CHAPTER 3

Convenient Categories of Spaces

3.0 Introduction

It is known that the category Top , of spaces and continuous functions, is not cartesian closed. However there are some cartesian closed subcategories of Top which are large enough for topologists to work in [B 2,3], ...

For ease of reference and completeness we will in this chapter give some known facts on generating a convenient category of spaces using a non-empty class of spaces, and some facts on the category SEQ of sequential spaces.

A category C is said to be cartesian closed if:

(i) for any X and Y in $|C|$ the product $X \times Y$ is in C ;

(ii) for $Y \in |C|$ the functor

$$- \times Y : C \to C$$

has a right adjoint. That is C has an exponentiation : for each $Z \in |C|$ there is an object $(Z)^Y$ and a morphism

$$e: (Z)^Y \times Y \to Z ,$$

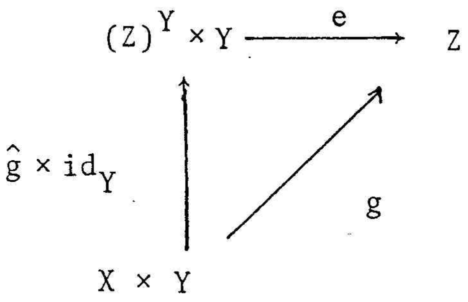
the evaluation morphism, for any object X and any

$$g: X \times Y \to Z$$

there is a unique

$$\hat{g}: X \to (Z)^Y$$

such that $e \circ (\hat{g} \times id_Y) = g$. That is the diagram



commutes.

In 1963, 4, R. Brown [B2,3] discussed the notion of a "convenient category of topological spaces" ([B3] p 245) essentially as a cartesian closed category of topological spaces which contains the usual spaces of homotopy theory, i.e. all CW-complexes. He suggested two equivalent candidates, namely the category of Hausdorff k-spaces and continuous maps, the the category of Hausdorff spaces and k-continuous maps.

In the mid-1960's D. Husemoller and J.C. Moore showed in unpublished work how to remove the Hausdorff condition. The convenience of the category of Hausdorff k-spaces was proclaimed in 1967 by N.E.Steenrod [St] . In section 1 we study in detail ways of generating a convenient category of spaces using a class of space.

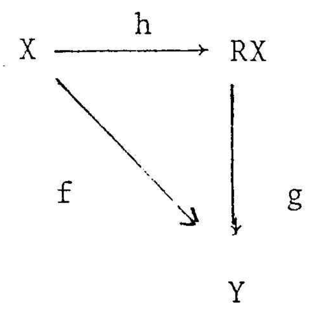
3.1 Generating a convenient subcategory of Top

In this section we will discuss in detail ways of generating a convenient subcategory of Top , using a non-empty class $\underline{C}$ of spaces. We will assume that $\underline{C}$ contains the terminal object $1 = \{*\}$. The results in this section may be found in Booth and Tillotsen [B-T] , Brown [B2,3], Day [D1,2] , Lamartin [La], Vogt [V] and Wyler [W2] .

By an epireflective subcategory C of Top we mean a full subcategory with the following property:

For object X of C, every subspace of X is in C and the product of spaces in C is also in C.

Lemma 3.1.1 Let C be a subcategory of Top. Then C is an epireflective subcategory if and only if for each space X there is an object RX of C and a surjective $h: X \to RX$ such that any morphism $f: X \to Y$, $Y \in |C|$, factors uniquely through h. That is $g \circ h = f$ for a unique $g: RX \to Y$



Examples. The following are epireflective subcategories of Top.

1) Top.
2) T_1 the category of T_1-spaces.
3) Haus the category of Hausdorff spaces.
4) R the category of regular spaces, assuming regular implies T_1.

Let C be a category of spaces. For X in C let aX be the set X with the final topology with respect to all continuous functions $A \to X$, $A \in \underline{C}$. That is, aX is the set X with the finest topology for which all continuous functions $A \to X$ are continuous.

The space X is called an <u>a-space</u> if $aX = X$, and a is a functor $C \to C$.

We denote by $\mathrm{Conv}_{\underline{C}}$ the category of a-spaces in C, the category generated by $\underline{C}$. It is easy to see that a is a right adjoint to the inclusion functor $i\colon \mathrm{Conv}_{\underline{C}} \to C$.

So $\mathrm{Conv}_{\underline{C}}$ is a coreflective subcategory of C. In particular if C = Top then $\mathrm{Conv}_{\underline{C}}$ is the <u>coreflective hull</u> of $\underline{C}$, that is the smallest coreflective subcategory of Top containing each object of $\underline{C}$.

The category $\mathrm{Conv}_{\underline{C}}$ is said to be a compactly generated category if it can be generated by a class of compact Hausdorff spaces in C and $\mathrm{Conv}_{\underline{C}}$ is coreflective in C. By letting $\underline{C}$ be the class of all compact Hausdorff spaces in C, $\mathrm{Conv}_{\underline{C}}$ is the largest compactly generated, coreflective, subcategory of C. Also if C is Haus then $\mathrm{Conv}_{\underline{C}}$ is the category of k-spaces.

Let $\underline{C}$ be a class of spaces. The category generated by $\underline{C}$, $\mathrm{Conv}_{\underline{C}}$, is said to be <u>convenient</u> if $\mathrm{Conv}_{\underline{C}}$ is cartesian closed and contains the underlying spaces of all CW-complexes, metric spaces and differentiable manifolds.

<u>Proposition 3.1.2</u> Let $\underline{C}$ be a class of spaces such that $\mathrm{Conv}_{\underline{C}}$ is a cartesian closed category. Then the following are equivalent.

1) N^+ is an object of $\mathrm{Conv}_{\underline{C}}$.
2) All sequential spaces are in $\mathrm{Conv}_{\underline{C}}$, (c.f. next section for definition).
3) $\mathrm{Conv}_{\underline{C}}$ is a convenient category of topological spaces.

Proposition 3.1.3 The category of sequential spaces is the smallest convenient category of topological spaces.

We now give some examples of categories of a-spaces which are cartesian closed, in fact some are convenient categories. In the following let C = Top .

1. The category of sequential spaces is generated by any of the following classes of spaces.

a) $\underset{\sim}{C}$ = the class of compact countable Hausdorff spaces.

b) $\underset{\sim}{C} = \{N^+\}$

c) $\underset{\sim}{C} = \{[0,1]\}$

2. Let $\underset{\sim}{C}$ be the class of finite discrete spaces. Then the cartesian closed category generated by $\underset{\sim}{C}$ is the category of discrete spaces, this category is not convenient.

3. If $\underset{\sim}{C}$ is the empty class, then the category generated by C is the category of discrete spaces.

4. If $\underset{\sim}{C}$ is the class of all n-cubes , I^n where n is a finite positive integer, then the category generated by $\underset{\sim}{C}$ is any one of the following:

a) The category of all topological sums of cubes I^n and the quotients of those sums.

b) The category of all underlying spaces of CW-complexes and their quotients.

5. Let $\underset{\sim}{C}$ be the class of all compact Hausdorff spaces. Then $Conv_{\underset{\sim}{C}}$ is the convenient category of compactly generated spaces.

Remark 3.1.4 The convenient category of 5. above is the now well-accepted category of compactly generated spaces which gives the non-Hausdorff version of k-spaces referred to in the introduction of this chapter. The topology on function space is the test open topology which is defined as follows:

Let X and Y be objects of $Conv_{\underline{C}}$. A sub-basis for the topology consists of all sets of the form

$$W(t,U) = \{f \in Conv_{\underline{C}}(X,Y) \mid ft(C) \subset U\} \quad \text{for all } C \in \underline{C}$$

continuous $t: C \to X$ and open U of Y .

3.2 The category of sequential spaces SEQ

In Chapter 4 we will discuss the embedding of SEQ into the quasitopos SuSEQ , of subsequential space, which contains SEQ as a reflective subcategory. Also in Chapter 5 we will investigate Johnstone [J2] embedding of SEQ in the topos Proc. Therefore we devote this section to recalling some facts about the category SEQ .

All facts given in this section may be found in [B-T] [B4], [Du] , [F1,2,3,4] and [Ti] .

Definition 3.2.1 Let X be a topological space. A subset $U \subset X$ is sequentially open if every sequence converges to a point of U is eventually in U . The space X is said to be a sequential space if every sequentially open set is open.

Let SEQ be the category of sequential spaces and continuous function. We now recall some facts about SEQ .

The category SEQ contains nice spaces : first countable spaces are sequential. In particular metric spaces, CW-complexes and discrete spaces are sequential.

Remark Some disadvantages of subsequential spaces are the following:

1. The continuous image of a sequential space need not be sequential.

2. There is a compact Hausdorff space which is not sequential.

3. The usual topological product of two sequential spaces need not be sequential.

4. Subspaces of a sequential space need not be sequential.

Proposition 3.2.2 1. Quotients of a sequential space are sequential.

2. The continuous open or closed image of a sequential space is sequential.

3. If a product space is sequential then so is each of its factors.

4. Let $\{X_i\}_{i \in I}$ be a family of sequential spaces. Then $X = \cup_{i \in I} X_i$, disjoint topological sum, is sequential.

5. The category SEQ is cocomplete.

6. A closed, or open, subspace of a sequential space is sequential.

7. Let X be a space with the final topology with respect to a family $(f_i : X_i \to X)_{i \in I}$, where $X_i \in |SEQ|$ for each i . Then X is sequential.

8. Let $X \in |Top|$. Then X has a unique sequential limit, that is each sequence has a unique limit, if and only if the diagonal is closed in $X \times X$.

Proposition 3.2.3 SEQ is a coreflective subcategory of Top .

Proof. The category SEQ is a full subcategory of Top . We need to define a right adjoint $S: Top \to SEQ$ to the inclusion functor I . Let (X,T) be a topological space. Then it is easy to see that the sequentially open sets form a new topology on X , which is the smallest sequential topology containing T . Thus we define S on objects.

Now let $Y \in |Top|$ and let $X \in |SEQ|$. If $f: X \to Y$ is continuous and U is sequentially open in Y , then clearly $f^{-1}(U)$ is sequentially open. That is $f: X \to S(Y)$ is continuous. This establishes the required isomorphism.

$$Top\,(IX,Y) \cong SEQ(X,S(Y)) . \quad \square$$

Recall, see the remark above, that subspaces and products of sequential spaces need not be sequential. However the functor S , proposition 3.2.3 , can be used to sequentialise the subspace and the product space. Then for a subset Y of a sequential space X, $S(Y)$ is a subspace of X , and for $Z \in |SEQ|$, $S(Y \times Z)$ is the categorical product in $|SEQ|$.

Proposition 3.2.4. SEQ is cartesian closed.

CHAPTER 4

The Category of Subsequential Spaces as a Quasitopos

4.0 Introduction

We have shown, Chapter 2, that the set PC(X,Y) of partial maps from a space X to a space Y can not be topologised nicely. So we gave further evidence that the notion of topology is not always appropriate and is even inadequate

One main aim of this chapter is to try to answer the following:

Can we solve the problem of finding a space which contains $P_c(X,Y)$ and $P_o(X,Y)$ as subspaces in a quasitopos or do we need a topos?

Our claim is that we need a topos. The following technique is used to justify this claim.

Let Y be a sequential space. Recall (2.3) the spaces $Y^{\sim}$ and $\hat{Y}$ which gave representability for partial maps with closed domain and for partial with open domain.

(1) We give an explicit description of the object $Y^{!}$ that represents strong partial morphisms in SuSEQ.

(2) We give a description of the subsequential structure on $Y^{\sim}$ and $\hat{Y}$. We conclude that

$$Y^{\sim} \longrightarrow Y^{!}$$

and

$$\hat{Y} \longrightarrow Y^{!}$$

are monics in SuSEQ which are not strong monics. That is neither $Y^{\sim}$ nor $\hat{Y}$ are subspaces of $Y^{!}$

Hence in SuSEQ , in general $P_c(X,Y)$ and $P_o(X,Y)$ are not subspaces of SPart(X,Y) $\cong$ SuSEQ $(X,Y^{!})$, where SPart(X,Y) denote the strong partial morphisms from X to Y . So in a quasitopos, in particular in SuSEQ , the problem remains unsolved.

However, in a topological topos and in particular in Proc (cf. Chapter 5-6) subobject , "subspace" , include all continuous injections, so the problem is solved in Proc. But the whole notion of "subspaces" is not so clear in Proc.

4.1 Toposes

In this section we will recall some basic definitions and facts about toposes. The following material may be found in [B-W] , [Fr] , [J1] , [K-W] , [K-M] , [M-R] or [Wr] .

Definition 4.1.1 An elementary topos is a category $\mathcal{E}$ satisfying the following conditions:

(i) $\mathcal{E}$ is finitely complete, has all finite limits;

(ii) $\mathcal{E}$ is Cartesian closed;

(iii) $\mathcal{E}$ has a subobject classifier.

By a subobject classifier we mean an object Ω together with a morphism true: $1 \to \Omega$, such that for any monic $f: X \rightarrowtail Y$ there is a unique morphism $\phi_f: Y \to \Omega$, called the classifying map, making

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \big\downarrow & & \big\downarrow \phi_f \\ 1 & \xrightarrow[\text{true}]{\qquad} & \Omega \end{array}$$

a pullback diagram.

The term "elementary" in the definition is just to distinguish between a Grothendieck topos and a topos in this sense. A Grothendieck topos is a category which is equivalent to shv(C,J) where (C,J) is a site, a small category C equipped with a Grothendieck topology J (definition in Chapter 5).

Definition 4.1.2 Let E be a category with finite products. A power object of an object X of E is an object $P(X)$ together with a monic morphism

$$\varepsilon : \varepsilon_X \longrightarrow P(X) \times X$$

such that for any monic

$$r : R \longrightarrow Y \times X$$

a relation, there is a unique morphism

$$\phi_r : \; Y \longrightarrow P(X)$$

making

$$\begin{array}{ccc} R & \xrightarrow{\quad r \quad} & Y \times X \\ \big\downarrow & & \big\downarrow \phi_r \times I_Y \\ \varepsilon_X & \xrightarrow[\varepsilon]{\qquad} & P(X) \times X \end{array}$$

a pullback diagram.

A category $\mathcal{E}$ is said to have power objects if a power object exists for each object of $\mathcal{E}$.

Proposition 4.1.3 [Wr] A category $\mathcal{E}$ is a topos if and only if $\mathcal{E}$ is finitely complete and has power objects.

Proof. Let $\mathcal{E}$ be a topos. Then for any object X let $P(X) = \Omega^X = \mathcal{E}(X,\Omega)$ and let

$$\varepsilon : \varepsilon_X \longrightarrow \Omega^X \times X$$

be the morphism whose classifying map is e , the evaluation morphism. That is the diagram

$$\begin{array}{ccc} \varepsilon_X & \xrightarrow{\ \varepsilon\ } & \Omega^X \times X \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle e} \\ 1 & \xrightarrow[\text{true}]{} & \Omega \end{array}$$

is a pullback. Then it is not hard to see that Ω^X and ε is a power object of X .

Conversely if $\mathcal{E}$ is finitely complete and has power objects, then it is easy to check that

$$\varepsilon_1 \longrightarrow \Omega^1 \times 1 \cong \Omega^1 = P(1)$$

is in fact the subobject classifier.

Also it is not hard to show that $\mathcal{E}$ is Cartesian closed. □

In the following proposition we list some basic properties of a topos. The proofs may be found in the references given above.

Proposition 4.1.4 Let $\mathcal{E}$ be a topos. Then the following are true:

1. A morphism f in $\mathcal{E}$ is an iso if and only if f is monic and epic; that is $\mathcal{E}$ is balanced.
2. Every monic is an equaliser.
3. Every epic is a coequaliser.
4. Every morphism has a unique factorisation as an epic followed by monic.
5. Partial morphisms are representable; a partial morphism $f: X \rightharpoonup Y$ in a category $\mathcal{E}$ is a diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & Y \\ {\scriptstyle m}\big\downarrow & & \\ X & & \end{array}$$

with m monic. Partial morphisms with codomain Y are said to be representable if there is a monic $\eta: Y \to \tilde{Y}$ such that for any partial morphism $f: X \rightharpoonup Y$ there exists a unique $\tilde{f}: X \to \tilde{Y}$ making

$$\begin{array}{ccc} X' & \xrightarrow{f} & Y \\ {\scriptstyle d}\big\downarrow & & \big\downarrow{\scriptstyle \eta} \\ X & \xrightarrow[\tilde{f}]{} & \tilde{Y} \end{array}$$

a pullback diagram.

6. For any topos $\mathcal{D}$, $\mathcal{E} \times \mathcal{D}$ is a topos.

7. For any object X of $\mathcal{E}$, the category $\mathcal{E} \downarrow X$ of objects of $\mathcal{E}$ over X is a topos.

8. The following are equivalent (see 4.2 for definitions):

   (i) Regular monic.

   (ii) Extremal monic.

   (iii) Strong monic.

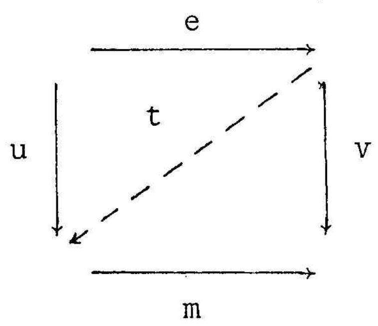
   (iv) Monic.

4.2 Regular monic, extremal monic and strong monic

Before defining quasitoposes and recalling some facts about them we will discuss various kinds of monics for the convenience of the reader, because these materials are not available in standard text books. We will show that in SEQ the notions of strong, regular and extremal monics are equivalent.

Definition 4.2.1 [K] Let m be a monic in a category C.

1. m is called a strong monic if for any commutative diagram of the form



with e epic, there exists a unique t such that $v = mt$ and $u = te$.

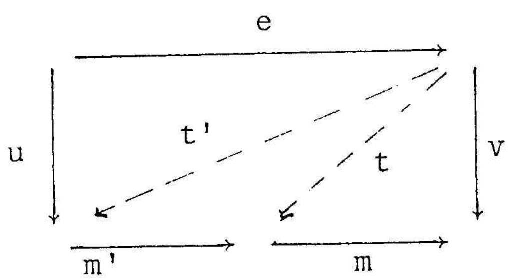
2. m is called an extremal monic if for $m = ue$ with e epic, then e is an iso.

3. m is called a <u>regular monic</u> if m is an equaliser in C .

Strong epic, extremal epic and regular epic are defined dually.

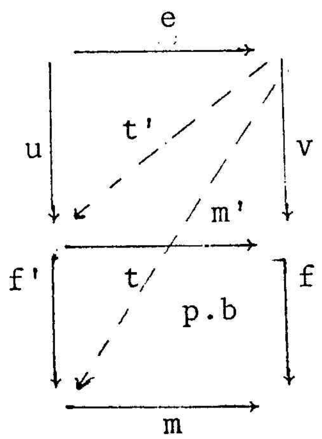
<u>Proposition 4.2.2</u> [K] Strong monics are closed under composition and pullback.

<u>Proof</u>. For composition, let m and m' be strong monics. Then for any commutative diagram



with e epic, there exists a unique t such that $v = mt$ and $m'u = te$. Again there is a unique t' such that $u = t'e$ and $t = m't'$. So t' is the unique morphism with $v = (mm')t'$ and $u = t'e$.

For pullback, let m' be the pullback of a strong monic m . Then for any epic e with $m'u = ve$, consider the commutative diagram

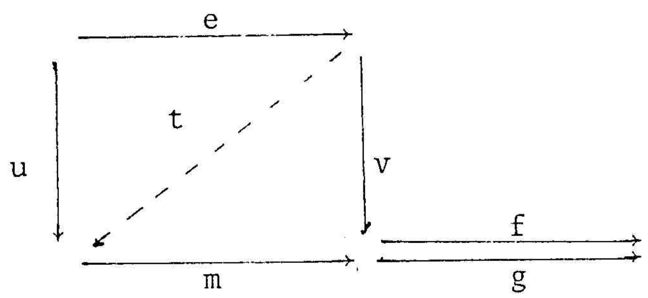


Then there exists a unique t with $fv = mt$. But the bottom square is a pullback. So there is a unique t' with $v = m't'$ and $f = f't'$. So $m'u = ve = m't'e$.
Hence $v = m't'$ and $u = t'e$. □

<u>Proposition 4.2.3</u> [K] (i) if m is a regular monic, then m is a strong monic.

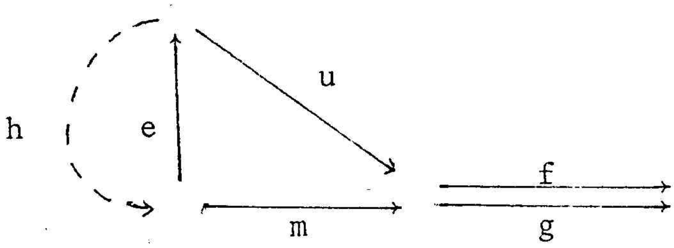
(ii) If m is a regular monic, then m is an extremal monic.

<u>Proof</u>. Let m be the equaliser of f and g . Let u , v and e be any morphisms such that $mu = ve$ and e is epic.



Now $fve = gve$, so $fv = gv$.
But m is an equaliser so v factors uniquely through m , that is $v = mt$ for a unique t . Also $mu = ve = mte$.
So $u = te$.

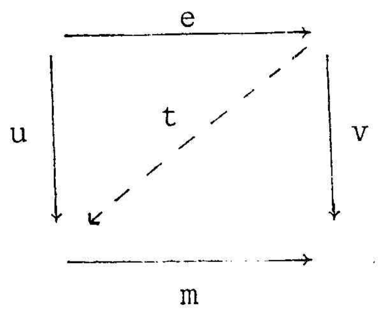
(ii) Again let m be the equaliser of f and g and assume $m = ue$ for some epic e . Then $fue = fm = gm = gue$, so $fu = gu$. But m is an equaliser so there is a unique h with $mh = u$.



Then m = ue = mhe , so he = id . That is e is an epic and has a left inverse, so e is an iso . □

Proposition 4.2.4 [K] If every epic in a category is a strong epic then every monic is a strong monic.

Proof. Let m be a monic such that mu = ve for some u , v and e with e epic. Then e is a strong epic and the existence of the unique t follows.



□

The following proposition gives a translation of topological terms into categorical terms and vice versa.

Proposition 4.2.5 [H1] Let f be a morphism in Top . Then

1) f is monic if and only if f is injective.

2) f is epic if and only if f is surjective.

3) The following are equivalent:

   a) f is a strong monic.

   b) f is a homeomorphism into .

c) f is an extremal monic.

d) f is a regular monic.

4) f is an iso if and only if f is a homeomorphism.

Similar translations for the category Haus , of Hausdorff spaces and continuous functions, are given in the following proposition.

<u>Proposition 4.2.6</u> [H1] Let f be a morphism in Haus. Then

1) f is a monic if and only if f is injective.

2) f is epic if and only if f is dense.

3) f is a strong monic if and only if f is a homeomorphism into .

4) f is an extremal monic if and only if f is a closed homeomorphism into .

5) f is an iso if and only if f is a homeomorphism.

The following, well known, example shows that in Haus various kinds of monic are not equivalent.

Let $i: Q \to \mathbb{R}$ be the inclusion, where Q is the set of rationals and $\mathbb{R}$ is the set of real numbers. Then i is not an extremal monic, since i is epic but not iso .

<u>Proposition 4.2.7</u> Let $m: X \to Y$ be a monic in SEQ . Then the following are equivalent:

1) m is a regular monic.

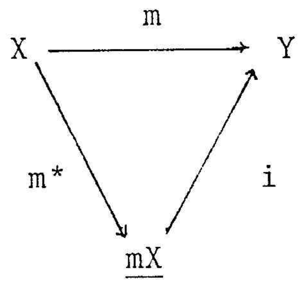
2) m is an extremal monic.

3) m is a homeomorphism into .

4) m is a strong monic.

<u>Proof</u>. In any category, a regular monic is extremal, (4.2.3 (ii)).

Assume m is an extremal monic. Then there is a commutative diagram

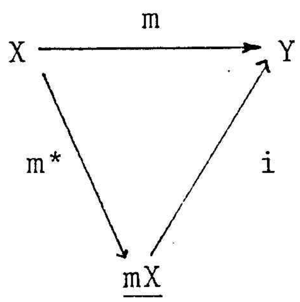


with m^* epic, i monic and $\underline{mX}$ has the smallest sequential topology containing the relative topology on mX . So m^* is an iso and i = m . Now $i: \underline{mX} \to Y$ is the equaliser of χ_i and c_1

$$\underline{mX} \xrightarrow{\ i\ } Y \underset{c_1}{\overset{\chi_i}{\rightrightarrows}} \{0,1\}$$

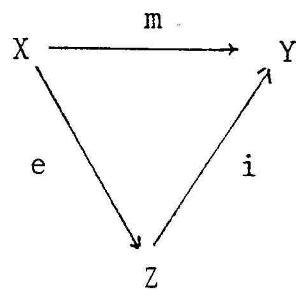
where c_1 is the constant function with value 1 and {0,1} is the indiscrete space. So m is a regular monic.

Let m be an extremal monic. Then the following diagram



commutes in SEQ . Since m^* is epic so m^* is an iso in SEQ . That is m is a homeomorphism into .

Assume m is a homeomorphism into. Let m = ie for any epic e and monic i , that is the diagram

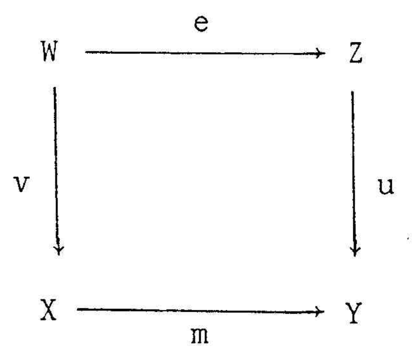


commutes. Then e is surjective, injective and

$$Z \xrightarrow{i^*} \underline{mX} \xrightarrow{(m^*)^{-1}} X$$

is an inverse of e . So e is an iso in SEQ .

Now assume m is a homeomorphism into. Let

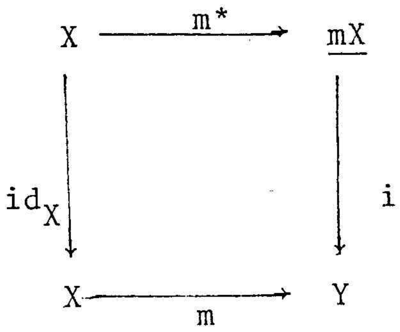


be any commutative diagram in SEQ , with e epic. Then $u(Z) \subseteq m(X)$, so the composite $t = (m^*)^{-1} u^*$

$$Z \xrightarrow{u^*} \underline{mX} \xrightarrow{(m^*)^{-1}} X$$

is the unique morphism with u = mt and v = te . So m is a strong monic.

Finally to show strong monic implies homeomorphism into; consider the following diagram



Then there exists a unique $t: \underline{mX} \to X$ which is the inverse of m. So m is a homeomorphism into . □

4.3 Quasitoposes

Quasitoposes were introduced by Penon [P] as a generalisation of toposes, and were also studied by Wyler [W3] .

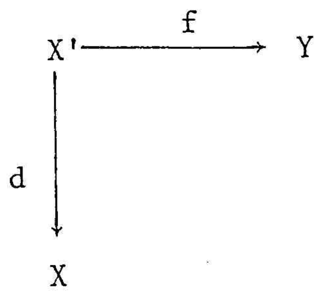
Definition 4.3.1 A quasitopos C is a category satisfying the following properties.

(i) C is finitely complete and cocomplete.

(ii) C is Cartesian closed.

(iii) Strong partial morphisms are representable in C .

By a strong partial morphism $f: X \to Y$ we mean a diagram



with d strong monic. Condition (iii) means for each Y in $|C|$, there is an object $Y^!$ and strong monic $\eta: Y \to Y^!$ such that for a strong partial morphism $f: X \to Y$ with codomain Y , there is a unique $f^!: X \to Y^!$ making

$$\begin{array}{ccc} X' & \xrightarrow{f} & Y \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle \eta} \\ X & \xrightarrow[f^{!}]{} & Y^{!} \end{array}$$

a pullback diagram.

Clearly a quasitopos C is a topos if and only if each monic in C is a strong monic.

We now give some examples of quasitoposes:

1) A relatively pseudo complemented lattice, known as a Heyting algebra [W3] .

2) The category of subsequential spaces (see next section).

In the following proposition we collect some properties of a quasitopos; these facts may be found in [P] or [W3] .

<u>Proposition 4.3.2</u> Let C be a quasitopos. Then the following are true:

(i) If f is a strong monic and epic in C , then f is an iso.

(ii) A strong monic is an equaliser.

(iii) A strong epic is a coequaliser.

(iv) Every morphism f has a factorisation as $f = mie$, where m is a strong monic , e is a strong epic, and i is an epic and monic.

(v) The factorisation above is preserved by pullback.

(vi) Colimits are preserved by pullback.

4.4 Subsequential spaces

The main concern of this chapter is to use the results of Kuratowski [Ku] to show that the category SuSEQ , of subsequential spaces, is Cartesian closed.

Recall [Ki] that an L-space is a pair (X,C) where X is a set and C is a family of apirs $((x_n),x_\infty)$ of a sequence (x_n) of X and a point x_∞ in X , thought of as a sequence (x_n) converging to x_∞ , such that

(i) For each $x \in X$, $((x),x)$ is in C , that is the constant sequence with value x converges to x ;

(ii) If $((x_n),x_\infty)$ is in C , then for any subsequence (x_{n_k}) of (x_n) , $((x_{n_k}),x_\infty)$ is in C .

An L-space which satisfies (*) , below, is called a subsequential space. This term is due to Johnstone [J2] .

(*) For any sequence (x_n) of X , if x_∞ is a point of X such that every subsequence (x_{n_k}) of (x_n) has a subsequence $(x_{n_{k_\ell}})$ such that $((x_{n_{k_\ell}}),x_\infty)$ is in C , then $((x_n),x_\infty)$ is in C .

Remark 4.4.1 a) A subsequential space was earlier called an L*-space by Kisynski [Ki] and Kuratowski [Ku] .

b) The notion of sequential convergence is not properly described by the notion of L-space. For example sequences which are eventually constant need not converge. Consider a set X with more than one point with $(x_n) \to x_\infty$ if and only if $x_n = x_\infty$ for all n , for x_∞ in X . Then X

is an L-space. However if $x, y \in X$, $y \neq x$, then the sequence $y,x,x,\ldots$ does not have a limit in X . This is an example of an L-space which is not a subsequential space.

We now state some facts about L-spaces and subsequential spaces due to Kisynski [Ki] .

Let X be an L-space with C as its family of convergent sequences. Let G be a subset of X . Then G is open if whenever (x_n) converges to a point of G , then all but finitely many terms of x_n are elements of G .

Proposition 4.4.3 [Ki] Let $T(C)$ be the set of all open sets of X . Then $T(C)$ is a topology on X , making X a T_0-space.

An L-space (X,C) is said to have unique limits or be sequentially Hausdorff if when $((x_n),x)$ and $((x_n),y)$ are in C , then $x = y$. Note that in Dudley [Du] L-spaces are sequentially Hausdorff. The following fact also may be found in [Du] .

Proposition 4.4.4 [Ki] If (X,C) is a sequentilly Hausdorff subsequential, then $C(T(C)) = C$ where $C(T(C))$ is the family of convergent sequences in $T(C)$.

The following example shows that the sequentially Hausdorff condition is necessary.

Example 4.4.5 [J2] Let (X,C) be such that $X = \{x,y,z\}$ and C is defined as follows:

For any sequence (x_n) , $((x_n),x)$ and $((x_n),y)$ are in C and $((x_n),z)$ is in C if finitely many terms of the sequence (x_n) are x. Then clearly (X,C) is a subsequential space, since condition (i) and (ii) are trivially satisfied. For (*) if $((x_n),x_\infty) \notin C$, then $x_\infty = z$ and the constant sequence (x) is a subsequence of (x_n) with no subsequence converging to $z = x_\infty$. Now $T(C)$ is indiscrete, so $C(T(C)) \neq C$. □

We now give some examples of subsequential spaces.

1) Sequential spaces. Recall that a topological space X is said to be sequential if every sequentially open set is open (cf. Chapter 3). For a sequential space X, let C be the family of all convergent sequences in X. Then (i) and (ii) are satisfied. For (*) see Proposition 4.4.9 below.

In particular all pseudo-metric spaces may be given the structure of subsequential space such that convergence is the usual convergence.

2) Pointwise convergence. Let Z be the set of functions $X \to Y$ where (Y,C_Y) is a subsequential and X an arbitrary topological space. Then (Z,C_p) is subsequential where in C_p, $f_n \to f_\infty$ if and only if for all $x \in X$, $((f_n(x)),f_\infty(x)) \in C_Y$.

Definition 4.4.6 [Ku] Let X be a subsequential space, $Y \subseteq X$. Define the closure $\bar{Y}$ of Y as

$$\bar{Y} = \{x \in X \mid \text{there exists } (x_n) \text{ in } Y \text{ with } x_n \to x\}.$$

Definition 4.4.7 [Ku] A function $f: X \to Y$, between subsequential spaces is continuous if it preserves convergent sequences.

Proposition 4.4.8 [Ku] Let X and Y be subsequential spaces. Then $f: X \to Y$ is continuous if and only if $f(\bar{Z}) \subseteq \overline{f(Z)}$ for each subset Z of X .

Let SuSEQ denote the category of subsequential spaces and continuous functions. Let $X \in |\text{SuSEQ}|$, and let U be a subset of X . Then U is sequentially open if and only if whenever a sequence (x_n) converging to a point in U then (x_n) is eventually in U . The following proposition relates the category SEQ , of sequential spaces, and SuSEQ .

Proposition 4.4.9 [Hy] SEQ is a reflective subcategory of SuSEQ .

Proof. Define

$$k: \text{SEQ} \to \text{SuSEQ}$$

as follows:

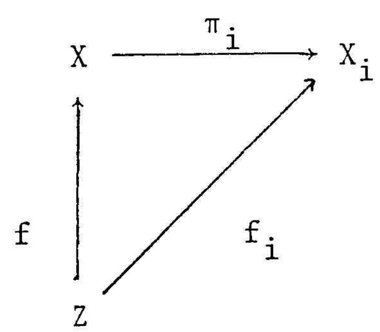
for (X,T) in $|\text{SEQ}|$, $k(X,T) = (X,C(T))$ and k is the identity on morphisms. We show $(X,C(T))$ is a subsequential space. Clearly condition (i) and (ii) are satisfied. For the proof of (*) , assume that $((x_n),x_\infty) \notin T(C)$. Then there exists a neighbourhood U of x_∞ such that $E = \{n \mid x_n \notin U\}$ is infinite. So E determines a subsequence of (x_n) with no convergent subsequence. Thus $(X,C(T))$ is subsequential, and so we have defined i on objects. Clearly k is full and faithful functor. So we can consider SEQ as a full subcategory of SuSEQ .

Now to define a left adjoint k' to k , let X be an object of SuSEQ . Then the sequentially open sets define a topology which is sequential. Clearly this defines a functor k' which is a left adjoint to k . □

Remark. For the rest of the thesis we will refer to the functors k: SEQ → SuSEQ and k': SuSEQ → SEQ of Proposition 4.4.9.

Proposition 4.4.10 [Ku] The category SuSEQ has arbitrary products.

Proof. Let (X_i, C_i) be objects of SuSEQ , $i \in I$. Let $X = \Pi_{i\in I} X_i$ and define convergence in X componentwise. That is a sequence $(x_n^{\ i})$ in X converges to $(x_\infty^{\ i})$ if and only if $(x_n^{\ i}) \to x_\infty^{\ i}$ for all $i \in I$. Then it is easy to see that X is a subsequential space and the projection $\pi_i: X \to X_i$ is continuous for all i . Also for a subsequential space Z and a family $f_i: Z \to X_i$ of SuSEQ morphisms, define $f: Z \to X$ as $f(x) = (f_i(x))_{i\in I}$.



Then trivially f is continuous and $f_i = \pi_i \circ f$ for all i . Thus SuSEQ has arbitrary products. □

Definition 4.4.11 Let (X,C) be an object of SuSEQ , let Y be a subset of X . Then (Y,C_Y) is said to be a subsequential subspace of X if C_Y consists of those and only those $((Y_n),y_\infty)$ which are in C_X .

Lemma 4.4.12 If (X,C) is an object of SuSEQ , then $C \cong$ SuSEQ (N^+,X) .

Proposition 4.4.13 [Ku] The category SuSEQ is Cartesian closed.

Proof. Let X and Y be objects of SuSEQ . Then the formula

$$\mathrm{SuSEQ}(N^+,Y^X) \cong \mathrm{SuSEQ}(N^+ \times X,Y)$$

gives the subsequential structure on Y^X , if it exists.

Claim. The subsequential structure C on Y^X can be defined as follows:

$$((f_n),f\ ) \text{ is in } C \text{ if and only if for each } ((x_n),x_\infty) \text{ in } C_X ,\ (f_n(x_n),f\ (x_\infty)) \in C_Y . \qquad (**)$$

Proof of the claim. We show that C defines a subsequential structure on Y^X . Clearly condition (i) is satisfied. For (ii) let (f_n) be a sequence which converges to f . Now for any subsequence (f_{n_k}) of (f_n) and $((x_n),x_\infty)$ in C_X , define a new sequence (y_m) as $y_m = x_n$ for all $k_{n-1} < m \leq k_n$. Then $((y_n),y_\infty)$ is in C_X . So $((f_n(y_n),f(x_\infty))$ is in C_Y . But $f_{n_k}(x_n)$ is a subsequence of $(f_n(y_n))$, so $(f_{n_k}(x_n),f(x_\infty))$ is in C_Y .

To show condition (*) . Let (f_n) be a sequence such that f_n does not converge to f . Then there exists $((x_n),x_\infty)$ in C_Y such that $(f_n(x_n),f(x_\infty))$ is not in C_Y . So there exists a subsequence $(f_{n_k}(x_{n_k}))$ such that none of its subsequences converge to $f(x_\infty)$. Now (f_{n_k}) is a subsequence of (f_n) such that none of its subsequences converge to f . For otherwise there is a subsequence $(f_{n_{k_i}})$ of (f_{n_k}) with $(f_{n_{k_i}}) \to f$. But then $((f_{n_{k_i}}(x_{n_{k_i}}))$, $f(x_\infty))$ is in C_Y .

Now we need to show that $C = \mathrm{SuSEQ}(N^+, Y^X) \cong \mathrm{SuSEQ}(N^+ \times X, Y)$.

Clearly $C \subseteq \mathrm{SuSEQ}(N^+ \times X, Y)$.

For the converse let $\hat{f} \in \mathrm{SuSEQ}(N^+ \times X, Y)$. That is for $i_n \to i_\infty$ and $x_n \to x_\infty$, $f_{i_n}(x_n) \to f_{i_\infty}(x_\infty)$. But (i_n) is a convergent sequence in N^+ . So (i_n) is eventually constant or contains a subsequence which is also a subsequence of (n) . If (i_n) is eventually constant, then (f_{i_n}) is eventually constant and so $((f_{i_n}), f_{i_\infty}) \in C$. Now assume (i_n) is not eventually constant, then $i_\infty = \infty$. Assume $((f_{i_n}), f_{i_\infty}) \notin C$. Then there is a subsequence $(f_{i_{n_k}})$ which contains no subsequence converging to f_{i_∞} . But (i_{n_k}) is a convergent sequence of N^+ , which is not eventually constant, so (i_{n_k}) contains a subsequence (i_{n_k}) which is also a subsequence of (n). That is $i_{n_{k_\ell}} \to \infty$. So $f_{i_{n_k}} \to f_\infty = f_{i_\infty}$, a contradiction. Hence $((f_{i_n}), f_{i_\infty}) \in C$. So $C = \mathrm{SuSEQ}(N^+ \times X, Y)$. This completes the proof of the claim.

Clearly the evaluation map

$$e\colon Y^X \times X \to Y$$

is continuous, where Y^X is given the subsequential structure (**) .

Let Z be any object of SuSEQ . We will now show that

$$\wedge\colon \mathrm{SuSEQ}(X\times Y,Z) \cong \mathrm{SuSEQ}(X,\mathrm{SuSEQ}(Y,Z))$$

where $\hat{f}(x)(y) = f(x,y)$. For $f \in \mathrm{SuSEQ}(X\times Y,Z)$ and $x \in X$, it is easy to see that $\hat{f}(x)$ and $\hat{f}$ are continuous. Also if f is continuous then $\hat{f}$ is continuous. It is easy to see that $\wedge$ is an isomorphism in SuSEQ . □

4.5 The category SuSEQ is a topological quasitopos

In this section we will show that SuSEQ is a topological category. First we recall the definition of a topological category.

A concrete category C is one with a faithful functor u: C → Set. We interpret this as a category whose objects are pairs (X,C) where X is a set and C is a C-structure on X and for objects X and Y C(X,Y) ⊆ Set(X,Y) . Also the operation of composition is the usual composition of functions, and u is the forgetful functor.

Definition 4.5.1 [H2] A topological category is a category C satisfying the following conditions:

(i) C is a concrete category;

(ii) Fibre-smallness:

$u^{-1}(X)$ is a set for each set X ;

(iii) Terminal separator property:

If X is a set with one element then there is a unique C-structure on X ;

(iv) Existence of initial structures:

For any C-object (Z,C_Z) any set Y and any function $g: Y \to Z$ there is a unique C-structure C_Y on Y such that for any object (X,C_X), a function $f: X \to Y$ is a C-morphism if and only if gf is a C-morphism.

In the following proposition we list some important properties of a topological category.

Proposition 4.5.2 [H4] Let C be a topological category. The following hold.

(i) There exists a final structure in C.

(ii) C is complete and cocomplete and $u: C \to Set$ preserves limits and colimits.

(iii) Let f be a C-morphism. Then f is monic if and only if f is injective. Also f is epic if and only if f is surjective.

(iv) Let $f: (X,C_X) \to (Y,C_Y)$ be a C-morphism. Then the following are equivalent:

1. f is an embedding, that is f is injective and C_X is initial with respect to (Y,C_Y).
2. f is extremal monic.
3. f is regular monic.

(v) The forgetful functor $u: C \to Set$ has a full and faithful left adjoint, as well as a full and faithful right adjoint.

(vi) For non-empty set X, any constant map $f: (X, C_X) \to (Y,C_Y)$ is a C-morphism.

Proposition 4.5.3 There exists an initial structure in SuSEQ .

Proof. Let (Z,C_Z) be an object of SuSEQ and let Y be a set. For a function $f: Y \to Z$ let

$$C_Y = \{ ((y_n),y_\infty) \mid ((f(y_n)) , f(y_\infty)) \text{ is in } C_Z \}.$$

Then (Y,C_Y) is an object of SuSEQ . To show this, clearly for each $y \in Y$, $((y),y)$ is in C_Y . Also if $((y_n),y_\infty)$ is in C_Y then $((y_{n_k}),y_\infty)$ is in C_Y .

For (*) , assume that $((y_n),y_\infty)$ is not in C_Y . Then $((f(y_n)),f(y_\infty))$ is not in C_Z , so there exists a subsequence $(f(y_{n_k}))$ of $(f(y_n))$ such that no subsequence of $(f(y_{n_k}))$ converges to $f(y_\infty)$. Then (y_{n_k}) has no subsequence converging to y . So (Y,C_Y) is in SuSEQ .

Now let (X,C_X) be any object of SuSEQ and let $g: X \to Y$ be a function . If fg is a morphism of SuSEQ and $((x_n),x_\infty)$ is in C_X , then $(f(g(x_n),f(g(x_\infty)))$ is in C_Z . That is $((g(x_n)),g(x_\infty))$ is in C_Y . Hence g is a C-morphism. □

Theorem 4.5.4 The category SuSEQ is a topological category.

Proof. Clearly SuSEQ is concrete, fibre small, satisfies the terminal separator property and by lemma 4.5.2 there exists an initial structure in SuSEQ . □

From the above theorem and Cartesian closedness of SuSEQ, we conclude that SuSEQ is a quasitopos.

4.6 Representability of strong partial morphisms

We have shown in the last section that SuSEQ is a quasitopos. In this section we give an explicit description of the subsequential structure on $Y^{!}$, the object that represents strong partial morphisms with codomain Y. Note that a strong monic in SuSEQ is an embedding (Proposition 4.5.2 (iv)). Finally we conclude that the space $P_c(X,Y)$ of partial maps with closed domain and the space $P_o(X,Y)$ of partial maps with open domain, in SEQ, are not subspaces of $\mathrm{SuSEQ}(X,Y^{!})$ in SuSEQ.

Proposition 4.6.1 For any subsequential space Y, there is a subsequential space $Y^{!}$ and strong monic $\eta: Y \to Y^{!}$ such that for any strong partial morphism $f: X \to Y$ there is a unique $f^{!}: X \to Y^{!}$ such that

$$\begin{array}{ccc} X' & \xrightarrow{\;f\;} & Y \\ \downarrow & & \downarrow \eta \\ X & \xrightarrow[\;f^{!}\;]{} & Y^{!} \end{array}$$

is a pullback square.

Proof. This is modelled on Booth-Brown [B-B1].

Let $Y^{!} = Y \cup \{\omega\}$, $\omega \notin Y$. Define a subsequential structure $C_{Y^!}$ on $Y^{!}$ as follows:

1) If $((y_n),y_\infty)$ is in C_Y, then $((y_n),y_\infty)$ is in $C_{Y^!}$.

2) If (y_n) is a sequence in $Y^{!}$, $y_\infty \in Y$, and the terms $y_n \neq \omega$ are either finite or form a subsequence (y_{n_k}) of (y_n) such that $((y_{n_k}),y_\infty)$ is in C_Y, then $((y_n),y_\infty)$ is in $C_{Y^!}$.

3) Every sequence converges to ω.

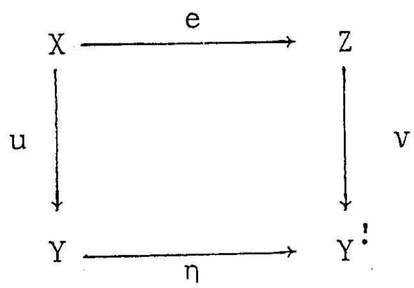
To check that $Y^{!}$ is in SuSEQ:

Condition (i) is clearly held. For (ii) let $((y_n),y_\infty)$ be in C_Y and (y_{n_k}) a subsequence of (y_n).

Assume that $y_\infty \neq \omega$ and let $E = \{n_k \mid y_{n_k} \neq \omega\}$. If E is finite then $((y_{n_k}),y_\infty)$ is in $C_{Y^!}$. Now if E is infinite then E determines a subsequence of (y_{n_k}) that converges to y_∞ . So $((y_{n_k}),y_\infty)$ is in $C_{Y^!}$, (2) .

For (*) , if $((y_n),y_\infty)$ does not belong to $C_{Y^!}$, then clearly $E = \{n \mid y_n \in Y\}$ is infinite. So E determines a subsequence (y_{n_k}) of (y_n) such that $((y_{n_k}),y_\infty)$ is not in C_Y . But then there is a subsequence $(y_{n_{k_\ell}})$ for which none of its subsequences converge to y_∞ . Hence $Y^!$ is a subsequential space.

Now the inclusion $\eta\colon Y \to Y^!$ is a strong monic. For consider any commutative diagram.

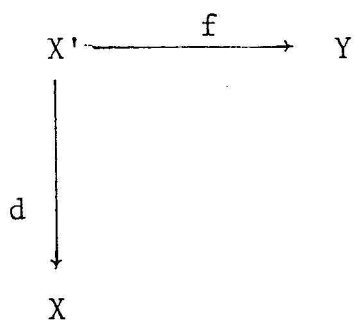


with e epic. Then e is surjective and for each $x \in X$

$$ve(x) = \eta u(x) \neq \omega .$$

Define $t\colon Z \to Y$ by $t(z) = v(z)$. Then t is a SuSEQ-morphism, since v is a SuSEQ-morphism. So η is a strong monic.

For a strong monic $f\colon X \to Y$, that is a diagram



where d is a strong monic, let $f^!\colon X \to Y^!$ be defined by

$$f^{!}(x) = \begin{cases} fx & x \in d(X') \\ \omega & \text{otherwise} . \end{cases}$$

To show that $f^{!}$ is continuous, let $((x_n), x_\infty)$ be in C_X. If $x_\infty \notin d(X')$ then $f^{!}(x_n) \to f^{!}x_\infty$. So assume $x_\infty \in d(X')$ and let $E = \{n \mid x_n \in d(X')\}$. If E is infinite then E determines a subsequence (x_{n_k}) that converges to x_∞. Now d is strong, so $(x_{n_k}) \to x_\infty$ in X'. Then $f(x_{n_k}) \to f(x_\infty)$, and $f^{!}(x_n) \to f^{!}(x_\infty)$. If E is finite then $f^{!}(x_n) = \omega$ for $n \geq$ some finite k. Hence $f^{!}(x_n) \to f(x_\infty)$. So $f^{!}$ is continuous. Now it is easy to see that $f^{!}$ is unique and the diagram

$$\begin{array}{ccc} X' & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle \eta} \\ Z & \xrightarrow[\quad f^{!} \quad]{} & Y^{!} \end{array}$$

is a pullback. □

Now when $Y = \{*\} = 1$, $1^{!}$ classifies strong monics (subspaces) in SuSEQ. The subsequential structure on $1^{!}$ is the indiscrete structure as shown below.

<u>Proposition 4.6.2</u> Let $Y = \{*\}$. Then $\{*\}^{!} = \{*\} \cup \{\omega\}$ is the indiscrete subsequential space.

Proof. Let (y_n) be a sequence in $\{*\}^!$. Then $(y_n) \to \omega$. To show that $(y_n) \to *$, let $E = \{n \mid y_n \neq \omega\}$. If E is finite then $(y_n) \to *$. Now if E is infinite then E determines the constant subsequence $(*)$. So $(y_n) \to *$. □

Let Y be a topological space. Recall from [B-B1], [A-B] (cf. Chapter 2), that there are spaces $Y^!$ and $\hat{Y}$, both with underlying sets $Y \cup \{\omega\}$ with $\omega \notin Y$, and with topologies:

$U \subseteq \tilde{Y}$ is closed if and only if $U = \tilde{Y}$ or U is closed in Y . Then $\{\omega\}$ is open but not closed.

Also the space $\hat{Y} = Y \cup \{\omega\}$ with the topology in which $U \subseteq \hat{Y}$ is open if and only if $U = \hat{Y}$ or U is open in Y .

We now give the subsequential structure on $\tilde{Y}$ and $\hat{Y}$.

Proposition 4.6.3 Let $Y \in |SEQ|$.

(i) The convergent sequences in $\tilde{Y}$ can be described as follows:

1) If $((y_n),y_\infty) \in C_Y$, then $((y_n),y_\infty) \in C_{Y^\sim}$;

2) $((y_n),\omega) \in C_{Y^\sim}$ if and only if $y_n = \omega$ for all $n \geq$ some finite k ;

3) If (y_n) is a sequence in $\tilde{Y}$, $y_\infty \in Y$ and the terms $y_n \neq \omega$ are either finite or form a subsequence (y_{n_k}) of (y_n) such that $((y_{n_k}),y_\infty) \in C_Y$ then $((y_n),y_\infty) \in C_{Y^\sim}$.

(ii) The convergent sequences in $\hat{Y}$ can be described as follows:

1) If $((y_n),y_\infty) \in C_Y$ then $((y_n),y_\infty) \in C_{\hat{Y}}$;

2) For any sequence (y_n) in $\hat{Y}$, $((y_n),\omega) \in C_{\hat{Y}}$;

3) Let (y_n) be such that $y_n = \omega$ finitely often and the terms $y_n \neq \omega$ form a subsequence converging to y_∞ in Y then $((y_n),y_\infty) \in C_{\hat{Y}}$.

(iii) The spaces $Y^\sim$ and $\hat{Y}$ are sequential.

Proof. The proof is easy. □

Corollary 4.6.4 Let Y be a sequential space. Then the functions

$$\mathrm{id}^\sim : Y^\sim \to Y^!$$
$$y \to y$$

and

$$\hat{\mathrm{id}} : \hat{Y} \to Y^!$$
$$y \to y$$

are monics in SuSEQ which are not strong monic.

In SuSEQ strong monic is an embedding, so for a sequential space Y , $Y^\sim$ and $\hat{Y}$ are not subspaces of $Y^!$.

The following result will be used to prove that the functor k: SEQ ⟶ SuSEQ preserves function spaces, and will be used to prove similar results in the next chapter.

Proposition 4.6.5 Let C and $\mathcal{D}$ be Cartesian closed categories. Let $k: C \to \mathcal{D}$ be a functor such that:

1) k is fully faithful;

2) k has a left adjoint k' ;

3) For $Y \in |\mathcal{C}|$, $k'(kY) \cong Y$.

If for $X, X' \in |\mathcal{D}|$, $k'(X \times X') = k'X \times k'X'$ then k preserves function spaces.

Proof. Let Y and $Z \in |\mathcal{C}|$. Then for $X \in |\mathcal{D}|$ there are natural isomorphisms

$$\begin{aligned} \mathcal{D}(X, k(Z^Y)) &\cong \mathcal{C}(k'X, (Z^Y)) \\ &\cong \mathcal{C}(k'X \times Y, Z) \\ &\cong \mathcal{C}(k'X \times k'(kY), Z) \\ &\cong \mathcal{C}(k'(X \times kY), Z) \\ &\cong \mathcal{D}(X \times kY, kZ) \\ &\cong \mathcal{D}(X, (kZ)^{kY}) \quad . \end{aligned}$$

Hence $k(Z^Y) \cong (kZ)^{kY}$. □

Proposition 4.6.6 The functor $k'\colon \text{SuSEQ} \to \text{SEQ}$ is such that for X and $Y \in |\text{SuSEQ}|$, $k'(X \times Y) = k'X \times k'Y$.

Proof. Let $Z = k'X \times k'Y$. Then Z is the sequentialized product Chapter 3 . That is $U \subseteq Z$ is open if and only if U is sequentially open. But it is easy to see that $z_n = (x_n, y_n)$ converges to $z_\infty = (x_\infty, y_\infty)$ if and only if $x_n \to x_\infty$ and $y_n \to y_\infty$.

Also the sequential topology on $k'(X \times Y)$ is induced by the sequentially open sets in $X \times Y$. But the sequentially open sets are then determined by the convergent sequences $C_X \times C_Y$. That is $x_n \to x_\infty$ and $y_n \to y_\infty$ if and only if $((x_n, y_n)) \to (x_\infty, y_\infty)$. So $k'(X \times Y) = k'X \times k'Y$. □

Proposition 4.6.7 The functor $k: SEQ \to SuSEQ$ preserves function spaces.

Proof. The functor k satisfies the conditions of Proposition 4.6.5. □

Now for sequential spaces X and Y,

$$P_c(X,Y) \cong SEQ(X,\tilde{Y})$$

and

$$P_o(X,Y) \cong SEQ(X,\hat{Y}) \quad .$$

So the next result follows.

Proposition 4.6.8 Let X and Y be sequential spaces. Then

$$k\,[P_c(X,Y)] \cong SuSEQ(kX,k\tilde{Y})$$

and

$$k\,[P_o(X,Y)] \cong SuSEQ(kX,k\hat{Y})$$

We now state one main conclusion of this chapter.

Proposition 4.6.9 Let X and Y be sequential spaces. Then $k[P_c(X,Y)]$ and $k[P_o(X,Y)]$ are not subspaces of

$$SuSEQ(X,Y^{!}) \cong SPart(X,Y)$$

in $SuSEQ$, where $SPart(X,Y)$ denote the space of strong partial morphisms with codomain Y.

Proof. Consider $X = \{*\}$ and Corollary 4.6.4 . □

We conclude, Proposition 4.6.9, that when we embed $P_o(X,Y)$ and $P_o(X,Y)$ in SuSEQ , then SuSEQ(X,Y) does not contain $P_c(X,Y)$ and $P_o(X,Y)$ as subspaces. Hence SuSEQ , quasitopos in general, does not solve the problem of finding a space which contains both $P_c(X,Y)$ and $P_o(X,Y)$ as subspaces.