Topos theoretic methods in general topology

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ABSTRACT

We consider the general problem of topologising spaces of partial maps. We explore this problem in the first place using standard methods of general topology, and then using newer sheaf theoretic methods. We are trying to approach the problem of finding the ''correct'' formal basis for our intuitive ideas of continuity by considering some of the basic properties of extensions which have been proposed of the idea of a topological space.

In Chapter 1 we study the basic propertis of the upper semi finite topology, which as far as we know have not been studied.

In Chapter 2 we show that the set PC(Y,Z) of all partial maps from a space Y to a space Z can not be topologised nicely. We introduce the compact connected open topology which contains the space of partial maps with closed domain and the space of partial maps with open domain as subspaces.

Chapter 3 is an easy reference on the category of squential spaces and generating a convenient category of spaces.

In Chapter 4 we show that the category of subsequential spaces is a quasitopos and give an explicit description of the strong partial morphisms classifier.

In Chapter 5 we give Johnstone's description of his topological topos.

In chapter 6 we give more properties of the embedding SEQ into Johnstone's topos, for instance we show that this embedding preserves function spaces and function spaces in the category of spaces over B, where B is Hausdorff. We also study the subobject classifier.

INTRODUCTION

We consider the general problem of topologising spaces of partial maps. We explore this problem in the first place using standard methods of general topology, and then using newer sheaf theoretic methods. We are trying to approach the problem of finding the "correct" formal basis for our intuitive ideas of continuity by considering some of the basic properties of extensions which have been proposed of the idea of a topological space.

Our aim is to consider the topos proposed by Johnstone in [J2] as an extension of the notion of sequential space. To this end we study in detail relations between this topos and the categories of sequential spaces and that of subsequential spaces.

We note that the history of mathematics demonstrates that when specific problems show up an inadequacy in current formulations, then the extended and revised theories usually turn out to have wide applications in mathematics. In the area of analysis and continuity, the study of various kinds of function spaces and their generalisation played a leading role in the impetus towards new methods.

In the study of topologising spaces of partial maps, we have an interesting case of an apparent breakdown in the standard approaches using the notion of topological spaces.

There is an extensive literature on topologies on spaces of continuous functions $Y \to Z$, in order to obtain a cartesian closed category of spaces. It seems reasonable to generalise these results to partial functions $Y \to Z$ because of the following reasons:

(i) The first reason is the wide appearance of partial functions in mathematics, $\log x$, $1/|x-2|$, $\sin^{-1}(e^x)$, are all examples of partial functions $\mathbb{R} \to \mathbb{R}$, and indeed in analysis it is a rare example which is defined everywhere.

(ii) There have been two generalisations to partial functions with closed domain [B-B1] and to partial functions with open domain [Eh], [A-B] . The first type is important in the theory of fibrations [B-B1,2] and the second type seems appropriate for uses in analysis and differential topology [A-B] . It therefore seems reasonable to ask for a topological space including both of these special cases.

We also want them to be subspaces because of the following reason:

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a partial map. Consider the adjoint $\hat{f}: \mathbb{R} \to \mathrm{Part}(\mathbb{R},\mathbb{R})$ $\hat{f}(x)(y) = f(x,y)$, where $\mathcal{D}_{\hat{f}(x)} = (\mathcal{D}_f)_x = \{y \mid (x,y) \in \mathcal{D}_f\}$. Then it is easy to see that there are situations where $\mathcal{D}_f$ is neither open nor closed and $(\mathcal{D}_f)_x$ is sometimes open, sometimes closed and sometimes neither open nor closed. For instance let

$$f: \mathbb{R}^2 \to \mathbb{R} \quad , \quad f(x,y) = \sqrt{x+y} - \frac{1}{x^2+y^2-1} \quad .$$

Then: $(\mathcal{D}_f)_x$ is open if $x = \pm\frac{\sqrt{2}}{2}$

$(\mathcal{D}_f)_x$ is closed if $x < -\frac{\sqrt{2}}{2}$ or $1 < x$

$(\mathcal{D}_f)_x$ is neither open nor closed otherwise.

However if $\mathcal{D}_f$ is open then $(\mathcal{D}_f)_x$ is open. So we expect to be able to factorise

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\hat{f}} & \text{Part}(\mathbb{R},\mathbb{R}) \\ & \searrow & \uparrow i \\ & & P_0(\mathbb{R},\mathbb{R}) \end{array} \quad ,$$

where $P_0(\mathbb{R},\mathbb{R})$ is the space of partial maps with open domain, (Chapter 2). It is possible to do so if i is strong. Similarly for the closed domain case. However, this cannot in general be done (Chapter 2).

In the above generalisations to spaces of maps with closed or with open domain, a particular trick is used in [B-B1] and [A-B] namely the representability of these types of partial map. Let $f: Y \to Z$ be a partial map, and let Z^+ be the union of Z and a point ω not in Z. Then we can define $f^+: Y \to Z^+$ by

$$f^+(y) = \begin{cases} f(y) & y \in \mathcal{D}_f \\ \omega & y \notin \mathcal{D}_f \end{cases} .$$

we would like a topology on Z^+ so that the continuous functions $Y \to Z^+$ determine particular partial functions $Y \to Z$. This is done in [B-B1] and [A-B] for partial functions with closed and with open domain respectively. Such techniques have not been found for spaces of all partial maps.

The category Top , of spaces and continuous functions, is not a convenient category for practising topologists to work with. One reason is that in Top there is no function space which satisfies an exponential law, that is there is no topology on the set Top(Y,Z) such that there is a natural equivalence

$$\mathrm{Top}(X \times Y, Z) \cong \mathrm{Top}(X, \mathrm{Top}(Y,Z)) ,$$

for any objects X, Y and Z of Top . Equivalently the functor

$$- \times Y \colon \mathrm{Top} \to \mathrm{Top}$$

does not always have a right adjoint

$$(\ )^Y \colon \mathrm{Top} \to \mathrm{Top} .$$

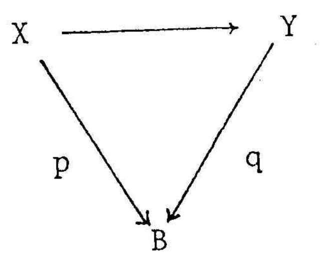
A category with a product in which the product functor has a right adjoint is known as a cartesian closed category, as defined by S. Eilenberg and G.M. Kelly [E-K] . Because of the important property, cartesian closedness, which Top fails to satisfy, working topologists have to restrict themselves to full subcategories of Top that are large enough to include all interesting spaces and are cartesian closed.

R. Brown [B2,3,5] proposed that the category of (Hausdorff) k-spaces might be adequate and convenient for all purposes of topology. The function space in this category is the k-ification of the function space C(Y,Z) with the compact open topology. These ideas were taken over and proclaimed by Steenrod in [St] .

Other categories are the category of quasitopological spaces by R. Thom [Th] and E. Spanier [Sp] . R. Vogt [V] , B. Day [D1] and O. Wyler [W2] suggested some coreflective subcategories of Top and Haus (the category of Hausdorff spaces) such as the category of compactly generated spaces which is a coreflective hull of C-Haus (the category of compact Hausdorff spaces) in Top ; the category of compactly generated Hausdorff spaces which is a coreflective hull of all compact Hausdorff spaces in Haus ; and the category of all sequential spaces which is the coreflective hull of all metrizable spaces in Top .

Some authors suggested embedding Top in a larger cartesian closed category, Edgar [E] , ...

Topologists also like to work with the category of spaces over a fixed space B , that is the category whose objects are continuous functions $p: X \to B$, $q: Y \to B$, ... and a morphism $h: p \to q$ is a commutative diagram



Then for suitable space B , namely Hausdorff or <u>weakly Hausdorff</u>, that is the diagonal is closed in the k-topology, $K \downarrow B$ is cartesian closed where K is the category of k-spaces. The origin of the cartesian closedness of $K \downarrow B$ goes back to

R. Thom [Th] . The theory was developed further by P.I. Booth [Bo1] [Bo2] , B. Day [D1] . P.I. Booth and R. Brown [B-B1] gave an explicit topology for $K \downarrow B\ (pq)$ and related the exponents (closedness) of $K \downarrow B$ to the space of partial maps with closed domain. G. Lewis [Lew] proved that the category of open maps from a compactly generated space into a compactly generated "base" space is a convenient category, that is he proved that for open maps $p: X \to B$ and $q: Y \to B$, $K \downarrow B\ (pq)$ is weakly Hausdorff.

There is an extensive literature on fibre bundles and fibrations; the various notions of fibrations are essential for solving many geometric problems. The crucial point about working in the category of spaces over B , $p: X \to B$, is that the set X is the disjoint union of the fibre $p^{-1}(b)$, $b \in B$. Also the interest is not only in maps $h: X \to Y$ but in all maps $p^{-1}(b) \to q^{-1}(b)$, for all $b \in B$, as well.

The rise of topos theory has encouraged mathematicians to look for a "super" convenient category of spaces, that is a cartesian closed topological category which is actually a topos.

Some of the pleasant features of toposes, additional to cartesian closedness and subobject classifier, are that in a topos all partial maps are representable and that for any object X of a topos $\mathcal{E}$ the category of objects of $\mathcal{E}$ over X is a topos.

Grothendieck considered a topos, in this sense a category of sheaves defined on a site, as a natural generalisation of a space [G-V] . J. Giraud [G-V] was the first mathematician to construct a topos of topological spaces, Lawvere [J2] was interested in finding better categories to do analysis but the topos Lawvere constructed did not have good colimits preservation as was shown by Isbell [J2] , Johnstone [J2] embedded the category of sequential spaces, which is the smallest convenient category, in a topos.

In Chapter 1 we study the basic properties of the upper semi-finite topology, which as far as we know have not been studied, on the set of all subsets of a topological space X .

In Chapter 2 we show that the set PC(Y,Z) of all partial maps from a space Y to a space Z cannot be topologised nicely even for $Y = Z = \mathbb{R}$. We also show that the modified compact open topology is admissible, jointly continuous. We introduce the compact connected open topology which contains the space of partial maps with open domain and the space of partial maps with closed domain as subspaces. This suggests that for the compact open topology on C(Y,Z) it is sufficient to consider all W(C,U) where C is compact connected in Y and U is open in Z as a sub-basis, provided that Y is a nice space.

In Chapter 3 we collect some known facts on generating a convenient category of spaces, and the category SEQ , of sequential spaces.

In Chapter 4 we give an explicit description of the strong partial morphism classifier in the category SuSEQ , of subsequential spaces. Also we investigate in detail the embedding of SEQ into the quasitopos SuSEQ .

In Chapter 5 we give Johnstone's description of his topological topos, however, we will give the proofs in more detail in order to make them readable for non-experts in topos theory.

An elementary topos is a cartesian closed category with finite limits and finite colimits and with a subobject classifer. Now cartesian closed categories are well studied in general topology, often under the name convenient categories, and limits and colimits are also of long standing use in this area. However, the notion of subobject classifer (or strong subobject classifer, for quasitopos) has not been well studied in general topology. One reason is the difficulty of defining this in a convenient category using the notion of topological spaces. One aim of Chapter 6 is an attempt to study the subobject classifier, in Johnstone's topos. We also give more properties of the embedding of SEQ and SuSEQ into Johnstone's topos, and discuss the relation between them.

CHAPTER 1

A TOPOLOGY ON THE POWER SET OF A TOPOLOGICAL SPACE

1.0 Introduction

In this chapter we will consider a topology on the set $P(X)$, the set of all subsets of a space X . One of the topologies that has been defined is the Vietoris topology, defined as the join of the upper-semi finite and the lower semi finite topology in the lattice of topologies.

However our interest is in the upper semi finite topology, of which the basic properties have not been studied. One reason that makes this topology a reasonable one to study is the following. Our main concern is to consider topologies on the set PC(Y,Z) , of all partial maps between spaces Y and Z . By considering the upper semi finite topology on $P(Y\times Z)$, it turns out that the graph topology on PC(Y,Z) , which we will introduce in section 4 as a generalisation of the graph topology on C(Y,Z) , is the initial topology with respect to the "graph function".

1.1 The upper semi finite topology on $P(X)$

Throughout this chapter we consider a topological space X with a topology T . As usual, we write $P(X)$ for the set of all subsets of X .

<u>Definition 1.1.1</u> [Mi] Let π_T be the topology on $P(X)$ which has a basis the family of sets $P(U)$ for all open U of X.

That this family is a basis follows from the rule

$$P(U) \cap P(V) = P(U \cap V) .$$

We assume that $P(X)$ has the topology π_T.

<u>Proposition 1.1.2</u> (i) If A is an open set of $P(X)$, then the union of the elements of A is an open set of X.

(ii) If C is a non-empty closed set of $P(X)$, then $X \in C$.

<u>Proof</u>. (i) Since A is open, it is the union of basic open sets $P(U_i)$, $i \in I$, say. Then the union of the elements of A is $\cup_{i \in I} U_i$, which is an open set of X.

(ii) If X does not belong to C then X belongs to the complement A of C. Since A is open there is a basic open set $P(U)$ such that $X \in P(U)$. Hence $U = X$ and so $C = \emptyset$. □

We now consider to what extent the lattice operations of union and intersection are continuous on $P(X)$.

<u>Proposition 1.1.3</u> The union function

$$(A_1, \ldots, A_n) \longrightarrow A_1 \cup \ldots \cup A_n$$

is continuous as a function

$$u\colon \Pi_{i=1}^{n} P(X) \longrightarrow P(X) .$$

Proof. Let U be an open set of X . Then clearly

$$u^{-1} [\mathcal{P}(U)] = \Pi_{i-1}^{n} \mathcal{P}(U) . \square$$

The intersection map $\cap$ on $\mathcal{P}(X)$ is not in general continuous, as is seen by considering the case $X = \mathbb{R}$, the real line; if $A = (0,1]$, $B = (0,1) \cup (1,2)$, then no neighbourhood of (A,B) in $\mathcal{P}(X) \times \mathcal{P}(X)$ is mapped into $\mathcal{P}((0,1))$ by $\cap$.

Let $\mathcal{O}(X)$, the set of open sets of X , have its topology as a subspace of $\mathcal{P}(X)$.

Proposition 1.1.4 The intersection map

$(A_1,\ldots,A_n) \longrightarrow A_1 \cap \ldots \cap A_n$ is continuous on

$\Pi_{i=1}^{n} \mathcal{O}(X)$.

Proof. A basis for the subspace topology on $\mathcal{O}(X)$ is clearly $\mathcal{O}(U)$ for all U open of X .

Let $\mathcal{O}(U)$ be a basic open set in $\mathcal{O}(X)$ such $\mathcal{O}(U)$ contains $A_1 \cap \ldots \cap A_n$. Then intersection maps $\mathcal{O}(A_1) \times \ldots \times \mathcal{O}(A_n)$ into $\mathcal{O}(U)$. $\square$

Remark 1.1.5 a) The empty set is an element of $\mathcal{P}(X)$ and clearly plays a special role. In fact let $\mathcal{P}'(X) = \mathcal{P}(X)\backslash\{\emptyset\}$. For any space Z , let $Z^{\sim}$ be $Z \cup \{w\}$ (where $w \notin Z$) with the topology in which C is closed in $Z^{\sim}$ if and only if $C = Z^{\sim}$ or C is closed in Z [B-B1] . Then we can identify $\mathcal{P}(X)$, given the above topology, with $(\mathcal{P}'(X))^{\sim}$.

b) This topology, on $\mathcal{P}(X)$, is not a topological topology. Recall that a topology on $\mathcal{P}(X)$ is called a topological topology if it makes finite intersection and arbitrary union continuous operations [I1] .

1.2 Basic properties of $P(X)$

In this section we will establish the basic properties of the topology π_T . Recall that we write $P'(X) = P(X)\setminus\{\emptyset\}$. Again, X will be a topological space with topology T and $P(X)$ will have the topology defined in 1.1, with $P'(X)$ as a subspace.

Proposition 1.2.1 (i) if X is non-empty, then $P(X)$ is not a T_1-space.

(ii) If X is a T_1-space, then $P'(X)$ is a T_0-space.

(iii) If X has more than one point then $P'(X)$ is not a T_1-space.

(iv) If X is separable, then $P'(X)$ is separable.

(v) If $P(X)$ is first countable, then X is first countable.

(vi) If $P(X)$ is second countable, then $O(X)$ is countable.

Proof. (i) This is obvious, since then $\{\emptyset\}$ is not closed.

(ii) Let A and B be subsets of X such that for some $p \in X$, $p \notin A \setminus B$. Then $P(X\setminus\{p\})$ is an open set containing B but not A .

(iii) Consider X and $\{p\}$ where p is a point of X .

(iv) Let $x \in X$. If $\{P(U_n)\}_{n\in N}$ is a countable basis for the neighbourhoods of $\{x\}$ in $P(X)$, then $\{U_n\}_{n\in N}$ is a countable basis for the neighbourhoods of x in X .

(v) If $\{(U_i)\}_{i \in I}$ is a basis for the topology π_T then any open set U of X is U_i for some i. In particular, if I is countable then the topology T of X is countable.

(vi) Let Y be a countable dense subset of X. Then it is easy to see that $\mathcal{Y} = \{\{y\} \mid y \in Y\}$ is a countable dense subset of $\mathcal{P}(X)$. □

<u>Definition 1.2.2</u> A space is said to be <u>unicompact</u> if whenever $\{U_i\}_{i \in I}$ is an open cover of X, then $X = U_i$ for some i.

An example of the unicompact space is $\tilde{Z}$ for any space Z, where $\tilde{Z}$ is as in remark 1.1.5.

<u>Proposition 1.2.3</u> If $\mathcal{K}$ is a subset of $\mathcal{P}(X)$ with the property that there is $Y \in \mathcal{K}$ such that $\cup \mathcal{K} = Y$, then $\mathcal{K}$ is unicompact.

<u>Proof</u>. Let $\{\mathcal{V}_i\}_{i \in I}$ be an open cover of $\mathcal{K}$. Then $Y \in \mathcal{V}_i$ for some i. But since $\mathcal{V}_i$ is open $\mathcal{V}_i = \cup_{i \in I} \mathcal{P}(U_i)$. So $Y \subseteq U_i$ for some i. Hence $\mathcal{K} \subseteq \mathcal{V}_i$. □

<u>Corollary 1.2.4</u> $\mathcal{P}'(X)$ is unicompact.

Let X^n be the n-fold product of X.

<u>Proposition 1.2.5</u> The function

$$\sigma_n : X^n \longrightarrow \mathcal{P}(X)$$

$$(x_1, x_2, \ldots, x_n) \longrightarrow \{x_1, \ldots, x_n\}$$

is continuous, and σ_1 is a homeomorphism onto a dense subset of $\mathcal{P}(X)$.

Proof. Let U be an open set of X . Then

$$\sigma_n^{-1}[P(U)] = \Pi_{i=1}^{n} U .$$

Clearly σ_1 is open and the image of σ_1 is a dense subset of $P(X)$. □

Corollary 1.2.6 If X is a space, then $(P(X), \sigma_1)$ is a compactification of X .

However the fact that $P'(X)$, is compact is because the set X , regarded as a point of $P'(X)$, plays a special role, as Definition 1.2.7 and Proposition 1.2.8 show.

Definition 1.2.7 A point x of a space X is said to be f-indiscrete if whenever $\{U_i\}_{i\in I}$ is a family of distinct neighbourhoods of x , then $X = U_{i1} \cup \ldots \cup U_{in}$ for finite n .

In the space $\tilde{Z} = Z \cup \{w\}$ the point w is f-indiscrete.

Proposition 1.2.8 If X has an f-indiscrete point then X is compact.

Proof. For a given open cover of X , consider the subfamily of neighbourhoods of an f-indiscrete point. □

Proposition 1.2.9 The space $P'(X)$ is sequentially compact.

Proof. Let $\{A_n\}_{n\in N}$ be a sequence in $P(X)$. Then

$$A_n \longrightarrow A \qquad \text{where} \qquad A = \cup_{n\in N} A_n ,$$

since any open set of $P'(X)$ containing A contains A_n for all $n \in N$. □

Proposition 1.2.10 $P'(X)$ is connected.

Proof. Assume $P'(X) = M \cup N$ where M and N are disjoint open sets of $P(X)$, so $X \in M$ say. But then $P(X) = M$. □

Remark 1.2.11 [Do] A space X is an R_0-space if for open U and for $x \in U$, $\overline{\{x\}} \subseteq U$. It follows, from 1.1.2 (ii), that $P(X)$ is not an R_0-space.

1.3 Some applications

In this section we will study certain functions from and to $P(X)$.

1.3.a Let X and Y be spaces such that X is compact. Consider the set $C(X,Y)$, of all continuous functions from X to Y. If $C(X,Y)$ is given the compact-open topology, then the range function

$$R: C(X,Y) \longrightarrow P(Y)$$
$$f \longrightarrow \text{Range}(f)$$

is continuous.

Proof. Clearly for open U in X,

$$R^{-1}[P(U)] = W(X,U) \; . \; \square$$

1.3.b Let $f: X \longrightarrow Y$ be a continuous function between two topological spaces. Then

$$f_*: P(X) \longrightarrow P(Y)$$
$$A \longrightarrow f(A)$$

is such that:

(i) f_* is continuous if and only if f is continuous;

(ii) f_* is bijective if and only if f is bijective;

(iii) f_* is a homeomorphism if and only if f is a homeomorphism.

Proof. Assume f_* is continuous. Let U be an open set of Y. Since $f_*^{-1}[P(U)] = P(f^{-1}(U))$ is open, so $P(f^{-1}(U)) = \bigcup_{i\in I} P(V_i)$ where $P(V_i)$ is a basic open set for all i. Hence $f^{-1}(U) = V_i$ for some i.

Now assume f is continuous. Then $f_*^{-1}(P(U)) = P(f^{-1}(U))$ is open.

The proof of (ii) and (iii) are trivial. □

1.3.c If $f:X \longrightarrow Y$ is an open injective continuous function then the inverse image function

$$f^*:P(Y) \longrightarrow P(X)$$
$$A \longrightarrow f^{-1}(A)$$

is continuous.

Proof. Clearly $(f^*)^{-1}(P(U)) = P(f(U))$, which is open if U is open in X. □

The following two examples show that the injectivity and openess of f cannot be dropped.

Example 1.3(i) Let $X = \{a,b\}$ be the Sierpinski space with $\{a\}$ open but not closed. Define $f:X \longrightarrow X$ by

$$f(a) = f(b) = a .$$

Then f is continuous, open but not injective and f^* is not continuous.

Example 1.3(ii) Let X be as in example (i) and X_d be the discrete space. Define $g:X_d \longrightarrow X$ by $g(a) = b$ and $g(b) = a$. Then g is injective but not open and g^* is not continuous.

Remark 1.3.1 Our idea of defining a topology on the set $P(X)$ for a space (X,T), was as follows: Consider any sub-basis S for a topology T. Then consider $P(S) = \{P(U) \mid U \in S\}$ as a sub-basis for a topology π_S on $P(X)$. If $S = T$ then π_T is the finest topology defined in this manner, and the topology π_T is just the upper semi finite topology, Definition 1.1.1.

For the rest of this section we will consider topologies on $P(X)$ by specifying a sub-basis for T.

1.3.d Let $\mathbb{R}$ be the set of all real numbers with

$$\{(-\infty,b),(a,\infty) \mid a,b \in \mathbb{R}\} \text{ as a sub-basis.}$$

Consider the function

$$f : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$$
$$(m,n) \longrightarrow m+\lambda n$$

$\lambda \in \mathbb{R}$. Let $A_\lambda = \text{Image}(f_\lambda)$. Then

A_λ is dense in $\mathbb{R}$ if λ is irrational

A_λ is closed and discrete in $\mathbb{R}$ if λ is rational,

in fact $A_\lambda = \frac{1}{q} Z$ where $\lambda = p/q$ in its lowest terms.

Let

$$A:\mathbb{R} \longrightarrow P(\mathbb{R})$$
$$\lambda \longrightarrow A_\lambda$$

Then A is continuous where $P(\mathbb{R})$ has as a sub-basis

$$\{P(-\infty,b),\ P(a,\infty) \mid a,b \in \mathbb{R}\}.$$

The proof is easy.

1.3.e (Let $(X,\leq)$ be an order space with the topology that has as a sub-basis all sets of the form

$$A_1 = \{ x \in X \mid a < x < b \} ,$$
$$A_2 = \{ x \in X \mid a < x \}$$
$$A_3 = \{ x \in X \mid x < b \} .$$

Let $P(X)$ have as a sub-basis for its topology all sets of the form $P(A_1)$, $P(A_2)$, $P(A_3)$.
Then the function $f:X \longrightarrow P(X)$

$$x \longrightarrow \{ y \in X \mid x \leq y \}$$

is continuous.
The proof is easy.

1.4 Relation to topologies on spaces of partial maps and to the Vietoris topology on spaces of subsets.

In this section we will discuss the relation of the topologies π_T to the graph topology, the compact-open topology, the pointwise convergence topology and the Vietoris topology.

We now introduce the graph topology on $PC(Y,Z)$, the set of all partial maps from Y to Z , as a generalisation of the graph topology on $C(Y,Z)$. The graph topology on $C(Y,Z)$ is defined in [N] .

Let Y and Z be topological space. For an open set U of $Y \times Z$ let $F_U = \{f \in PC(Y,Z) \mid \text{graph } (f) \subseteq U\}$.

Definition 1.4.1 The graph topology Γ on $PC(Y,Z)$ is the one which has as a basis $\{F_U \mid U$ is open in $Y \times Z\}$.

Remark 1.4.2 Let $P(Y \times Z)$ be given the topology in which a basis is $\{P(U) \mid U$ is open in $Y \times Z\}$, and consider the graph function

$$G: PC(Y,Z) \longrightarrow P(Y \times Z)$$
$$f \longrightarrow G(f) = \text{graph}(f)$$

Then the initial topology, G, with respect to G coincides with the graph topology, since $G^{-1}[P(U)] = F_U$.
Also $C(Y,Z)$ with the graph topology is a subspace of $PC(Y,Z)$.

Proposition 1.4.3 [N] If Y is Hausdorff and Z is an arbitrary space, then the compact-open topology on $C(Y,Z)$ is contained in G. Moreover if Y is compact then G coincides with the compact open-topology.

Proof. Let $W(C,U) = \{f \in C(Y,Z) \mid f(C) \subset U\}$ be a sub-basic open set of the compact open-topology. Then

$$V = [(Y \setminus C) \times Z] \cup (Y \times U)$$

is an open set of $Y \times Z$. So $W(C,U) = G^{-1}[(V)]$.

The proof of the second part is trivial. □

Proposition 1.4.4 Let $PC(Y,Z)$ and $C(Y,Z)$ be given the graph topology. Then

(i) the range function $R: PC(Y,Z) \longrightarrow P(Z)$ is continuous.

(ii) the range function $R: C(Y,Z) \longrightarrow P(Z)$ is continuous.

Proof. The proof follows from the fact that

$$G^{-1}[P(U \times V)] = R^{-1}[P(V)] \quad . \quad \square$$

The pointwise convergence topology on PC(Y,Z) is defined as having a sub-basis all sets of the form

$$W(x,U) = \{f \in PC(Y,Z) \mid f\,x \in U \quad \text{or} \quad x \notin \mathrm{Dom}(f)\} \; .$$

Proposition 1.4.5 If Y is a T_1-space then the pointwise convergence topology is contained in the graph topology on PC(Y,Z) .

Proof. Let W(x,U) be a sub-basic open set of the pointwise convergence topology. Then clearly $W(x,U) = F_V$ where V is the open set $(Y \times U) \cup [(Y \setminus \{x\}) \times Z]$. $\square$

The Vietoris topology:

We recall the definition and some basic facts about the Vietoris topology on $P'(X)$ [Mi].

Definition 1.4.6 [Mi] Let X be a topological space. For a collection $\{U_i \mid i \in I\}$ of open sets of X , let $\langle U_i \rangle_{i \in I}$ denote $\{Y \subseteq X \mid Y \subseteq \cup_{i \in I} U_i$ and $Y \cap U_i \neq \emptyset$ for all $i \in I\}$, and for finite I we write $\langle U_i \rangle_{i \in I}$ as $\langle U_i \rangle_{i=1}^{n}$ or $\langle U_1, U_2, \ldots, U_n \rangle$. The Vietoris topology on $P'(X)$ is defined as the one with sub-basis all sets of the form $\langle U_1, U_2, \ldots, U_n \rangle$, where $U_1, U_2, \ldots, U_n$ are open in X .

Proposition 1.4.7 [Mi] Let $U_1, U_2, \ldots, U_n$, be open in X . Then the set of all elements of the form $\langle U_i \rangle_{i=I}^{n}$, form a basis for the Vietoris topology.

<u>Proof</u>. The set X is open since $\mathcal{P}'(X) = \langle X \rangle$. Now let $\mathcal{U} = \langle U_i \rangle_{i=I}^{n}$ and $\mathcal{V} = \langle V_i \rangle_{i=I}^{n}$. Then clearly

$$\mathcal{U} \cap \mathcal{V} = \langle U_1 \cap B, \ldots, U_n \cap B,\ V_1 \cap A, \ldots, V_n \cap A \rangle \quad \text{where}$$

$$A = \cup_{i=I}^{n} U_i \quad \text{and} \quad B = \cup_{i=I}^{n} V_i \ . \ \square$$

As we mentioned before, the topology which we considered on $\mathcal{P}'(X)$ is called the upper semi-finite topology. On the other hand the lower semi-finite topology on $\mathcal{P}'(X)$ is defined as the one which has as a sub-basis all sets of the form

$$\{Y \subseteq X \mid Y \cap U \neq \emptyset\} \quad \text{where } U \text{ is an open set of } X \ .$$

Clearly the Vietoris topology is the join of the upper semi finite and the lower semi finite in the lattice of topologies on $\mathcal{P}'(X)$.

<u>Proposition 1.4.8</u> [Mi] Let X be a topological space and let $\mathcal{P}'(X)$ have the Vietoris topology. Then a function

$$f : X \longrightarrow \mathcal{P}'(X)$$

is continuous if and only if the set $\{x \in X \mid f^{-1}x \cap U \neq \emptyset\}$ is open whenever U is open and is closed whenever U is closed.

The following two examples show that the upper semi finite is more appropriate, for our purpose, than the Vietoris topology.

<u>Example 1.4.9</u> Consider $PC(Y,Z)$ with the graph topology, let $P(Y \times Z)$ be given the Vietoris topology. Then the graph function

$$G : PC(Y,Z) \longrightarrow P(Y \times Z)$$
$$f \longmapsto \text{graph}(f)$$

is not continuous , even if $Y = Z = \mathbb{R}$.

Proof. Let U and V be disjoint open sets of $Y \times Z$. Then we claim that $G^{-1}(\langle U,V\rangle)$ is not open. To show this let $f \in G^{-1}(\langle U,V\rangle)$. Then $G(f) \subseteq U \cup V$ and

$$G(f) \cap U \neq \emptyset \quad \text{and} \quad G(f) \cap V \neq \emptyset .$$

Now for any open W of $Y \times Z$ with $f \in F_W$, $F_W \nsubseteq G^{-1}(\langle U,V\rangle)$. For otherwise, $F_{W \cap U} \subseteq F_W \subseteq G^{-1}(\langle U,V\rangle)$ which is impossible, because the function whose graph is a subset of U would be an element of $G^{-1}(\langle U,V\rangle)$. □

Example 1.4.10 Let $PC(Y,Z)$ be given the graph topology and $P'(Z)$ the Vietoris topology. Then the range function

$$R: PC(Y,Z) \longrightarrow P(Z)$$

is not continuous, even if $Z = \mathbb{R}$.

Proof. Let U and V be open sets of $Y \times Z$. Let $f \in R^{-1}(\langle U,V\rangle)$ where $f(x) \in U \setminus V$ for some $x \in Y$. Then there is no open W with $f \in F_W \subseteq R^{-1}(\langle U,V\rangle)$, since the constant function with value $f(x)$ for all $y \in Y$ is an element of F_W but not of $R^{-1}(\langle U,V\rangle)$. □

Similar argument shows that the graph function is not continuous as a function $G: C(Y,Z) \longrightarrow P'(Z)$, where $P'(Z)$ has the Vietoris topology and $C(Y,Z)$ the graph topology.

Example 1.4.11 Let Y be a Hausdorff space. Then the compact-open topology is contained in G, the graph topology. So the range function $R: C(Y,Z) \longrightarrow P(Z)$ is not continuous, if $P'(Z)$ is given the Vietoris topology and $C(Y,Z)$ is given the compact-open topology.

CHAPTER 2

TOPOLOGIES ON THE SET OF CONTINUOUS PARTIAL FUNCTIONS

2.0 Introduction

In this chapter we consider topologies on PC(Y,Z), the set of all partial maps, partial continuous functions, from Y to Z. We will show that there is no reasonable topology on PC(Y,Z) even for nice spaces $Y = Z = \mathbb{R}$; more precisely there is no splitting, proper, and jointly continuous, admissible, topology on PC(Y,Z).

The technique we use is the following.

(i) Generalise the results on function spaces to the partial maps case, in particular, if U is splitting and T is coarser than U then T is splitting.

(ii) Give a modified compact-open topology on PC(Y,Z), and show that this topology is the smallest topology which is jointly continuous on compacta on PC(Y,Z) when Y is regular or Hausdorff.

(iii) Show that even a small topology, such as the point-wise convergence topology, on PC(Y,Z), is not splitting.

We also introduce the compact connected open topology on PC(Y,Z) which contains the space of partial maps with closed domain and the space of partial maps with open domain as subspaces. We will also conclude that the graph topology on PC(Y,Z), defined in Chapter 1, is jointly continuous.

2.1 Basic results

In this section we will generalise the results of [A-D] on $C(Y,Z)$ to the set of partial maps. We now give the definition of splitting and jointly continuous topology on $PC(Y,Z)$.

Definition 2.1.1 Let T be a topology on $PC(Y,Z)$. T is said to be splitting or proper if for any X the continuity of

$$f : X \times Y \longrightarrow Z$$

implies that of its adjoint

$$\hat{f} : X \longrightarrow PC(Y,Z) , \quad \hat{f}(x)(y) = f(x,y)$$

where $\text{Dom } \hat{f} = \{x \in X \mid (x,y) \in \text{Dom } f \text{ for some } y \in Y\}$ and $\text{Dom } \hat{f}(x) = \{y \in Y \mid (x,y) \in \text{Dom } f \text{ for some } y \in Y\}$.

T is said to be jointly continuous or admissible if the evaluation map

$$e : PC(Y,Z) \times Y \longrightarrow Z$$
$$(f , y) \longrightarrow f(y)$$

is continuous, where e is defined at (f,y) if and only if f is defined at y .

Proposition 2.1.2 A topology T on $PC(Y,Z)$ is jointly continuous if and only if for any space X the continuity of

$$\hat{g} : X \longrightarrow PC(Y,Z)$$

implies that of its adjoint

$$g : X \times Y \longrightarrow Z .$$

Proof. Define $h : X \times Y \longrightarrow PC(Y,Z) \times Y$

$$(x,y) \longrightarrow (\hat{g}(x),y) .$$

Now $e: PC(Y,Z) \times Y \longrightarrow Z$

is continuous and h is continuous so eh is continuous, but $g = eh$. □

It is well known, for $C(Y,Z)$ that if T is jointly continuous and U is finer than T then U is jointly continuous. Also if U is splitting and T is coarser than U then T is splitting. And for topologies T and U on $C(Y,Z)$ where T is splitting and U is jointly continuous then U is finer than T .

The following propositions show that these results are in fact true for partial maps as well.

Proposition 2.1.3 Let T and U be topologies on $PC(Y,Z)$.

(i) If T is jointly continuous and U is finer than T then U is jointly continuous.

(ii) If U is splitting and T is coarser than U then T is splitting.

Proof. (i) Let T be jointly continuous and let U be finer than T .

Then $id_{PC}: PC_U(Y,Z) \longrightarrow PC_T(Y,Z)$

is continuous. So the composite

$$PC_U(Y,Z) \times Y \xrightarrow{id_{PC} \times id_Y} PC_T(Y,Z) \times Y \xrightarrow{e} Z$$

is continuous, that is

$$e: PC_U(Y,Z) \times Y \longrightarrow Z$$

is continuous.

(ii) The proof is similar to (i) . □

Proposition 2.1.4 If T is splitting on $PC(Y,Z)$ and U is jointly continuous then U is finer than T.

Proof. Let U be jointly continuous. Then

$$e: PC_U(Y,Z) \times Y \longrightarrow Z$$

is continuous. But T is splitting so

$$id_{PC} = \hat{e} \ : \ PC_U(Y,Z) \longrightarrow PC_T(Y,Z)$$

is continuous. □

2.2 A jointly continuous topology on $PC(Y,Z)$

In this section we will show that the modified compact open topology on $PC(Y,Z)$ is jointly continuous, in fact is the smallest topology jointly continuous on compacta. Using the results of the last chapter it will be shown that for Hausdorff spaces Y, the graph topology is jointly continuous.

We will also introduce the compact connected open topology. It turns out that for nice spaces Y the compact connected open topology is jointly continuous.

The main result of this chapter is Proposition 2.2.6, which states that the set of partial maps cannot be topologised nicely.

Let U be a topology on $PC(Y,Z)$. Then U is said to be jointly continuous on compacta if $e_C: PC(Y,Z) \times C \longrightarrow Z$ is continuous for all compact C of Y, where e_C is the restriction of the evaluation map.

The compact open topology T on $PC(Y,Z)$ is the one which has as a sub-basis all sets of the form

$$W(C,U) = \{f \in PC(Y,Z) \mid f(C) \subseteq U\}$$

for all compact subsets C of Y and open U subsets of Z, where $f(C) = f(C \cap D_f)$. The topology T is jointly continuous on compacta.

Proposition 2.2.1 If Y is regular or Hausdorff the topology T is the smallest jointly continuous on compacta for $PC(Y,Z)$.

Proof. Let C be a compact subset of Y and

$$e_C : PC(Y,Z) \times C \to Z$$

be the restriction of e.

If U is an open set of Z containing $e_C(f,c) = fc$ then there is a neighbourhood V of c in C with $f(V) \subseteq U$. But then there is a compact neighbourhood A of c in C such that $A \subset V$. Hence $f \in W(A,U)$, $c \in A$ and $e_C(W(A,U) \times A) \subseteq U$.

To show that T is the smallest such topology, assume S is jointly continuous on compacta.

Let $W(C,U)$ be a sub-basic open set of T containing f. Now e_C is continuous so $e_C^{-1}(U)$ is open in $PC(Y,Z) \times C$. But $\{f\} \times C$ is a compact subset of $e_C^{-1}(U)$, so there exists $S' \in S$ containing f and $S' \times C \subseteq e_C^{-1}(U)$. That is $f \in S' \subset W(C,U)$. □

Corollary 2.2.2 If Y is Hausdorff then the graph topology on $PC(Y,Z)$ is jointly continuous.

Remark. It is better to use the test open topology, 3.1.4, to avoid the Hausdorffness condition.

Now we introduce the compact connected-open topology on $PC(Y,Z)$.

Definition 2.2.3 The compact connected open topology on $PC(Y,Z)$ is defined as the one which has as a sub-basis all sets of the form $W(C,U) = \{f \in PC(Y,Z) \mid f(C) \subseteq U$ and $C \cap \mathrm{Dom}\, f$ is compact$\}$ for all compact C of Y and all open U of Z.

Corollary 2.2.4 If Y is locally compact locally connected and Hausdorff then the compact connected open topology is jointly continuous on $PC(Y,Z)$.

One might expect that there is a splitting jointly continuous topology on $PC(Y,Z)$. However this is not the case. The following examples show that the pointwise convergence topology is not splitting. Then by 2.1.3 (ii) any finer topology is not splitting.

Example 2.2.5 (a) Let $PC(Y,Z)$ be given the topology P^o which has as a sub-basis all sets of the form

$$W(c,U) = \{f \in PC(Y,Z) \mid f(c) \in U, f \text{ is defined at } c\}$$

for all $c \in Y$ and all open U of Z . Then P^o is not splitting even if $Z = Y = \mathbb{R}$. To see this consider the function

$$f\colon \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

where $f = P_2 | G$, the restriction of the second projection to G ,
and $G = \mathbb{R}^2 \setminus \{(x,0) \mid x > 0\}$.

Now f is continuous but $\hat{f}\colon \mathbb{R} \longrightarrow PC(\mathbb{R},\mathbb{R})$ is not continuous, since if $c = 0$ and $U = (-a,a)$, then $\hat{f}(0) \in W(0,U)$ and clearly no neighbourhood V of 0 will satisfy that $f(V) \subseteq W(0,U)$.

Example 2.2.5 (b) Let $PC(Y,Z)$ be given the topology P^c which has as a sub-basis all sets of the form

$$W(c,U) = \{f \in PC(Y,Z) \mid f(c) \in U\} ,$$

where $f(c) = f(\{c\} \cap D_f)$, for all $c \in Y$ and open U of Z .

Then P^C is not splitting even for $Z = Y = \mathbb{R}$. To show this let

$f: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, be the second projection

restricted to $G^C = \mathbb{R} \setminus \{0\}$.

Then $\hat{f}: \mathbb{R} \longrightarrow PC(\mathbb{R},\mathbb{R})$ is not continuous.

We now state the main result of this chapter.

Proposition 2.2.6 There is no splitting jointly continuous topology on $PC(Y,Z)$, even for the case $Y = Z = \mathbb{R}$.

Proof. This follows easily from 2.2.5(a) , (b) and 2.2.1. □

Note that the situation for partial maps contrasts with that for all maps in the sense that it seems more difficult to obtain the proper condition.

2.3 Partial maps with open domain and with closed domain.

In the next section we will define a topology on $PC(Y,Z)$, namely the compact open connected topology, which contains the space of partial maps with open domain and the space of partial maps with closed domain as subspaces. So we devote this section to recall some facts about these two spaces, all the results in this section can be found in [B-B1] and [A-B] .

1. Partial maps with closed domain.

A closed domain partial map from Y to Z , called parc map, is a continuous function from a closed subspace C of Y to Z . The set of all parc maps from Y to Z is written as $P_c(Y,Z)$. The topology on $P_c(Y,Z)$ is the compact open topology in which a sub-basis consists of sets of the form

$$W(C,U) = \{ f \in P_c(Y,Z) \mid f[C] = f(C \cap D_f) \subseteq U \}$$

for all compact C of Y and open U of Z .

For a space Z let $Z^{\sim} = Z \cup \{w\}$ where $w \notin Z$. Define a topology on $Z^{\sim}$ as follows:

A set $C \subseteq Z^{\sim}$ is closed in $Z^{\sim}$ if and only if $C = Z^{\sim}$ or C is closed in Z. Then $\{w\}$ is open but not closed and the functor

$$P_c(-,Y) : \text{Top} \longrightarrow \text{Top}$$
$$X \longrightarrow P_c(x,Y)$$

is representable.

<u>Proposition 2.3.1</u> Let Y and Z be any spaces. Then $C(Y,Z^{\sim})$ is homeomorphic to $P_c(Y,Z)$.

<u>Proof</u>. Define $\phi: C(Y,Z^{\sim}) \longrightarrow P_c(Y,Z)$

as
$$\phi(f) = f \mid f^{-1}(Z) .$$

Then it is easy to see that ϕ is a homeomorphism. □

Recall that a pair (X,Y) is called an <u>exponential pair</u> if for any space Z the exponential function

$$\theta: C(X \times Y,Z) \longrightarrow C(X,C(Y,Z)) ,$$

where $\theta(f)(x)(y) = f(x,y)$, is surjective where the topology on $C(Y,Z)$ is the compact open topology.

It is known that (X,Y) is an exponential pair if

(i) $X \times Y$ is a Hausdorff k-space or

(ii) Y is locally compact.

Note that if Y is an initial object of Top, $Y = \emptyset$, then the only element of $C(Y,Z)$ is the empty map and if $Z = \emptyset$ then $C(Y,Z) = \emptyset$. Also the empty map is an element of $P_c(Y,Z)$.

Proposition 2.3.2 (Exponential law for parc maps).

The exponential function

$$\theta: P_c(X\times Y,Z) \longrightarrow C(X,P_c(Y,Z))$$

$$\theta(f)(x)(y) = f(x,y) \ , \quad f \in P_c(X\times Y,Z),\ x \in X,\ y \in Y \ ,$$

is well defined. Moreover

(i) if (X,Y) is an exponential pair then θ is surjective;

(ii) the function θ is continuous if X is Hausdorff and is a homeomorphism into if both X and Y are Hausdorff.

2. Partial map with open domain

A partial map with open domain is a continuous function from an open subspace of Y to Z. Such a function is called paro map, and the set of all paro maps from Y to Z is written as $P_o(Y,Z)$.

Similarly the case of partial map with closed domain, a topology on $P_o(Y,Z)$ is defined as the one which has as a sub-basis all sets of the form

$$W(C,U) = \{ f \in P_o(Y,Z) \mid f(C) \subseteq U \text{ and } C \subseteq D_f \}$$

for all compact C of Y and open U of Z .

For a space Y let $\hat{Y}$ be the space $Y \cup \{w\}$, $w \notin Y$, with the topology in which $U \subseteq \hat{Y}$ is open if and only if $U = \hat{Y}$ or U is open in Y . Then $\{w\}$ is closed but not open in $\hat{Y}$.

Proposition 2.3.3 The function

$$\mu: P_o(Y,Z) \longrightarrow C(Y,\hat{Z})$$

$$\mu(f)(x) = \begin{cases} f x & \text{if } x \in D_f \\ w & \text{otherwise} \end{cases}$$

is a homeomorphism.

Proposition 2.3.4 Exponential law for paro maps.

The exponential function

$$\lambda: P_o(X \times Y, Z) \longrightarrow C(X, P_o(Y,Z))$$
$$\lambda(f)(x)(y) = f(x,y)$$

is well-defined. Furthermore

(i) if (X,Y) is an exponential pair then λ is surjective;

(ii) the function λ is continuous if X is Hausdorff and is a homeomorphism if both X and Y are Hausdorff.

Proposition 2.3.5 (i) If Y is locally compact then the evaluation map

$$e: P_o(Y,Z) \times Y \longrightarrow Z$$
$$(f,y) \longrightarrow f(y)$$

has an open domain.

(ii) If both X and Y are locally compact then the composite mapping

$$P_o(Z,Y) \times P_o(Y,Z) \longrightarrow P_o(X,Z)$$
$$(f,y) \longrightarrow gf$$

is continuous.

2.4 The compact connected open topology on $PC(Y,Z)$

The topology T^c on $PC(Y,Z)$ that we will consider for the rest of this chapter is the compact connected-open topology, which has as a sub-basis all sets of the form

$$W(C,U) = \{f \in PC(Y,Z) \mid f(C) \subseteq U \text{ and } C \cap D_f \text{ is compact}\}$$

for all compact connected C of Y and open U of Z.

It will be shown that for nice space Y, namely locally compact, locally connected and Hausdorff, $(PC(Y,Z),T^c)$ contains $P_c(Y,Z)$ and $P_o(Y,Z)$ as subspaces. Write $PC^c(Y,Z)$ for $(PC(Y,Z),T^c)$.

The compact connected-open topology on $P_c(Y,Z)$, the underlying set, is defined as the one which has a sub-basis all sets of the form

$$W(C,U) = \{ f \in P_c(Y,Z) \mid f(C) \subseteq U \}$$

for all compact connected C of Y and open U of Z, where $f(C) = f(C \cap D_f)$.

Proposition 2.4.1 If Y is locally compact, locally connected and Hausdorff then the compact connected-open topology is splitting and jointly continuous on $P_c(Y,Z)$.

Proof. Since the compact-open topology is splitting and finer than the compact connected-open topology, so the compact connected-open topology is splitting. Also it is easy to see that the compact connected-open topology is jointly continuous. □

Corollary 2.4.2 If Y is locally compact, locally connected and Hausdorff then the compact-open topology coincides with the compact connected-open topology on $P_c(Y,Z)$.

Proposition 2.4.3 If Y is locally compact, locally connected and Hausdorff then $PC^c(Y,Z)$ contains $P_c(Y,Z)$ as a subspace.

Proof. The proof is easy. □

Let F be a subset of $P_o(Y,Z)$. The compact connected open topology on F is defined as the topology with sub-basis all sets of the form

$$W(C,U) = \{ f \in F \mid f(C) \subseteq U \text{ and } C \subseteq D_f \}$$

for all compact connected C of Y and open U of Z .

Proposition 2.4.4 If Y is locally compact, locally connected and Hausdorff then the compact-open topology coincides with the compact connected-open topology on F .

Proof. Let $f \in W(C,U)$ where C is compact and U is open. Then for each $y \in C$ there exists an open connected neighbourhood V_y of y such that $\bar{V}_y$ is compact and $f(\bar{V}_y) \subseteq U$. By compactness of C there exists $y_1,\ldots,y_n$ with $C \subseteq \cup_{i=1}^{n} V_{y_i}$ and $f(\cup_{i=1}^{n} \bar{V}_{y_i}) \subseteq U$. But then $f \in \cap_{i=1}^{n} W(V_{u_i},U) \subseteq W(C,U)$. Hence $W(C,U)$ is open in the compact connected-open topology. □

Lemma 2.4.5 Let U be an open set of Y , C is closed connected. If $U \cap C$ is closed then $U \cap C = \emptyset$ or $C \subseteq U$.

Proof. If $U \cap C$ is non-empty and $C \subseteq U$, then $C \subseteq U^c \cup (C \cap U)$. But this is a contradiction to the connectedness of C . □

Proposition 2.4.6 If Y is locally compact, locally connected and Hausdorff then $PC^c(Y,Z)$ contains $P_o(Y,Z)$ as a subspace.

Proof. Let $f \in P_o(Y,Z)$ and let C be a compact Hausdorff space. If $D_f \cap C$ is compact then $C \subseteq D_f$. □

Proposition 2.4.4 suggests that for locally compact, locally connected Hausdorff space Y it is sufficient to consider as a sub-basis for the compact-open topology on $C(Y,Z)$ sets of the form $W(C,U)$ where C is compact connected and U open.

It is clear that the compact connected-open topology contains the pointwise convergence topology, since the family of all sets $\{f \mid fx \in U\}$ for $x \in Y$ and open U of Z is a sub-basis for the pointwise convergence topology on $C(Y,Z)$.

The following two results are well known as true for the compact-open topology, in fact they are true for the compact connected-open topology.

Proposition 2.4.7 If Z is T_0, T_1 or T_2 then so is $C(Y,Z)$ with the compact connected-open topology.

Proof. If $f(a) \neq g(a)$ and U,V are separating open sets of $f(a)$ and $g(a)$, then $W(\{a\},U)$ and $W(\{a\},V)$ are separating open sets of f and g .

Proposition 2.4.8 If Z is regular then $C(Y,Z)$ is regular, with the compact connected-open topology.

Proof. This is proved by similar arguments to that for the compact-open topology.