular complex of X by the relation of filter-homotopy.

The category ω-Gpd of ω-groupoids is a convenient algebraic model for certain geometric constructions. In particular it is well-suited for the discussion of homotopies and higher homotopies and their composition. The internal hom functor for cubical sets generalises immediately to ω-groupoids as follows.

For any ω-groupoid H, considered as a cubical set, the n-fold left path complex P^nH has $(P^nH)_r = H_{n+r}$, with cubical operators $\partial_{n+1}^\alpha, \partial_{n+2}^\alpha, \dots, \partial_{n+r}^\alpha : (P^nH)_r \to (P^nH)_{r-1}$ and $\varepsilon_{n+1}, \varepsilon_{n+2}, \dots, \varepsilon_{n+r} : (P^nH)_{r-1} \to (P^nH)_r$. The connections $\Gamma_{n+1}, \Gamma_{n+2}, \dots, \Gamma_{n+r-1} : (P^nH)_{r-1} \to (P^nH)_r$ and the compositions $+_{n+1}, +_{n+2}, \dots, +_{n+r}$ on $(P^nH)_r$ make P^nH an ω-groupoid since the laws to be checked are just a subset of the ω-groupoid laws of H. We call P^nH the *n-fold (left) path ω-groupoid* of H. The operators of H not used in P^mH give maps

$$\begin{aligned} \partial_1^\alpha, \dots, \partial_m^\alpha : P^mH &\longrightarrow P^{m-1}H, \\ \varepsilon_1, \dots, \varepsilon_m : P^{m-1}H &\to P^mH, \\ \Gamma_1, \dots, \Gamma_{m-1} : P^{m-1}H &\to P^mH \end{aligned}$$

which are morphisms of ω-groupoids and obey the cubical laws. The unused additions of H define partial compositions $+_1, +_2, \dots, +_m$ on P^mH which, by the ω-groupoid laws for H, are compatible with the ω-groupoid structure of P^mH. Hence

$$\mathbf{P}H = (H, PH, P^2H, \dots,)$$

with the above operators and additions is an internal ω-groupoid in the category of ω-groupoids.

For any ω-groupoids G, H, we now define

$$K = \omega\text{-GPD}(G, H) = \omega\text{-Gpd}(G, PH),$$

that is, $\omega\text{-GPD}_m(G, H) = \omega\text{-Gpd}(G, P^m H)$, and it is clear that the internal ω-groupoid structure on PH induces an ω-groupoid structure on K with operators $\partial_1^\alpha, \dots, \partial_m^\alpha : K_m \to K_{m-1}, \quad \varepsilon_1, \dots, \varepsilon_m, \Gamma_1, \dots, \Gamma_{m-1} : K_{m-1} \to K_m$ and compositions $+_1, \dots, +_m$ on K_m all induced by the similarly numbered operations on H. Thus in dimension 0, $\omega\text{-GPD}(G, H)$ consists of all morphisms $G \to H$,while in dimension n it consists of n*-fold (left) homotopies* $G \to H$. We make $\omega\text{-GPD}(G, H)$ a functor in G and H (contravariant in G) in the obvious way: if $g : G \to G'$ and $h : H \to H'$ are morphisms, the corresponding morphism

$$k : \omega\text{-GPD}(G, H) \to \omega\text{-GPD}(G', H')$$

is given, in dimension r, by

$$k_r(f) = (P^r h) \circ f \circ g, \quad \text{where } f : G \to P^r H.$$

The definition of tensor products of ω-groupoids is harder. We require that $- \otimes G$ be left adjoint to $\omega\text{-GPD}(G, -)$ as a functor from ω-Gpd to ω-Gpd, and this determines $\otimes$ up to natural isomorphism. Its existence, that is, the representability of the functor $\omega\text{-Gpd}(F,\omega\text{-GPD}(G, -))$ can be asserted on general grounds. The point is that ω-Gpd is an equationally defined category of many sorted algebras in which the domains of the operations are defined by finite limit diagrams. General theorems on such algebraic categories (see [12–15, 22, 27]) imply that ω-Gpd is complete and cocomplete and that it is monadic over the category Cub of cubical sets. In particular the underlying cubical set functor $U : \omega\text{-Gpd} \to \mathsf{Cub}$ has a left adjoint $\rho : \mathsf{Cub} \to \omega\text{-Gpd}$, and we call $\rho(K)$ the *free* ω-Gpd *on the cubical set* K. (This notation is consistent with our previous use of $\rho(K)$ as the fundamental ω-Gpd of the filtered space X because, for any cubical set K, $\rho(K) \cong \rho(|\mathbf{K}|)$. See Note (i) at the end of this section.) We may also specify an ω-groupoid by a *presentation*, that is, by giving a set of generators in each dimension and a set of defining relations of the form $u = v$, where u, v are well-formed formulae of the same dimension made from generators and the operators $\partial_i^\alpha, \varepsilon_i, \Gamma_i, +_i, -_i$. (For example, $\rho(K)$ is the ω-groupoid generated by the elements of K with defining relations given by the face and degeneracy maps $\partial_i^\alpha : K_n \to K_{n-1}$ and $\varepsilon_i : K_{n-1} \to K_n$.)

The validity of using presentations in this context enables us follow the standard procedure for defining tensor products of modules. Given ω-groupoids F, G, we define $F \otimes G$ by giving a presentation of it as an ω-groupoid. The universal property of the presentation will then give the required adjointness. Gray [16, 17] has given a related definition of tensor products of 2-categories, but his definition is complicated by the fact that he restricts attention to 2-categories. The tensor product of two 2-categories is more naturally defined as a 4-category, so Gray has to introduce extra defining relations to ensure that elements in dimensions 3 and 4 are trivial.

Let F, G be ω-groupoids. We define $F \otimes G$ to be the ω-groupoid generated by elements in dimension $n \geqslant 0$ of the form $x \otimes y$ where $x \in F_p$, $y \in G_q$ and $p + q = n$, subject to the following defining relations (plus, of course, the laws for ω-groupoids)

2.1 (i) $\partial_i^\alpha(x \otimes y) = \begin{cases} (\partial_i^\alpha x) \otimes y & \text{if } 1 \leqslant i \leqslant p, \\ x \otimes (\partial_{i-p}^\alpha y) & \text{if } p + 1 \leqslant i \leqslant n; \end{cases}$

(ii) $\varepsilon_i(x \otimes y) = \begin{cases} (\varepsilon_i x) \otimes y & \textit{if } 1 \leqslant i \leqslant p+1, \\ x \otimes (\varepsilon_{i-p} y) & \textit{if } p+1 \leqslant i \leqslant n+1; \end{cases}$

(iii) $\Gamma_i(x \otimes y) = \begin{cases} (\Gamma_i x) \otimes y & \textit{if } 1 \leqslant i \leqslant p, \\ x \otimes (\Gamma_{i-p} y) & \textit{if } p+1 \leqslant i \leqslant n; \end{cases}$

(iv) $(x +_i x') \otimes y = (x \otimes y) +_i (x' \otimes y)$ *if* $1 \leqslant i \leqslant p$ *and* $x +_i x'$ *is defined in* F*;*

(v) $x \otimes (y +_j y') = (x \otimes y) +_{p+j} (x \otimes y')$ *if* $1 \leqslant j \leqslant q$ *and* $y +_j y'$ *is defined in* G*;*

We note that the relation

(vi) $-_i(x \otimes y) = \begin{cases} (-_i x) \otimes y & \text{if } 1 \leqslant i \leqslant p, \\ x \otimes (-_{i-p} y) & \text{if } p+1 \leqslant i \leqslant n \end{cases}$

follows from (iv) and (v). Also the relation

(vii) $(\varepsilon_{p+1} x) \otimes y = x \otimes (\varepsilon_1 y)$

follows from (ii).

An alternative way of stating this definition is to define a *bimorphism* $(F, G) \to A$, where F, G, A are ω-groupoids, to be a family of maps $F_p \times G_q \to A_{p+q}$ $(p, q \geqslant 0)$, denoted by $(x, y) \mapsto \chi(x, y)$ such that

(a) for each $x \in F_p$, the map $y \mapsto \chi(x, y)$ is a morphism of ω-groupoids $G \to P^p A$;

(b) for each $g \in G_q$ the map $x \mapsto \chi(x, y)$ is a morphism of ω-groupoids $F \to TP^q TA$, where as in section 1, the ω-groupoid TX has the same elements as X but has its cubical operations, connections and compositions numbered in reverse order.

The ω-groupoid $F \otimes G$ is now defined up to natural isomorphism by the two properties:

(i) the map $(x, y) \mapsto x \otimes y$ is a bimorphism $(F, G) \to F \otimes G$;

(ii) every bimorphism $(F, G) \to A$ is uniquely of the form $(x, y) \mapsto \sigma(x \otimes y)$ where $\sigma : F \otimes G \to A$ is a morphism of ω-groupoids.

In the definition of a bimorphism $(F, G) \to A$, condition (a) gives maps $F_p \to \omega\text{-GPD}_p(G, A)$ for each p, and condition (b) states that these combine to give a morphism of ω-groupoids $F \to \omega\text{-GPD}(G, A)$. This observation yields a natural bijection between bimorphisms $(F, G) \to A$ and morphisms $F \to \omega\text{-GPD}(G, A)$. Since there is also a natural bijection between bimorphisms $(F, G) \to A$ and morphisms $F \otimes G \to A$, we have

Proposition 2.2 *The functor* $- \otimes G$ *is left adjoint to the functor* $\omega\text{-GPD}(G, -)$ *from* $\omega\text{-Gpd}$*to* $\omega\text{-Gpd}$. □

Proposition 2.3 *For ω-groupoids* F, G, H*, there are natural isomorphisms of ω-groupoids*

(i) $(F \otimes G) \otimes H \cong F \otimes (G \otimes H)$*, and*

(ii) $\omega\text{-GPD}(F \otimes G, H) \cong \omega\text{-GPD}(F, \omega\text{-GPD}(G, H))$

giving ω-Gpd *the structure of a monoidal closed category.*

Proof (ii) In dimension r there is, by adjointness, a natural bijection

$$\begin{aligned}\omega\text{-GPD}_r(F \otimes G, H) &= \omega\text{-Gpd}(F \otimes G, P^r H)\\ &\cong \omega\text{-Gpd}(F, \omega\text{-GPD}(G, P^r H))\\ &= \omega\text{-Gpd}(F, P^r(\omega\text{-GPD}(G, H)))\\ &= \omega\text{-GPD}_r(F, \omega\text{-GPD}(G, H)).\end{aligned}$$

These bijections combine to form the natural isomorphism (ii) of ω-groupoids because, on both sides, the ω-groupoid structures are given by $\partial_i^\alpha, \varepsilon_j, \Gamma_k, +_l$ induced by the corresponding operators $\partial_i^\alpha, \varepsilon_j$, etc. in H. For example, if $u \in \omega\text{-GPD}_r(F \otimes G, H)$ corresponds to $\hat{u} \in \omega\text{-GPD}_r(F, \omega\text{-GPD}(G, H))$, then

$$u(f \otimes g) = \hat{u}(f)(g) \in H_{p+q+r} \text{ for all } f \in F_p, g \in G_q;$$

hence, for $1 \leqslant i \leqslant r$,

$$(\partial_i^\alpha u)(f \otimes g) := \partial_i^\alpha(u(f \otimes g)) = \partial_i^\alpha(\hat{u}(f)(g))) := \; d_i^\alpha(\hat{u})(f)(g)$$

so $\partial_i^\alpha u$ also corresponds to $\partial_i^\alpha \hat{u}$.

(i) This isomorphism can be proved directly or deduced from (ii) by means of the natural isomorphisms

$$\begin{aligned}\omega\text{-GPD}((F \otimes G) \otimes H, K) &\cong \omega\text{-GPD}(F \otimes G, \omega\text{-GPD}(H, K))\\ &\cong \omega\text{-GPD}(F, \omega\text{-GPD}(G, \omega\text{-GPD}(H, K)))\\ &\cong \omega\text{-GPD}(F, \omega\text{-GPD}(G \otimes H, K))\\ &\cong \omega\text{-GPD}(F \otimes (G \otimes H), K).\end{aligned}$$

Again, coherence is easily established. $\square$

Proposition 2.4 *For a cubical set* L *and an* ω*-groupoid* G*, there is a natural isomorphism of cubical sets*

$$U(\omega\text{-GPD}(\rho(L), G)) \cong \text{CUB}(L, UG).$$

Proof The functor $\rho : \text{Cub} \to \omega\text{-Gpd}$ is left adjoint to $U : \omega\text{-Gpd} \to \text{Cub}$, and this is what the proposition says in dimension 0. In dimension r we have a natural bijection

$$\begin{aligned}\omega\text{-GPD}_r(\rho(L), G) &= \omega\text{-Gpd}(\rho(L), P^r G)\\ &\cong \text{Cub}(L, UP^r G) = \text{CUB}_r(L, UG)\end{aligned}$$

and these bijections are compatible with the cubical operators as in the proof of (2.3)(ii). $\square$

Proposition 2.5 *If* K, L *are cubical sets, there is a natural isomorphism of* ω*-groupoids*

$$\rho(\mathsf{K} \otimes \mathsf{L}) \cong \rho(\mathsf{K}) \otimes \rho(\mathsf{L}).$$

Proof By (2.3) and (2.4), for any ω-groupoid G, there are natural isomorphisms of cubical sets

$$\begin{aligned} \mathsf{U}(\omega\text{-}\mathsf{GPD}(\rho(\mathsf{K} \otimes \mathsf{L}), \mathsf{G})) &\cong \mathsf{CUB}(\mathsf{K} \otimes \mathsf{L}, \mathsf{UG}) \\ &\cong \mathsf{CUB}(\mathsf{K}, \mathsf{CUB}(\mathsf{L}, \mathsf{UG})) \\ &\cong \mathsf{CUB}(\mathsf{K}, \mathsf{U}(\omega\text{-}\mathsf{GPD}(\rho(\mathsf{L}), \mathsf{G}))) \\ &\cong \mathsf{U}(\omega\text{-}\mathsf{GPD}(\rho(\mathsf{K}), \omega\text{-}\mathsf{GPD}(\rho(\mathsf{L}), \mathsf{G}))) \\ &\cong \mathsf{U}(\omega\text{-}\mathsf{GPD}(\rho(\mathsf{K}) \otimes \rho(\mathsf{L}), \mathsf{G})). \end{aligned}$$

The proposition follows from the information in dimension 0, namely

$$\omega\text{-}\mathsf{Gpd}(\rho(\mathsf{K} \otimes \mathsf{L}), \mathsf{G}) \cong \omega\text{-}\mathsf{Gpd}(\rho(\mathsf{K}) \otimes \rho(\mathsf{L}), \mathsf{G}).$$

□

Writing $\mathbb{G}(n)$ for $\rho(\mathbb{I}^n)$, the ω-groupoid freely generated by one element in dimension n, and using (1.3)(iii), we have

Corollary 2.6 *There are natural isomorphisms of* ω*-groupoids*

$$\mathbb{G}(m) \otimes \mathbb{G}(n) \cong \mathbb{G}(m+n).$$

□

Proposition 2.7 (i) $\mathbb{G}(n) \otimes -$ *is left adjoint to* $\mathsf{P}^n : \omega\text{-}\mathsf{Gpd} \to \omega\text{-}\mathsf{Gpd}$.

(ii) $- \otimes \mathbb{G}(n)$ *is left adjoint to* $\omega\text{-}\mathsf{GPD}(\mathbb{G}(n), -)$.

(iii) $\omega\text{-}\mathsf{GPD}(\mathbb{G}(n), -)$ *is naturally isomorphic to* $\mathsf{TP}^n\mathsf{T}$.

Proof (i) There are natural bijections

$$\begin{aligned} \omega\text{-}\mathsf{Gpd}(\mathbb{G}(n) \otimes \mathsf{H}, \mathsf{K}) &\cong \omega\text{-}\mathsf{Gpd}(\mathbb{G}(n), \omega\text{-}\mathsf{GPD}(\mathsf{H}, \mathsf{K})) \\ &\cong \omega\text{-}\mathsf{GPD}_n(\mathsf{H}, \mathsf{K}) = \omega\text{-}\mathsf{Gpd}(\mathsf{H}, \mathsf{P}^n\mathsf{K}). \end{aligned}$$

(ii) This is a special case of (2.2).

(iii) It follows from (i) that $\mathsf{TP}^n\mathsf{T} : \omega\text{-}\mathsf{Gpd} \to \omega\text{-}\mathsf{Gpd}$has left adjoint $\mathsf{T}(\mathbb{G}(n) \otimes \mathsf{T}(-)) \cong - \otimes \mathsf{T}\mathbb{G}(n)$. But the obvious isomorphism $\mathsf{T}\mathbb{I} \to \mathbb{I}$ induces an isomorphism $\mathsf{T}\mathbb{G}(n) \cong \mathbb{G}(n)$, so $-\otimes\mathsf{T}\mathbb{G}(n)$ is naturally isomorphic to $- \otimes \mathbb{G}(n)$. The result now follows from (ii). □

Notes. (i) It was proved in [7] that $\mathbb{G}(n)$ is the fundamental ω-groupoid $\rho(I^n_*)$ of the n-cube with its skeletal filtration. We will show in [10][2], by similar methods, that for any cubical set K, there

[2]This paper was submitted in a cubical setting, but was not welcomed by some referees ('not exciting'). So the final paper was simplicial, and this result did not appear in that version. This result will appear in a new exposition 'Nonabelian algebraic topology', by R. Brown, P.J. Higgins, and R. Sivera.

is a natural isomorphism $\rho(K) \cong \rho(|\mathbf{K}|)$, where $|\mathbf{K}|$ is the geometric realisation of K, with its skeletal filtration. Thus (2.5) gives an isomorphism

$$\rho(|\mathbf{K}| \otimes |\mathbf{L}|) \cong \rho(|\mathbf{K}|) \otimes \rho(|\mathbf{L}|)$$

which can be generalised to an isomorphism

$$\rho(X \otimes Y) \cong \rho(X) \otimes \rho(Y)$$

for arbitrary CW-complexes X, Y.

(ii) We will show in Section 4 that the tensor product of ω-groupoids is symmetric, although the isomorphism $G \otimes H \cong H \otimes G$ is not an obvious one.

3 Crossed complexes

A crossed complex C consists of a set C_0, a groupoid C_1 over C_0, a crossed module C_2 over C_1 and, for $n \geqslant 3$, modules C_n over C_1, with boundary maps $\delta : C_n \to C_{n-1}$ satisfying certain identities. The details are in [6], where it is proved that the category Crs of crossed complexes is equivalent to the category ω-Gpd. The equivalence γ :ω-$\mathsf{Gpd} \to \mathsf{Crs}$ is straightforward: if G is an ω-groupoid, then γG consists of those cubes x in G all of whose faces $\partial_i^\alpha x$ except $\partial_1^0 x$ are concentrated at a point $p \in G_0$. The inverse equivalence $\lambda : \mathsf{Crs} \to \omega$-$\mathsf{Gpd}$ was defined in [6] using a complicated folding operation Φ and the homotopy addition lemma. This definition of λ involves certain choices, so different conventions are possible; we use those adopted in [6]. (With hindsight one can show that $(\lambda C)_n = \mathsf{Crs}(\Pi \mathbb{I}^n, C)$, but there are difficulties in using this more canonical description as a definition.)

The monoidal closed structure defined on ω-Gpd in section 2 can clearly be transferred to the category Crs by the equivalences λ and γ. One simply defines $A \otimes B = \gamma(\lambda A \otimes \lambda B)$ and $\mathsf{CRS}(A, B) = \gamma(\omega\text{-}\mathsf{GPD}(\lambda A, \lambda B))$, for arbitrary crossed complexes A and B. The problem is then to describe these constructions internally in the category Crs. The difficulty in passing from presentations in ω-Gpd to presentations in Crs may be illustrated by the example $\mathbb{G}(n)$. In ω-Gpdthis is free on one generator in dimension n; however, the corresponding crossed complex $\gamma\mathbb{G}(n) \cong \pi(\mathbb{I}^n)$ requires, for each r-dimensional face d of I^n, a generator $x(d)$ in dimension r, with defining relations of the form

$$\delta(x(d)) = \sum_{(\alpha,i)} \{x(\partial_i^\alpha d)\},$$

where the formula for the 'sum of the faces' on the right is given by the Homotopy Addition Lemma ((7.1) of [6]).

The key to the translation is the notion of (left or right) m-fold homotopy for crossed complexes, analogous to the corresponding notion for ω-groupoids and related to it by means of the folding map Φ. Let B, C be crossed complexes and let $m \geqslant 1$. Then an *m-fold left homotopy* $B \to C$ is a pair (h, f), where $f : B \to C$ is a morphism of crossed complexes (the base morphism of the homotopy) and h is a map of degree m from B to C (i.e., $h : B_n \to C_{m+n}$ for each $n \geqslant 0$) satisfying

3.1 (i) $\beta h(b) = \beta f(b)$*for all* $b \in B$;

(ii) *if* $b, b' \in B_1$*and* $b + b'$*is defined, then*

$$h(b+b') = h(b)^{f(b')} + h(b');$$

(iii) *if* $b, b' \in B_n(n \geqslant 2)$*and* $b+b'$*is defined, then*

$$h(b+b') = h(b) + h(b');$$

(iv) *if* $b \in B_n(n \geqslant 2), b_1 \in B_1$*and* b^{b_1}*is defined, then*

$$h(b^{b_1}) = h(b)^{f(b_1)}.$$

Here, in any crossed complex C, βc denotes the base-point of c, that is, if $c \in C_0$ then $\beta c = c$, if $c \in C_1(p,q)$ or $c \in C_n(q)$ for $n \geqslant 2$, then $\beta c = q$. (In an ω-groupoid G, the base-point of a cube x of dimension n is $\beta x = \partial_1^1 \partial_2^1 \cdots \partial_n^1 x$.) Condition (3.1)(ii) (in combination with (i)) says that, on B_1, h is a derivation over f. Conditions (iii) and (iv) say that, on $B_n(n \geqslant 2), h$ is a morphism of modules over the morphism $f : B_1 \to C_1$ of groupoids. Thus, in each dimension, h and f preserve structure in the only reasonable way. (However, there is no requirement that h should be compatible with the boundary maps $\delta : B_n \to B_{n-1}$ and $\delta : C_n \to C_{n-1}$.)

There is a corresponding notion of m*-fold right homotopy* $B \to C$ in which (3.1)(ii) is replaced by

(ii)′ $h(b+b') = h(b') + h(b)^{f(b')}$,

that is, $h : B_1 \to C_{m+1}$ is an anti-derivation over f. Since C_n is abelian for $n \geqslant 3$, the m-fold left and right homotopies coincide except when $m = 1$.

Proposition 3.2 *Let* G, H *be* ω*-groupoids and let* $B = \gamma G, C = \gamma H$ *be the corresponding crossed complexes. Let* $\psi : G \to H$ *be an* m*-fold left homotopy and, for* $b \in B$*, define*

(i) $h(b) = \Phi\psi(b)$,

(ii) $f(b) = \partial_1^1 \partial_2^1 \cdots \partial_m^1 \psi(b)$.

Then (h, f) *is an* m*-fold left homotopy of crossed complexes* $B \to C$.

Moreover, if (h, f) *is any* m*-fold left homotopy from* B *to* C *there is a unique* m*-fold left homotopy* $\psi : G \to H$ *satisfying (i), (ii) for* $b \in B$ *and the extra condition*

(iii) $\partial_i^\alpha \psi(x) = \varepsilon_1^{m-1} \hat{f}(x)$ *for* $1 \leqslant i \leqslant m, (\alpha, i) \neq (0,1)$ *and all* $x \in G$,

where $\hat{f}$ *denotes the unique morphism of* ω*-groupoids* $G \to H$ *extending the morphism* $f : B \to C$ *of crossed complexes.*

Proof Since $\partial_1^1 \partial_2^1 \cdots \partial_m^1 \psi : G \to H$ is a morphism of ω-groupoids it maps B to C and induces a morphism $f : B \to C$ of crossed complexes. Also, for $b \in B_n$,

$$\begin{aligned} \beta h(b) &= \beta\Phi\psi(b) = \beta\psi(b) = \partial_1^1 \partial_2^1 \cdots \partial_{m+n}^1 \psi(b) \\ &= \partial_1^1 \cdots \partial_m^1 \psi(\partial_1^1 \cdots \partial_n^1 b) = f(\beta b). \end{aligned}$$

The other conditions for (h, f) to be a homotopy are consequences of the formulae for $\Phi(x +_i y)$ in (4.9) of [6]. Firstly, if $b + b'$ is defined in $B_1 = G_1$, then

$$\begin{aligned} h(b + b') &= \Phi\psi(b +_1 b') = \Phi(\psi(b) +_{m+1} \psi(b')) \\ &= (\Phi\psi(b))^u + \Phi\psi(b') = h(b)^u + h(b'), \end{aligned}$$

where $u = u_{m+1}\psi(b') = \partial_1^1 \cdots \partial_m^1 \psi(b') = f(b')$. (We recall that, in the notation of [6], $u_i x$, for a k-dimensional cube x, denotes the edge $\partial_1^1 \cdots \partial_{i-1}^1 \partial_{i+1}^1 \cdots \partial_k^1 x$.) Similarly, if $n \geqslant 2$ and $b + b'$ is defined in B_n, then

$$h(b + b') = h(b)^u + h(b'),$$

where

$$\begin{aligned} u = u_{m+n}\psi(b') &= \partial_1^1 \cdots \partial_m^1 \partial_{m+1}^1 \cdots \partial_{m+n-1}^1 \psi(b') \\ &= \partial_1^1 \cdots \partial_m^1 \psi(\partial_1^1 \cdots \partial_{n-1}^1 b'). \end{aligned}$$

But since $b' \in B_n = \gamma_n G$, the element $\partial_1^1 \cdots \partial_{n-1}^1 b'$ of B_1 is the identity element $\varepsilon_1 \beta b'$; so $u = f(\varepsilon_1 \beta b')$ is also an identity element and $h(b + b') = h(b) + h(b')$. Finally, if b' is defined, where $b \in B_n (n \geqslant 2)$ and $t \in B_1$, then

$$\begin{aligned} h(b') = \Phi\psi(b') &= \Phi\psi(-_n \varepsilon_1^{n-1} t +_n b +_n \varepsilon_1^{n-1} t) \\ &= (-_{m+n} \varepsilon_{m+1}^{n-1} \psi(t) +_{m+n} \psi(b) +_{m+n} \varepsilon_{m+1}^{n-1} \psi(t)) \\ &= -(\Phi \varepsilon_{m+1}^{n-1} \psi(t))^u + (\Phi\psi(b))^v + \Phi \varepsilon_{m+1}^{n-1} \psi(t) \end{aligned}$$

for certain edges $u, v \in C_1$. But $n \geqslant 2$, so $\varepsilon_{m+1}^{n-1} \psi(t)$ is degenerate and $\Phi \varepsilon_{m+1}^{n-1} \psi(t) = 0$ (see (4.12) of [6]). Hence $h(b^t) = h(b)^v$, where

$$\begin{aligned} v &= u_{m+n}(\varepsilon_{m+1}^{n-1} \psi(t)) \\ &= \partial_1^1 \cdots \partial_{m+n-1}^1 \varepsilon_{m+1}^{n-1} \psi(t) = \partial_1^1 \cdots \partial_m^1 \psi(t) = f(t). \end{aligned}$$

Suppose now that (h, f) is any m-fold left homotopy $B \to C$. If ψ is an m-fold homotopy $G \to H$ satisfying conditions (i),(ii) and (iii), then it must also satisfy, for $x \in G_n$,

$$\begin{aligned} h(\Phi x) &= \Phi\psi(\Phi x) = \Phi\psi(\Phi_1 \cdots \Phi_{n-1} x) \\ &= \Phi \Phi_{m+1} \cdots \Phi_{m+n-1} \psi(x) = \Phi\psi(x) \text{ by (4.10) of [6].} \end{aligned}$$

Thus in proving existence and uniqueness of ψ we may replace condition (i) by the stronger condition

$$h(\Phi x) = \Phi\psi(x) \text{ for all } x \in G.$$

We may also omit condition (ii) which is a consequence of (iii). So we look for $\psi : G \to H$ of degree m satisfying, for all $x \in G_n, n \geqslant 0$,

(iii) $\partial_i^\alpha \psi(x) = \varepsilon_1^{m-1}\hat{f}(x)$ for $i \leqslant m$, $(\alpha, i) \neq (0,1)$, and

(iv) $\Phi\psi(x) = h(\Phi x)$,

the right-hand sides being specified in advance, and we proceed by induction on n.

When $n = 0$, all faces but one of $\psi(x)$ are specified by (iii). The elements $z_i^\alpha = \varepsilon_1^{m-1}\hat{f}(x) = \varepsilon_1^{m-1}f(x)$ of H_{m-1} for $(\alpha, i) \neq (0,1)$ form a box and the Homotopy Addition Lemma ((7.1 of [6]) gives a unique last face z_1^0 such that $\delta\Phi\mathbf{z} = \Sigma\mathbf{z}$ has the value $\delta h(\Phi x) \in C_{m-1}$. Proposition 5.6 of [6] then gives a unique filler $\psi(x)$ for the box such that $\Phi(\psi(x))$ has the value $h(\Phi x)$. (Of course, one must verify that $\delta h(\Phi x) = \delta h(x)$ has the same basepoint as the given box, but this is clear since $\beta h(x) = \beta f(x)$.)

Now suppose that $n \geqslant 1$ and assume that $\psi(x)$ is already defined for all x of dimension $< n$ and that it satisfies (iii) and (iv) for all such x. Assume further that ψ satisfies all the conditions for an m-fold left homotopy in so far as they apply to elements of dimension $< n$. Then, for $x \in G_n$ we need to find $\psi(x) \in H_{m+n}$ satisfying (amongst others) the conditions

$$(\dagger) \qquad \begin{cases} \partial_j^\alpha \psi(x) = \varepsilon_1^{m-1}\hat{f}(x) & \text{for } 1 \leqslant j \leqslant m, (\alpha, j) \neq (0,1), \\ \partial_{m+j}^\alpha \psi(x) = \psi(\partial_j^\alpha x) & \text{for } 1 \leqslant j \leqslant n, \\ \Phi\psi(x) = h(\Phi x). \end{cases}$$

One verifies that the specified faces of $\psi(x)$ form a box whose basepoint is $\beta\hat{f}(x) = f(\Phi x) = h(\Phi x)$ and therefore, as in the case $n = 0$, there is a unique $\psi(x)$ satisfying these conditions. To complete the induction we need only verify that this $\psi(x)$ has all defining properties of an m-fold homotopy. For example, to prove that

$$\psi(x +_i y) = \psi(x) +_{m+i} \psi(y),$$

we first note that $\partial_{m+i}^1 \psi(x) = \psi(\partial_i^1 x) = \psi(\partial_i^0 y) = \partial_{m+i}^0 \psi(y)$ so that $z = \psi(x) +_{m+i} \psi(y)$ is defined. We then verify easily, using the induction hypotheses, that the faces of z other than $\partial_1^0 z$ are given by

$$\begin{cases} \partial_j^\alpha z = \varepsilon_1^{m-1}\hat{f}(x +_i y) & \text{for } 1 \leqslant j \leqslant m, \\ (\alpha, j) \neq (0,1), & \\ \partial_{m+j}^\alpha z = \psi(\partial_j^\alpha(x +_i y)) & \text{for } 1 \leqslant j \leqslant n. \end{cases}$$

Also

$$\begin{aligned} \Phi z &= \Phi(\psi(x) +_{m+i} \psi(y)) \\ &= (\Phi\psi(x))^u + \Phi\psi(y), \text{ by (4.9) of [6]} \end{aligned}$$

where $u = u_{m+i}\psi(y) = \partial_1^1 \cdots \partial_m^1 \psi(u_i y) = \hat{f}(u_i y) = f(u_i y)$. But it may be verified that

$$h(\Phi(x +_i y)) = h(\Phi x)^{f(u_i y)} + h(\Phi y)$$

using the defining properties of h and formulae (4.9) of [6]. (In the case $n = 1, i = 1$ one needs to observe that addition in C_{m+n} is commutative.) Hence

$$\Phi z = h(\Phi(x +_i y))$$

in all cases. It follows, by the uniqueness of $\psi(x)$ satisfying conditions (†), that $z = (x +_i y)$. The other properties of ψ are proved in a similar way. $\square$

This proposition sets up a bijection between m-fold left homotopies $B \to C$ and certain special m-fold left homotopies $\psi : G \to H$, namely those satisfying

$$\partial_i^\alpha \psi(x) = \varepsilon_1^{m-1} \partial_1^1 \partial_2^1 \cdots \partial_m^1 \psi(x) \text{ for } i \leqslant m, (\alpha, i) \neq (0, 1).$$

(Note that if $\partial_i^\alpha u = \varepsilon_1^{m-1} v$, then v must be $\partial_1^1 \cdots \partial_m^1 u$.) These are precisely the elements of $\gamma(\omega\text{-}\mathsf{GPD}(G, H)) = \mathsf{CRS}(B, C)$ in dimension m, where $m \geqslant 1$. We complete this correspondence by defining a 0-fold left (or right) homotopy of crossed complexes $B \to C$ to be a morphism $f : B \to C$. We then have:

Proposition 3.3 *The elements of* $\mathsf{CRS}(B, C)$ *in dimension* $m \geqslant 0$ *are in natural one-one correspondence with the* m*-fold left homotopies from* B *to* C. $\square$

In view of this result we will, from now on, identify $\mathsf{CRS}(B, C)$ with the collection of morphisms and left homotopies from B to C. The operations which give this collection the structure of a crossed complex can be deduced from the correspondence in Proposition 3.2. They will be described later, but we note here that the basepoint of the homotopy (h, f) is the morphism $f : B \to C$.

We now introduce the analogue for crossed complexes of bimorphisms $(F, G) \to H$ of ω-groupoids. A *bimorphism* $\theta : (A, B) \to C$ of crossed complexes is a family of maps $\theta : A_m \times B_n \to C_{m+n}$ satisfying the following conditions, where $a \in A_m, b \in B_n$:

3.4 (i) $\beta(\theta((a, b)) = \theta(\beta a, \beta b))$ *for all* $a \in A, b \in B$.

(ii) $\theta(a, b^{b_1}) = \theta(a, b)^{\theta(\beta a, b_1)}$ *if* $m \geqslant 0, n \geqslant 2$.

(ii)' $\theta(a^{a_1}, b) = \theta(a, b)^{\theta(a_1, \beta b)}$ *if* $m \geqslant 2, n \geqslant 0$.

(iii)

$$\theta(a, b + b') = \begin{cases} \theta(a, b) + \theta(a, b') & \text{if } m = 0, n \geqslant 1 \text{ or } m \geqslant 1, n \geqslant 2, \\ \theta(a, b)^{\theta(\beta a, b')} + \theta(a, b') & \text{if } m \geqslant 1, n = 1. \end{cases}$$

(iii)'

$$\theta(a + a', b) = \begin{cases} \theta(a, b) + \theta(a', b) & \text{if } m \geqslant 1, n = 0 \text{ or } m \geqslant 2, n \geqslant 1, \\ \theta(a', b) + \theta(a, b)^{\theta(a', \beta b)} & \text{if } m = 1, n \geqslant 1. \end{cases}$$

(iv)

$$\delta(\theta(a, b)) = \begin{cases} \theta(\delta a, b) + (-)^m \theta(a, \delta b) & \text{if } m \geqslant 2, n \geqslant 2, \\ -\theta(a, \delta b) - \theta(\delta^1 a, b) + \theta(\delta^0 a, b)^{\theta(a, \beta b)} & \text{if } m = 1, n \geqslant 2, \\ (-)^{m+1} \theta(a, \delta^1 b) + (-)^m \theta(a, \delta^0 b)^{\theta(\beta a, b)} + \theta(\delta a, b) & \text{if } m \geqslant 2, n = 1, \\ -\theta(\delta^1 a, b) - \theta(a, \delta^0 b) + \theta(\delta^0 a, b) + \theta(a, \delta^1 b) & \text{if } m = n = 1. \end{cases}$$

(v)

$$\delta(\theta(a,b)) = \begin{cases} \theta(a,\delta b) & \textit{if } m=0, n \geqslant 2, \\ \theta(\delta a, b) & \textit{if } m \geqslant 2, n=0. \end{cases}$$

$$\delta^\alpha(\theta(a,b)) = \begin{cases} \theta(a,\delta^\alpha b) & \textit{if } m=0, n=1(\alpha=0,1), \\ \theta(\delta^\alpha a, b) & \textit{if } m=1, n=0(\alpha=0,1). \end{cases}$$

These equations have been displayed for future reference; they can be summed up as follows:

3.5 (i) *For each* $a \in A_m (m \geqslant 1)$ *the maps*

$$\begin{cases} h_a : b \mapsto \theta(a,b), \\ f_a : b \mapsto \theta(\beta a, b) \end{cases}$$

form an m*-fold left homotopy* $: B \to C$*.(If* $m = 0$*, then* $h_a = f_a$ *is a morphism of crossed complexes* $: B \to C$.)

(ii) *For each* $b \in B_n$ *the maps*

$$\begin{cases} h'_b : a \mapsto \theta(a,b), \\ f'_b : a \mapsto \theta(a, \beta b) \end{cases}$$

form an n*-fold right homotopy* $: A \to C$*.(If* $n = 0$*, then* $h'_b = f'_b$ *is a morphism* $: A \to C$.)

(iii) $\delta(\theta(a,b))$ *is given by equations (3.4) (iv) for* $m \geqslant 1, n \geqslant 1$.

(Note that (3.4)(v) follows from (3.5)(i) and (ii)).

Proposition 3.6 *Let* F, G, H *be* ω*-groupoids with corresponding crossed complexes* $A = \gamma F, B = \gamma G$*,* $C = \gamma H$*. If* $\chi : (F,G) \to H$ *is any bimorphism of* ω*-groupoids, then* $\theta : (A,B) \to C$ *defined by*

(*) $$\theta : (a,b) = \Phi\chi(a,b) \textit{ for } a \in A, b \in B,$$

is a bimorphism of crossed complexes. Conversely, given any bimorphism $\theta : (A,B) \to C$ *of crossed complexes, there is a unique bimorphism* $\chi : (F,G) \to H$ *of* ω*-groupoids satisfying (*).*

Proof First let χ be given. Then

$$\beta\theta(a,b) = \beta\Phi\chi(a,b) = \beta\chi(a,b) = \chi(\beta a, \beta b) = \theta(\beta a, \beta b).$$

If $a \in A_0$ is fixed, then $x \mapsto \chi(a,x)$ is a morphism of ω-groupoids $G \to H$, so $b \mapsto \Phi\chi(a,b)$ is a morphism of crossed complexes $B \to C$. Similarly, by Proposition 3.2, if $a \in A_m$ is fixed, then $x \mapsto \chi(a,x)$ is an m-fold left homotopy of ω-groupoids, so $b \mapsto \Phi\chi(a,b)$ is an m-fold left homotopy $B \to C$ over the morphism $b \mapsto \partial_1^1 \cdots \partial_m^1 \chi(a,b) = \chi(\beta a, b)$. This morphism maps B into $C = \gamma H$, so $\Phi\chi(\beta a, b) = \chi(\beta a, b)$ and the morphism can also be written $b \mapsto \theta(\beta a, b)$. This proves (3.5)(i).

Condition (3.5)(ii) follows in the same way from the right-homotopy version of Proposition 3.2. (Note that in this version for n-fold right homotopies $A \to C$ the formula $f(b) = \partial_1^1 \cdots \partial_m^1 \psi(b)$ is replaced by $f(a) = \partial_{n+1}^1 \cdots \partial_{n+m}^1 \psi(a)$. Hence, if $b \in B_n$, the right homotopy $a \mapsto \Phi\chi(a,b) : A \to C$ has base morphism $a \mapsto \partial_{n+1}^1 \cdots \partial_{n+m}^1 \chi(a,b) = \chi(a, \beta b)$.)

To prove (3.4)(iv) we use the Homotopy Addition Lemma; in order to compute $\delta\theta(a,b) = \delta\Phi\chi(a,b)$ we need to compute $\Phi\partial_i^\alpha \chi(a,b)$ for each face of $\chi(a,b)$ and sum them according to the formulae in (7.1) of [6], which we will refer to as HAL.

Lemma 3.7 *If $\chi : (F,G) \to H$ is a bimorphism of ω-groupoids, then $\chi(x,y)$ is thin whenever x or y is thin.*

Proof For any fixed $y \in G_n$, the map $x \mapsto \chi(x,y)$ is a morphism of ω-groupoids from F to $P^n H$. If x is thin in F, it follows that $\chi(x,y)$ is a thin element of $P^n H$. But the thin elements of $P^n H$ are a subset of the thin elements of H. $\Box$

To compute $\delta\theta(a,b)$ in the general case $m \geqslant 2, n \geqslant 2$ we note that the faces of a and b other than $\partial_1^0 a, \partial_1^0 b$ are all thin, so all but two faces of $\chi(a,b)$ are thin by Lemma 3.7, and we conclude that $\Phi\partial_i^\alpha \chi(a,b) = 0$ except when $\alpha = 0$ and $i = 1$ or $m+1$. The appropriate formula of HAL now gives

$$\begin{aligned}\delta\theta(a,b) = \delta\Phi\chi(a,b) &= (\Phi\chi(\partial_1^0 a, b))^v + (-)^m (\Phi\chi(a, \partial_1^0 b))^w \\ &= \theta(\delta a, b)^v + (-)^m \theta(a, \delta b)^w,\end{aligned}$$

where $v = u_1\chi(a,b) = \chi(u_1 a, \beta b)$ and $w = u_{m+1}\chi(a,b) = \chi(\beta a, u_1 b)$. Since $a \in A, b \in B$, both $u_1 a$ and $u_1 b$ are identities, so v, w act trivially and we obtain the formula

$$\delta\theta(a,b) = \theta(\delta a, b) + (-)^m \theta(a, \delta b).$$

The other formulae of (3.4)(iv) are proved in the same way using the different forms of HAL in various cases. Thus θ is a bimorphism of crossed complexes.

Now suppose that $theta$ is given; we wish to reconstruct χ. For each $a \in A_m$ we have an m-fold left homotopy $(h_a, f_a) : B \to C$. By Proposition 3.2, there is a unique m-fold left homotopy $\psi_a : G \to H$ satisfying the conditions

3.8

$$\begin{cases} \Phi\psi_a(b) = h_a(b) = \theta(a,b) & \textit{for } b \in B, \\ \psi_a \in \gamma(\omega\text{-GPD}(G,H)) = D, & \textit{say.} \end{cases}$$

The required bimorphism χ must yield such an n-fold left homotopy $y \mapsto \chi(a,y)$, so the definition $\chi(a,y) = \psi_a(y)$ is forced. Furthermore, since A generates F as ω-groupoid (see (5.8) of [6]) and $\chi(x,y)$ must preserve ω-groupoid operations on the first variable x, for fixed y, the values $\chi(a,y)$ for $a \in A, y \in G$ determine χ completely. Thus χ is unique if it exists.

To prove that the required bimorphism χ exists we first note that we have a map $a \mapsto \psi_a$ from A to D of degree 0 and we will show that it is a morphism of crossed complexes. The crossed complex structure of D is defined, as in §3 of [6], by the ω-groupoid structure of ω-GPD(G,H) which in turn comes from the ω-groupoid structure of H. The operations of D are therefore as follows. If $\xi, \eta \in D_m$ and $\tau \in D_1$ are such that $\xi + \eta$ and ξ^τ are defined, then, for any $y \in G$,

$$\begin{aligned}(\xi+\eta)(y) &= \xi(y) +_m \eta(y),\\ \xi^\tau(y) &= -_m \varepsilon_1^{m-1}\tau(y) +_m \xi(y) +_m \varepsilon_1^{m-1}\tau(y),\\ (\delta\xi)(y) &= \partial_1^0(\xi(y)) \quad (m \geqslant 2),\\ (\delta^\alpha\xi)(y) &= \partial_1^\alpha(\xi(y)) \quad (m = 1).\end{aligned}$$

We need to show that $\psi_{a+a'} = \psi_a + \psi_{a'}, \psi_{a^t} = \psi_a^{\psi_t}, \psi_{\delta a} = \delta\psi_a$ if $a \in A_m (m \geqslant 2)$, and $\psi_{\delta^\alpha a} = \delta^\alpha\psi_a$ if $a \in A_1$. Using the characterisation (3.8) of ψ_a and the fact that $\psi_a + \psi_{a'}, \psi_a^{\psi_t}$ etc. are all elements of D, it is enough to prove that, for $b \in B$,

3.9 (i) $\Phi(\psi_a(b) +_m \psi_{a'}(b)) = \theta(a + a', b)$ *if* $a + a'$ *is defined in* A_m,

(ii) $\Phi(-_m\varepsilon_1^{m-1}\psi_t(b) +_m \psi_a(b) +_m \varepsilon_1^{m-1}\psi_t(b)) = \theta(a^t, b)$ *if* $t \in A_1$ *and* a^t *is defined in* $A_m (m \geqslant 2)$,

(iii) $\Phi(\partial_1^0\psi_a(b)) = \theta(\delta a, b)$ *if* $a \in A_m, m \geqslant 2$,

(iv) $\Phi(\partial_1^\alpha\psi_a(b)) = \theta(\delta^\alpha a, b)$ *if* $a \in A_1, \alpha = 0, 1$.

The calculations for (i) and (ii) are similar to calculations done in the proof of Proposition 3.2. For example, in (3.9)(ii), if $a \in A_m, b \in B_n$, then $\Phi(\varepsilon_1^{m-1}\psi_t(b)) = 0$, so

$$\Phi(-_m\varepsilon_1^{m-1}\psi_t(b) +_m \psi_a(b) +_m \varepsilon_1^{m-1}\psi_t(b) = (\Phi\psi_a(b))^\nu = \theta(a,b)^\nu$$

where

$$\begin{aligned}\nu = u_m\varepsilon_1^{m-1}\psi_t(b) &= \partial_1^1 \cdots \partial_{m-1}^1\partial_{m+1}^1 \cdots \partial_{m+n}^1\varepsilon_1^{m-1}\psi_t(b)\\ &= \partial_2^1 \cdots \partial_{n+1}^1\psi_t(b) = \psi_t(\partial_1^1 \cdots \partial_n^1 b)\\ &= \psi_t(\beta b) = \theta(t, \beta b) \quad \text{(since } \Phi = \text{id in dimension 1).}\end{aligned}$$

Hence $\theta(a,b)^\nu = \theta(a,b)^{\theta(t,\beta b)} = \theta(a^t, b)$ since $a \mapsto \theta(a,b)$ is an n-fold right homotopy with base morphism $a \mapsto \theta(a,b)$. The calculations for (3.9)(iii) and (iv) use HAL and the extra defining property (3.4)(iv) for θ. For example, to prove (3.9)(iii) we observe that $\Phi\psi_a(b) = \theta(a,b)$ and $\delta\Phi\psi_a(b) = \Sigma\{\Phi\partial_i^\alpha\psi_a(b)\}$, the sum of the folded faces on the right being calculated by the appropriate formula of HAL, depending on the dimensions of a and b. Now $a \in A$ and $b \in B$ so, as in the proof of (3.4)(iv), most terms in this sum are 0. In the case $m \geqslant 2, n \geqslant 2$, two terms survive and one of these, $\Phi\partial_{m+1}^0\psi_a(b)$, we can calculate: because ψ_a is an m-fold left homotopy of ω-groupoids, $\Phi\partial_{m+1}^0\psi_a(b) = \Phi\psi_a(\partial_1^0 b) = \theta(a, \delta b)$. Hence HAL says

$$\delta\theta(a,b) = \Phi\partial_1^0\psi_a(b) + (-)^m\theta(a, \delta b).$$

Comparing this with the defining property

$$\delta\theta(a,b) = \theta(\delta a, b) + (-)^m\theta(a, \delta b)$$

we obtain (3.9)(iii). The other cases are similar.

This proves that $a \mapsto \psi_a$ is a morphism of crossed complexes from $A = \gamma F$ to $D = \gamma(\omega\text{-}\mathsf{GPD}(G, H))$. It therefore extends uniquely to a morphism of ω-groupoids $x \mapsto \psi_x$, say, from F to $\omega\text{-}\mathsf{GPD}(G, H)$. But now the definition $\chi(x, y) = \psi_x(y)$ gives a bimorphism of ω-groupoids $(F, G) \to H$ such that $\Phi\chi(a, b) = \Phi\psi_a(b) = \theta(a, b)$ for $a \in A, b \in B$, and this completes the proof of Proposition 3.6. □

It is now easy to describe tensor products of crossed complexes. Taking $A = \gamma F, B = \gamma G, C = \gamma H$ as above, we have $A \otimes B = \gamma(F \otimes G)$ by definition. Any morphism of crossed complexes $\eta : A \otimes B \to C$ extends uniquely to a morphism of ω-groupoids $\hat\eta : F \otimes G \to H$, giving a bimorphism $\chi : (F, G) \to H$ defined by $\chi(x, y) = \hat\eta(x \otimes y)$. This induces a bimorphism of crossed complexes $\theta : (A, B) \to C$ by $\theta(a, b) = \Phi\chi(a, b) = \Phi\hat\eta(a\hat\otimes b)$ where $a\hat\otimes b$ denotes the tensor product in $F \otimes G$. We write $a \otimes b$ for $\Phi(a\hat\otimes b) \in A \otimes B$ and deduce that $\theta(a, b) = \hat\eta(\Phi(a\hat\otimes b)) = \eta(a \otimes b)$. This correspondence is one-one by (3.5), so $A \otimes B$ is the universal object in Crs for bimorphisms from (A, B) to arbitrary crossed complexes. The definition (3.4) of a bimorphism now gives the following presentation of $A \otimes B$:

Proposition 3.10 *Let* A, B *be crossed complexes. Then* $A \otimes B$ *is the crossed complex generated by elements* $a \otimes b$ *in dimension* $m + n$*, where* $a \in A_m, b \in B_n$*, with the following defining relations (plus, of course, the laws for crossed complexes):*

3.11 (i) $\beta(a \otimes b) = \beta a \otimes \beta b$.

(ii) $a \otimes b^t = (a \otimes b)^{\beta a \otimes t}$ *if* $m \geqslant 0, n \geqslant 2, t \in B_1$.

(ii)′ $a^s \otimes b = (a \otimes b)^{s \otimes \beta b}$ *if* $m \geqslant 2, n \geqslant 0, s \in A_1$.

(iii) *If* $b + b'$ *is defined in* B_n*, then*

$$a \otimes (b + b') = \begin{cases} a \otimes b + a \otimes b', & \textit{if } m = 0, n \geqslant 1 \textit{ or if } m \geqslant 1, n \geqslant 2, \\ (a \otimes b)^{\beta a \otimes b'} + a \otimes b', & \textit{if } m \geqslant 1, n = 1. \end{cases}$$

(iii)′ *If* $a + a'$ *is defined in* A_m*, then*

$$(a + a') \otimes b = \begin{cases} a \otimes b + a' \otimes b, & \textit{if } m \geqslant 1, n = 0 \textit{ or if } m \geqslant 2, n \geqslant 1, \\ a' \otimes b + (a \otimes b)^{a' \otimes \beta b}, & \textit{if } m = 1, n \geqslant 1. \end{cases}$$

(The reversal of addition is significant only when $m = n = 1$*.)*

(iv)

$$\delta(a \otimes b) = \begin{cases} \delta a \otimes b + (-)^m (a \otimes \delta b) & \textit{if } m \geqslant 2, n \geqslant 2, \\ -(a \otimes \delta b) - (\delta^1 a \otimes b) + (\delta^0 a \otimes b)^{a \otimes \beta b} & \textit{if } m = 1, n \geqslant 2, \\ (-)^{m+1}(a \otimes \delta^1 b) + (-)^m (a \otimes \delta^0 b)^{\beta a \otimes b} + \delta a \otimes b & \textit{if } m \geqslant 2, n = 1, \\ -\delta^1 a \otimes b - a \otimes \delta^0 b + \delta^0 a \otimes b + a \otimes \delta^1 b & \textit{if } m = 1, n = 1, \\ a \otimes \delta b & \textit{if } m = 0, n \geqslant 2, \\ \delta a \otimes b & \textit{if } m \geqslant 2, n = 0. \end{cases}$$

$$\delta^\alpha(a \otimes b) = \begin{cases} a \otimes \delta^\alpha b & \text{if } m = 0, n = 1, \\ \delta^\alpha a \otimes b & \text{if } m = 1, n = 0. \end{cases}$$

□

Proposition 3.12 *There is a natural bijection between*

(i) *morphisms of crossed complexes* $\xi : A \to \mathsf{CRS}(B, S)$*,*

(ii) *morphisms of crossed complexes* $\hat{\xi} : A \otimes B \to C$*, and*

(iii) *bimorphisms of crossed complexes* $\theta : (A, B) \to C$*, given by*

$$\xi(a)(b) = \hat{\xi}(a \otimes b) = \theta(a, b).$$

□

We now return to $\mathsf{CRS}(B, C)$ and give a description of its crossed complex structure in terms of the crossed complex structures of B and C. We write $\mathbb{C}(m)$ for the crossed complex freely generated by one generator a in dimension m. Any given element of $\mathsf{CRS}_m(B, C)$ induces a morphism $\mathbb{C}(m) \to \mathsf{CRS}(B, C)$, which is equivalent to a bimorphism $\theta : (\mathbb{C}(m), B) \to C$. If $m = 0$ the given element is the morphism

$$\psi_a : B \to C \text{ defined by } \psi_a(b) = \theta(a, b).$$

If $m \geqslant 1$ it is the homotopy $\psi_a = (h_a, f_a)$ defined by

$$h_a(b) = \theta(a, b), \quad f_a(b) = \theta(\beta a, b).$$

Similarly, if two elements of $\mathsf{CRS}(B, C)$ are given, we may choose A to be the free crossed complex on two generators of appropriate dimensions and represent both the given elements as induced by the same bimorphism $\theta : (A, B) \to C$ for suitable fixed values of the first variable. We have seen that the map $a \mapsto \psi_a$ from A to $\mathsf{CRS}(B, C)$ given in this way by θ is a morphism of crossed complexes, so we can now read off the crossed complex operations on $\mathsf{CRS}(B, C)$ from the bimorphism laws (3.4) for θ. For example, given $(h, f) \in \mathsf{CRS}_m(B, C)(m \geqslant 2)$ we determine $\delta(h, f)$ as follows. Write $(h, f) = (h_a, f_a)$ for suitable $a \in A$ as above, where $h_a(b) = \theta(a, b), f_a(b) = \theta(\beta a, b)$. Then $\delta(h, f) = (h_{\delta a}, f_{\delta a})$. We note that $f_{\delta a} = f$ since $\delta\beta a = \beta a$. We write δh for $h_{\delta a}$, so that $\delta(h, f) = (\delta h, f)$. Now $(\delta h)(b) = \theta(\delta a, b)$ is given by formula (3.4)(iv) in terms of known elements, namely (assuming $m \geqslant 2$)

$$\theta(\delta a, b) = \begin{cases} \delta(\theta(a, b)) + (-)^{m+1}\theta(a, \delta b) & \text{if } b \in B_n (n \geqslant 2), \\ (-)^{m+1}\theta(a, \delta^0 b)^{\theta(\beta a, b)} + (-)^m \theta(a, \delta^1 b) + \delta(\theta(a, b)) & \text{if } b \in B_1, \\ \delta(\theta(a, b)) & \text{if } b \in B_0. \end{cases}$$

In other words

3.13

$$(\delta h)(b) = \begin{cases} \delta(h(b)) + (-)^{m+1} h(\delta b) & \textit{if } b \in B_n (n \geqslant 2), \\ (-)^{m+1} h(\delta^0 b)^{f(b)} + (-)^m h(\delta^1 b) + \delta(h(b)) & \textit{if } b \in B_1, \\ \delta(h(b)) & \textit{if } b \in B_0. \end{cases}$$

This automatic procedure gives

Proposition 3.14 *The crossed complex structure of* $\mathsf{CRS}(B, C)$ *is defined as follows:*

Dimension 1. *If* (h, f) *is a* 1*-fold left homotopy* $B \to C$*, then* $\delta^1(h, f) = \beta(h, f) = f$ *and* $\delta^0(h, f) = f^0$*, where*

$$f^0(b) = \begin{cases} [f(b) + h(\delta b) + \delta(hb)]^{-h(\beta b)} & \text{if } b \in B_n (n \geqslant 2), \\ h(\delta^0 b) + f(b) + \delta(hb) - h(\delta^1 b) & \text{if } b \in B_1, \\ \delta^0(hb) & \text{if } b \in B_0. \end{cases}$$

If (k, g) *is another* 1*-fold left homotopy with* $\delta^0(k, g) = \delta^1(h, f) = f$*, then*

$$(h, f) + (k, g) = (h + k, g)$$

where

$$(h + k)(b) = \begin{cases} k(b) + h(b)^{k(\beta b)} & \text{if } b \in B_n, n \geqslant 1, \\ h(b) + k(b) & \text{if } b \in B_0. \end{cases}$$

Dimension $\geqslant 2$. *If* $(h, f), (k, f)$ *are* m*-fold homotopies* $B \to C$ *over the same base morphism* f*, where* $m \geqslant 2$*, and if* (h_1, f_1) *is a* 1*-fold left homotopy such that* $\delta^0(h_1, f_1) = f$*, then*

(i) $\delta(h, f) = (\delta h, f)$ *where* δh *is given by (3.13).*

(ii) $(h, f) + (k, f) = (h + k, f)$ *where* $(h + k)(b) = h(b) + k(b)$ *for all* $b \in B$.

(iii) $(h, f)^{(h_1, f_1)} = (h^{h_1}, f_1)$ *where* $h^{h_1}(b) = h(b)^{h_1(\beta b)}$ *for all* $b \in B$. □

To end this section we summarise the basic properties of $\otimes$ and CRS in the category Crs.

Theorem 3.15 *(i) The functor* $- \otimes B$ *is left adjoint to the functor* $\mathsf{CRS}(B, -)$ *from* Crs *to* Crs.

(ii) For crossed complexes A, B, C*, there are natural isomorphisms of crossed complexes*

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C),$$

$$\mathsf{CRS}(A \otimes B, C) \cong \mathsf{CRS}(A, \mathsf{CRS}(B, C))$$

giving Crs *the structure of a monoidal closed category.* □

For any cubical set K we define the *fundamental crossed complex* of K to be $\pi(K) = \gamma\rho(K)$. Propositions 2.5 and 2.6 then give immediately

Theorem 3.16 *If* K, L *are cubical sets, there is a natural isomorphism of crossed complexes*

$$\pi(K \otimes L) \cong \pi(K) \otimes \pi(L).$$

In particular

$$\pi(\mathbb{I}^m) \otimes \pi(\mathbb{I}^n) \cong \pi(\mathbb{I}^{m+n}).$$

□

For any crossed complex A we define the *cubical nerve* of A to be $NA = U\lambda A$, which is a cubical set. Since ρ is left adjoint to $U, \pi = \gamma\rho$ is left adjoint to $N = U\lambda$, but we now prove a stronger result. We observe that, for any ω-groupoid G and any cubical set $L, \mathsf{CUB}(L, UG)$ has a canonical ω-groupoid structure induced by the structure of G (see Proposition 2.4). In particular $\mathsf{CUB}(L, NA)$ is an ω-groupoid and Proposition 2.4 gives

Theorem 3.17 *For any cubical set L and any crossed complex A, there are natural isomorphisms of crossed complexes*

$$\mathsf{CRS}(\Pi L, A) \cong \gamma(\omega\text{-}\mathsf{GPD}(\rho L, \lambda A)) \cong \gamma(\mathsf{CUB}(L, NA)).$$

□

By taking cubical nerves and connected components we obtain

Corollary 3.18 *Let L be a cubical set and A be a crossed complex.*

(i) *There is a natural isomorphism of cubical sets*

$$\mathsf{CUB}(L, NA) \cong N(\mathsf{CRS}(\Pi L, A)).$$

(ii) *There is a natural bijection*

$$[L, NA] \cong [\Pi L, A],$$

where $[-,-]$ denotes the set of homotopy classes of morphisms in Cub *or in* Crs*, as the case may be.*

□

4 The symmetry of tensor products

We have seen that in the category Cub, the map $x \otimes y \mapsto y \otimes x$ does not give an isomorphism $X \otimes Y \to Y \otimes X$; indeed it is easy to construct examples of cubical sets X, Y such that $X \otimes Y$ and $Y \otimes X$ are not isomorphic. However, in ω-Gpdand Crs the situation is different. Although the map $x \otimes y \mapsto y \otimes x$ still does not give an isomorphism $X \otimes Y \to Y \otimes X$, there is a less obvious map which does. This is easiest to see in Crs.

Theorem 4.1 *Let A, B be crossed complexes. Then there is a natural isomorphism $A \otimes B \to B \otimes A$ which, for $a \in A_m, b \in B_n$, sends the generator $a \otimes b$ to $(-)^{mn} b \otimes a$. This isomorphism, combined with the structure given in Proposition 3.14, makes the category of crossed complexes a symmetric monoidal closed category.*

Proof One merely checks that the defining relations (3.11)(i)-(iv) satisfied by the generators $a \otimes b$ are preserved by the map $a \otimes b \mapsto (-)^{mn} b \otimes a$. The necessary coherence and naturality conditions are obviously satisfied. □

This proof is unsatisfactory because, although it is clear that $a \otimes b \mapsto b \otimes a$ does not preserve the relations (3.11), the fact that $a \otimes b \mapsto (-)^{mn} b \otimes a$ does preserve them seems like a happy accident. A better explanation is provided by the transposing functor T (see Sections 1 and 2).

For a cubical set K, TK is not in general isomorphic to K. But for any ω-groupoid G and any crossed complex B we will construct isomorphisms $G \to TG$ and $B \to TB$. Since in all these categories we have obvious natural isomorphisms $T(X \otimes Y) \cong TY \otimes TX$, this implies the symmetry $X \otimes Y \cong Y \otimes X$.

For an ω-groupoid G, TG has the same elements as G but has all its operations $\partial_i^\alpha, \varepsilon_i, \Gamma_i, +_i, -_i$ numbered in reverse order with respect to i (but not with respect to $\alpha = 0, 1$). For a crossed complex B, TB is defined, of course, as $\gamma(T\lambda B)$. The calculation expressing this crossed complex in terms of the crossed complex structure of B is straightforward (though it needs a clear head).

Proposition 4.2 *The crossed complex* TB *is defined, up to natural isomorphism, in the following way:*

(i) $(TB)_0 = B_0$ *as a set;*

(ii) $(TB)_2 = B_2^{op}$ *as a groupoid;*

(iii) $(TB)_n = B_n$ *as a groupoid for* $n = 1$ *and* $n \geqslant 3$*;*

(iv) *the action of* $(TB)_1$ *on* $(TB)_n (n \geqslant 2)$ *is the same as the action of* B_1 *on* B_n*;*

(v) *the boundary map* $T\delta : (TB)_{n+1} \to (TB)_n$ *is given by*

$$T\delta = (-)^n \delta : B_{n+1} \to B_n.$$

□

(We note that $-\delta : B_2 \to B_1$ is an anti-homomorphism, that is a homomorphism $B_2^{op} \to B_1$, as required; the map $+\delta : B_3 \to B_2^{op}$ is also a homomorphism because the image is in the centre of B_2. In higher dimensions the groupoids B_n and B_n^{op} are the same.)

Corollary 4.3 *For any crossed complex* B *there is a natural isomorphism* $\tau : B \to TB$ *given by*

$$\tau(b) = (-)^{[n/2]} b \text{ for } b \in B_n.$$

□

The somewhat surprising sign $(-)^{[n/2]}$ is forced by the signs in (4.2); it is less surprising when one notices that it is the signature of the permutation which reverses the order of $(1, 2, \dots, n)$. The symmetry map of Theorem 4.1 now comes from the map

$$a \otimes b \to \tau^{-1}(\tau b \otimes \tau a) = (-)^k b \otimes a,$$

where $k = [m/2] + [n/2] - [(m+n)/2]$, which is 0 if m or n is even, and 1 if both are odd.

The isomorphism $\tau : B \to TB$ extends uniquely to an isomorphism $\tau : G \to TG$, where $G = \lambda B, B = \gamma G$. This isomorphism can be viewed as a 'reversing automorphism' $x \mapsto x^*$ of G, that is, a map of degree 0 from G to itself which preserves the operations while reversing their order (e.g. $(x +_i y)^* = x^* +_{n-i+1} y^*$ in dimension n). The isomorphism $G \otimes H \to H \otimes G$ for ω-groupoids is then given by

$$x \otimes y \mapsto (y^* \otimes x^*)^*.$$

Note. The element x^* should be viewed as a transpose of the cube x, and in the geometric case $G = \rho(X)$, it is induced from x by the map $(t_1, \dots, t_n) \mapsto (t_n, \dots, t_1)$ of the unit n-cube. The operation * is preserved by morphisms of ω-groupoids, because of the naturalness of $\tau : 1toT$. It follows that the operation * can be written in terms of the ω-groupoid operations $\partial_i^\alpha, \varepsilon_i, \Gamma_i, +_i, -_i$, but the formulae needed for this are rather complicated.

5 The pointed case

We consider briefly the notions of tensor product and homotopy in the categories ω-Gpd_* and Crs_* of pointed ω-groupoids and pointed crossed complexes. Here the objects have a distinguished element $*$ in dimension 0 and only morphisms preserving this basepoint are included.

For any ω-groupoid H with basepoint $*$, the ω-groupoid $P^m H$ has basepoint $0_* = \varepsilon_1^m(*)$, the constant cube at $*$. An m*-fold pointed (left) homotopy* $h : G \to H$ is a morphism $h : G \to P^m H$ preserving basepoints, that is, a homotopy h with $h(*) = 0_*$. Clearly, all such pointed homotopies form an ω-subgroupoid ω-$\mathsf{GPD}_*(G, H)$ of ω-$\mathsf{GPD}(G, H)$ since $0_* = \varepsilon_1^m(*)$ is an identity for all the compositions $+_i (1 \leqslant i \leqslant m)$. This ω-subgroupoid has as basepoint the trivial morphism $G \to H$ which sends each element of dimension n to $0_* = \varepsilon_1^n(*)$. Thus we have an internal hom functor ω-$\mathsf{GPD}_*(G, H)$ in the pointed category ω-Gpd_*. The pointed morphisms from F to ω-$\mathsf{GPD}_*(G, H)$ are in one-one correspondence with the *pointed bimorphisms* $\chi : (F, G) \to H$, that is, bimorphisms χ satisfying the extra conditions

5.1

$$\begin{cases} \chi(x, *) = 0_* & \textit{for all } x \in F, \\ \chi(*, y) = 0_* & \textit{for all } y \in G. \end{cases}$$

To retain the correspondence between bimorphisms $(F, G) \to H$ and morphisms $F \otimes G \to H$, we must therefore add corresponding relations to the definition of the tensor product. Thus, for pointed ω-groupoids F, G, we define $F \otimes_* G$ to be the ω-groupoid with generators $x \otimes_* y (x \in F, y \in G)$, basepoint $* = * \otimes_* *$, and defining relations (2.1) together with

5.2

$$\begin{cases} x \otimes_* * = 0_* & \textit{for all } x \in F, \\ * \otimes_* y = 0_* & \textit{for all } y \in G. \end{cases}$$

These equations are to be interpreted as $x \otimes_* * = * \otimes_* y = *$ when x, y have dimension 0, so that $(F \otimes_* G)_0 = F_0 \wedge G_0$.

Similar remarks apply to crossed complexes. In the pointed category Crs_* we define an m-fold pointed left homotopy $B \to C$ to be an m-fold left homotopy (h, f) satisfying $f(*) = *$ and $h(*) = 0_* \in C_m$. The collection of all these is a sub-crossed complex $\mathsf{CRS}_*(B, C)$ of $\mathsf{CRS}(B, C)$ with basepoint the zero morphism $b \mapsto 0_*$ and, clearly, $\mathsf{CRS}_*(B, C) = \gamma(\omega\text{-}\mathsf{GPD}_*(\lambda B, \lambda C))$. A pointed bimorphism $\theta : (A, B) \to C$ is a bimorphism satisfying

5.3

$$\begin{cases} \theta(a, *) = 0_* & \textit{for } a \in A, \\ \theta(*, b) = 0_* & \textit{for } b \in B, \end{cases}$$

and $A \otimes_* B$ is the pointed crossed complex generated by all $a \otimes_* b$ with defining relations (3.11) and

5.4

$$\begin{cases} a \otimes_* * = 0_* & \textit{for } a \in A, \\ * \otimes_* b = 0_* & \textit{for } b \in B. \end{cases}$$

The symmetry $A \otimes B \cong B \otimes A$ preserves the relations (5.4) and so gives a symmetry $A \otimes_* B \cong B \otimes_* A$ which can be carried over to the tensor product in ω-Gpd_*.

Theorem 5.5 *The pointed tensor products and hom functors described above define symmetric monoidal closed structures on the pointed categories* ω-Gpd_* *and* Crs_*. $\Box$

6 Computations

Any group G can be viewed as a crossed complex A with $A_0 = \{\cdot\}, A_1 = G, A_n = 0 (n \geqslant 2)$. The tensor product of two such crossed complexes will have one vertex and will be trivial in dimension $\geqslant 3$, that is, it will be a crossed module. We use multiplicative notation for G for reasons which will appear later.

Proposition 6.1 *The tensor product of groups* G, H*, viewed as crossed complexes of rank* 1*, is the crossed module* $G \Box H \to G * H$*, where* $G \Box H$ *denotes the Cartesian subgroup of the free product* $G * H$*, that is, the kernel of the map* $G * H \to G \times H$*. If* $g \in G, h \in H$*, then* $g \otimes h$ *is the commutator* $[h, g] = h^{-1}g^{-1}hg = [g, h]^{-1}$ *in* $G * H$.

Proof $G \Box H$ is a normal subgroup of $G * H$, so $G \Box H \mapsto G * H$ is a crossed module which we view as a crossed complex C, trivial in dimension $\geqslant 3$. One verifies easily that the equations $\theta(g, h) = [h, g], \theta(g, \cdot) = g, \theta(\cdot, h) = h$ define a bimorphism $\theta : (G, H) \to C$; the equations (3.4)(iii),(iii)' reduce to the standard commutator identities

$$[hh_1, g] = [h, g]^{h_1}[h_1, g],$$

$$[h, gg_1] = [h, g_1][h, g]^{g_1},$$

and the rest are trivial.

Now $G \Box H$ is a free group with basis consisting of all $[g, h] (g \in G, h \in H, g, h \neq 1)$ (see Gruenberg [18], Levi [23]). It follows that if $\phi : (G, H) \to D$ is any bimorphism, there is a unique morphism of groups $\phi_2 : G \Box H \to D_2$ such that $\phi_2([h, g]) = \phi(g, h)$ for all $g \in G, h \in H$. (Note that the definition of bimorphism implies that $\phi(g, h) = 1$ if either $g = 1$ or $h = 1$.) There is also a unique morphism $\phi_1 : G * H \to D_1$ such that $\phi_1(g) = \phi(g, \cdot)$ and $\phi_1(h) = \phi(\cdot, h)$ for all $g \in G, h \in H$. These morphisms combine to give a morphism $\phi_* : C \to D$ of crossed complexes as we show below, and this proves the universal property making C the tensor product of G and H, with $g \otimes h = [h, g]$. We need to verify that (i) ϕ_* is compatible with $\delta : G \Box H \hookrightarrow G * H$ and that (ii) ϕ_* preserves the actions of $G * H$ and D_1. Now

$$\begin{aligned} \delta\phi_2([h, g]) &= \delta\phi(g, h) \\ &= -\phi(\cdot, h) - \phi(g, \cdot) + \phi(\cdot, h) + \phi(g, \cdot) \quad \text{by (3.3)(iv)} \\ &= [\phi(\cdot, h), \phi(g, \cdot)] = [\phi_1(h), \phi_1(g)] = \phi_1[h, g] \end{aligned}$$

and this implies (i). As for (ii):

$$\begin{aligned}\phi_2([h,g]^{g_1}) &= \phi_2([h,g_1]^{-1}[h,gg_1]) \\ &= -\phi(g_1,h)+\phi(gg_1,h) \\ &= \phi(g,h)^{\phi(g_1,\cdot)} \quad \text{by (3.3)(iii)'} \\ &= \phi_2([h,g])^{\phi_1(g_1)}.\end{aligned}$$

There is a similar calculation for the action of $h_1 \in H$, and the result follows. □

Note. This tensor product of (non-Abelian) groups is related to, but not the same as, the tensor product defined by Brown and Loday and used in their construction of universal crossed squares of groups [11]. The Brown-Loday product is defined for two groups acting compatibly on each other. It also satisfies the standard commutator identities displayed above. The relation between the two tensor products is clarified in [28].

We can easily extend Proposition 6.1 to groupoids $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$, viewed as crossed complexes of rank 1. For this we need a new construction to take the place of the free product $G * H$. We define the groupoid $G \# H$ to be the pushout in the category Gpd of the diagram

$$\begin{tikzcd} & G \times \mathrm{id}(H) \\ \mathrm{id}(G) \times \mathrm{id}(H) \arrow[ur] \arrow[dr] & \\ & \mathrm{id}(G) \times H \end{tikzcd}$$

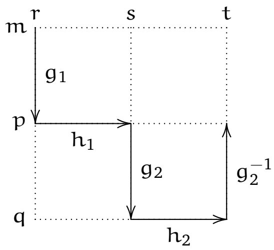
where, for any groupoid K, $\mathrm{id}(K)$ denotes the trivial sub-groupoid consisting of all identity elements of K. Thus $G \# H$ is a groupoid over $G_0 \times H_0$ and, in the category of groupoids over $G_0 \times H_0$, it is the coproduct of $G\times$ $\mathrm{id}(H)$ and $\mathrm{id}(G) \times H$. It is generated by all elements $(1_p, h), (g, 1_q)$ where $g \in G, h \in H, p \in G_0, q \in H_0$. We will sometimes write g for $(g, 1_q)$ and h for $(1_p, h)$. This may seem to be wilful ambiguity, but when composites are specified in $G \# H$, the ambiguity is resolved; for example, if gh is defined in $G \# H$, then g must refer to $(g, 1_q)$, where $q = \delta^0 h$, and h must refer to $(1_p, h)$, where $p = \delta^1 g$. This convention simplifies the notation and there is an easily stated solution to the word problem for $G \# H$. Every element of $G \# H$ is uniquely expressible in one of the following forms:

(i) an identity element $(1_p, 1_q)$;

(ii) a generating element $(g, 1_q)$ or $(1_p, h)$, where $p \in G_0, q \in H_0, g \in G, h \in H$ and g, h are not identities;

(iii) a composite $k_1 k_2 \cdots k_n (n \geqslant 2)$ of non-identity elements of G or H in which the k_i lie alternately in G and H, and the odd and even products $k_1 k_3 k_5 \cdots$ and $k_2 k_4 k_6 \cdots$ are defined in G or H.

For example, if $g_1 : m \to p, g_2 : p \to q$ in G, and $h_1 : r \to s, h_2 : s \to t$ in H, then the word $g_1h_1g_2h_2g_2^{-1}$ represents an element of $G \# H$ from (m, r) to (p, t).



Note that the two occurrences of g_2 refer to different elements of $G \# H$, namely $(g_2, 1_s)$ and $(g_2, 1_t)$.

The similarity with the free product of groups is obvious and the normal form can be verified in the same way; for example, one can use 'van der Waerden's trick'. We omit the details.

There is a canonical morphism $\sigma : G \# H \to G \times H$ induced by the inclusions $\mathrm{id}(G) \times H \to G \times H$ and $G \times \mathrm{id}(H) \to G \times H$. This morphism separates the odd and even products $k_1k_3 \cdots$ and $k_2k_4 \cdots$ from each word $k_1k_2k_3 \cdots$, that is, it introduces a sort of commutativity between G and H. The kernel of σ will be called the Cartesian subgroupoid of $G \# H$ and denoted by $G \square H$. It consists of all elements of type (i) and those of type (iii) for which both odd and even products are trivial. Clearly, it is generated by all 'commutators' $[g, h] = g^{-1}h^{-1}gh$, where $g \in G, h \in H$ and g, h are not identities. (Note that $[g, h]$ is uniquely defined in $G \# H$ for any such pair of elements g, h, but the two occurrences of g (or of h) do not refer to the same element of $G \# H$.)

Proposition 6.2 *(i) The Cartesian subgroupoid $G \square H$ of $G \# H$ is freely generated, as a groupoid, by all elements $[g, h]$ where g, h are non-identity elements of G, H, respectively. Thus, $G \square H$ is the disjoint union of free groups, one at each vertex, and the group at vertex (p, q) has a basis consisting of all $[g, h]$ with $\delta^1 g = p$ and $\delta^1 h = q$ (g and h not identity elements).*

(ii) The tensor product of the groupoids G and H, considered as crossed complexes of rank 1, is the crossed complex

$$\cdots \to 0 \to \cdots \to 0 \to G \square H \to G \# H \rightrightarrows G_0 \times H_0,$$

with $g \otimes h = [h, g], p \otimes h = (1_p, h), g \otimes q = (g, 1_q)$ for $g \in G, h \in H, p \in G_0, q \in H_0$.

Proof In the notation introduced above the 'commutators' $[h, g]$ satisfy the same formal identities as in the group case:

$$[h, g] = [g, h]^{-1},$$

$$[hh_1, g] = [h, g]^{h_1}[h_1, g],$$

$$[h, gg_1] = [h, g_1][h, g]^{g_1}$$

whenever gg_1, hh_1 are defined in G, H. These identities are to be interpreted as equations in $G \# H$, with the obvious meaning for conjugates: $[h, g]^{h_1}$ means $h_1^{-1}h^{-1}g^{-1}hgh_1$, which represents a unique element of $G \# H$. The proof of the proposition is now formally the same as the proof of (6.1). □

We are now in a position to analyse the structure of $A \otimes H$, where A is any crossed complex and H is a groupoid (viewed as a crossed complex). We write $\vec{\mathbb{Z}}H$ for the abelian groupoid (family of abelian groups) over H_0 in which $\vec{\mathbb{Z}}H(q)$ is the free abelian group on all $h \in H$ with $\delta^1 h = q$. Then $\vec{\mathbb{Z}}H$ becomes a (right) H-module under composition on the right:

$$(\sum \alpha_h h)^k = \sum \alpha_h (hk)$$

when $\sum \alpha_h h \in \vec{\mathbb{Z}}H(q)$ and $k \in H(q, r)$. This is the *right regular* H*-module*. The *right augmentation module* $\vec{I}H$ of H is the submodule generated by all $h - h'$, where $h, h' \in H$ and $\delta^1 h = \delta^1 h'$. Then $\vec{I}H(q)$ has a basis consisting of all $h - 1_q$ where $\delta^1 h = q$. (These constructions are the reason for writing H multiplicatively.)

We observe that if M, N are modules over the groupoids G, H, respectively, we may form $M \otimes_{\mathbb{Z}} N$, an abelian groupoid over $G_0 \times H_0$ with the group $M(p) \otimes_{\mathbb{Z}} N(q)$ lying over (p, q). Then $M \otimes_{\mathbb{Z}} N$ is a $(G \times H)$-module, with diagonal action

$$(m \otimes_{\mathbb{Z}} n)^{(g,h)} = m^g \otimes_{\mathbb{Z}} n^h,$$

and hence it is a $(G \# H)$-module via the canonical map $G\#H \to G \times H$.

We also need two constructions from the theory of crossed modules. If A is a crossed complex and H is a groupoid, then A_2 is a crossed module over the groupoid A_1 and hence $A_2 \times H_0$ is a crossed module over $A_1 \times \mathrm{id}(H)$. The embedding $\mu : A_1 \times \mathrm{id}(H) \to A_1 \# H$ now induces a crossed module $\mu_*(A_2 \times H_0)$ over $A_1 \# H$ (see [5, 7]) and we denote this crossed module by $\hat{A_2}$. The Cartesian subgroup $A_1 \square H$ is also a crossed module over $A_1 \# H$ and we need the coproduct of these two crossed $(A_1 \# H)$-modules. In the case when P is a group, the construction of the coproduct $X \circ_P Y$ of crossed P-modules X and Y is described in [4]. This construction works equally well when P is a groupoid. The family of groups X acts on Y via P, so one can form the semidirect product $X \ltimes Y$. It consists of a semidirect product of groups $X_i \ltimes Y_i$ at each vertex i of P and it is a pre-crossed module over P. One then obtains the crossed P-module $X \circ_P Y$ from $X \ltimes Y$ by factoring out its Peiffer groupoid.

We write A_1 multiplicatively but A_n additively for $n \geqslant 2$.

Theorem 6.3 *Let* A *be a crossed complex and* H *be a groupoid. Let* $C = A \otimes H$, *where* H *is viewed as a crossed complex of rank* 1. *Then (up to natural isomorphism)*

$$\begin{aligned}
C_0 &= A_0 \times H_0,\\
C_1 &= A_1 \# H,\\
C_2 &= \hat{A_2} \circ_{C_1} (A_1 \square H),\\
C_3 &= A_3 \otimes_{\mathbb{Z}} \vec{\mathbb{Z}}H \oplus A_2^{ab} \otimes_{\mathbb{Z}} \vec{I}H,\\
C_n &= A_n \otimes_{\mathbb{Z}} \vec{\mathbb{Z}}H \oplus A_{n-1} \otimes_{\mathbb{Z}} \vec{I}H (n \geqslant 4).
\end{aligned}$$

In this model for C, $C_1 = A_1 \# H$ *acts diagonally, via* $A_1\#A \to A_1 \times H$, *on each of the* $\mathbb{Z}$*-tensor products in* C_n *for* $n \geqslant 3$ *as described above. Its action on* C_2 *is given as part of the structure of* C_2 *as a coproduct of* C_1*-crossed modules. The canonical generators* $a \otimes h$ *for* $A \otimes H$ *are defined as follows. Let* $a_n \in A_n, p \in H_0, h \in H, \delta^1 h = q$ *and let* $\bar{a_2}$ *be the image of* a_2 *in* A_2^{ab}. *Then*

$$\begin{aligned}
a_0 \otimes p &= (a_0, p) \in A \times H_0,\\
a_1 \otimes p &= (a_1, 1_p) \in A_1 \# H,\\
a_2 \otimes p &= \sigma(a_2, p)
\end{aligned}$$

where σ is the canonical map $\sigma : A_2 \times H_0 \to \mu_*(A_2 \times H_0) = \hat{A_2} \to C_2$,

$$
\begin{aligned}
a_n \otimes p &= a_n \otimes_{\mathbb{Z}} 1_p \in A_n \otimes_{\mathbb{Z}} \vec{\mathbb{Z}}H \subseteq C_n,\\
a_0 \otimes h &= (1_{a_0}, h) \in A_1 \# H,\\
a_1 \otimes h &= \tau([h, a])
\end{aligned}
$$

where τ is the canonical map $A_1 \circ H \to C_2$,

$$
\begin{aligned}
a_2 \otimes h &= \bar{a_2} \otimes_{\mathbb{Z}} (h - 1_q) \in A_2^{ab} \otimes_{\mathbb{Z}} \vec{I}H \subseteq C_3,\\
a_n \otimes h &= a_n \otimes_{\mathbb{Z}} (h - 1_q) \in A_n \otimes_{\mathbb{Z}} \vec{I}H \subseteq C_n (n \geqslant 3).
\end{aligned}
$$

Finally, the boundary maps are defined as follows. The map $\delta : C_2 \to C_1$ is given as part of the crossed module structure. The map $\delta : C_3 \to C_2$ is given by

6.4
$$
\begin{cases}
\delta(a_3 \otimes_{\mathbb{Z}} h) = (\delta a_3 \otimes p)^h \in \hat{A_2}, & \text{where } p = \delta^0 h,\\
\delta(\bar{a_2} \otimes_{\mathbb{Z}} (h - 1_q)) = -a_2 \otimes q + (a_2 \otimes p)^h + \delta a_2 \otimes h,
\end{cases}
$$

(where h acts as the appropriate $(1_s, h) \in C_1$). The map $\delta : C_n \to C_{n-1}$ for $n \geqslant 5$ is given by

6.5
$$
\begin{cases}
\delta(a_n \otimes_{\mathbb{Z}} u) = \delta a_n \otimes_{\mathbb{Z}} u & (a_n \in A_n, u \in \vec{\mathbb{Z}}H),\\
\delta(a_{n-1} \otimes_{\mathbb{Z}} v) = (-)^{n-1} a_{n-1} \otimes_{\mathbb{Z}} \delta v + \delta a_{n-1} \otimes_{\mathbb{Z}} v & (a_{n-1} \in A_{n-1}, v \in \vec{I}H),
\end{cases}
$$

where δv denotes the image of v under inclusion $\vec{I}H \to \vec{\mathbb{Z}}H$. When $n = 4$, δ is given by the same formulae with δa_3 replaced by $\delta \bar{a}_3 \in A_2^{ab}$.

Proof We first verify that C is a crossed complex. The formulae (6.5) define, for $n \geqslant 4$, a unique morphism $\delta : C_n \to C_{n-1}$ of C_1-modules. The definition of C_2 ensures that $\delta : C_2 \to C_1$ is a crossed module. However, the map $\delta : C_3 \to C_2$ is more problematic since C_3 is abelian, but C_2 is not. We have to show that the relations imposed on C_2 by the definitions of induced crossed module and coproduct of crossed modules are sufficient to ensure the existence of a morphism $\delta : C_3 \to C_2$ of C_1-modules satisfying (6.4).

We write

$$
\begin{aligned}
x = x(a_3, h) &= (\delta a_3 \otimes p)^h = (\delta a_3 \otimes p)^{(1_s, h)},\\
y = y(a_2, h) &= -a_2 \otimes q + (a_2 \otimes p)^h + \delta a_2 \otimes h
\end{aligned}
$$

for the right-hand sides of (6.4), where $p = \delta^0 h, q = \delta^1 h, s = \beta a_3$. Then x and y are elements of C_2, and we easily verify that

$$
\delta x = (1_s, h)^{-1} \delta(\delta a_3 \otimes p)(1_s, h) = (1_s, h)^{-1}(1_s, 1_p)(1_s, h) = 1_{(s,q)}.
$$

Also

$$
\delta y = \delta(a_2 \otimes q)^{-1}(1_t, h)^{-1} \delta(a_2 \otimes p)(1_t, h) \delta(\delta a_2 \otimes h),
$$

where $t = \beta a_2$. In our abbreviated notation for $A_1 \# H$, this is

$$\delta y = (\delta a_2)^{-1} h^{-1} (\delta a_2) h [h, \delta a_2] = 1_{(t,q)}.$$

Since $\delta : C_2 \to C_1$ is a crossed module, these equations imply that x and y are in the centre of C_2 (that is, in the centres of the appropriate groups of C_2). It now follows that

$$\begin{aligned}
y(a_2 + a_2', h) &= -(a_2 + a_2') \otimes ((a_2 + a_2') \otimes p)^h + (\delta a_2 + \delta a_2') \otimes h \\
&= -a_2' \otimes q - a_2 \otimes q + (a_2 \otimes p)^h + (a_2' \otimes p)^h + \delta a_2' \otimes h + (\delta a_2 \otimes h)^{\delta(a_2' \otimes q)} \\
&= -a_2' \otimes q + y(a_2, h) - \delta a_2 \otimes h + a_2' \otimes q + y(a_2', h) - a_2' \otimes q + \delta a_2 \otimes h + a_2' \otimes q \\
&= y(a_2, h) + y(a_2', h) = y(a_2', h) + y(a_2, h).
\end{aligned}$$

Thus $y(a_2, h)$ is additive as a function of a_2 and depends only on $\bar{a_2} \in A_2^{ab}$. It preserves the action of $A_1 \times \mathrm{id}(H) \subseteq A_1 \# H$ since, for $a \in A_1$ (acting as $(a, 1_q) \in A_1 \# H$),

$$\begin{aligned}
y(a_2, h)^a &= -(a_2 \otimes q)^a + (a_2 \otimes p)^{ha} + (\delta a_2 \otimes h)^a \\
&= -a_2^a \otimes q + (a_2 \otimes p)^{ah\delta(a \otimes h)} - (a \otimes h) + (\delta a_2 + a) \otimes h \\
&= -a_2^a \otimes q - a \otimes h + (a_2^a \otimes p)^h + (a + \delta(a_2^2)) \otimes h \\
&= -a_2^a \otimes q - a \otimes h + (a_2^a \otimes p)^h + \delta(a_2^a) \otimes h + (a \otimes h)^{\delta(a_2^a)} \\
&= -a_2^a \otimes q - a \otimes h + a_2^a \otimes q + y(a_2^a, h) - a_2^a \otimes q + a \otimes h + a_2^a \otimes q \\
&= y(a_2^a, h).
\end{aligned}$$

For each $h \in H$, we now have a morphism of A_1-modules $\bar{a_2} \mapsto y(a_2, h)$ from A_2^{ab} to the centre of C_2. Since $\vec{I}H$ has $\mathbb{Z}$-basis consisting of all $h - 1_q (h \neq 1_q)$ and since $y(a_2, 1_q) = 0$ for all a_2, these maps combine to give a morphism $\delta : A_2^{ab} \otimes_{\mathbb{Z}} \vec{I}H \to C_2$ of groups with $\delta(\bar{a_2} \otimes_{\mathbb{Z}} (h - 1_q)) = y(a_2, h)$, and δ preserves the action of $A_1 \times \mathrm{id}(H)$. The reason for the appearance of $\vec{I}H$ is that its H-module structure is such that the action of $\mathrm{id}(A_1) \times H$ is also preserved by δ. For let $k \in H$ with $\delta^0 k = q, \delta^1 k = r$. Then

$$(h - 1_q)^k = hk - k = (hk - 1_r) - (k - 1_r)$$

while

$$\begin{aligned}
y(a_2, h)^k &= -(a_2 \otimes q)^k + (a_2 \otimes p)^{hk} + (\delta a_2 \otimes h)^k \\
&= -(a_2 \otimes q)^k + (a_2 \otimes p)^{hk} + \delta a_2 \otimes hk - \delta a_2 \otimes k \\
&= -(a_2 \otimes q)^k + a_2 \otimes r + y(a_2, hk) - \delta a_2 \otimes k \\
&= \delta a_2 \otimes k - y(a_2, k) + y(a_2, hk) - \delta a_2 \otimes k \\
&= y(a_2, hk) - y(a_2, k).
\end{aligned}$$

Thus the second equation of (6.4) defines a unique morphism δ of C_1-modules from $A_2^{ab} \otimes_{\mathbb{Z}} \vec{I}H$ to the centre of C_2. A similar, but much easier calculation shows that the first equation of (6.4) defines a unique C_1-morphism from $A_3 \otimes_{\mathbb{Z}} \vec{\mathbb{Z}}H$ to C_2. Hence we have a C_1-morphism $\delta : C_3 \to C_2$.

Now $C_2 \to C_1$ is, by definition, a crossed module, so the only other crossed complex axioms to be checked are: (i) $\delta\delta = 0$; (ii) δC_2 acts trivially on C_n for $n \geqslant 3$. We leave the first of these to the reader (having already checked the case $C_3 \to C_2 \to C_1$). As for (ii), we know that C_2 is generated, as C_1-crossed module, by all elements $a_2 \otimes q \in \hat{A}_2$ and $a_1 \otimes h \in A_1 \Box H$. Hence δC_2 is generated by conjugates of elements $\delta(a_2 \otimes q) = (\delta a_2, 1_q)$ and $\delta(a_1 \otimes h) = [h, a_1]$ in $A_1 \# H$. But δa_2 acts trivially on A_n for $n \geqslant 3$ and also on A_2^{ab} (since it acts by conjugation on A_2), and the commutator $[h, a_1]$ acts trivially on $C_n (n \geqslant 3)$ since the action of $A_1 \# H$ is diagonally defined.

It is now an easy matter to verify that the given definition of $a \otimes h$ satisfies all the laws (3.11) (some of the calculations have already been done) and it is universal. □

Corollary 6.6 *The canonical morphism* $A \otimes H_0 \to A \otimes H$ *is an injection.*

Proof It is clear that $A \otimes H_0$ (where H_0 stands for the crossed complex with vertices H_0 and trivial groupoids in all dimensions) has the groupoid $A_n \times H_0$ in dimension n. Indeed, this is a special case of the theorem. The canonical map sends the element $a_n \otimes p$ to a $a_n \otimes_{\mathbb{Z}} 1_p \in A_n \otimes_{\mathbb{Z}} \vec{\mathbb{Z}}H \subseteq C_n$ when $n \geqslant 3$, and this is clearly an injection. We have already shown that $A_1 \times H_0$ is embedded in $A_1 \# H = C_1$, so it remains to examine the map in dimension 2.

Now $A_2 \times H_0$ has vertices $A_0 \times H_0$ and can be made into a crossed $(A_1 \# H)$-module by the rules $\delta(a_2, p) = (\delta a_2, 1_p), (a_2, p)^{(a_1, 1_p)} = (a_2^{a_1}, p), (a_2, p)^{(1_s, h)} = (a_2, q)$, where $p = \delta^0 h, q = \delta^1 h, s = \beta a_2$. The identity map on $A_2 \times H_0$ therefore induces a morphism $\hat{A}_2 \to A_2 \times H_0$ of crossed $(A_1 \# H)$-modules. There is also a morphism $A_1 \Box H \to A_2 \times H_0$ of crossed $(A_1 \# H)$-modules in which each element is mapped to the zero element at the same vertex. These two morphisms induce a morphism from $C_2 = \hat{A}_2 \circ_{C_1} (A_1 \Box H)$ to $A_2 \times H_0$. The composite $A_2 \times H_0 \to C_2 \to A_2 \times H_0$ is the identity map, and the corollary follows. □

We recall that the crossed complex $\mathbb{C}(1) = \pi(\mathbb{I})$ has vertices p^0, p^1 and is freely generated by an edge e_1 from p^0 to p^1. When viewed as a groupoid it is often denoted J and is called the unit interval groupoid.

Corollary 6.7 *For any crossed complex* A *the canonical maps* $i^0, i^1 : A \to A \otimes J$ *defined by* $i^\alpha(a) = a \otimes p^\alpha (\alpha = 0, 1)$ *are injections.* □

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