Free crossed resolutions for graph products and amalgamated sums of groups

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Abstract

The category of crossed complexes gives an algebraic model of the category of CW-complexes and cellular maps. We explain basic results on crossed complexes which allow the computation of free crossed resolutions of graph products of groups, and of free products with amalgamation, given free crossed resolutions of the individual groups.

Introduction

The aim of this paper is to advertise the category of crossed complexes, and the notion of free crossed resolution, as a working tool for computing identities among relations [9] and higher homotopical syzygies [26] for certain constructions on groups.

Crossed complexes are an analogue of chain complexes of modules over a group ring, but with a non abelian part, a *crossed module*, at the root. This allows for crossed complexes to contain the data for a presentation of a group, as well as higher homological data. The non abelian nature, and also the generalisation to groupoids rather than just groups, allows for a closer representation of geometry, and this, combined with very convenient properties of the category of crossed complexes, allows for more and easier calculations than are available in the purely abelian theory.

The notion of *crossed complex* of groups was defined by A.L.Blakers in 1946 [3] (under the term 'group system') and Whitehead [34], under the term 'homotopy system' (except that he

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restricted to the free case). Blakers used these as a way of systematising known properties of relative homotopy groups $\pi_n(X_n, X_{n-1}, x)$ of a filtered space

$$X_* : X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X_\infty.$$

It is significant that he used the notion to establish relations between homotopy and homology of a space. Whitehead was strongly concerned with realisability, that is with the passage between algebra and geometry and back again. He explored the relations between crossed complexes and chain complexes with a group of operators and established some remarkable realisability properties, which we explain later.

There was another stream of interest in crossed complexes, but in a broader algebraic framework, in work of Frohlich [19] and Lue [27]. This gave a general formulation of cohomology groups relative to a variety in terms of equivalence classes of certain exact sequences. However the relation with the usual cohomology of groups was not made explicit till papers of Holt [23] and Huebschmann [24]. The situation is described in Lue's paper [28].

Since our interest is in the relation with homotopy theory, we are interested in the case of groups rather than other algebraic systems. However there is one key change we have to make, as stated above, namely that we have to generalise to groupoids rather than groups. This makes for a more effective modelling of the geometry, since we need to use CW-complexes which are non reduced, i.e. have more than one 0-cell, for example universal covering spaces, and simplices. This also gives the category of crossed complexes better algebraic properties, principally that it is a monoidal closed category in the sense of having an internal hom which is adjoint to a tensor product. This is a generalisation of a standard property of groupoids: if Gpd denotes the category of groupoids, then for any groupoids A, B, C there is a natural bijection

$$\mathsf{Gpd}(A \times B, C) \cong \mathsf{Gpd}(A, \mathsf{GPD}(B, C))$$

where $A \times B$ is the usual product of groupoids, and $\mathsf{GPD}(B, C)$ is the groupoid whose objects are the morphisms $B \to C$ and whose arrows are the natural equivalences (or conjugacies) of morphisms.

A more computational feature of the use of groupoids is that we need the notion of *homotopy pushout* of groups, and the result of this construction is a groupoid. It turns out that we can more easily construct free crossed resolutions of the homotopy pushout than of the pushout. A key ingredient of these homotopical notions is the groupoid $\mathcal{I}$ which has two objects $0, 1$ and exactly one arrow between any two objects. This groupoid is a model of the unit interval in topology. Once this apparently trivial groupoid is allowed as a natural extension of the family of groups, then essentially all groupoids are allowed, since any groupoid is obtained by identifications from a disjoint union of copies of $\mathcal{I}$.

1 Definitions and basic properties

A *crossed complex* C (of groupoids) is a sequence of morphisms of groupoids over C_0

$$\begin{array}{ccccccccc} \cdots \longrightarrow & C_n & \xrightarrow{\delta_n} & C_{n-1} & \longrightarrow \cdots \longrightarrow & C_2 & \xrightarrow{\delta_2} & C_1 \\ & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} & & {\scriptstyle\delta^0}\downarrow\downarrow{\scriptstyle\delta^1} \\ & C_0 & & C_0 & & C_0 & & C_0. \end{array}$$

Here $\{C_n\}_{n\geqslant 2}$ is a family of groups with base point map β, so that for $p \in C_0$, we have groups $C_n(p) = \beta^{-1}(p)$, and δ^0, δ^1 are the source and targets for the groupoid C_1. We further require given an operation of the groupoid C_1 on each family of groups C_n for $n \geqslant 2$ such that:

(i) each δ_n is a morphism over the identity on C_0;

(ii) $C_2 \to C_1$ is a crossed module over C_1;

(iii) C_n is a C_1-module for $n \geqslant 3$;

(iv) $\delta : C_n \to C_{n-1}$ is an operator morphism for $n \geqslant 3$;

(v) $\delta\delta : C_n \to C_{n-2}$ is trivial for $n \geqslant 3$;

(vi) δC_2 acts trivially on C_n for $n \geqslant 3$.

Because of axiom (iii) we shall write the composition in C_n additively for $n \geqslant 3$, but we will use multiplicative notation in dimensions 1 and 2 (except when giving the rules for the tensor product). Note that if $a : p \to q, b : q \to r$ in C_1 then the composition is written $ab : p \to r$. If further $x \in C_n(p)$ then $x^a \in C_n(q)$ and the usual laws of an action apply. We write $C_1(p) = C_1(p,p)$, and C_1 operates on this family of groups by conjugation. The condition (iv) then implies that $\delta_2(x^a) = a^{-1}\delta_2(x)a$, while condition (ii) gives further that $x^{-1}yx = y^{\delta_2(x)}$ for $x, y \in C_2(p), a \in C_1(p,q)$. Consequently $\delta_2(C_2)$ is normal in C_1, and $Ker\ \delta_2$ is central in C_2 and is operated on trivially by $\delta_2(C_2)$.

Let C be a crossed complex. Its *fundamental groupoid* $\pi_1 C$ is the quotient of the groupoid C_1 by the normal, totally disconnected subgroupoid δC_2. The rules for a crossed complex give C_n, for $n \geqslant 3$, and also $Ker\ \delta_2$, the induced structure of $\pi_1 C$-module.

The crossed complex C is *reduced* if C_0 is a singleton, so that all the groupoids $C_n, n \geqslant 1$ are groups. This was the case considered in [3, 34] and many other sources.

A *morphism* $f : C \to D$ of crossed complexes is a family of groupoid morphisms $f_n : C_n \to D_n$ $(n \geqslant 0)$ which preserves all the structure. This defines the category Crs of crossed complexes. The fundamental groupoid now gives a functor $\pi_1 : \mathsf{Crs} \to \mathsf{Gpd}$. This functor is left adjoint to the functor $i : \mathsf{Gpd} \to \mathsf{Crs}$ where for a groupoid G the crossed complex iG agrees with G in dimensions 0 and 1, and is otherwise trivial.

An *m-truncated* crossed complex C consists of all the structure defined above but only for $n \leqslant m$. In particular, an m-truncated crossed complex is for $m = 0, 1, 2$ simply a set, a groupoid, and a crossed module respectively.

One basic algebraic example of crossed complex comes from the notion of *identities among relations*. Let $\mathcal{P} = \langle X_1 | w \rangle$ be a presentation of a group G where w is a function from a set X_2 to $F(X_1)$, the free group on the set X_1 of generators of G. This gives an epimorphism $\varphi : F(X_1) \to G$, with kernel $N(R)$, the normal closure in $F(X_1)$ of the set $R = w(X_2)$.

Let H be the free $F(X_1)$-operator group on the set X_2, so that H is the free group on the elements $(x, u) \in X_2 \times F(X_1)$. Let $\delta' : H \to F(X_1)$ be determined by $(x, u) \mapsto u^{-1}(wx)u$, so that the image of δ' is exactly $N(R)$. Note that $F(X_1)$ operates on H by $(x, u)^v = (x, uv)$, and for all $h \in H, u \in F(X_1)$ we have
CM1) $\delta'(h^u) = u^{-1}\delta'(h)u$.
We say that $\delta' : H \to F(X_1)$ is a *precrossed module.*

We now define *Peiffer commutators* for $h, k \in H$

$$\langle h, k \rangle = h^{-1}k^{-1}hk^{\delta' h}.$$

Then δ' vanishes on Peiffer commutators. Also the subgroup $P = \langle H, H \rangle$ generated by the Peiffer commutators is a normal $F(X_1)$-invariant subgroup of H. So we can define $C(w) = H/P$ and obtain the exact sequence

$$C(w) \xrightarrow{\delta_2} F(X_1) \xrightarrow{\varphi} G \to 1.$$

The morphism δ_2 satisfies
CM2) $c^{-1}dc = d^{\delta_2 c}$ for all $c, d \in C(w)$.
The rules CM1), CM2) are the laws for a *crossed module*, and $\delta_2 : C(w) \to F(X_1)$ is known as the *free crossed $F(X_1)$-module on w.* The injection $i : X_2 \to C(w)$ has the universal property that if $\mu : M \to F(X_1)$ is a crossed module and $v : X_2 \to M$ is a function such that $\mu v = w$, then there is a unique crossed module morphism $\eta : C(w) \to M$ such that $\eta i = w$. The elements of $C(w)$ are 'formal consequences'

$$c = \prod_{i=1}^{n} (x_i^{\varepsilon_i})^{u_i}$$

where $n \geqslant 0, x_i \in X_2, \varepsilon_i = \pm 1, u_i \in F(X_1)$, $\delta_2(x^\varepsilon)^u = u^{-1}(wx)^\varepsilon u$, subject to the crossed module rule $cd = dc^{\delta_2 d}, c, d \in C(w)$. (For more details on the above, see [9].)

The kernel $\pi(\mathcal{P})$ of δ_2 is abelian and in fact obtains the structure of G-module – it is known as the G-module of *identities among relations* for the presentation.

We can now splice to the free crossed module any resolution of $\pi(\mathcal{P})$ by free G-modules, and so obtain what is called a *free crossed resolution* of the group G.

This construction is analogous to the usual construction of higher order syzygies and free resolutions for modules, but taking into account the non abelian nature of the group and its presentation, and in particular the action of $F(X_1)$ on $N(R)$.

There is a notion of homotopy for morphisms of crossed complexes which we will explain later. Assuming this we can state one of the basic homological results, namely the uniqueness up to homotopy equivalence of free crossed resolutions of a group G.

There is a *standard free crossed resolution* $F_*^{st}(G)$ of a group G in which $F_1^{st}(G)$ is the free group on the set G with generators $[a], a \in G$; $F_2^{st}(G)$ is the free crossed $F_1^{st}(G)$-module on $w : G \times G \to F_1^{st}(G)$ given by

$$w(a, b) = [a][b][ab]^{-1}, a, b \in G;$$

for $n \geqslant 3$, $F_n^{st}(G)$ is the free G-module on G^n, with

$$\delta_3[a, b, c] = [a, bc][ab, c]^{-1}[a, b]^{-1}[b, c]^{[a]^{-1}},$$

and for $n \geqslant 4$

$$\delta_n[a_1, a_2, \ldots, a_n] = [a_2, \ldots, a_n]^{a_1^{-1}} + \sum_{i=1}^{n-1}(-1)^i[a_1, a_2, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_n] + \\ + (-1)^n[a_1, a_2, \ldots, a_{n-1}]. \quad (1)$$

We can now see the advantage of this setup in considering the notion of non abelian 2-cocycle on the group G with values in the group K. This is a pair of functions $k^1 : T \to Aut(K), k^2 : T \times T \to K$ satisfying certain properties. But suppose G is infinite. Then it is difficult to know how to specify these functions and check the required properties.

However we can regard k^1, k^2 as specifying a morphism

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & F_3^{st}(G) & \xrightarrow{\delta_3} & F_2^{st}(G) & \xrightarrow{\delta_2} & F_1^{st}(G) & \xrightarrow{\varphi} & G \\
& \downarrow & & \downarrow{\scriptstyle k^2} & & \downarrow{\scriptstyle k^1} & & \\
\cdots \longrightarrow & 0 & \longrightarrow & K & \xrightarrow[\partial]{} & Aut(K) & &
\end{array}$$

such that $\partial k^2 = k^1\delta_2$, $k^2\delta_3 = 0$, where $K \xrightarrow{\partial} Aut(K)$ is the inner automorphism map and so is a crossed module. Further, equivalent cocycles are just homotopic morphisms.

Data equivalent to the above up to homotopy is obtained by replacing the standard free crossed resolution by any homotopy equivalent free crossed resolution. For example, if T is the *trefoil group* with presentation $\langle x, y | x^3y^{-2}\rangle$ then we show in our final section that there is a free crossed resolution of T of the form

$$\cdots \to 1 \to C(r) \xrightarrow{\delta} F\{x, y\}$$

where $\delta(r) = x^3y^{-2}$. Hence a 2-cocycle on T with values in K can also be specified totally by elements $s \in K, c, d \in Aut(K)$ such that $\partial(s) = c^3d^{-2}$, which is essentially a finite description. It is also easy to specify equivalence of cocycles.

It is shown in [12] that the extension $1 \to K \to E \to T \to 1$ determined by this 2-cocycle is obtained by taking E to be the quotient of the semidirect product $F\{x, y\} \ltimes K$ by the relations $(\delta r, k^2(r^{-1}))$. This is a case where there are no identities among relations. The general necessity to refer to identities among relations in this context was first observed by Turing [31].

A similar method can be used to determine the 3-dimensional obstruction class $l^3 \in H^3(G, A)$ corresponding to a crossed module $\mu : M \to P$ with $Coker\ \mu = G, Ker\ \mu = A$. For this we need a small free crossed resolution of the group G. This method is successfully applied to the case with G finite cyclic in [14, 15].

2 Relation with topology

In order to give the basic geometric example of a crossed complex we first define a *filtered space* X_*. By this we mean a topological space X_∞ and an increasing sequence of subspaces

$$X_* : X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X_\infty.$$

A *map* $f : X_* \to Y_*$ of filtered spaces consists of a map $f : X_\infty \to Y_\infty$ of spaces such that for all $i \geqslant 0, f(X_i) \subseteq Y_i$. This defines the category FTop of filtered spaces and their maps. This category has a monoidal structure in which

$$(X_* \otimes Y_*)_n = \bigcup_{p+q=n} X_p \times Y_q,$$

where it is best for later purposes to take the product in the convenient category of compactly generated spaces, so that if X_*, Y_* are CW-spaces, then so also is $X_* \otimes Y_*$.

We now define the *fundamental, or homotopy, crossed complex* functor

$$\pi : \mathsf{FTop} \to \mathsf{Crs}.$$

If $C = \pi(X_*)$, then $C_0 = X_0$, and C_1 is the fundamental groupoid $\pi_1(X_1, X_0)$. For $n \geqslant 2$, $C_n = \pi_n X_*$ is the family of relative homotopy groups $\pi_n(X_n, X_{n-1}, p)$ for all $p \in X_0$. These come equipped with the standard operations of $\pi_1 X_*$ on $\pi_n X_*$ and boundary maps $\delta : \pi_n X_* \to \pi_{n-1} X_*$, namely the boundary of the homotopy exact sequence of the triple (X_n, X_{n-1}, X_{n-2}). The axioms for crossed complexes are in fact those universally satisfied by this example, though this cannot be proved at this stage.

This construction also explains why we want to consider crossed complexes of groupoids rather than just groups. One reason is exactly analogous to that for considering non reduced CW-complexes, namely that we wish to consider covering spaces, which automatically have more than one vertex in the non trivial case. Similarly, we wish to consider covering morphisms of crossed complexes as a tool for analysing presentations of groups, analogously to the way covering morphisms of groupoids were used for group theory applications by P.J. Higgins in 1964 in [20]. A key tool in this is the use of paths in a Cayley graph as giving elements of the free groupoid on the Cayley graph, so that one moves to consider presentations of groupoids. Further, as is shown by Brown and Razak in [13], higher dimensional information is obtained by regarding the free generators of the universal cover of a free crossed resolution as giving a higher order Cayley graph, i.e. a Cayley graph with higher order syzygies. This method actually yields computational methods, by using the geometry of the Cayley graph, and the notion of deformation retraction of this universal cover.

A second reason which we will see in section 4.2 is that we need to consider a construction analogous to double mapping cylinders of spaces, namely homotopy pushouts of crossed complexes. Such homotopy pushouts are naturally considered as crossed complexes of groupoids.

Thus crossed complexes give a useful algebraic model of the category of CW-complexes and cellular maps. This model does lose a lot of information, but its corresponding advantage is that it allows for algebraic description and computation, for example of morphisms and homotopies. This is the key aspect of the methods of [13].

Many geometric and algebraic situations are specified by related, and often more complicated, non abelian information, not readily computable by traditional means. As an example, we mention the non abelian tensor product of groups due to Brown and Loday [11].

Thus we can say that crossed complexes:

(i) give a first step to a full non abelian theory;

(ii) have good categorical properties;

(iii) give a 'linear' model of homotopy types, which includes all homotopy 2-types;

(iv) are amenable to computation;

(v) give one form of 'higher dimensional group'.

A further advantage of using crossed complexes of groupoids is that this allows for the category Crs to be monoidal closed: there is a tensor product $-\otimes-$ and internal hom $\mathsf{CRS}(-,-)$ such that there is a natural isomorphism

$$\mathsf{Crs}(C \otimes D, E) \cong \mathsf{Crs}(C, \mathsf{CRS}(D, E))$$

for all crossed complexes C, D, E. Here $\mathsf{CRS}(D,E)_0 = \mathsf{Crs}(D,E)$, the set of morphisms $D \to E$, while $\mathsf{CRS}(D,E)_1$ is the set of '1-fold left homotopies' $D \to E$. Note that while the tensor product can be defined directly in terms of generators and relations, such a definition makes it not easy to verify essential properties of the tensor product, such as that the tensor product of free crossed complexes is free. The proof of this fact in [8] uses the above adjointness as a necessary step to prove that $-\otimes D$ preserves colimits.

An important result is that if X_*, Y_* are filtered spaces, then there is a natural transformation

$$\eta : \pi(X_*) \otimes \pi(Y_*) \to \pi(X_* \otimes Y_*)$$

which is an isomorphism if X_*, Y_* are CW-complexes [8] (and in fact more generally [2]). In particular, the basic rules for the tensor product are modelled on the geometry of the product of cells $E^m \otimes E^n$ where E^0 is the singleton space, E^1 is the interval $[-1,1]$ with two 0-cells and one 1-cell, while for $m \geqslant 2$ $E^m = e^0 \cup e^{m-1} \cup e^m$. This leads to defining relations for the tensor product. To give these we first define a bimorphism of crossed complexes (using additive notation throughout).

A *bimorphism* $\theta : (A, B) \to C$ of crossed complexes is a family of maps $\theta : A_m \times B_n \to C_{m+n}$ satisfying the following conditions, where $a \in A_m, b \in B_n$:

(i) $\beta(\theta((a,b)) = \theta(\beta a, \beta b))$ for all $a \in A, b \in B$.

(ii) $\theta(a, b^{b_1}) = \theta(a,b)^{\theta(\beta a, b_1)}$ if $m \geqslant 0, n \geqslant 2$.

(ii)′ $\theta(a^{a_1}, b) = \theta(a,b)^{\theta(a_1, \beta b)}$ if $m \geqslant 2, n \geqslant 0$.

(iii)

$$\theta(a, b+b') = \begin{cases} \theta(a,b) + \theta(a,b') & \text{if } m=0, n \geqslant 1 \text{ or } m \geqslant 1, n \geqslant 2, \\ \theta(a,b)^{\theta(\beta a, b')} + \theta(a,b') & \text{if } m \geqslant 1, n = 1. \end{cases}$$

(iii)′

$$\theta(a+a', b) = \begin{cases} \theta(a,b) + \theta(a',b) & \text{if } m \geqslant 1, n=0 \text{ or } m \geqslant 2, n \geqslant 1, \\ \theta(a',b) + \theta(a,b)^{\theta(a', \beta b)} & \text{if } m = 1, n \geqslant 1. \end{cases}$$

(iv)

$$\delta(\theta(a,b)) = \begin{cases} \theta(\delta a, b) + (-)^m \theta(a, \delta b) & \text{if } m \geqslant 2, n \geqslant 2, \\ -\theta(a, \delta b) - \theta(\delta^1 a, b) + \theta(\delta^0 a, b)^{\theta(a, \beta b)} & \text{if } m = 1, n \geqslant 2, \\ (-)^{m+1}\theta(a, \delta^1 b) + (-)^m \theta(a, \delta^0 b)^{\theta(\beta a, b)} + \theta(\delta a, b) & \text{if } m \geqslant 2, n = 1, \\ -\theta(\delta^1 a, b) - \theta(a, \delta^0 b) + \theta(\delta^0 a, b) + \theta(a, \delta^1 b) & \text{if } m = n = 1. \end{cases}$$

(v)

$$\delta(\theta(a,b)) = \begin{cases} \theta(a, \delta b) & \text{if } m = 0, n \geqslant 2, \\ \theta(\delta a, b) & \text{if } m \geqslant 2, n = 0. \end{cases}$$

$$\delta^\alpha(\theta(a,b)) = \begin{cases} \theta(a, \delta^\alpha b) & \text{if } m = 0, n = 1 \ (\alpha = 0, 1), \\ \theta(\delta^\alpha a, b) & \text{if } m = 1, n = 0 \ (\alpha = 0, 1). \end{cases}$$

The tensor product of crossed complexes A, B is then given by the universal bimorphism $(A, B) \to A \otimes B$, $(a, b) \mapsto a \otimes b$. So the rules for the tensor product are obtained by replacing $\theta(a, b)$ by $a \otimes b$ in the above.

Example 2.1 Let $\langle X|R\rangle, \langle Y|S\rangle$ be presentations of groups G, H respectively and let $C(R) \to F(X)$, $C(S) \to F(Y)$ be the corresponding free crossed modules, regarded as crossed complexes of length 2. Their tensor product T is of length 4 and is given as follows:
T_1 is the free group on generating set $X \sqcup Y$;
T_2 is the free crossed T_1-module on $R \sqcup S \sqcup (X \otimes Y)$ with the boundaries on R, S as given before but $\delta_2(x \otimes y) = y^{-1}x^{-1}yx$;
T_3 is the free $(G \times H)$-module on generators $r \otimes y, x \otimes s$ with boundaries

$$\delta_3(r \otimes y) = r^{-1}r^y(\delta_2 r \otimes y), \qquad \delta_3(x \otimes s) = (x \otimes \delta_2 s)^{-1}s^{-1}s^x;$$

T_4 is the free $(G \times H)$-module on generators $r \otimes s$, with boundary

$$\delta_4(r \otimes s) = (\delta_2 r \otimes s) + (r \otimes \delta_2 s).$$

The conventions here may seem (even are) awkward. They arise from the derivation of the tensor product via another cubical category, and the result is forced by our conventions for the equivalence of the two categories [4, 6]. The important point is that we can if necessary calculate with these formulae, because elements such as $\delta_2 r \otimes y$ may be expanded using the rules for the tensor product. Alternatively, the form $\delta_2 r \otimes y$ may be left as it is since it naturally represents, for example if $\dim y = 1$, a subdivided cylinder.

A related result is that if C, D are free crossed resolutions of groups C, D then $C \otimes D$ is a free crossed resolution of $G \times H$ [30]. This allows for presentations of modules of identities among relations for a product of groups to be read off from the presentations of the individual modules. There is a lot of work on generators for modules of identities among relations (see for example [22]) but not so much on higher order syzygies.

It should be said that some of these last results are not proved directly, but instead use a category equivalent to Crs, namely a category of 'cubical ω-groupoids with connections'. It is in the latter category that the exponential law is easy to formulate and prove, as is the construction of the natural transformation η. However the proof of all the properties of the equivalence is a long story.

In particular if we set $\mathcal{I} = \pi(\mathsf{E}^1)$, then a '1-fold left homotopy' is defined to be a morphism $\mathcal{I} \otimes D \to E$. The existence of this 'cylinder object' allows a lot of abstract homotopy theory [25] to be applied immediately to the category Crs. This is useful in constructing homotopy equivalences of crossed complexes, using for example gluing lemmas.

An important construction is the simplicial nerve NC of a crossed complex C. This is the simplicial set defined by

$$(NC)_n = \mathsf{Crs}(\pi\Delta^n, C).$$

It directly generalises the nerve of a group. In particular this can be applied to the internal hom functor $\mathsf{CRS}(D, E)$ to give a simplicial set $N(\mathsf{CRS}(D, E))$ and so turn the category Crs into a simplicially enriched category. This allows the full force of the methods of homotopy coherence to be used [17].

The *classifying space* BC of a crossed complex C is simply the geometric realisation $|NC|$ of the nerve of C. This construction generalises at the same time: the classifying space of a group; an Eilenberg-Mac Lane space $K(A, n),\ n \geqslant 2$; the classifying space for local coefficients. It also includes the notion of classifying space $B\mathcal{M}$ of a crossed module $\mathcal{M} = (\mu : M \to P)$. Every connected CW-space has the homotopy 2-type of such a space, and so crossed modules classify all connected homotopy 2-types. This is one way in which crossed modules are naturally seen as 2-dimensional analogues of groups.

3 A Generalised Van Kampen Theorem

This theorem states roughly that the functor $\pi : \mathsf{FTop} \to \mathsf{Crs}$ preserves certain colimits. This allows the calculation of certain crossed complexes, and in particular to see how free crossed complexes arise from CW-complexes.

Definition 3.1 *A filtered space X_* is called* connected *if the following conditions $\varphi(X, m)$ hold for each $m \geqslant 0$:*

> $\varphi(X, 0)$*: If $j > 0$, the map $\pi_0 X_0 \to \pi_0 X_j$, induced by inclusion, is surjective*
>
> $\varphi(X, m),\ (m \geqslant 1)$ *: If $j > m$ and $\nu \in X_0$, then the map*
>
> $$\pi_m(X_m, X_{m-1}, \nu) \to \pi_m(X_j, X_{m-1}, \nu)$$
>
> *induced by inclusion, is surjective.*

The following result gives another useful formulation of this condition. We omit the proof.

Proposition 3.2 *A filtered space X is connected if and only if for all $n > 0$ the induced map $\pi_0 X_0 \to \pi_0 X_n$ is surjective and for all $r > n > 0$ and $\nu \in X_0, \pi_n(X_r, X_n, \nu) = 0$.*

The filtration of a CW-complex by skeleta is a standard example of a connected filtered space.

Suppose for the rest of this section that X_* is a filtered space. Let $X = X_\infty$.

We suppose given a cover $\mathcal{U} = \{U^\lambda\}_{\lambda \in \Lambda}$ of X such that the interiors of the sets of $\mathcal{U}$ cover X. For each $\zeta \in \Lambda^n$ we set

$$U^\zeta = U^{\zeta_1} \cap \cdots \cap U^{\zeta_n}, U_i^\zeta = U^\zeta \cap X_i.$$

Then $U_0^\zeta \subseteq U_1^\zeta \subseteq \cdots$ is called the *induced filtration* U_*^ζ of U^ζ. Consider the following π-*diagram* of the cover:

$$\bigsqcup_{\zeta\in\Lambda^2} \pi U_*^\zeta \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_{\lambda\in\Lambda} \pi U_*^\lambda \xrightarrow{c} \pi X_* \qquad (2)$$

Here $\bigsqcup$ denotes disjoint union (which is the same as coproduct in the category of crossed complexes); a, b are determined by the inclusions $a_\zeta : U^\lambda \cap U^\mu \to U^\lambda, b_\zeta : U^\lambda \cap U^\mu \to U^\mu$ for each $\zeta = (\lambda, \mu) \in \Lambda^2$; and c is determined by the inclusions $c_\lambda : U^\lambda \to X$.

The following result constitutes a generalisation of the Van Kampen Theorem for the fundamental group (or groupoid).

Theorem 3.3 (The coequaliser theorem for crossed complexes: Brown and Higgins [5]) *Suppose that for every finite intersection U^ζ of elements of $\mathcal{U}$ the induced filtration U_*^ζ is connected. Then*

(C) *X_* is connected, and*

(I) *in the above π-diagram of the cover, c is the coequaliser of a, b in the category of crossed complexes.*

The proof of this theorem is not at all straightforward, and uses another category equivalent to that of crossed complexes, called the category of cubical ω-groupoids with connections [4]. It is this category which is adequate for two key elements of the proof, the notion of 'algebraic inverse to subdivision', and the 'multiple compositions of homotopy addition lemmas' [5]. The setting up of this machinery takes considerable effort.

In this paper we shall take as a corollary that the coequaliser theorem applies to the case when X is a CW-complex with skeletal filtration and the U^λ form a family of subcomplexes which cover X.

In order to apply this result to free crossed resolutions, we need to replace free crossed resolutions by CW-complexes. A fundamental result for this is the following, which goes back to Whitehead [35] and Wall [32]:

Theorem 3.4 *Let X_* be a CW-filtered space, and let $\varphi : \pi X_* \to C$ be a homotopy equivalence to a free crossed complex with a preferred free basis. Then there is a CW-filtered space Y_* and an isomorphism $\pi Y_* \cong C$ of crossed complexes with preferred basis such that φ is realised by a homotopy equivalence $X_* \to Y_*$.*

In fact Wall states his result in terms of chain complexes, but the crossed complex formulation seems more natural, and avoids questions of realisability in dimension 2, which are unsolved for chain complexes.

Corollary 3.5 *If C is a free crossed resolution of a group G, then C is realised as free crossed complex with preferred basis by some CW-filtered space Y_*.*

Proof We only have to note that the group G has a classifying CW-space BG whose fundamental crossed complex πBG is homotopy equivalent to C. □

These results now give a strategy of weaving between spaces and crossed complexes. The key problem is to prove that a construction on free crossed resolutions yields an aspherical free crossed complex, and so also a resolution. The previous result allows us to replace the free crossed resolutions by spaces. We have a result of Whitehead [33] which allows us to build up $K(G,1)$s as pushouts of other $K(G,1)$s provided the induced morphisms of fundamental groups are injective. The Coequaliser Theorem now gives that the resulting fundamental crossed complex is exactly the one we want.

Note also an important feature of this method: *we use colimits rather than exact sequences.* This enables precise results in situations where exact sequences might be inadequate, since they often give information only up to extension.

The relation of crossed complex methods to those of the more usual chain complexes with operators is studied in [7], developing work of Whitehead [34].

4 Applications

We give two applications as 'case studies' in the use of the above methodology.

4.1 Graph products of groups

Graph products of groups have been studied for example in [1, 16, 18]. The paper [1] obtains generators for the module of identities among relations, while [16] obtains a projective resolution of the graph product given projective resolutions of the individual groups. Of course our aim is for the information to be given in the non-abelian form, where appropriate.

Let Γ be a finite, undirected graph without loops or multiple edges. Suppose the vertices of Γ are well ordered, and suppose given for each vertex p of Γ a group $G^{(p)}$. The *graph product* of the groups

$$G_\Gamma = \prod_{\Gamma}^{p} G^{(p)}$$

is obtained from the free product of the groups $G^{(p)}$ (in the given order) by adding the relations $[G^{(p)}, G^{(q)}] = \{1\}$ whenever (p,q) is an edge of Γ.

We now consider the problem of constructing a free crossed resolution of the graph product G_Γ given free crossed resolutions $C^{(p)}$ of each group $G^{(p)}$. To this end we define graph products in two other contexts.

First of all we define the nerve $N(\Gamma)$ to be the simplicial complex whose (ordered) simplices are the complete subgraphs of Γ. If $\sigma = (p_0, \dots, p_n)$ is in $N(\Gamma)$ and we set

$$G^{(\sigma)} = \prod_{i=0}^{n} G^{(p_i)}$$

then G_Γ is the colimit of a diagram of these groups over the poset corresponding to $N(\Gamma)$.

Next suppose we are given for each vertex p of Γ a pointed crossed complex $C^{(p)}$. For each simplex $\sigma = (p_0, \dots, p_n)$ of $N(\Gamma)$ let $C^{(\sigma)}$ be the subcomplex of the tensor product of all the $C^{(p)}$ (in the given vertex order) such that the pth coordinate is the base point of $C^{(p)}$ if $p \notin \sigma$ and is otherwise arbitrary. Let C_Γ be the subcomplex generated by all the $C^{(\sigma)}$ for all $\sigma \in N(\Gamma)$. This we call the *graph tensor product*

$$\otimes^p_\Gamma C^{(p)}$$

of the crossed complexes $C^{(p)}$. This is also the colimit of the $C^{(\sigma)}$ over $N(\Gamma)$ in the same way as for the groups.

Next suppose we are given for each vertex p of Γ a pointed CW-space $X^{(p)}$. For each simplex $\sigma = (p_0, \dots, p_n)$ of $N(\Gamma)$ let $X^{(\sigma)}$ be the subset of the product of all the $X^{(p)}$ (in the given vertex order) such that the pth coordinate is the base point of $X^{(p)}$ if $p \notin \sigma$ and is otherwise arbitrary. Let X_Γ be the union of all the $X^{(\sigma)}$ for all $\sigma \in N(\Gamma)$. This we call the *graph product* of the spaces $X^{(p)}$.

The main results on these spaces are:

Theorem 4.1 *If each space $X^{(p)}$ is aspherical, then so also is their graph product.*

Proof The proof is modelled on that given by Cohen in [16].

If Γ is a complete graph the result is clear, since finite products of aspherical spaces are aspherical. We now work by induction on the number of vertices of Γ.

Suppose Γ is not complete. Then there are vertices p, q of Γ which do not form an edge of Γ. Let V be the vertex set of Γ, and let $\Gamma_0, \Gamma_1, \Gamma_2$ be the full subgraphs of Γ on the complements in V of $\{p, q\}, \{p\}, \{q\}$ respectively. Let X_0, X_1, X_2 be the corresponding graph products. By the inductive assumption, these are aspherical. It is clear that

$$X_\Gamma = X_1 \cup_{X_0} X_2.$$

But the maps on fundamental groups induced by the inclusions $X_0 \to X_1, X_0 \to X_2$ are injective. A theorem of Whitehead [33] now implies that X_Γ is aspherical. □

Theorem 4.2 *The fundamental crossed complex of the graph product of CW-spaces is the graph tensor product of their fundamental crossed complexes.*

Proof This is immediate from previous results on the fundamental crossed complex of a product of CW-spaces, and the Generalised Van Kampen Theorem. □

Corollary 4.3 *If each $C^{(p)}$ is a free crossed resolution of the group $G^{(p)}$ then the graph tensor product of the crossed resolutions is a free crossed resolution of the graph product of the groups.*

Note that our result is stronger than that of Cohen in [16] in that we obtain non abelian information. On the other hand he obtains information on projective resolutions over arbitrary rings which cannot currently be obtained by our methods.

Example 4.4 In the case of a direct product $A \times B \times C$ of groups, generators x, y, z of A, B, C respectively give rise to an element of dimension 3 of the corresponding tensor product of free crossed resolutions, namely $x \otimes y \otimes z$. The boundary of this is an identity among relations, and can be worked out explicitly from the formulae for the tensor product. It in fact corresponds to the Homotopy Addition Lemma in [4]. This gives another view of Loday's 'favourite example' in [26], which is the case $A = B = C = \mathbb{Z}$.

4.2 Free products with amalgamation

We illustrate the use of crossed complexes of groupoids with the construction of a free crossed resolution of a free product with amalgamation, given free crossed resolutions of the individual groups. This is a special case of results on graphs of groups which will be given in [10, 29] but this special case nicely shows the advantage of the method and in particular the necessary use of groupoids.

Suppose the group G is given as a free product with amalgamation

$$G = A *_C B,$$

which we can alternatively describe as a pushout of groups

$$\begin{array}{ccc} C & \xrightarrow{j} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ A & \xrightarrow[j']{} & G \end{array}$$

We are assuming the maps i, j are injective so that by standard results so also are the maps i', j'. Suppose we are given free crossed resolutions $F(Q)$ of Q for $Q = C, A, B$. The morphisms i, j may then be lifted (non uniquely) to morphisms $F(i) : F(C) \to F(A), F(j) : F(C) \to F(B)$. However we cannot expect that the pushout of these morphisms in the category Crs gives a free crossed resolution of G.

To see this, suppose that these crossed resolutions are realised by CW-filtrations $X(Q)$ for $Q = C, A, B$ and $F(i), F(j)$ are realised by cellular maps $X(i) : X(C) \to X(A), X(j) : X(c) \to X(B)$. However, the pushout in topological spaces of cellular maps does not in general yield a CW-complex – for this it is required that one of the maps is an inclusion of a subcomplex, and there is no reason why this should be true in this case. The standard construction instead is to take the double mapping $M = M(X(i), X(j))$ cylinder given by the *homotopy pushout*

$$\begin{array}{ccc} X(C) & \xrightarrow{X(j)} & X(B) \\ {\scriptstyle X(i)}\big\downarrow & \simeq & \big\downarrow \\ X(A) & \longrightarrow & M \end{array}$$

which is obtained from the disjoint union of $X(A), X(B), X(C) \times I$ by identifying $(x, 0) \sim X(i)(x), (x, 1) \sim X(j)(x)$ for $x \in X(C)$. This ensures that M is a CW-complex and that the composite maps $X(C) \to M$ given by the two ways round the square are homotopic cellular maps.

It follows that the appropriate construction to take for crossed complexes is obtained by applying π to this homotopy pushout: this yields a homotopy pushout in Crs

$$\begin{array}{ccc} F(C) & \xrightarrow{F(j)} & F(B) \\ {\scriptstyle F(i)}\big\downarrow & \simeq & \big\downarrow \\ F(A) & \longrightarrow & F(M) \end{array}$$

Since M is aspherical we know that $F(M)$ is aspherical and so is a free crossed resolution. Of course $F(M)$ has two vertices $0, 1$. Thus it is not a free crossed resolution of G but is a free crossed resolution of the homotopy pushout in the category Gpd

$$\begin{array}{ccc} C & \xrightarrow{j} & B \\ {\scriptstyle i}\big\downarrow & \simeq & \big\downarrow \\ A & \longrightarrow & G(M) \end{array}$$

which is obtained from the disjoint union of the groupoids $A, B, C \times \mathcal{I}$ by adding the relations $(c, 0) \sim i(c), (c, 1) \sim j(c)$ for $c \in C$. The groupoid $G(M)$ has two objects $0, 1$ and each of its object groups is isomorphic to the amalgamated product group G, but we need to keep these two object groups distinct. This idea for forming a fundamental groupoid for a graph of groups is due to Higgins [21], who shows that it leads to convenient normal forms for elements of the fundamental groupoid. This view is pursued in the recent thesis of Emma Moore [29], from which this section is largely taken.

The two crossed complexes of groups $F(M)(0), F(M)(1)$ which are the parts of $F(M)$ lying over $0, 1$ respectively are free crossed resolutions of the groups $G(M)(0), G(M)(1)$. From the formulae for the tensor product of crossed complexes we can identify free generators for $F(M)$: in dimension n we get
free generators x at 0 where x runs through free generators of $F(A)_n$;
free generators y at 1 where y runs through free generators of $F(B)_n$;
free generators $z \otimes \iota$ at 1 where z runs through free generators of $F(C)_{n-1}$.

Example 4.5 The trefoil group T given in section 1 can be presented as a free product with amalgamation $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$ where the two morphisms $\mathbb{Z} \to \mathbb{Z}$ have cokernels of orders 3 and 2 respectively. The resulting homotopy pushout we call the *trefoil groupoid*. We immediately get a free crossed resolution of length 2 for the trefoil groupoid, whence we can deduce the free crossed resolution of the trefoil group T stated in section 1.

More elaborate examples and discussion are given in [10, 29].

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