Higher dimensional modelling and rewriting in algebraic topology *

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1 Introduction

Rewriting usually involves transforming some formula into another, for example $2 \times (5+3)$ into $2 \times 5 + 2 \times 3$. So presumably "higher dimensional rewriting" should involve "higher dimensional formulae". What could these be?

Note that a "higher dimensional view" can be useful. For example the above rewrite can be seen more simply as two views of the diagram

$$\begin{matrix} ||||| & ||| \\ ||||| & ||| \end{matrix}$$

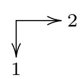
as could be understood in ancient Rome.

Again the usual interchange law for two binary operations $\circ_1, \circ_2$ on a set X can be written on a line as

$$(x \circ_1 z) \circ_2 (y \circ_1 w) = (x \circ_2 y) \circ_1 (z \circ_2 w)$$

but in a 2-dimensional form as

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix}$$



It is easy to verify in this formulation that if $\circ_1, \circ_2$ both have identities[1] e_1, e_2 then $e_1 = e_2$, and hence $\circ_1 = \circ_2$, which are abelian (and associative). This argument seemed to put paid to any idea of higher dimensional nonabelian versions of the fundamental group.

This triviality argument does not work if either of $\circ_1, \circ_2$ are partial binary operations, which then have a possible family of identities. This idea was nicely expressed by Ehresmann in terms of double categories and groupoids, an idea which clearly extended to higher dimensions. So we really do have proper inclusions

$$(\text{groups}) \subset (\text{groupoids}) \subset (\text{higher groupoids}).$$

That is, the extension from total to partial algebraic operations leads to a vast new range of algebraic structures; these can have a clear geometric content since often the domains of the operations are defined by geometric conditions.

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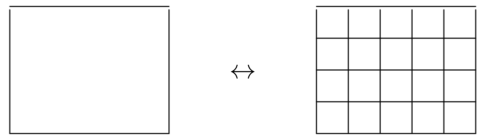
[1]The complications of double structures without identities are shown in for example [2].

Part of the impetus for my own work since 1965 was the intuition that in the diagram

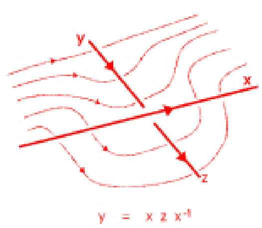


while from left to right clearly gives *subdivision*, from right to left should give *composition*.

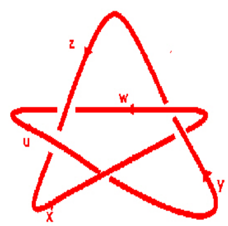
The aim of this note is to sketch the kind of 2-dimensional rewriting that has been involved in the approach to algebraic topology in [5], which itself is an exposition of work over some 50 years.

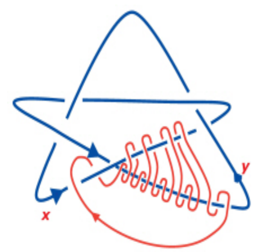
The start of this search goes back to the idea that since we can move from groups to higher dimensions, then what other ideas do we need to exploit this?

2 Group rewriting and knots



Relation at a crossing





The first diagram illustrates for the fundamental group of a knot the basic Wirtinger relation $y = xzx^{-1}$. You can Tietze transform the relations from the five crossings of a pentoil knot as in the middle diagram to

$$xyxyxy^{-1}x^{-1}y^{-1}x^{-1}y^{-1} = 1.$$

You can then tie string onto a copper knot according to the lefthand side of this equation to give the right hand diagram and it turns out that the loop comes off the knot! (This would be demonstrated in a talk!) (of course the real life problem with string is not exactly the same as the fundamental group problem, since string cannot cross itself in a move.)

The higher dimensional problem is to classify the ways in which the loop comes off the knot.

The idea of moving from techniques of group theory to ideas in homotopy theory was started by J.H.C. Whitehead in his paper [10], reformulated after the war as [11]. In particular the rewriting ideas of Tietze transformations of presentations of groups was developed geometrically in terms of expansions and collapses of cell complexes, and he showed that there could be a genuine obstruction, lying in what is now called the Whitehead group. These techniques of simple homotopy theory are now rather basic in geometric topology.

Some of his ideas here were an inspiration for the work of [5], particularly the Higher Homotopy Seifert-Van Kampen Theorem, which can be thought of allowing "higher dimensional relations', and so the need for higher dimensional rewriting.

3 Modelling homotopy types

The general philosophy for modelling homotopy types in dimension > 1 which has developed with many writers over the years, see [4], is to consider categories and functors

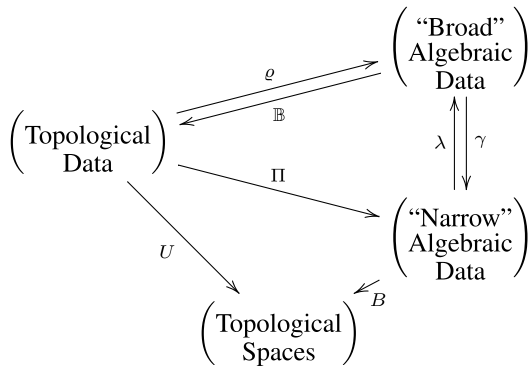


Figure 1: Broad and Narrow Algebraic Data

The functors γ, λ form an adjoint equivalence.

The functors ϱ, Π are homotopically defined, and $\gamma\varrho \simeq \Pi$.

The “Broad” Algebraic Data has a geometric type of axioms, is expressive, and useful for conjecturing and proving theorems, particularly colimit theorems, and for constructing classifying spaces. Examples are the cubical ω-groupoids of [5].

The “Narrow” algebraic data has complicated algebraic axioms, but relates to classical theory, is useful for explicit calculation, and colimit examples have led to new algebraic constructions. Examples given later will be crossed modules and crossed complexes of [5].

The necessarily *algebraic* proof of equivalence of these two structures allows one to get the best of both worlds. The hardest part of the equivalence is usually the natural equivalence $\lambda\gamma \simeq 1$, which shows that the broad data can be recovered from the narrow data. Also the functor γ usually involves choices. The whole relationship is a key to the power of the theory, and is important in developing aspects of it. Intermediate models may also have specific uses. Of course the use of various geometric models such as discs, globes, simplices, cubes, for defining invariants and constructions is standard in homotopy theory.

The functor ϱ satisfies a Higher Van Kampen theorem, allowing specific colimit calculations of some homotopy types. It may not always be clear how to compute specific invariants such as homotopy groups from the calculation of the bigger structure.

The above philosophy is exemplified in the book [5]: the Topological Data is that of filtered spaces; the Narrow Algebraic Data is that of crossed complexes; the Broad Algebraic Data is that of cubical ω-groupoids with connections. The equivalence between the last two categories very much uses 2-dimensional rewriting involving the connections, and we give a taster for this in the next section.

The nontrivial construction of the functor

$$\varrho : (\text{filtered spaces}) \to (\omega - \text{groupoids}),$$

and the derivation of some major properties, involves Whitehead's methods of expansions and collapsings.

4 Connections

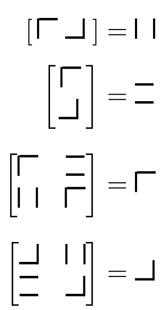
The notion of *connection* as an additional kind of degeneracy structure on cubical sets came first from considering possible extra structure on those double groupoids which arose from crossed modules. It also arose from the need to define "commutative cubes" in a double groupoid, analogous to commutative squares in a groupoid. More discussion on this development on cubical sets is in the article [4].

We need some special squares called *thin*, all of which are commutative, for some of which we use a special notations in which thick lines denote constant edges. So for example ⚌, | |, □ are all identities while we also have

$$\begin{pmatrix} & a & \\ a & & 1 \\ & 1 & \end{pmatrix} \qquad \begin{pmatrix} & 1 & \\ 1 & & a \\ & a & \end{pmatrix}$$

┘ or Γa ┌ or $\Gamma' a$ These are *connections*

The laws on connections are as follows:



$$[\,\Gamma' a \ \Gamma a\,] = \varepsilon_1 a \qquad \text{(cancellation)}$$

$$\begin{bmatrix} \Gamma' a \\ \Gamma a \end{bmatrix} = \varepsilon_2 a \qquad \text{(cancellation)}$$

$$\begin{bmatrix} \Gamma' a & \varepsilon_2 a \\ \varepsilon_1 a & \Gamma' b \end{bmatrix} = \Gamma'(ab) \qquad \text{(transport)}$$

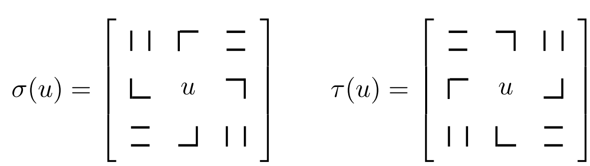
$$\begin{bmatrix} \Gamma a & \varepsilon_1 b \\ \varepsilon_2 b & \Gamma b \end{bmatrix} = \Gamma(ab) \qquad \text{(transport)}$$

The cancellation identities can be interpreted as "turning left (right) and then right (left) leaves you facing the same way" while the transport laws say "turning left (right) with your arm outstretched is the same as turning left (right)".

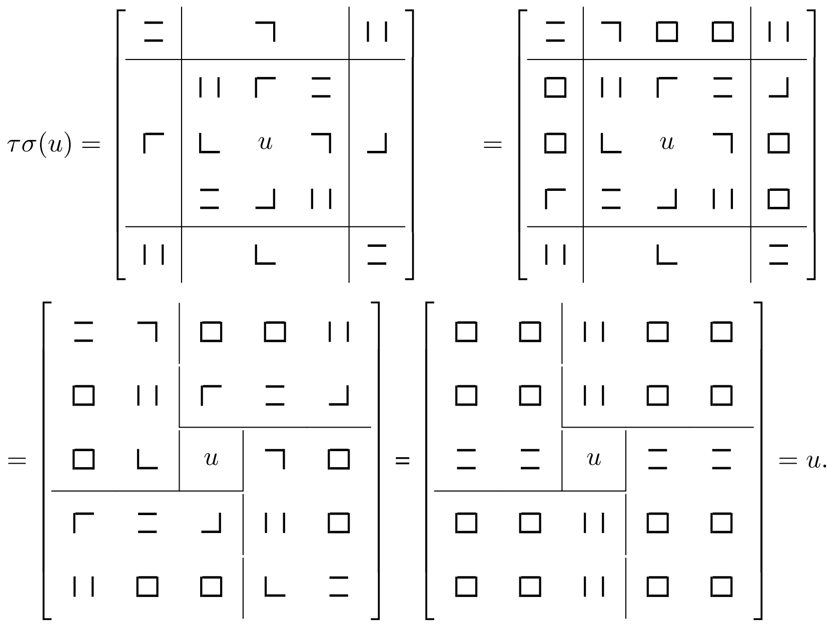
The formulation of the transport law, but in more general terms, was derived from a relation with path connections in differential geometry, as explained in [6].

5 Rotations

Here we illustrate the use of connections in terms of rotations, as in the following definitions:

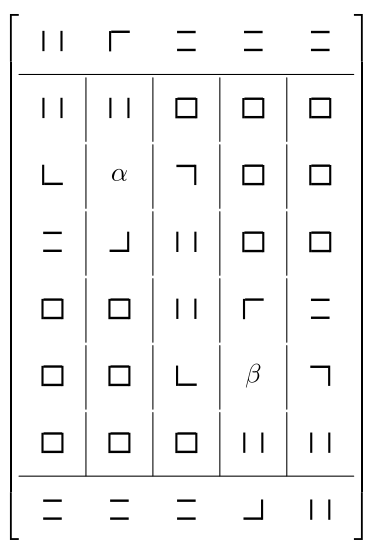


Now we prove $\tau\sigma(u) = u$ using 2-dimensional rewriting:



Further work shows $\sigma^2 u = -_1 -_2 u$, so $\sigma^4 u = u$. This algebra applied to $\varrho(X, A, C)$ shows the existence of specific homotopies.

As a further taster to "2-dimensional rewriting" we note that making two interpretations of the following diagram



(1)

shows that $\sigma \begin{bmatrix} \alpha & \beta \end{bmatrix} = \begin{bmatrix} \sigma\alpha \\ \sigma\beta \end{bmatrix}$. See [5, Theorem 6.4.10]; interpreting this proof shows in principle how to construct explicit homotopies for 2-dimensional representatives of the homotopy double groupoid $\varrho(X, A, C)$ of a pair (X, A) of spaces with a set C of base points. A homotopy can be thought of as a reversible rewrite! Thus the monoidal closed structure on crossed complexes given in [5] may be useful for higher rewriting.

Diagrammatic arguments using the rules for connections are a central feature of that book. The notation for connections was introduced in [3]. It should also be said that the verification of such rules as above may also be done in the category of crossed modules over groupoids, which is equivalent to that of double groupoids with connections; but such a method would be more boring!

Fuller details and justification of these rewriting methods are given in the paper [7], with essentially a use of 3-dimensional rewriting, which is used also in [1, Theorem 5.2] to obtain for $i > 1$ a braid type relation

$$\psi_i\psi_{i-1}\psi_i = \psi_{i-1}\psi_i\psi_{i-1},$$

where ψ_i is a "folding operation" defined using connections. This suggests that the use of connections should be related to the use of braid type diagrams in higher category theory.

6 Further possibilities?

A main feature of the book [5] is the use of multiple compositions of cubes, and the explicit nontrivial construction of these compositions in homotopically defined functors with values in certain strict multiple groupoids. A major step in this work was the use not of "bare" topological spaces, but of structured spaces, in particular pairs of spaces or filtered spaces. This contrasts to much development of work on weak or lax forms of ∞-categories or groupoids, where the structure derives easily from the Kan extension condition, and these functors are defined on unstructured topological spaces.

This leads to the question of whether such compositions can be defined in more general geometries than that of cubes. For example, in group theory considerable use is made of Van Kampen diagrams for showing deductions from relations, see for example [5, Section 3.1.ii], [9]. A general theory of this "polyhedral" type is described in [8]; the "marked complexes" described there are in fact equivalent to methods of the discrete Morse theory of R. Forman, which have been well used for computations.

References

[1] Al-Agl, F,, Brown, R. and Steiner, R., 'Multiple categories: the equivalence between a globular and cubical approach', *Advances in Mathematics*, 170 (2002) 71–118. 6

[2] Bagherzadeh, F. and Bremner, M. 'Commutativity in double interchange semigroups', arXiv:1706.04693 1

[3] Brown, R. 'Higher dimensional group theory'. In 'Low-dimensional topology (Bangor, 1979)', *London Math. Soc. Lecture Note Ser.*, Volume 48. Cambridge Univ. Press, Cambridge (1982), 215–238. 6

[4] R. Brown 'Modelling and computing homotopy types: I', *Indagationes Math: Special issue in honor of L.E.J. Brouwer*, (2017) https://doi.org/10.1016/j.indag.2017.01.009 arXiv:1610.07421. 3, 4

[5] Brown, R., Higgins, P. J. and Sivera, R. *Nonabelian algebraic topology: filtered spaces, crossed complexes, cubical homotopy groupoids*. EMS Tracts in Mathematics Vol 15. European Mathematical Society (2011). 2, 3, 5, 6

[6] Brown, R. and Spencer, C. B. 'Double groupoids and crossed modules'. *Cahiers Topologie Géom. Différentielle* **17** (4) (1976) 343–362. 4

[7] Higgins, P. J. 'Thin elements and commutative shells in cubical ω-categories'. *Theory Appl. Categ.* **14** (2005) No. 4, 60–74 6

[8] Jones, D. W. 'A general theory of polyhedral sets and the corresponding T-complexes'. *Dissertationes Math. (Rozprawy Mat.)* **266** (1988) 1–110. 6

[9] Ol'shanskii, A.Y. *Geometry of defining relations in groups*, Kluwer (1991). 6

[10] Whitehead, J. H. C. 'On incidence matrices, nuclei and homotopy types'. *Ann. of Math. (2)* **42** (1941) 1197–1239. 2

[11] Whitehead, J. H. C. 'Simple homotopy types'. *Amer. J. Math.* **72** (1950) 1–57. 2