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Nonabelian Algebraic Topology

Filtered Spaces, Crossed Complexes, Cubical Homotopy Groupoids

With contributions by
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European Mathematical Society

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Preface

This book is an exposition of some basic algebraic topology at the border between homology and homotopy. The emphasis is explained as follows. A Leverhulme Emeritus Fellowship was awarded to the first author for 2002–2004 to support making available, in one volume and in a consistent style, the work on crossed complexes and related higher homotopy groupoids carried out from 1974 to 2005 mainly by the first two authors. This work resulted in 12 joint papers as well as a number of other collaborations.

The project arose from the question formulated in about 1965 as to whether or not groupoids could be useful in higher homotopy theory. Could one develop theories and applications of higher groupoids in a spirit similar to that of combinatorial group theory, enabling both understanding and calculation, and thus continuing J. H. C. Whitehead's project of 'Combinatorial homotopy'? This aim also explains the term 'Higher dimensional group theory' in [Bro82].

This account also elucidates fully, as did [BH81a], a paragraph near the end of the Introduction to [Bro67] which mentioned an n-dimensional version of the Seifert–van Kampen Theorem, but really referred speculatively to an intuition of a proof, still then in search of a theorem. The necessary machinery of strict cubical higher homotopy groupoids was set up over the years to express that intuition. Surprisingly, it worked out as intended, though the task involved many new ideas and inputs from many people. However the key notion of *higher homotopy groupoid* has not attracted much attention in algebraic topology. So a full exposition is needed.

We also try to include references to work directly relevant to the main themes, and hope this book will also be useful as a reference on related work. It is not intended to be a survey of all work on, for example, crossed modules. Also we cannot claim that the historical references are full and definitive, but we hope they will give a useful entrée to the literature.

The organisation and some new details have been worked out by Brown and Sivera who carry full responsibility for the final result and in particular for errors and obscurities. However the main thrust of this exposition comes from the joint papers of Brown and Higgins; the contribution of Philip Higgins to this research by imagination, powers of organisation of material, algebraic insight and expository skills are seen throughout this book, and so he is rightly a joint author.

Obtaining these results depended on other fortunate collaborations, particularly initial work on double groupoids and crossed modules with Chris Spencer in 1971–73 under SERC support, see [BS76a], [BS76b]. Collaboration at Bangor over the years with Tim Porter and Chris Wensley has been especially important. Other collaborators on joint papers relevant to the 'groupoid project' were: Lew Hardy, Jean-Louis Loday, Sid Morris, Phil Heath, Peter Booth, Johannes Huebschmann, Graham Ellis, Heiner Kamps, Nick Gilbert, Tim Porter, David Johnson, Edmund Robertson, Hans Baues,

Razak Salleh, Kirill Mackenzie, Marek Golasinski, Mohammed Aof, Rafael Sivera, Osman Mucuk, George Janelidze, Ilhan Icen, James Glazebrook; research students at Bangor (with date of completion, supervised by Brown unless marked P for Porter or W for Wensley): Lew Hardy (1974), Tony Seda (1974), A. Razak Salleh (1975), Keith Dakin (1976), Nick Ashley (1978), David Jones (1984), Graham Ellis (1984), Fahmi Korkes (1985, P), Ghafar H. Mosa (1987), Mohammed Aof (1988), Fahd Al-Agl (1988), Osman Mucuk (1993), Andy Tonks (1993), Ilhan Icen (1996), Phil Ehlers (1994, P), J. Shrimpton (1990, with W), Zaki Arvasi (1995, P), Murat Alp (1997, W), Ali Mutlu (1998, W), Anne Heyworth (1998 with W), Emma Moore (2001, with W).

It is a pleasure to acknowledge also:

(i) The influence of the work of Henry Whitehead, who was Brown's supervisor until 1960, when Henry died suddenly in Princeton at the age of 55. It was then that Michael Barratt guided Brown's doctoral work towards the homotopy type of function spaces, a study whose methods are in the background of much of this book; Michael's splendid example of how to go about mathematical research is gratefully acknowledged. In writing the first edition of the book which has now become 'Topology and Groupoids', [Bro06], Brown turned to Whitehead's work on 'Combinatorial Homotopy'; that term shows the influence on Whitehead of the analogy between ideas in homotopy theory and in combinatorial group theory.

(ii) The further contributions of research students at Bangor, and of other colleagues, who all contributed key ideas to the whole programme.

(iii) The stimulus of a correspondence with Alexander Grothendieck in the years 1982–1991. This correspondence is to be published by the Société Mathématique de France, as an Appendix to the visionary manuscript by Grothendieck, 'Pursuing Stacks', which was distributed from Bangor in 1983, see [GroPS1], [GroPS2].[1]

(iv) The support of the Leverhulme Trust through an Emeritus Fellowship for Brown in 2002–2004, and to my chosen referees for that, Professors A. Bak (Bielefeld) and J. P. May (Chicago). This Fellowship provided the support for: the revision and publication of 'Topology and Groupoids'; meetings of Brown with Sivera and with Higgins; LaTeX work, well done by Genevieve Tan and Peilang Wu; other travel and visitors; and also a moral impetus to complete this project.

(v) Support by the University of Wales and by the SERC for the collaboration with Higgins in the years 1974–1985, and the SERC/EPSRC for support of 11 of the above research students, including Keith Dakin, whose invention of T-complexes, [Dak77], made it clear that there did exist acceptable generalisations to all dimensions of double groupoids with connections.

(vi) Chris Wensley for careful reading of many parts of drafts of this book, and seeing errors and obscurities; for the joint published work involved in Chapter 5; and supplying Section 5.9 on the use of symbolic computation for computing induced crossed modules.

(vii) Sergei Soloviev for work with Brown on Section C.7 of Appendix C on monoidal closed categories.

(viii) Jean-Louis Loday for a happy collaboration starting in 1981 and resulting in three joint papers which showed how aspects of these ideas could be taken further.

(ix) Johannes Huebschmann for helpful remarks on the history of identities among relations and the influence of Seifert and Reidemeister; Urs Schreiber for comments on the Dold–Kan Theorem; and Ulrich Diez for writing the macro which gives the link between endnotes and hyperref.

(x) Manfred Karbe for guidance of the book through to this form, and Irene Zimmermann for tremendous and perceptive help at the copy editing stage.

We have of course tried to avoid typos, mistakes, obscurities and infelicities, and failure to reference appropriate material, but are unlikely to have done so completely. We therefore thank in advance those readers who inform us of such, and intend to keep a note of these on the web.

We thank many for support and interest, and also those whose unexplained scepticism, even disbelief, has over decades been a challenge and stimulus.

One aim of this account has been to give appropriate honour to the structures which arise naturally from the geometry, and thus make it easier to find further developments of these methods. For this reason, the final Chapter 16 on 'Future directions?' suggests a number of problems and questions.

Notes

1 p. xiv In a letter dated 02/05/1983 Alexander Grothendieck wrote:

> Don't be surprised by my supposed efficiency in digging out the right kind of notions – I have just been following, rather let myself be pulled ahead, by that very strong thread (roughly: understand non commutative cohomology of topoi!) which I kept trying to sell for about ten or twenty years now, without anyone ready to "buy" it, namely to do the work. So finally I got mad and decided to work out at least an outline by myself.

Prerequisites and reading plan

The aim is for the major parts of this book to be readable by a graduate student acquainted with general topology, the fundamental group, notions of homotopy, and some basic methods of category theory. Many of these areas, including the concept of groupoid and its uses, are covered in Brown's text 'Topology and Groupoids', [Bro06]. The only theory we have to assume for the Homotopy Classification Theorem in Chapter 11 is some results on the geometric realisation of cubical sets.

Some aspects of category theory perhaps less familiar to a graduate student, or for which we wish to emphasise a viewpoint, are given in Appendices, particularly the notion of representable functor, the notion of dense subcategory, and the preservation of colimits by a left adjoint functor. This last fact is a simple but basic tool of algebraic computation for those algebraic structures which are built up in several levels, since it can often show that a colimit of such a structure can be built up level by level. We also give an account of fibrations and cofibrations of categories, which give a general background to the notions and techniques for dealing with colimits of mathematical structures with structures at various levels. Indeed it is *the use in algebraic topology of algebraic colimit arguments rather than exact sequences that is a key feature of this book*.

We make no use of classical tools such as simplicial approximation, but some knowledge of homology and homotopy of chain complexes could be useful at a few points, to help motivate some definitions.

We feel it is important for readers to understand how this theory derives from the basic intuitions and history of algebraic topology, and so we start Part I with some history. After that, historical comments are given in Notes at the end of each chapter.

This book is designed to cater for a variety of readers.

Those with some familiarity with traditional accounts of relative homotopy theory could skip through the first two chapters, and then turn to Chapter 6, and its key account of the homotopy double groupoid $\rho(X, A, C)$ of a pair of spaces (X, A) with a set C of base points. This construction avoids two problems with second relative homotopy groups, namely that a choice has to be made in their definition, and all their group compositions are on a single line. The quite natural construction of the homotopy double groupoid is the key to proving a 2-dimensional Seifert–van Kampen Theorem, and so giving the applications in Chapters 4 and 5. Part I does develop a lot of the algebra and applications of crossed modules (particularly coproducts and induced crossed modules) and the full story of these can be skipped over.

Part II gives the major applications of crossed complexes, with the proofs of key results given in Part III, using the techniques of cubical ω-groupoids.

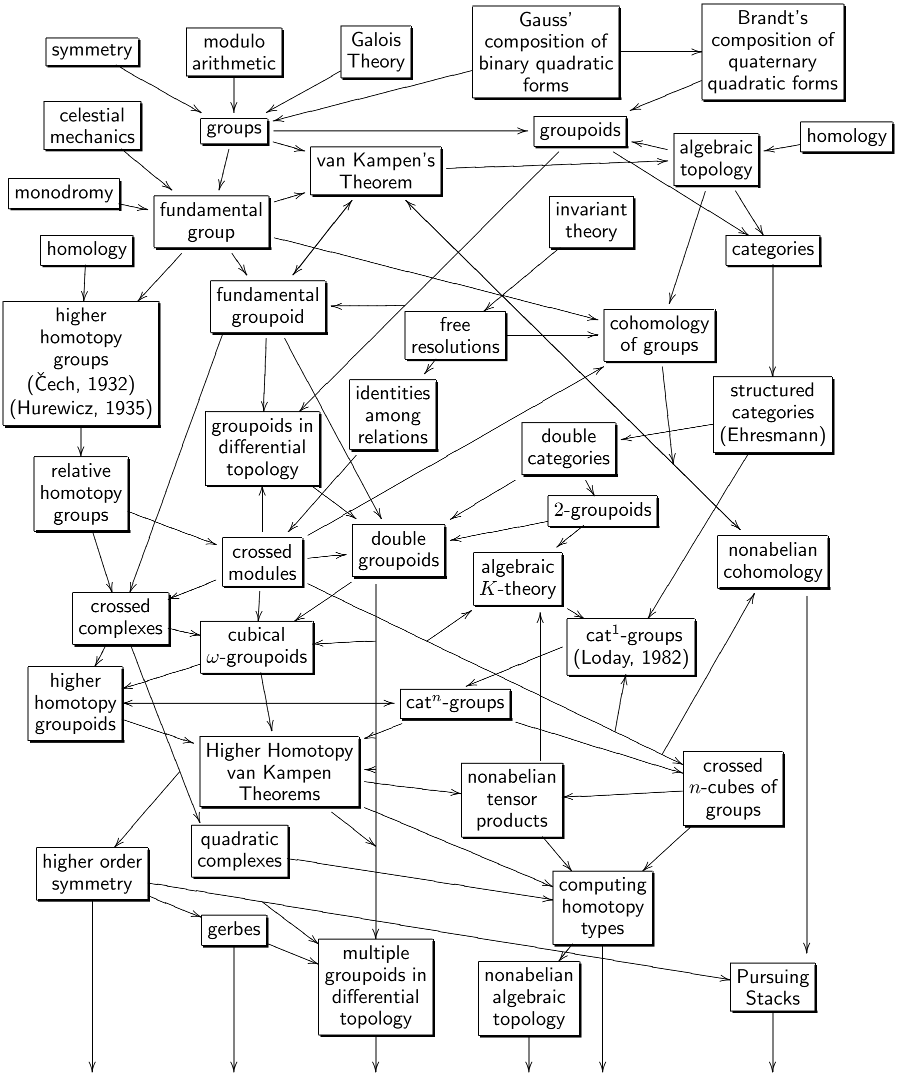
Finally, those who want the pure logical order could read the book starting with Part III, and referring back for basic definitions where necessary; or, perhaps better, start with Chapter 6 of Part I for the pictures and intuitions, and then turn to Part III.

The book [Bro06] is used as a basic source for background material on groupoids in homotopy theory. Otherwise, in order not to interrupt the flow of the text, and to give an opportunity for wider comment, we have put most background comments and references to the literature in notes at the end of each chapter. Nonetheless we must make the usual apology that we might not have been fair to all contributors to the subject and area.

We have tried to make the index as useful as possible, by indexing multi-part terms under each part. To make the Bibliography more useful, we have used hyperref to list the pages on which an item is cited; thus the Bibliography serves to some extent as a name index.

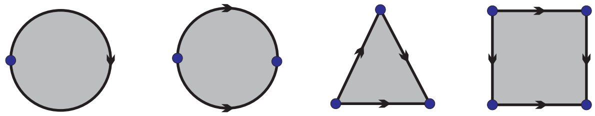
Because of the complexity and intricacy of the structure we present, readers may find it useful to have the e-version with hyperref as well as the printed version.

Historical context diagram



This diagram aims to give a sketch of some influences and interactions leading to the development of nonabelian algebraic topology, and higher dimensional algebra, so that this exposition is seen as part of a continuing development. There are a number of other inputs and directions which were not easy to fit in the diagram, for example the contribution of Seifert to the Seifert–van Kampen Theorem, and the work of C. Ehresmann on n-fold categories. He was a pioneer in this, and his definition and examples of double categories in [Ehr65] were in 1965 a starting input for this project on groupoids in higher homotopy theory.

The theory of groupoids and categories gets more complicated in higher dimensions basically because of the complexity of the basic geometric objects. Thus in dimension 2 we might take as basic objects the 2-disk, 2-globe, 2-simplex, or 2-cube as in the following pictures:



In this book we will use principally the 2-disks, which give us crossed modules, and the 2-cubes, which give us double groupoids, while in higher dimensions the disks and cubes give us crossed complexes and cubical ω-groupoids respectively. In essence, the cubical model leads to conjectures and then theorems, partly through the ease of expressing multiple compositions, see p. xxii, and Remark 6.3.2, while the disk model leads to calculations, and clear relations to classical work.

In category theory rather than groupoid theory the disk model is not available. There is however important work, even a majority, in higher category theory which takes a globular or simplicial rather than cubical route, so there is still much work to be done to relate and evaluate all these models, for the aims of this treatise, or for current and future applications.

Introduction

The theory we describe in this book was developed over a long period, starting about 1965, and always with the aim of developing groupoid methods in homotopy theory of dimension greater than 1. Algebraic work made substantial progress in the early 1970s, in work with Chris Spencer. A substantial step forward in 1974 by Brown and Higgins led us over the years into many fruitful areas of homotopy theory and what is now called 'higher dimensional algebra'[2]. We published detailed reports on all we found as the journey proceeded, but the overall picture of the theory is still not well known. So the aim of this book is to give a full, connected account of this work in one place, so that it can be more readily evaluated, used appropriately, and, we hope, developed.

Structure of the subject

There are several features of the theory and so of our exposition which divert from standard practice in algebraic topology, but are essential for the full success of our methods.

Sets of base points: Enter groupoids

The notion of a 'space with base point' is standard in algebraic topology and homotopy theory, but in many situations we are unsure which base point to choose. One example is if $p\colon Y \to X$ is a covering map of spaces. Then X may have a chosen base point x, but it is not clear which base point to choose in the discrete inverse image space $p^{-1}(x)$. It makes sense then to take $p^{-1}(x)$ as a set of base points.

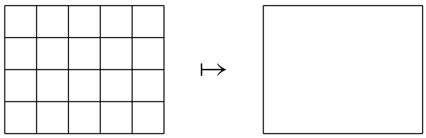
Choosing a set of base points according to the geometry of the situation has the implication that we deal with fundamental groupoids $\pi_1(X, X_0)$ on a *set* X_0 *of base points* rather than with the family of fundamental groups $\pi_1(X, x)$, $x \in X_0$. The intuitive idea is to consider X as a country with railway stations at the points of X_0; we then want to consider all the journeys *between* the stations and not just what is usually called 'change of base point', the somewhat bizarre concept of the set of return journeys from the individual stations, together with ways of moving from a return journey at one station to a return journey at another.

Sets of base points are used freely in what we call 'Seifert–van Kampen type situations' in [Bro06], when two connected open sets U, V have a disconnected intersection $U \cap V$. In such case it is sensible to choose a set X_0 of base points, say one point in each component of the intersection.[3]

The method is to use a Seifert–van Kampen type theorem to pass from topology to algebra by determining the fundamental groupoid $\pi_1(U \cup V, X_0)$ of a union, and then to compute a particular fundamental group $\pi_1(U \cup V, x)$ by using what we call 'combinatorial groupoid methods', i.e. using graphs and trees in combination with the groupoid theory. This follows the principle of keeping track of structure for as long as is reasonable.

Groupoids in 2-dimensional homotopy theory

The successful use of groupoids in 1-dimensional homotopy theory in [Bro68] suggested the desirability of investigating the use of groupoids in higher homotopy theory. One aspect was to find a mathematics which allowed 'algebraic inverse to subdivision', in the sense that it could represent multiple compositions as in the following diagram

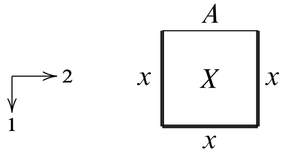


(multcomp)

in a manner analogous to the use of $(a_1, a_2, \dots, a_n) \mapsto a_1 a_2 \dots a_n$ in categories and groupoids, but in dimension 2. Note that going from right to left in the diagram is subdivision, a standard technique in mathematics.

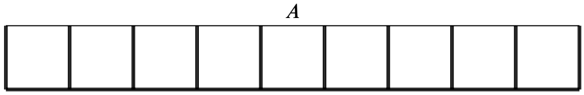
Traditional homotopy theory described the family $\pi_2(X, x)$ of homotopy groups, consisting of homotopy classes of maps $I^2 \to X$ which take the edges of the square I^2 to x, but this did not incorporate the groupoid idea, except under 'change of base point'.

Also considered were the relative homotopy groups $\pi_n(X, A, x)$ of a based pair (X, A, x) where $x \in A \subseteq X$. In dimension 2 the picture is as follows, where thick lines denote constant maps:

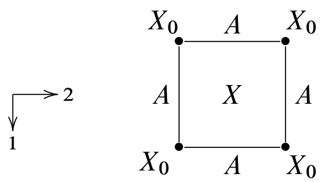


That is, we have homotopy classes of maps from the square I^2 to X which take the edge ∂_1^- to A, and the remaining three edges to the base point.

This definition involves choices, is unsymmetrical with respect to directions, and so is unaesthetic. The composition in $\pi_2(X, A, x)$ is the clear horizontal composition, and does give a group structure, but even large compositions are still 1-dimensional, i.e. in a line:



In 1974 Brown and Higgins found a new construction, finally published in [BH78a], which we called $\rho_2(X, A, X_0)$: it involves no such choices, and really does enable multiple compositions as wished for in Diagram (multcomp). We considered homotopy classes *rel vertices* of maps $[0,1]^2 \to X$ which map edges to A and vertices to X_0:



Part of the geometric structure held by this construction is shown in the diagram:

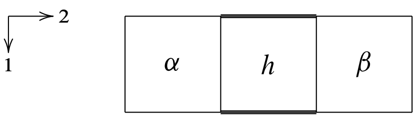
$$\rho_2(X, A, X_0) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \pi_1(A, X_0) \;\substack{\longrightarrow\\ \longrightarrow}\; X_0$$

where the arrows denote boundary maps.

A horizontal composition in $\rho_2(X, A, X_0)$ is given by

$$\langle\!\langle \alpha \rangle\!\rangle +_2 \langle\!\langle \beta \rangle\!\rangle = \langle\!\langle \alpha +_2 h +_2 \beta \rangle\!\rangle$$

as shown in the following diagram, where h is a homotopy rel end points in A between an edge of α and an edge of β, and thick lines show constant paths.



The proof that this composition is well defined on homotopy classes is not entirely trivial and is given in Chapter 6. With a similar vertical composition, we obtain the structure of *double groupoid*, which enables multiple compositions as asked for in Diagram (multcomp).

There is still more structure which can be given to ρ_2, namely that of 'connections', which we describe in the section on cubical sets with connections on p. xxviii.

Crossed modules

A surprise was that the investigation of double groupoids led back to a concept due to Henry Whitehead when investigating the properties of second relative homotopy groups, that of crossed module. Analogous ideas were developed independently by Peiffer and Reidemeister in [Pei49], [Rei49], the war having led to zero contact between mathematicians in Germany and the UK[4]. It is interesting that Peiffer's paper was submitted in June, 1944. Work by Brown with C. B. Spencer in 1971–73 led to the discovery of a close relation between double groupoids and crossed modules. This, with the construction in the previous section, led to a 2-dimensional Seifert–van Kampen Theorem, making possible some new computations of nonabelian second relative homotopy groups which we give in detail in Chapters 4, 5.

A *crossed module* is a morphism

$$\mu\colon M \to P$$

of groups together with an action of the group P on the right of the group M, written $(m, p) \mapsto m^p$, satisfying the two rules:

CM1) $\mu(m^p) = p^{-1}(\mu m)p$;

CM2) $m^{-1}nm = n^{\mu m}$,

for all $p \in P, m, n \in M$. Algebraic examples of crossed modules include normal subgroups M of P; P-modules; the inner automorphism crossed module $M \to \operatorname{Aut} M$; and many others. There is the beginnings of a combinatorial, and also a related computational, crossed module theory.

The standard geometric example of crossed module is the boundary morphism of the second relative homotopy group

$$\partial\colon \pi_2(X, X_1, x) \to \pi_1(X_1, x)$$

where X_1 is a subspace of the topological space X and $x \in X_1$.

Our 2-dimensional Seifert–van Kampen Theorem, Theorem 2.3.1, yields computations of this crossed module in many useful conditions when X is a union of open sets, with special cases dealt with in Chapters 4 and 5. These results deal with nonabelian structures in dimension 2, and so are not available by the more standard methods of homology and covering spaces.

The traditional focus in homotopy theory has been on the second homotopy group, sometimes with its structure as a module over the fundamental group. However Mac Lane and Whitehead showed in [MLW50] that crossed modules model weak pointed homotopy 2-types; thus the 2-dimensional Seifert–van Kampen Theorem allowed new computations of some homotopy 2-types. It is not always straightforward to compute the second homotopy group from a description of the 2-type, but this can be done in some cases.

An aim to compute a second homotopy group is thus reached by computing a larger structure, the homotopy 2-type. This is not too surprising: a determination of the 2-type of a union should require information on the 2-types of the pieces and on the way these fit together. The 2-type also in principle determines the second homotopy group as a module over the fundamental group.

For all these reasons, crossed modules are commonly seen as good candidates for 2-*dimensional groups*. The algebra of crossed modules and their homotopical applications are the themes of Part I of this book.

In the proof of the 2-dimensional Seifert–van Kampen Theorem we use double groupoid structures which are related to *crossed modules of groupoids*; the latter are part of the structure of crossed complexes defined later.

Filtered spaces

Once the 2-dimensional theory had been developed it was easy to conjecture, particularly considering work of J. H. C. Whitehead in [Whi49b], that the theory in all dimensions should involve filtered spaces, a concept central to this book. An approach to algebraic topology via filtered spaces is unusual, so it is worth explaining here what is a filtered space and how this notion fits into algebraic topology.

A *filtered space* X_* is simply a topological space X and a sequence of subspaces:

$$X_*\colon\ X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X_\infty = X.$$

A standard example is the filtration of a geometric simplicial complex by its skeleta: X_n is the union of all the simplices in X of dimension $\leqslant n$. More generally, X would be a CW-complex, the generalisation of the finite cell complexes in [Bro06], and X_n is the union of all the cells of dimension $\leqslant n$. Here X_{n+1} is obtained from X_n by attaching cells of dimension $n+1$.

There are other simple examples, which are important for us. One is when (X, A, x) is a pointed pair of spaces, i.e. $x \in A \subseteq X$, and $n \geqslant 2$. Then we have a filtered space $X_*^{[n]}$ in which $X_i^{[n]}$ is $\{x\}$ for $i = 0$, A for $0 < i < n$ and is X for $i \geqslant n$. It may be asked: why go to this bother? Why not just stick to the pair (X, A, x)? The answer is that for $n \geqslant 3$ we want to use conditions such as $\pi_i(X, A, x) = 0,\ 1 < i < n$, and to this end we in some sense 'climb up' the above filtration $X_*^{[n]}$.

Another geometric example of filtered space is when X is a smooth manifold and $f\colon X \to \mathbb{R}$ is a smooth map. Morse theory shows that f may be deformed into a map g which induces what is called a handlebody decomposition of X, which is a filtration of X in which X_{n+1} is obtained from X_n by attaching 'handles' of type $n+1$. This area is explored by methods related to ours in Chapter VI of [Sha93]. A further refinement of filtered space is the notion of *topologically stratified space*, which occurs in singularity theory – see the entry in Wikipedia, for example, and also [Gro97], Section 5, which is especially interesting for Grothendieck's comments on the foundations of general topology. But the methods of this book have not yet been applied in that area.

It is of course standard to consider the simplicial singular complex SX of a topological space X, to obtain invariants from this, and then if X has a filtration to make further developments to get information on the filtered invariants. An example of this kind is when X is a CW-complex and we use the skeletal filtration. These ideas were developed by Blakers in [Bla48] for relating homology and homotopy groups, following work of Eilenberg in [Eil44] and Eilenberg–Mac Lane in [EML45b], and are related to the use of what are commonly called *Eilenberg subcomplexes*, see for example [Sch91].

In conclusion, we use filtered spaces because with them we can make this theory work, for understanding and for calculation.

Crossed complexes

Central to our work is the association to any filtered space X_* of its *fundamental crossed complex* ΠX_*. This is defined using the fundamental groupoid $\pi_1(X_1, X_0)$ and the family of relative homotopy groups $\pi_n(X_n, X_{n-1}, x)$ for all $x \in X_0$ and $n \geqslant 2$, and generalises the crossed module of a pointed pair of spaces.

A *crossed complex* C over C_1, where C_1 is a groupoid with object set C_0, is a sequence

$$\cdots \longrightarrow C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots\cdots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1$$

of morphisms of groupoids over C_0 such that for $n \geqslant 2$ C_n is just a family of groups, abelian if $n \geqslant 3$; C_1 operates on C_n for $n \geqslant 2$; $\delta_{n-1}\delta_n = 0$ for $n \geqslant 3$; and other axioms hold which we give in full in Section 7.1.iii. The axioms are in fact those universally satisfied by ΠX_*, as we prove in Corollary 14.5.4.

One crucial point is that $\delta_2 \colon C_2 \to C_1$ is a crossed module (over the groupoid C_1). The whole structure has analogies to a chain complex with a groupoid of operators; this analogy is worked out in terms of a pair of adjoint functors in Section 7.4. However in passing from a crossed complex to its associated chain complex with operators some structure is lost. Crossed complexes have better realisation properties than these chain complexes: the crossed module part in dimensions 1 and 2 in crossed complexes allows the modelling of homotopy 2-types, unlike the chain complexes.

In the case X_0 is a singleton, which we call the *reduced* case, the construction of ΠX_* is longstanding, but the general case was defined by Brown and Higgins in [BH81], [BH81a].

Why crossed complexes?

• They generalise groupoids and crossed modules to all dimensions, and the functor Π is classical, involving relative homotopy groups.

• They are good for modelling CW-complexes.

• Free crossed resolutions enable calculations with small CW-models of $K(G,1)$s *and their maps* (Whitehead, Wall, Baues).

• Crossed complexes give a kind of ‘linear model’ of homotopy types which includes all 2-types. Thus although they are not the most general model by any means (they do no contain quadratic information such as Whitehead products), this simplicity makes them easier to handle and to relate to classical tools. The new methods and results obtained for crossed complexes can be used as a model for more complicated situations. This is how a general n-adic Hurewicz Theorem was found in [BL87a], [Bro89].

• They are convenient for some *calculations* generalising methods of computational group theory, e.g. trees in Cayley graphs. We explain some results of this kind in Chapter 10.

• They are close to the traditional chain complexes with a group(oid) of operators, as shown in MD6) on p. xxxii, and are related to some classical homological algebra (e.g. *identities among relations for groups*). Further, if SX is the simplicial singular complex of a space, with its skeletal filtration, then the crossed complex $\Pi(SX)$ can be considered as a slightly noncommutative version of the singular chains of a space. However crossed complexes have better realisation properties than the related chain complexes.

• The category of crossed complexes has a monoidal structure suggestive of further developments (e.g. *crossed differential algebras*).

• They have a good homotopy theory, with a *cylinder object, and homotopy colimits*. There are homotopy classification results (see Equation (MD9)) generalising a classical theorem of Eilenberg–Mac Lane.

• They have an interesting relation with the Moore complex of simplicial groups and of simplicial groupoids, [Ash88], [NT89a], [EP97].

• They are useful for calculations in situations where the operations of fundamental groups are involved. As an example, in Example 12.3.13 we consider the spaces $K = \mathbb{R}P^2 \times \mathbb{R}P^2$ and Z, the space $\mathbb{R}P^3$ with higher homotopy groups killed, and give a part calculation of the based homotopy classes of maps from $K \to Z$ which induce the morphism $(1,1)\colon \mathsf{C}_2 \times \mathsf{C}_2 \to \mathsf{C}_2$ on fundamental groups. This calculation uses most of the techniques developed here for crossed complexes.

Higher Homotopy Seifert–van Kampen Theorem

The reason why we deal with the filtered spaces defined in the section on p. xxv of this Introduction is the following. It is well known that many useful and geometrically interesting topological spaces are built by processes of gluing, or what we call colimits, from simpler spaces. Very often these simpler spaces have a natural, perhaps simple, filtration so that we often get an induced filtration on the colimit. One of our central results is a Higher Homotopy Seifert–van Kampen Theorem (HHSvKT), which involves the fundamental crossed complex functor Π of previous sections. The theorem shows that for a filtered space built as a 'nice' colimit of so called *connected* filtered spaces, not only is the colimit also connected but we can compute the homotopical invariant Π of the colimit as a colimit of the Π of the individual pieces from which the colimit is built, and the morphisms between them.

From this result we deduce, for example:

(i) the Brouwer Degree Theorem (the n-sphere S^n is $(n-1)$-connected and the homotopy classes of maps of S^n to itself are classified by an integer called the *degree* of the map);

(ii) the Relative Hurewicz Theorem, which is seen here as describing the morphism

$$\pi_n(X, A, x) \to \pi_n(X \cup CA, CA, x) \xrightarrow{\cong} \pi_n(X \cup CA, x)$$

when (X, A) is $(n-1)$-connected, and so does not require the usual involvement of homology groups;

(iii) Whitehead's theorem (1949) that $\pi_2(X \cup \{e^2_\lambda\}, X, x)$ is a free crossed $\pi_1(X, x)$-module;

(iv) a generalisation of that theorem to describe the crossed module

$$\pi_2(X \cup_f CA, X, x) \to \pi_1(X, x)$$

as induced by the morphism $f_*\colon \pi_1(A, a) \to \pi_1(X, x)$ from the identity crossed module $\pi_1(A, a) \to \pi_1(A, a)$; and

(v) a coproduct description of the crossed module $\pi_2(K \cup L, M, x) \to \pi_1(M, x)$ when $M = K \cap L$ is connected and (K, M), (L, M) are 1-connected and cofibred.

Note that (iii)–(v) are about nonabelian structures in dimensions 1 and 2. Of course proofs of the Brouwer Degree Theorem and Relative Hurewicz Theorem are standard in algebraic topology texts, and the theorem of Whitehead on free crossed modules is sometimes stated, but rarely proved. However it is not so well known that all of (i)–(v) are applications of colimit results for relative homotopy groups published before 1985. So one of our aims is to make such colimit arguments more familiar and accessible in algebraic topology, and so perhaps lead to wider applications.

We explain later other applications of crossed complexes in algebraic topology. However we are unable to prove our major results in the sole context of crossed complexes, and have to venture into new structures on *cubical sets*. The next section begins the explanation of the background which leads to cubical higher homotopy groupoids.

Cubical sets with connections

An extra structure which we needed for $\rho_2(X, A, X_0)$ in order to express the notion of cube with commutative boundary was what Chris Spencer and I called *connections*, because of a relation with path-connections in differential geometry. The background is as follows.

Even in ordinary category theory we need the 2-dimensional notion of commutative square:

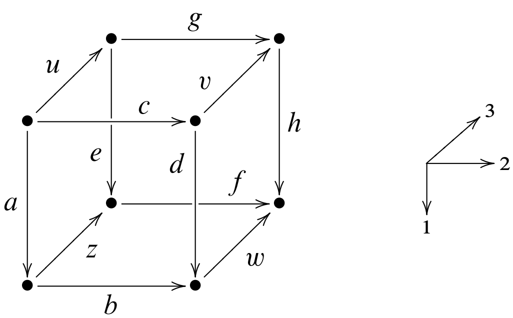
$$\begin{tikzcd} \cdot \arrow[r, "c"] \arrow[d, "a"'] & \cdot \arrow[d, "d"] \\ \cdot \arrow[r, "b"'] & \cdot \end{tikzcd} \qquad ab = cd \quad (a = cdb^{-1} \text{ in the groupoid case}).$$

An easy result is that any composition of commutative squares is commutative. For example, in ordinary equations:

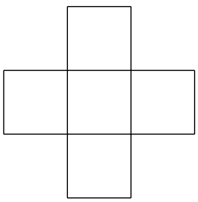
$$ab = cd,\ ef = bg \ \text{ implies } \ aef = abg = cdg.$$

The commutative squares in a category form a *double category*, and this fits with Diagram (multcomp).

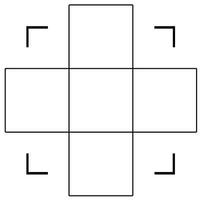
What is a commutative cube, or, more precisely, what is a cube with commutative boundary? Here is a diagram of a 3-cube with labelled and directed edges:



A prospective 'commutativity formula' involving just the edges is easy to write down. However, we want a 2-dimensional notion of the 'commutativity of the faces'. We want to say what it means for the *faces* to commute! We might try to say 'the top face is the composite of the other faces': so fold the other faces flat to give



which makes no sense as a composition! But notice that the two edges adjacent to a corner 'hole' are the same, since we have cut the cube to fold it. So we need canonical fillers to express this as in the diagram:



These extra kind of degeneracies were called *connections*, because of a relation with path connections in differential geometry, as explained in [BS76a]. They may also be thought of as 'turning left or right'. So we can obtain a formula which makes sense for a particular kind of double groupoid with this extra structure. These connections also need to satisfy enough axioms to ensure that composites of 'commutative cubes' in any of three directions are also commutative. It turns out that the axioms are sufficient for this and other purposes, including relating these kinds of double groupoids closely to a concept well established in the literature, that of *crossed module*. This led to the general concept of 'cubical set with connections', which is a key to the theory in all dimensions.

We also need sufficient axioms to be able to prove that any well-defined composition of commutative cubes is commutative. We give these axioms for this dimension in Chapter 6. The idea has then to be carried through in all dimensions. This is part of the work of Chapter 13, and clearly needs new ideas to avoid what might seem impossible complications. While cubical sets have been used since 1955, the use of cubical sets with connections and compositions is another departure from tradition.

Why cubical homotopy omega-groupoids with connections?

Standard algebraic topology uses a singular complex SX of a topological space, develops homology, and then if X has a filtration, needs to relate the algebraic topology of X to that of the filtered structure. Our approach is to take a singular complex which depends on the filtration; it is also necessary to work cubically.[5]

It was easy to conjecture that to generalise the construction $\rho_2(X, A, X_0)$ given above, we should consider a filtered space X_* and the family $R_n X_*$ of sets of maps $I^n \to X$ which map the r-skeleton of I^n into X_r, i.e. the filtered maps $I^n_* \to X_*$; and then take homotopy classes of such maps relative to the vertices of I^n, giving a quotient map $p\colon RX_* \to \rho X_*$. Both RX_* and ρX_* have easily the structure of cubical set, using well-known face and degeneracy maps. Cubical theory was initiated by D. M. Kan in 1955, but was abandoned for the simplicial theory, on which there is now an enormous literature. Nonetheless, multiple compositions are difficult simplicially, while the natural context for them is cubical. Such a cubical approach does move away from standard algebraic topology. Also it was necessary to introduce into the cubical theory the notion of connections in all dimensions.

It was not found easy to prove a central feature of our work that the easily defined multiple compositions in RX_* were inherited by ρX_*. A further difficulty was to relate the structure held by ρX_* to the crossed complex ΠX_* traditional in algebraic topology. These proofs needed new ideas and are stated and proved in Chapter 14.

Here are the basic elements of the construction.

I^n_*: the n-cube with its skeletal filtration.

Set $R_n X_* = \mathsf{FTop}(I^n_*, X_*)$. This is a *cubical set with compositions, connections, and inversions*.

For $i = 1, \dots, n$ there are standard:

face maps $\partial_i^\pm \colon R_n X_* \to R_{n-1} X_*$;

degeneracy maps $\varepsilon_i \colon R_{n-1} X_* \to R_n X_*$;

connections $\Gamma_i^\pm \colon R_{n-1} X_* \to R_n X_*$;

compositions $a \circ_i b$ defined for $a, b \in R_n X_*$ such that $\partial_i^+ a = \partial_i^- b$;

inversions $-_i \colon R_n \to R_n$.

The connections are induced by $\gamma_i^\pm \colon I^n \to I^{n-1}$ defined using the monoid structures $\max, \min \colon I^2 \to I$. They are essential for many reasons, e.g. to discuss the notion of *commutative cube*.

These operations have certain algebraic properties which are easily derived from the geometry and which we do not itemise here – see for example [AABS02]. These were listed first in the Bangor thesis of Al-Agl [AA89]. (In the paper [BH81] the only basic connections needed are the Γ_i^+, from which the Γ_i^- are derived using the inverses of the groupoid structures.)

Here we explain why we need to introduce such new structures.

• The functor ρ gives a form of *higher homotopy groupoid*, thus confirming the visions of topologists of the early 20th century of higher dimensional nonabelian forms of the fundamental group.

• They are equivalent to crossed complexes, and this equivalence is a kind of cubical and nonabelian form of the Dold–Kan Theorem, relating chain complexes with simplicial abelian groups.

• They have a clear *monoidal closed structure*, and notion of homotopy, from which one can deduce analogous structures on crossed complexes, with detailed formulae, using the equivalence of categories.

• It is easy to relate the functor ρ to tensor products, but quite difficult to do this for Π.

• Cubical methods, unlike globular or simplicial methods, allow for a simple *algebraic inverse to subdivision*, involving multiple compositions in many directions, see p. xxii, and Remarks 6.3.2 and 13.1.11, which are crucial for the proof of our HHSvKT in Chapter 14; see also the arguments in the proof of say Theorem 6.4.10.

• The additional structure of ‘connections’, and the equivalence with crossed complexes, allows the notion of *thin cube*, Section 13.7, which subsumes the idea of commutative cube, and yields the proof that *multiple compositions of thin cubes are thin*. This last fact is another key component of the proof of the HHSvKT, see Theorem 14.2.9.

• The cubical theory gives a construction of a (*cubical*) *classifying space*

$$BC = (\mathscr{B}C)_\infty$$

of a crossed complex C, which generalises (cubical) versions of Eilenberg–Mac Lane spaces, including the local coefficient case.

• Many papers, including [BJT10], [BP02], [PRP09], [Mal09], [Gou03], [Koc10], [FMP11], [Živ06], [HW08] show a *resurgence of the use of cubes* in for example algebraic K-theory, algebraic topology, concurrency, differential geometry, combinatorics, and group theory.

Diagram of the relations between the main structures

The complete and intricate story has its main facts summarised in the following diagram and comments:

$$\begin{tikzcd}
 & & \text{filtered spaces} \arrow[dll, bend right, "C_*=\nabla\circ\Pi"'] \arrow[dl, shift right, "\Pi"'] \arrow[dr, shift left, "\rho"] & \text{filtered cubical sets} \arrow[l, "|\,|"'] \\
\text{operator chain complexes} \arrow[r, shift right, "\Theta"'] & \text{crossed complexes} \arrow[ur, shift right, "\mathscr{B}"'] \arrow[l, shift right, "\nabla"'] \arrow[rr, shift left, "\lambda"] & & \text{cubical } \omega\text{-groupoids with connections} \arrow[u, "U_*"'] \arrow[ul, shift left, "|\,|\circ U_*"] \arrow[ll, shift left, "\gamma"]
\end{tikzcd}$$

Main Diagram

in which

MD 1) the categories FTop of filtered spaces, Crs of crossed complexes and ω-Gpds of ω-groupoids, are monoidal closed, and have a notion of homotopy using $\otimes$ and unit interval objects;

MD 2) ρ, Π are homotopical functors (that is they are defined in terms of homotopy classes of certain maps), and preserve homotopies;

MD 3) λ, γ are inverse adjoint equivalences of monoidal closed categories, and λ is a kind of 'nerve' functor;

MD 4) there is a natural equivalence $\gamma\rho \simeq \Pi$, so that either ρ or Π can be used as appropriate;

MD 5) ρ preserves certain colimits and certain tensor products, and hence so also does Π;

MD 6) the category Chn of chain complexes with a groupoid of operators is monoidal closed, and ∇ is a monoidal functor which has a right adjoint Θ;

MD 7) by definition, the *cubical filtered classifying space* is $\mathcal{B} = |\;| \circ U_* \circ \lambda$ where U_* is the forgetful functor to filtered cubical sets using the filtration of an ω-groupoid by skeleta, and $|\;|$ is geometric realisation of a cubical set;

MD 8) there is a natural equivalence $\Pi \circ \mathcal{B} \simeq 1$;

MD 9) if C is a crossed complex and its cubical classifying space is defined as $BC = (\mathcal{B}C)_\infty$, then for a CW-complex X, and using homotopy as in MD1) for crossed complexes, there is a natural bijection of sets of homotopy classes

$$[X, BC] \cong [\Pi X_*, C]. \tag{MD9}$$

Structure of the book

Because of the complications set out above in the Main Diagram, and in order to communicate the basic intuitions, we divide our account into three parts, each with an introduction giving the chapter structure of that part.

Part I is on the history and proofs of the 1- and 2-dimensional Seifert–van Kampen Theorems, and the applications of the 2-dimensional theorem to crossed modules of groups. This part covers the main nonabelian colimit results and is intended to convey the context and intuitions in a case where one can easily draw pictures.

Part II is on the theory and applications of crossed complexes over groupoids, using the fundamental crossed complex Π of a filtered space, and giving a full account of applications. The principal tools are: the Higher Homotopy Seifert–van Kampen Theorem for Π; the monoidal closed structure on the category of crossed complexes, which gives a full context for homotopies and higher homotopies; and the cubical classifying space of a crossed complex. A recurring theme is the relation of crossed complexes with chain complexes with a groupoid of operators, which thus relates the material to more classical considerations. An aim of the theory is Chapter 12, which deals with cohomology and the homotopy classification of maps, and the relations of crossed complexes with group and groupoid cohomology.

Part III justifies the theorems on crossed complexes by proving an equivalence between crossed complexes and cubical ω-groupoids, and then proving the main results in the latter context. These main theorems were essentially, and maybe only have been, conjectured in the latter context. Thus this part realises the intuitions behind the main results.

Part III ends with a chapter on ‘Further directions?’ suggesting a number of open areas and questions.

There are also three Appendices giving accounts of various aspects of category theory which are helpful for understanding of the topics, and to give wider context. This account of category theory does not claim to be complete but hopefully gives a useful and somewhat different emphasis from other texts. There is an extended account of fibrations and cofibrations of categories, to give background to the general use of pushouts and pullbacks, and as more examples of 'categories for the working mathematician', in showing analogies between different areas of mathematics.

Notes

2 p. xxi The paper [Bro87], p. 124, suggested that "n-dimensional phenomena require for their description n-dimensional algebra", and this led to the term 'higher dimensional algebra', which widens the term 'higher dimensional group theory' used in [Bro82].

3 p. xxi Here we give some history on this theorem. The first result describing the fundamental group of a union was that of Seifert in [Sei31], for the union of two connected subcomplexes, with connected intersection, of a simplicial complex. The next result was that of van Kampen in [Kam33]. He also gives a formula for the case of nonconnected intersection. His proofs are difficult to follow. Some further history of the subject is given in [Gra92].

The start of the modern approach is the paper of Crowell [Cro59], based on lectures of R.H. Fox, which used the term colimit and the proof was by verification of the universal property. The paper deals with arbitrary unions.

Olum in [Olu58] gave a proof for the case of a union of two sets with connected intersection using nonabelian cohomology with coefficients in a group, and he also carefully analyses Seifert–van Kampen's local conditions. The Mayer–Vietoris type sequence given by Olum was extended in [Bro65a], so that the fundamental group of the circle, or a wedge of circles, could be computed.

It was then found that a more powerful result with simpler proof could be obtained using groupoids, [Bro67]; this gave the fundamental groupoid on a set of base points for the case of nonconnected intersection of two open sets. This result was suggested by the use by Higgins in [Hig64] of free product with amalgamation of groupoids. Thus an aim to compute a fundamental group was reached by first computing a larger structure, a fundamental groupoid on a set of base points, and then giving methods of a combinatorial character for computing the group from the larger structure.

It was also noticed that this possibility ran contrary to the general scope of methods in homological algebra and algebraic topology, which often used exact sequences which did not give such complete results, since an invariant relating close dimensions could often be described immediately only up to extension.

A generalisation to unions of families was given in [BRS84]. A general result for the nonconnected case but still only for groups is in [Wei61], using the notion of the nerve of the cover to describe graph theoretic properties of the components of the intersections of the open sets. A combination of the method of Olum with the use of groupoids is given in [BHK83].

All these insights have been important for the generalisations to higher dimensions. Thus we find it convenient to refer to theorems of these types as *Seifert–van Kampen Theorems*.

We say more later on other extensions and analogues of the 1-dimensional theorem.

We note that the basic results here are referred to in the literature either as Seifert–van Kampen Theorems or as van Kampen Theorems.

We feel it is important to recognise the great contribution of Seifert and the German school of topology. The classic book, [ST80], first published in 1934, had an influence well into the 1950s, and is still worth consulting for the geometric background. It is also worth stating that Seifert was politically in opposition to the Nazi regime in Germany, and was never officially nominated as a full Professor at Heidelberg during the Nazi period. He was nominated after the war, and was then the only scientist at Heidelberg University the American administration would accept to become the Dean of the newly introduced Faculty of Natural Sciences, see [Pup99], [Pup97].

4 p. xxiv Reidemeister, like Seifert, was in opposition to the Nazi ideology and lost his Professorship in Königsberg in 1933, but did, however, become Professor at Marburg, [Art72], [Seg99]. By contrast, the British topologists M. H. A. Newman, J. H. C. Whitehead, and S. Wylie were all working at Bletchley Park during the war, along with many other mathematicians.

5 p. xxx This work progressed in the 1970s when we abandoned the attempt to define a 'higher homotopy groupoid' for a space and instead worked with pairs of spaces and for higher dimensions with filtered spaces. This enabled us to construct the cubical homotopy ω-groupoid $\rho(X_*)$ which is at the heart of this work. Nowadays this would be called a 'strict' ω-groupoid. There is a tendency to call the simplicial complex SX the 'fundamental ∞-groupoid' of the space X, and even to label it ΠX, see for example [Lur09]. Our notation ΠX_* is intended to reflect the close relation to traditional concepts in homotopy theory, the relative homotopy groups. In a similar manner, the notation $\Pi \mathbf{X}$ is used in [BL87] to denote the strict structure of what is there termed the fundamental cat^n-group of an n-cube of spaces $\mathbf{X}$.

Part I

1- and 2-dimensional results

Introduction to Part I

Part I develops that aspect of nonabelian algebraic topology related to the Seifert–van Kampen Theorem (SvKT) in dimensions 1 and 2. The surprising fact is that in this part we are able in this way to obtain in homotopy theory many nonabelian calculations in dimension 2 which seem unavailable without this theory, and without any of the standard machinery of algebraic topology, such as simplicial complexes or simplicial sets, simplicial approximation, chain complexes, or homology theory.

We start in Chapter 1 by giving a historical background, and outline the proof of the Seifert–van Kampen Theorem in dimension 1. It was an analysis of this proof which suggested the higher dimensional possibilities.

We then explain in Chapter 2 the functor

$$\Pi_2 \colon \text{(pointed pairs of spaces)} \to \text{(crossed modules)}$$

in terms of second relative homotopy groups, state a 2-dimensional Seifert–van Kampen Theorem (2dSvKT) for this, and give applications.

Chapter 3 explains the basic algebra of crossed modules and their relations to other topics. The more standard structures of abelian groups or modules over a group are but pale shadows of the structure of a crossed module, as we see over the next two chapters.

Two important constructions for calculations with crossed modules, are *coproducts of crossed modules* on a fixed base group (Chapter 4) and *induced crossed modules* (Chapter 5). Both of these chapters of Part I illustrate how some nonabelian calculations in homotopy theory may be carried out using crossed modules. Induced crossed modules illustrate well the way in which low dimensional identifications in a space can influence higher dimensional homotopical information; they also include free crossed modules, which are important in applications to defining and determining identities among relations for presentations of groups. This last concept has a relation to the cohomology theory of groups, which will become clear in Chapters 10 and 12.

Finally in this part, Chapter 6 gives the proof of the Seifert–van Kampen Theorem for the functor Π_2, a theorem which gives precise situations where Π_2 preserves colimits. A major interest here is that this proof requires another structure, namely that of *double groupoid with connection*, which we abbreviate to *double groupoid*. We therefore construct in a simple way as suggested on p. xxii a functor

$$\rho_2 \colon \text{(triples of spaces)} \to \text{(double groupoids)},$$

and show that this is equivalent in a clear sense to a functor

$$\Pi_2 \colon \text{(triples of spaces)} \to \text{(crossed modules of groupoids)},$$

which is a natural generalisation of our earlier Π_2 functor. Here a triple of spaces is of the form (X, X_1, X_0), where $X_0 \subseteq X_1 \subseteq X$, and the pointed case is when X_0 is a singleton. In Part I we do not make much use of the many pointed case, but it becomes crucial in Part II. This final substantial chapter of Part I thus develops the 2-dimensional groupoid theory which is then used in the proof of Theorem 6.8.2.

Note that all the results contained in Chapters 2–5 are about crossed modules over groups, while in Chapter 6 we generalise to crossed modules over groupoids to prove the 2-dimensional Seifert–van Kampen Theorem. The fact that pushouts, and coequalisers, give the same results in these two contexts follows from the fact that these two types of colimit are defined by connected diagrams, and then applying Theorem B.1.7 of Appendix B.

All this theory generalises to higher dimensions, as we show in Parts II and III, but the ideas and basic intuitions are more easily explained and pictures drawn in dimension 2.

Chapter 1
History

In this chapter we give some of the context and historical background to the main work of this book, in order to show the traditions from which the eventually intricate structures we describe have been found and developed. We think this background is necessary to understand the direction of the research, and to help evaluate the successes and the work still needed to be done for further developments and applications.

We will show how the extensions first from groups to groupoids, and then to certain multiple groupoids and other related structures, enable new results and understanding in algebraic topology.

It is generally accepted that the notion of abstract group is a central concept of mathematics, and one which allows the successful expression of the intuitions of reversible processes. In order to obtain the higher dimensional, nonabelian, local to global results described briefly in the Introduction, the concept of group has:

A) to be 'widened' to that of groupoid, which in a sense generalises the notion of group to allow a spatial component, and

B) to be 'increased in height' to higher dimensions.

Step A) is an essential requirement for step B).

A major stimulus for this view was work of Philip Higgins in his 1964 paper [Hig64], and this book is based largely on his resulting collaboration with Brown. Higgins writes in the Preface to [Hig71] that:

> The main advantage of the transition [from groups to groupoids] is that the category of groupoids provides a good model for certain aspects of homotopy theory. In it there are algebraic models for such notions as path, homotopy, deformation, covering and fibration. Most of these become vacuous when restricted to groups, although they are clearly relevant to group-theoretic problems. ... There is another side of the coin: in applications of group theory to other topics it is often the case that the natural object of study is a groupoid rather than a group, and the algebra of groupoids may provide a more concrete tool for handling concrete problems.

In fact there is a range of intuitions which abstract groups are unable to express, and for which other concepts such as groupoid, pseudogroup and inverse semigroup have turned out to be more appropriate.

As Mackenzie writes in [Mac87]:

> The concept of groupoid is one of the means by which the twentieth century reclaims the original domain of applications of the group concept. The modern, rigorous concept of group is far too restrictive for the range

> of geometrical applications envisaged in the work of Lie. There have thus arisen the concepts of Lie pseudogroup, of differentiable and of Lie groupoid, and of principal bundle – as well as various related infinitesimal concepts such as Lie equation, graded Lie algebra and Lie algebroid – by which mathematics seeks to acquire a precise and rigorous language in which to study the symmetry phenomena associated with geometrical transformations which are only locally defined.

A number of these concepts related to groupoids were initiated by C. Ehresmann over many years, particularly the notion of differential or Lie groupoid ([Ehr83], [Bro07], [Pra07]). The last paper shows how the work of Ehresmann extends the famous Erlangen Programme of Felix Klein.

A failure to accept a relaxation of the concept of group made it difficult to develop a higher dimensional theory modelling some key aspects of homotopy theory. To see the reasons for this we need to understand the basic intuitions which a higher dimensional theory is trying to express, and to see how these intuitions were dealt with historically. This study will confirm a view that it is reasonable to examine and develop the algebra which arises in a natural way from the geometry rather than insist that the geometry has to be expressed within the current available concepts, schemata and paradigms.

1.1 Basic intuitions

There were two simple intuitions involved: One was that of an

algebraic inverse to subdivision.

The other was that of a

commutative cube.

To explain the first intuition, we know how to cut things up, but do we have available an algebraic control over the way we put them together again? This is of course a general problem in mathematics, science and engineering, where we want to represent and determine the behaviour of complex objects from the way they are put together from standard pieces. This is the ‘local-to-global problem’. Any algebra which gives new insights into questions of this form, and yields new computations, clearly has arguments in its favour.

We explain this a bit more in a very simple situation. We often translate geometry into algebra. For example, a figure as follows,

$$\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \xleftarrow{c} \bullet \xrightarrow{d} \bullet$$

is easily translated into

$$abc^{-1}d.$$

For the second intuition, we first consider the easy notion of a commutative square, which means that in the following diagram, going round one way agrees with going round the other:

$$\begin{tikzcd}
\bullet \arrow[r, "a"] \arrow[d, "c"'] & \bullet \arrow[d, "b"] \\
\bullet \arrow[r, "d"'] & \bullet
\end{tikzcd} \tag{1.1.1}$$

it is easy to write

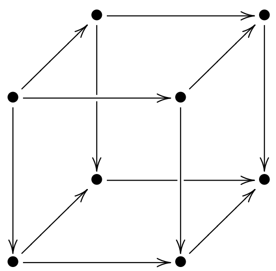
$$ab = cd, \quad \text{or} \quad a = cdb^{-1}.$$

All this is part of the standard repertoire of mathematics. The formulae given make excellent sense as part of say the theory of groups. We also know how to calculate with such formulae.

The problem comes when we try to express similar ideas in one dimension higher. How can one write down algebraically the following picture, where each small square is supposed labeled?

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & \bullet \arrow[l] \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & \bullet \arrow[l] \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet & \bullet \arrow[l] \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[u] & \bullet \arrow[l] \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u]
\end{tikzcd} \tag{1.1.2}$$

Again, how can one write down algebraically the formulae corresponding to the above commutative square (1.1.1) but now for the cube:



(1.1.3)

What does it mean for the *faces* of the cube to commute, or for the top face to be the composition, in some sense, of the other faces?

It is interesting that the step from a linear statement to a 2-dimensional statement should need a lot of apparatus; it also took a lot of experimentation to find an appropriate formulation. As we shall see later, the 2-dimensional composition (1.1.2) requires double categories (Section 6.1), while the second (1.1.3) requires double groupoids with *thin structure*, or with *connections* (Section 6.4).

Thus the step from dimension 1 to dimension 2 is the critical one, and for this reason most of Part I of this book is devoted to the 2-dimensional case. Further reasons are that: the theory in dimension 2 is more straightforward than it becomes in higher dimensions; illustrative pictures are easier to give; and the novel features of the 2-dimensional theory need to be well understood before passing to higher dimensions. It is also intriguing that so much can be done once one has the mathematics to express the intuitions, and that then the mathematical structures control the ways the calculations have to go. This requires an emphasis on *universal properties*, which are afterwards interpreted to give formulae.

1.2 The fundamental group and homology

The above questions on 2-dimensional compositions did not arise out of the void but from a historical context which we now explain.

The intuition for a *Nonabelian Algebraic Topology* was seen early on in algebraic topology, after the ideas of homology and of the fundamental group $\pi_1(X,x)$ of a space X at a base point x of X were developed.

The motivation for Poincaré's definition of the fundamental group in his 1895 paper [Poi96], see also [Poi10], seems to be from the notion of monodromy, that is the change in the value of a meromorphic function of many complex variables as it is analytically continued along a loop avoiding the singularities. This change in value depends only on the homotopy class of the loop, and this consideration led to the notion of the group $\pi_1(X,x)$ of homotopy classes of loops at x, where the group structure arises from composition of loops. Poincaré called this group the *fundamental group*; this fundamental group $\pi_1(X,x)$, with its relation to covering spaces, surface theory, and the later combinatorial group theory, came to play an increasing rôle in the geometry, complex analysis and algebra of the next hundred years. Indeed Poincaré is also keen on generators and relations for groups, and reading them off from a fundamental region.

It also seems possible that an additional motivation arose from dynamics, in the classification of orbits in a phase space.

The utility of the group concept in homotopy theory is increased by the relations between the fundamental group considered as a functor from based topological spaces to groups

$$\pi_1 : \mathsf{Top}_* \to \mathsf{Groups}$$

and another functor called the *classifying space*

$$B : \mathsf{Groups} \to \mathsf{Top}_*,$$

which is the composite of the *geometric realisation* and the *nerve functor* N from groups to simplicial sets.

We shall review the properties of B in Section 2.4. Now let us note that B and π_1 are inverses in some sense. To be more precise, BP is a based space that has all homotopy groups trivial except the fundamental group, which itself is isomorphic to P. Moreover, if X is a connected based CW-complex and P is a group, then there is a natural bijection

$$[X, BP]_* \cong \mathsf{Hom}(\pi_1 X, P),$$

where the square brackets denote pointed homotopy classes of maps.

It follows that there is a map

$$X \to B\pi_1 X$$

inducing an isomorphism of fundamental groups. It is in this sense that *groups are said to model homotopy* 1*-types*, and a computation of a group P is also regarded as a computation of the 1-type of the classifying space BP.

The fundamental group of a space may be calculated in many cases using the Seifert–van Kampen Theorem (see Section 1.5), and in other cases using fibrations of spaces. The main result on the latter, for those familiar with fibrations of spaces, and the classifying space BP of a group P, is that if $1 \to K \to E \to P \to 1$ is a short exact sequence of groups, then the induced sequence $BK \to BE \to BP$ is a fibration sequence of spaces. Conversely, if $F \xrightarrow{i} X \xrightarrow{p} Y$ is a fibration sequence of spaces, and $x \in F$ then there is an induced exact sequence of groups and based sets

$$\cdots \to \pi_1(F, x) \xrightarrow{i_*} \pi_1(X, x) \xrightarrow{p_*} \pi_1(Y, px) \xrightarrow{\partial} \pi_0(F) \to \pi_0(X) \to \pi_0(Y).$$

This result gives some information on $\pi_1(X, x)$ if the other groups are known and even more if the various spaces are connected. We shall return to this sequence in Section 2.6, and it will be used in other contexts, with more information on exactness at the last few terms, in Section 12.1.ii. The classifying space of a group will be generalised to the classifying space of a crossed complex in Chapter 11.

Much earlier than the definition of the fundamental group, higher dimensional topological information had been obtained in terms of 'Betti numbers' and 'torsion coefficients'. These numbers were combined into the powerful idea of the abelian *homology groups* $H_n(X)$ of a space X defined for all $n \geqslant 0$, and which gave very useful topological information on the space. They measured the presence of 'holes' in X of various dimensions and of various types. The origins of homology theory lie in integration, the theorems of Green and Stokes, and complex variable theory.

The notion of 'boundary' and of a 'cycle' as having zero boundary is crucial in the methods and results of this theory, but was always difficult to express precisely until Poincaré brought in simplicial decompositions, and the notion of a 'chain' as a formal sum of oriented simplices. It seems that the earlier writers thought of a cycle as in some sense a 'composition' of the pieces of which it was made, but this 'composition' was, and still is, difficult to express precisely. Dieudonné in [Die89] suggests that the key intuitions can be expressed in terms of cobordism. In any case, the notion of 'formal sum' fitted well with integration, where it was required to integrate over a formal sum of domains of integration, with the correct orientation for these:

$$\int_C f\, dz + \int_{C'} f\, dz = \int_{C+C'} f\, dz.$$

It was also found that if X is connected then the group $H_1(X)$ is the fundamental group $\pi_1(X, x)$ made abelian:

$$H_1(X) \cong \pi_1(X, x)^{\mathrm{ab}}.$$

So the nonabelian fundamental group gave much more information than the first homology group. However, the homology groups were defined in all dimensions. So there was pressure to find a generalisation to all dimensions of the fundamental group.

1.3 The search for higher dimensional versions of the fundamental group

According to [Die89], Dehn had some ideas on this search in the 1920s, as would not be surprising. The first published attack on this question was the work of Čech, using the idea of classes of maps of spheres instead of maps of circles. He submitted his paper on higher homotopy groups $\pi_n(X, x)$ to the International Congress of Mathematicians at Zurich in 1932. The story is that Alexandrov and Hopf quickly proved that these groups were abelian for $n \geqslant 2$, and so on these grounds persuaded Čech to withdraw his paper. All that appeared in the Proceedings of the Congress was a brief paragraph, [Čec32].

The main algebraic reason for this abelian nature was the following result, in which the two compositions $\circ_1$, $\circ_2$ are thought of as compositions of 2-spheres in two directions. The proof is often known as the Eckmann–Hilton argument.

Theorem 1.3.1. *Let S be a set with two monoid structures $\circ_1$, $\circ_2$ each of which is a morphism for the other. Then the two monoid structures coincide and are abelian.*

Proof. The condition that the structure $\circ_1$ is a morphism for $\circ_2$ is that the function

$$\circ_1 \colon (S, \circ_2) \times (S, \circ_2) \to (S, \circ_2)$$

is a morphism of monoids, where $(S, \circ_2)$ denotes S with the monoid structure $\circ_2$. This condition is equivalent to the statement that for all $x, y, z, w \in S$,

$$(x \circ_2 y) \circ_1 (z \circ_2 w) = (x \circ_1 z) \circ_2 (y \circ_1 w).$$

This can be interpreted as saying that the diagram

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} \qquad \begin{tikzcd}[sep=small] {} \arrow[r] \arrow[d] & 2 \\ 1 & \end{tikzcd}$$

has only one composition. Here the arrows indicate that we are using matrix conventions in which the first coordinate gives the rows, and the second coordinate gives the columns. This law is commonly called the *interchange law*.

We now use some special cases of the interchange law. Let e_1, e_2 denote the identities for the structures $\circ_1$, $\circ_2$. Consider the matrix

$$\begin{bmatrix} e_1 & e_2 \\ e_2 & e_1 \end{bmatrix}.$$

This yields easily that $e_1 = e_2$. We write then e for e_1.

Now we consider the matrix composition

$$\begin{bmatrix} x & e \\ e & w \end{bmatrix}.$$

Interpreting this in two ways yields

$$x \circ_1 w = x \circ_2 w.$$

So we write $\circ$ for $\circ_1$.

Finally we consider the matrix composition

$$\begin{bmatrix} e & y \\ z & e \end{bmatrix}$$

and find easily that $y \circ z = z \circ y$. This completes the proof.

Incidentally, it will also be found that associativity comes for free. We leave this to the reader. □

Remark 1.3.2. The above argument was formulated in generality in [EH62] and has proved very useful. But the intuition that originally showed the higher homotopy groups were abelian was probably something like the following. Consider two maps α, β of a square I^2 to X which map the edges to the base point of X. Their 'composite' may be considered as a map of a square into X in which most of it goes to the base point but two disjoint small squares inside are mapped essentially by α and β. But now we see that they can be moved around in the big square, so that, up to homotopy, the order of composition does not matter. □

This result seemed to kill any possibility of 'nonabelian algebraic topology', or of any generalisations to higher dimensions of the fundamental group. In 1935, Hurewicz, without referring to Čech, published the first of his celebrated notes on higher homotopy groups, [Hur35], and the latter groups are often referred to as the 'Hurewicz homotopy groups'. As the abelian higher homotopy groups came to be accepted, a considerable amount of work in homotopy theory moved as far as possible from group theory and the nonabelian fundamental group, and the original concern about the abelian nature of the higher homotopy groups came to be seen as a quirk of history, an unwillingness to accept a basic fact of life. Indeed, Alexandrov and Finikov in their Obituary Notice for Čech, [AF61], referred to the unfortunate lack of appreciation of Čech's work on higher homotopy groups, resulting from too much attention to the disadvantage of their abelian nature.

However J. H. C. Whitehead[6] published important nonabelian work in dimension 2 in 1941, 1946 and 1949, with the second paper introducing the term crossed module – these crossed modules are a central theme of this book. Brown remembers Henry Whitehead remarking in 1957 that early workers in homotopy theory were fascinated by the action of the fundamental group on higher homotopy groups. Many also were dissatisfied with the fact that the composition in higher homotopy groups was independent of the direction. Deeper reasons for this independence are contained in the theory of iterated loop spaces (see the book by Adams, [Ada78], or the books and survey articles by May *et al* [May72], [May77a], [May77b], [May82]).

A new possibility eventually arose in 1967 through the notion of *groupoid*, which we discuss in the next section.

1.4 The origin of the concept of abstract groupoid

A *groupoid* G is defined formally as a small category in which every arrow is invertible. For more details see the surveys [Bro84a], [Wei01], and the books [Bro06], [Hig71]. The category of groupoids and their morphisms will be written Gpds.

There are two important, related and relevant differences between groups and groupoids. One is that groupoids have a partial multiplication, and the other is that the condition for two elements of a groupoid to be composable is a geometric one, namely the end point of one is the starting point of the other. This partial multiplication allows for groupoids to be thought of as 'groups with many identities'. The other is that the geometry underlying groupoids is that of directed graphs, whereas the geometry underlying groups is that of based sets, i.e. sets with a chosen base point. It is clear that graphs are more interesting than sets with base point, and can reflect more geometry. Hence people find in practice that groupoids can reflect more geometry than can groups alone. It seems that the objects of a groupoid allow the addition of a spatial component to group theory.

An argument usually made for groups is that they give the mathematics of reversible processes, and hence have a strong connection with symmetry. This argument applies even more strongly for groupoids. For groups, the processes all start and return to the same position. This is like considering only journeys which start at and end at the same place. However to *analyse* a reversible process, such as a journey, we must describe the intermediate steps, the stopping places. This description requires groupoids, since in this setting the processes described are allowed to start at one point and finish at another. Groupoids clearly allow a more flexible and powerful analysis, and this confirms a basic intuition that, in dimension 1, groupoids are more convenient than groups for writing down an 'algebraic inverse to subdivision'.

The definition of groupoid arose from Brandt's attempts to extend to quaternary forms Gauss' work on a composition law of binary quadratic forms, which has a strong place in *Disquisitiones Arithmeticae*. It is of interest here that Bourbaki [Bou70], p. 153, cites this composition law as an influential early example of a composition law which arose not from numbers, even taken in a broad sense, but from distant analogues:

> C'est vers cette même époque que, pour le premier fois en Algèbre, la notion de loi de composition s'étend, dans deux directions différents, à des élements qui ne présentent plus avec les ⟨⟨nombres⟩⟩ (au sens le plus large donné jusque-là à ce mot) que des analogies lointaines. La première de ces extensions est due à C. F. Gauss, à l'occasion de ses recherches arithmétiques sur les formes quadratiques ...

Brandt found that each quaternary quadratic form had a left unit and a right unit, and that two forms were composable if and only if the left unit of one was the right unit of the other. This led to his 1926 paper on groupoids [Bra26]. (A modern account of this work on composition of forms is given by Kneser *et al.* [KOK$^+$86].).

Groupoids were then used in the theory of orders of algebras, see for example [Jac43], Chapter 6, §11. Curiously, groupoids did not form an example in Eilenberg and Mac Lane's basic 1945 paper on category theory, [EML45b]. Groupoids appear in Reidemeister's 1932 book on topology, [Rei32], for handling the change of generators of the (combinatorially defined) fundamental group of a closed surface induced by the change of normal form of the surface, and for handling isomorphisms of a family of structures. The fundamental groupoid $\pi_1(X)$ of a space X was well known by the 1950s, but Crowell and Fox write in [CF63]:

> A few [definitions], like that of a group or of a topological space, have a fundamental importance to the whole of mathematics that can hardly be exaggerated. Others are more in the nature of convenient, and often highly specialised, labels which serve principally to pigeonhole ideas. As far as this book is concerned, the notions of category and groupoid belong to the latter class. It is an interesting curiosity that they provide a convenient systematisation of the ideas involved in developing the fundamental group.

By contrast, we referred earlier to the extensive work of C. Ehresmann on groupoids in differential topology, see again [Pra07]. One motivation for this work was his strong interest in local-to-global situations. Problems of this kind are often central in mathematics and in science.

The *fundamental groupoid* $\pi_1(X, A)$ on a set $A \subseteq X$ of base points was introduced in [Bro67] and the idea was developed in the 1968, 1988, and the final edition of [Bro06]. It generalises both the fundamental group (when $A = \{x\}$) and the fundamental groupoid (when $A = X$). Its successes suggest the value of an aesthetic approach to mathematics, and that the concept which feels right and gives 'a convenient systematisation' is likely to be the most powerful one, and may become quite generally useful. Indeed groupoids are more general than groups and so are not 'highly specialised'. In this viewpoint, much good mathematics enables difficult things to become easy, and an important part of the development of good mathematics is finding: (i) the appropriate underlying structures, (ii) the appropriate language to describe these structures, and (iii) means of calculating with these structures. Without the appropriate structures to guide us, we may take many wrong turnings.

There is no benefit today in arithmetic in Roman numerals. There is also no benefit today in insisting that the group concept is more fundamental than that of groupoid; one uses each at the appropriate place. It is as well to distinguish the sociology of the use of a mathematical concept from the scientific consideration of its relevance to the progress of mathematics.

It should also be said that the development of new concepts and language is a different activity from the successful employment of a range of known techniques to solve already formulated problems.

The notion that groupoids give a more flexible tool than groups in some situations is only beginning to be widely appreciated. One of the most significant of the books which use the notion seriously is Connes book "Noncommutative geometry", published in 1994, [Con94]. He states that Heisenberg discovered quantum mechanics by considering the *groupoid of transitions* for the hydrogen spectrum, rather than the usually considered group of symmetry of an individual state. This fits with the previously expounded philosophy. The main examples of groupoids in his book are equivalence relations and holonomy groupoids of foliations.

On the other hand, in books on category theory the role of groupoids is often fundamental (see for example Mac Lane and Moerdijk [MLM96]). In foliation theory, which is a part of differential topology and geometry, the notion of *holonomy groupoid* is widely used. For surveys of the use of groupoids, see [Bro87], [Hig71], [Wei01], [Mac05], [Bro07], [GS06].

1.5 The Seifert–van Kampen Theorem

We believe a change of prospects for homotopy theory came about in a roundabout way, in the mid 1960s. R. Brown was writing the first edition of the book [Bro06] and became dissatisfied with the standard treatments of the Seifert–van Kampen Theorem. This basic tool computes the fundamental group of a space X given as the union of two connected open subsets U_1, U_2 with connected intersection U_{12}. For those familiar with the concepts, the result is that the natural morphism

$$\pi_1(U_1, x) *_{\pi_1(U_{12},x)} \pi_1(U_2, x) \to \pi_1(X, x) \tag{1.5.1}$$

induced by inclusions is an isomorphism. The group on the left-hand side of the above arrow is the *free product with amalgamation*; it is the construction for groups corresponding to $U_1 \cup U_2$ for spaces, as we shall see later in discussing pushouts. This version of the theorem was given by Crowell [Cro71], based on lectures by R.H. Fox. One important consequence is that the fundamental group shared the same possibilities and the same difficulties of computation as general abstract groups.

The problem was with the connectivity assumption on U_{12}, since this prevented the use of the theorem for deducing the result that the fundamental group of the circle S^1 is isomorphic to the group $\mathbb{Z}$ of integers. (See Section 1.7 where $\pi_1(S^1)$ is calculated.) If S^1 is the union of two connected open sets, then their intersection cannot be connected, compare [Bro06], Section 9.2. So the fundamental group of the circle is usually determined by the method of covering spaces. Of course this method is basic stuff anyway, and needs to be explained, but having to make this detour, however attractive the route, is unaesthetic. It was regarding this situation as an anomaly needing correction which in effect led to the whole of the work of this book.

It was found that a uniform method including the fundamental group of the circle could be given using nonabelian cohomology, [Bro65a], but a full exposition of this became turgid. Then Brown came across the paper by Philip Higgins entitled ‘Presentations of groupoids with applications to groups’ [Hig64], which among other things defined free products with amalgamation of groupoids. We will explain something about groupoids in Section 1.7. It seemed reasonable to insert an exercise in the book on an analogous result to (1.5.1) for the fundamental groupoid $\pi_1(X)$, namely that the natural morphism of groupoids

$$\pi_1(U_1) *_{\pi_1(U_{12})} \pi_1(U_2) \to \pi_1(X) \tag{1.5.2}$$

is an isomorphism. It then seemed desirable to write out a solution to the exercise, and lo and behold! the solution was much clearer *and more powerful* than all the turgid stuff on nonabelian cohomology. Further work showed that computations required the generalisation from the fundamental group $\pi_1(X, x)$ on one base point x to the fundamental groupoid $\pi_1(X, A)$ on a *set A of base points* chosen freely according to a given geometric situation. In particular if U_{12} is not connected it is not clear from which component of U_{12} a base point should be chosen. So one hedges one’s bets, and chooses a set $A \subseteq U_{12}$ of base points, at least one in each component of U_{12}. The following picture shows a union of two open sets whose intersection has four components.

One finds that the natural morphism of groupoids

$$\pi_1(U_1, A) *_{\pi_1(U_{12},A)} \pi_1(U_2, A) \to \pi_1(X, A) \tag{1.5.3}$$

is also an isomorphism and that the proof of this result using groupoids is simpler than the original proof of (1.5.1) for groups. One also obtains a new range of calculations.

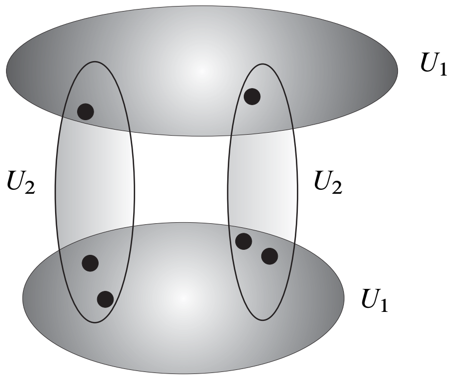


Figure 1.1. Example of spaces in a Seifert–van Kampen type situation.

For example, U_1, U_2, U_{12} may have respectively 27, 63, and 283 components, and yet X could be connected – a description of the fundamental group of this situation in terms of groups alone is not really sensible.[7]

A feature of this groupoid version of the Seifert–van Kampen Theorem was that it yielded quite precise results on fundamental groups, even in nonconnected situations. This was surprising since in algebraic topology, invariants in situations involving a pair of adjacent dimensions, in this case 0 and 1, are often related by exact sequences, and so are not determined precisely: this reasoning showed that the result in terms of groupoids was more powerful than the previous results referred to above obtained using nonabelian cohomology. The underlying basis for this power seemed to be that groupoids had structure in dimensions 0 and 1, and so could model well the gluing process of spaces. It was therefore natural to seek for homotopically defined invariants with structure in dimensions 0, 1, 2, …, n, again with a version of the Seifert–van Kampen Theorem, but they took a number of years to find; they are the subject matter of this book.

In view of these results on groupoids, the writing of the first edition, [Bro68], of the book [Bro06] was redirected to give a full account of groupoids and the Seifert–van Kampen Theorem. A conversation with G. W. Mackey in 1967 informed Brown of Mackey's work on ergodic groupoids (see the references in [Bro87]). It seemed that if the idea of groupoid arose in two separate fields, then there was more in this than met the eye. Mackey's use of the relation between group actions and groupoids suggested the importance of strengthening the book with an account of covering spaces in terms of groupoids, following the initial lead of Higgins in [Hig64] for applications to group theory, and of Gabriel and Zisman in [GZ67], for applications to topology.

Later Grothendieck was to write (1985):

> The idea of making systematic use of groupoids (notably fundamental groupoids of spaces, based on a given set of base points), however evident as it may look today, is to be seen as a significant conceptual advance,

> which has spread into the most manifold areas of mathematics. …In my own work in algebraic geometry, I have made extensive use of groupoids - the first one being the theory of the passage to quotient by a 'pre-equivalence relation' (which may be viewed as being no more, no less than a groupoid in the category one is working in, the category of schemes say), which at once led me to the notion (nowadays quite popular) of the nerve of a category. The last time has been in my work on the Teichmüller tower, where working with a 'Teichmüller groupoid' (rather than a 'Teichmüller group') is a must, and part of the very crux of the matter …

The relevance of groupoids to the notion of quotient structure is discussed in Appendix C, Section C.8.

1.6 Proof of the Seifert–van Kampen Theorem (groupoid case)

In this section we give the full proof that the morphism of groupoids induced by inclusions

$$\eta\colon \pi_1(U_1, A_1) *_{\pi_1(U_{12}, A_{12})} \pi_1(U_2, A_2) \to \pi_1(X, A) \tag{1.6.1}$$

is an isomorphism when U_1, U_2 are open subsets of $X = U_1 \cup U_2$ and A meets each path component of U_1, U_2 and $U_{12} = U_1 \cap U_2$. Here we write $A_\lambda = U_\lambda \cap A$ for $\lambda = 1, 2, 12$.

What one would expect is that the proof would construct directly an inverse to η. Alternatively, the proof would verify in turn that η is surjective and injective.

The proof we give might at first seem roundabout, but in fact it follows the important procedure of *verifying a universal property*. One advantage of this procedure is that we do not need to show that the free product with amalgamation of groupoids exists in general, nor do we need to give a construction of it at this stage. Instead we define the free product with amalgamation by its universal property, which enables us to go directly to an efficient proof of the Seifert–van Kampen Theorem. It also turns out that the universal property guides many explicit calculations. More importantly, the proof guides other results, such the higher dimensional ones in this book.

We use the notion of pushout in Example A.4.4 of Appendix A. Here is the definition for groupoids. We say that the groupoid G and the two morphisms of groupoids $b_1\colon G_1 \to G$ and $b_2\colon G_2 \to G$ are the *pushout* of the two morphisms of groupoids $a_1\colon G \to G_1$ and $a_2\colon G \to G_2$ if they satisfy the following two axioms:

Pushout 1. The diagram

$$\begin{tikzcd} G_0 \arrow[r, "a_1"] \arrow[d, "a_2"'] & G_1 \arrow[d, "b_1"] \\ G_2 \arrow[r, "b_2"'] & G \end{tikzcd}$$

is a commutative square, i.e. $b_1 a_1 = b_2 a_2$.

Pushout 2. The previous diagram is universal with respect to this type of diagram, i.e. for any groupoid K and morphisms of groupoids $k_1 \colon G_1 \to K$ and $k_2 \colon G_2 \to K$ such that the following diagram is commutative

$$\begin{tikzcd} G_0 \arrow[r, "a_1"] \arrow[d, "a_2"'] & G_1 \arrow[d, "k_1"] \\ G_2 \arrow[r, "k_2"'] & K \end{tikzcd}$$

there is a unique morphism of groupoids $k \colon G \to K$ such that $kb_1 = k_1$, $kb_2 = k_2$. The two diagrams are often combined into one as follows:

$$\begin{tikzcd} G_0 \arrow[r, "a_1"] \arrow[d, "a_2"'] & G_1 \arrow[d, "b_1"'] \arrow[ddr, bend left, "k_1"] & \\ G_2 \arrow[r, "b_2"] \arrow[drr, bend right, "k_2"'] & G \arrow[dr, dotted, "k"] & \\ & & K. \end{tikzcd}$$

We think of a pushout square as given by a standard input, the pair (a_1, a_2), and a standard output, the pair (b_1, b_2). The properties of this standard output are defined by reference to *all other* commutative squares with the same (a_1, a_2). At first sight this might seem strange, and logically invalid. However a pushout square is somewhat like a computer program: given the data of another commutative square of the right type, then the output will be a morphism (k in the above diagram) with certain defined properties.

It is a basic feature of universal properties that the standard output, in this case the pair (b_1, b_2) making the diagram commute, is determined up to isomorphism by the standard input (a_1, a_2). See [Bro06], Section 6.6, and further discussion of the more general colimits will be found in Appendix A.4.

We now state and prove the Seifert–van Kampen Theorem for the fundamental groupoid on a set of base points in the case of a cover by two open sets. The reason for giving this in detail is that the proofs of the analogous theorems in higher dimensions are modelled on this one, but need new gadgets of higher homotopy groupoids to realise them, see Chapters 6 and 14.[8]

Theorem 1.6.1. *Let U_1, U_2 be open subsets of X whose union is X and let A be a subset of $U_{12} = U_1 \cap U_2$ meeting each path component of U_1, U_2, U_{12} (and therefore of X). Let $A_i = U_i \cap A$ for $i = 1, 2, 12$. Then the following diagram of morphisms* *induced by inclusion*

$$\begin{tikzcd} \pi_1(U_{12}, A_{12}) \arrow[r, "a_1"] \arrow[d, "a_2"'] & \pi_1(U_1, A_1) \arrow[d, "b_1"] \\ \pi_1(U_2, A_2) \arrow[r, "b_2"'] & \pi_1(X, A) \end{tikzcd}$$

is a pushout of groupoids.

Proof. We suppose given a commutative diagram of morphisms of groupoids

$$\begin{tikzcd} \pi_1(U_{12}, A_{12}) \arrow[r, "a_1"] \arrow[d, "a_2"'] & \pi_1(U_1, A_1) \arrow[d, "k_1"] \\ \pi_1(U_2, A_2) \arrow[r, "k_2"'] & K. \end{tikzcd}$$

We have to prove that there is a unique morphism $k\colon \pi_1(X, A) \to K$ such that $kb_1 = k_1, kb_2 = k_2$.

We write $b_{12}\colon \pi_1(U_{12}, A_{12}) \to \pi_1(X, A)$ for the composite $b_1a_1 = b_2a_2$, write $k_{12} = k_1b_1 = k_2b_2$, and also write b_i for the map of spaces $U_i \to X$.

Let us take an element $[\alpha] \in \pi_1(X, A)$ with representative $\alpha\colon (I, \partial I) \to (X, A)$. Suppose first α has image in U_λ for $\lambda = 1$ or 2. Then $\alpha = a_\lambda\beta$ for $\beta\colon (I, \partial I) \to (U_\lambda, A_\lambda)$ and we define $k[\alpha] = k_\lambda[\beta]$. The condition $k_1a_1 = k_2a_2$ ensures this definition is independent of the choice of λ if α has image in $U_1 \cap U_2$, but it still has to be shown the definition is independent of the choice of α in its class.

We now consider a general $[\alpha]$. By the Lebesgue Covering Lemma ([Bro06], 3.6.4) there is a subdivision

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$$

of I into intervals by equidistant points such that α maps each $[t_i, t_{i+1}]$ into U_1 or U_2 (possibly in both). Choose one of these written U^i for each i. The subdivision determines a decomposition

$$\alpha = \alpha_0\alpha_1 \ldots \alpha_{n-1}$$

such that α_i has image in U^i. Of course the point $\alpha(t_i)$ need not lie in A, but it lies in $U^i \cap U^{i-1}$ and this intersection may be U_1, U_2 or U_{12}. By the connectivity conditions, for each $i = 0, 1, \ldots, n-1$, we may choose a path γ_i in $U^i \cap U^{i-1}$ joining $\alpha(t_i)$ to A. Moreover, if $\alpha(t_i)$ already lies in A we choose γ_i to be the constant path at $\alpha(t_i)$. In particular γ_0 and γ_n are constant paths. The following figure shows the path α in black and the paths γ_i in white.

Now for each $0 \leqslant i < n$ the path $\beta_i = \gamma_i^{-1}\alpha_i\gamma_{i+1}$ lies in U^i and joins points of A. Notice that β_i also represents a class in $\pi_1(U^i, A)$, which maps by b^i (which may be b_1, b_2, b_{12}) to $\pi_1(X, A)$. It is clear that

$$[\alpha] = b^0[\beta_0]b^1[\beta_1] \ldots b^{n-1}[\beta_{n-1}] \in \pi_1(X, A).$$

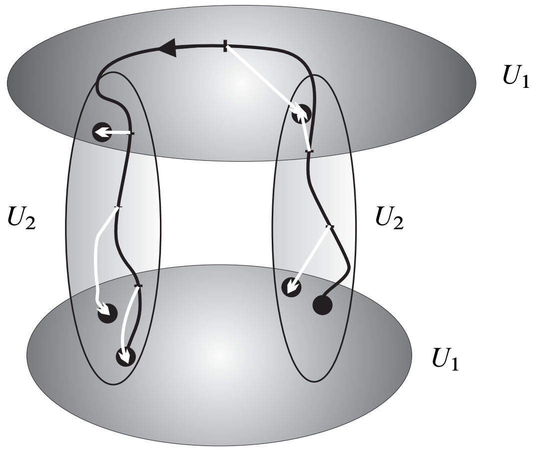


Figure 1.2. A decomposition of a path α in a Seifert–van Kampen type situation.

If there exists the homomorphism k of groupoids that makes the external square commute then the value of $k([\alpha])$ is determined by the above subdivision as

$$\begin{aligned} k([\alpha]) &= k(b^0[\beta_0]b^1[\beta_1]\dots b^{n-1}[\beta_{n-1}]) \\ &= k^0[\beta_0]k^1[\beta_1]\dots k^{n-1}[\beta_{n-1}]. \end{aligned}$$

This proves uniqueness of k, and also proves that $\pi_1(X, A)$ is generated as a groupoid by the images of $\pi_1(U_1, A_1)$, $\pi_1(U_2, A_2)$ by b_1, b_2 respectively.

We have yet to prove that the element $k([\alpha])$ is independent of all the choices made. Before going into that, notice that the construction we have just made can be interpreted diagrammatically as follows. The starting situation looks like the bottom side of the diagram

$$\begin{tikzcd} \bullet \arrow[r, dashed, "\beta_0"] \arrow[d, no head, "\gamma_0"'] & \bullet \arrow[r, dashed, "\beta_1"] & \bullet \arrow[r, dotted, no head] & \bullet \arrow[r, dashed, "\beta_{n-2}"] & \bullet \arrow[r, dashed, "\beta_{n-1}"] & \bullet \arrow[d, no head, "\gamma_n"] \\ \bullet \arrow[r, "\alpha_0"'] & \circ \arrow[r, "\alpha_1"'] \arrow[u, "\gamma_1"'] & \circ \arrow[r, dotted, no head] \arrow[u, "\gamma_2"'] & \circ \arrow[r, "\alpha_{n-2}"'] \arrow[u, "\gamma_{n-2}"'] & \circ \arrow[r, "\alpha_{n-1}"'] \arrow[u, "\gamma_{n-1}"'] & \bullet \end{tikzcd} \tag{1.6.2}$$

where the solid circles denote points which definitely lie in A, and in which γ_0, γ_n are constant paths. The path β_i may be obtained from the other three paths in its square by composing with a retraction from above, as shown in Figure 1.3.

This retraction also provides a homotopy

$$u : \alpha \simeq \beta = (b^0\beta_0)(b^1\beta_1)\dots(b^{n-1}\beta_{n-1}) \tag{1.6.3}$$

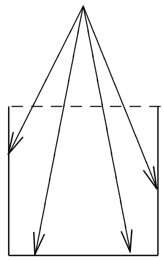
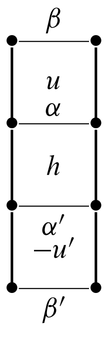


Figure 1.3. Retraction from above-centre.

rel end points. This is the first of many filling arguments where we define a map on parts of the boundary of a cube and extend the map to the whole cube using appropriate retractions. This technique is studied in all dimensions in Section 11.3.i, using ‘expansions’ and ‘collapses’, and is essential for the main results of this book, which is why we emphasise it here.[9]

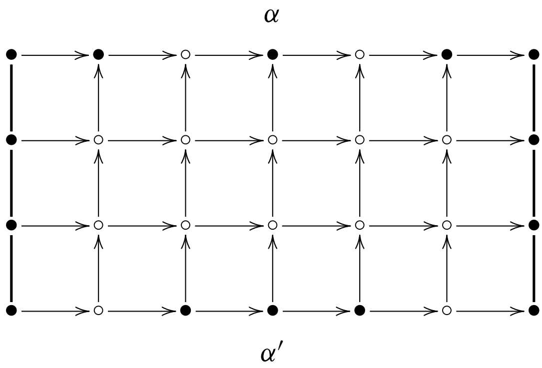
We shall use another filling argument in I^3 to prove independence of choices. Suppose that we have a homotopy rel end points $h\colon \alpha \simeq \alpha'$ of two maps $(I, \partial I) \to (X, A)$. We can perform the construction of a homotopy in (1.6.3) for each of α, α', and then glue the three homotopies together. Here thick lines denote constant paths.



(1.6.4)

So, replacing βs by αs, we can assume the maps α, α' have subdivisions $\alpha = [\alpha_i]$, $\alpha' = [\alpha'_j]$ such that each α_i, α'_j has end points in A and has image in one of U_1, U_2. Since h is a map $I^2 \to X$, we may again by the Lebesgue covering lemma make a subdivision $h = [h_{lm}]$ such that each h_{lm} lies in one of U_1, U_2. Also by further subdivision as necessary, we may assume this subdivision of h refines on $I \times \partial I$ the given subdivisions of α, α'.

The problem is that none of the vertices of this subdivision are necessarily mapped into A, except those on $\partial I \times I$ (since the homotopy is rel vertices and α, α' both map ∂I to A) and those on $I \times \partial I$ determined by the initial subdivisions of α, α'. So the situation looks like the following:



(1.6.5)

Again thick lines denote constant paths. We want to deform the homotopy h to a new homotopy $\bar{h} : \bar{\alpha} \simeq \bar{\alpha}'$ again rel end points such that:

$[\alpha] = [\bar{\alpha}]$, $[\alpha'] = [\bar{\alpha}']$ in $\pi_1(X, A)$;

h' has the same subdivision as does h;

any subsquare mapped by h into U_i, $i = 1, 2, 12$ remains so in h';

and any vertex already in A is not moved.

This deformation is constructed inductively on dimension of cells of the subdivision by what we call 'filling arguments' in the cube I^3.

Let us imagine the 3-dimensional cube I^3 as $I^2 \times I$ where I^2 has the subdivision we are working with in h. Define the bottom map to be h. We have to fill I^3 so that in the top face we get a similar diagram but with all the vertices solid, i.e. in A, and each subsquare in the top face lies in the same U_i as the corresponding in the bottom one.

The following picture shows the initial stage of the deformation needed.

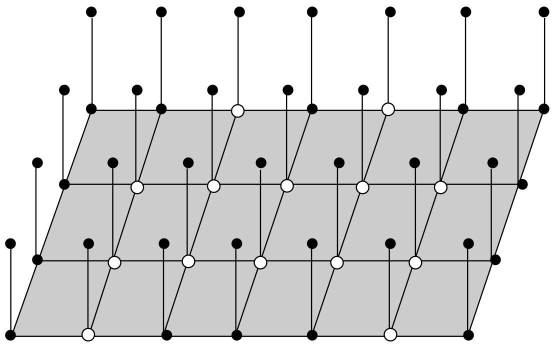


Figure 1.4. Extending to the edges.

We start by defining the deformation on all 'vertical' edges $\{v\} \times I$ arising from vertices v in the partition of I^2. If the image of a vertex lies in A, then v is to be deformed by a constant deformation; otherwise, we consider the 4, 2, or 1 squares of which v is a vertex, let U^v be the intersection of the sets of the cover into which these are mapped, and choose a path in U^v joining $h(v)$ to a point of A. Let us write e_{lm} for the path we have chosen between the vertex $h(s_l, t_m)$ and A. (These e_{lm} are constant if $h(s_l, t_m)$ lies already in A). This gives us the map on the vertical edges of I^3 as in Figure 1.4.

From now on, we restrict our construction to the part of I^3 over the square $S_{lm} = [s_l, s_{l+1}] \times [t_m, t_{m+1}]$ and fix some notation for the restriction of h to its sides, $\sigma_{lm} = h|_{[s_l, s_{l+1}] \times \{t_m\}}$ and $\tau_{lm} = h|_{\{s_l\} \times [t_m, t_{m+1}]}$. Then, using the retraction of Figure 1.3 on each lateral face, we can fill all the faces of a 3-cube except the top one. Now, using the retraction from a point on a line perpendicular to the centre of the top face, as in the following Figure 1.5

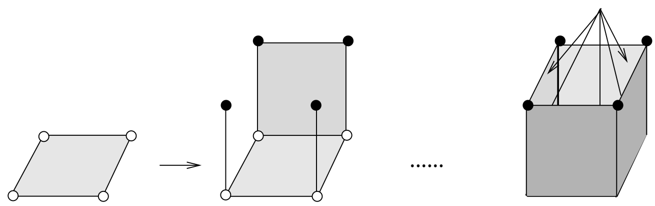
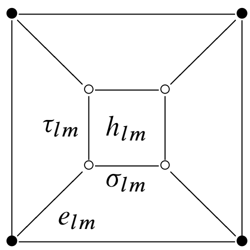


Figure 1.5. Extending to the lateral faces.

we get at the top face a map that looks like



(1.6.6)

and in particular is a map into U^i sending all vertices into A.

If we do the above construction in each square of the subdivision, we get a top face of the cube that is a homotopy $\bar{h}$ rel end points between two paths in the same classes as α and α', and subdivided in such a way that each subsquare goes into some U_λ and sends all vertices of the subsquare into A. Each of these squares produces a commutative square say σ_{ij} of path classes in one of $\pi_1(U_\lambda, A)$, $\lambda = 1, 2$. Thus the diagram can be pictured as follows:

$$\bar{\alpha}$$

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[u, dash] \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u, dash] \\
\bullet \arrow[u, dash] \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u, dash] \\
\bullet \arrow[u, dash] \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u, dash]
\end{tikzcd}$$

$$\bar{\alpha}'$$

Applying the appropriate k_λ to a subsquare σ_{ij} we get a commutative square l_{ij} in K. Since $k_1 a_1 = k_2 a_2$, we get that the l_{ij} compose in K to give a square l in K.

Now comes the vital point. Since *any composite of commutative squares in a groupoid is itself a commutative square*, the composite square l is commutative.

But because of the way we constructed it, two sides of this composite commutative square l in K are identities, as the images of the class of constant paths. Therefore the opposite sides of l are equal. This shows that our element $k([\alpha])$ is independent of the choices made, and so proves that k is well defined as a function on arrows of the fundamental groupoid $\pi_1(X, A)$.

The proof that k is a morphism is now quite simple, while uniqueness has already been shown. So we have shown that the diagram in the statement of the theorem is a pushout of groupoids.

This completes the proof. □

There is another way of expressing the above argument on the composition of commutative squares being a commutative square, namely by working on formulae for each individual square as in the expression $a = cdb^{-1}$ for (1.1.1). Putting together two such squares as in

$$\begin{tikzcd}
\bullet \arrow[r, "a"] \arrow[d, "c"'] & \bullet \arrow[r, "e"] \arrow[d, "b"'] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r, "d"'] & \bullet \arrow[r, "g"'] & \bullet
\end{tikzcd} \tag{1.6.7}$$

allows cancelation of the middle term

$$ae = (cdb^{-1})(bgf^{-1}) = cdgf^{-1}$$

which if $c = 1,\ f = 1$ reduces to $ae = dg$. This argument extends to longer gluings of commutative squares, and hence extends, by induction, and in the other direction, to a subdivision of a square.

We would like to extend the above argument to the faces of a cube, and then to an n-dimensional cube.

For a cube, the expression of one of the faces in terms of the others can be done (see the Homotopy Commutativity Lemma 6.7.6) and then can be used to prove a 2-dimensional Seifert–van Kampen Theorem. That will be done in Section 6.8.

It is much more difficult to follow this route in the general case and a more round-about method is developed in Chapter 14. We need an algebra of 'thin $(n+1)$-cubes', each of which can be seen to have 'commuting boundary', and for which it is automatic that compositions of thin cubes are thin. The algebra to carry out this argument in all dimension is given in Chapter 13. It is interesting that such a complicated and subtle algebra seems to be needed to make it all work. We emphasise that the purely algebraic work of Chapter 13 is essential for the applications in the following two chapters of Part III, and that Part III gives the theoretical underpinning for the whole of Part II.

Remark 1.6.2. One of the nice things about proving the theorem by verifying the universal property is that the proof uses some calculations in a general groupoid K, and groupoids have, in some sense, the minimal set of properties needed for the result. This avoids a calculation in $\pi_1(X, A)$, and somehow makes the calculations in the proof as easy as possible. Also the proof does not require a knowledge of how to compute pushouts, or more general colimits, of groupoids, or even that they exist in general. The same advantages hold in some other verifications of universal properties, for example in the computation of the fundamental groupoid of an orbit space in [Bro06], Chapter 11. We will see a similar situation later for double groupoids in Chapter 6, and in all dimensions in Chapter 14. □

1.7 The fundamental group of the circle

In order to interpret the last theorem, one has to set up the basic algebra of computational groupoid theory. In particular, one needs to be able to deal with presentations of groupoids. This is done to a good extent in [Hig71], [Bro06]. Here we can give only the indications of the theory.

The theory of groupoids may be thought of as an algebraic analogue of the theory of groups, but based on directed graphs rather than on sets.[10]

Let us explain some basic definitions in groupoid theory. A groupoid G is called *connected* if $G(a, b)$ is nonempty for all $a, b \in \mathrm{Ob}(G)$.[11] The maximal connected subgroupoids of G are called the (*connected*) *components* of G .

If a is an object of the groupoid G, then the set $G(a, a)$ inherits a group structure from the composition on G, and this is called the *object group* of G at a and is written also $G(a)$. The groupoid G is called *simply connected* if all its object groups are trivial. If it is connected and simply connected, it is called 1-*connected*, or an *indiscrete groupoid* .

A standard example of an indiscrete groupoid is the groupoid $I(S)$ on a set S. This has object set S and arrow set $S \times S$, with $s, t\colon S \times S \to S$ being the first and second projections. The composition on $I(S)$ is given by

$$(a,b)(b,c) = (a,c), \quad \text{for all } a,b,c \in S.$$

A special case is the groupoid we will write $\mathcal{I} = I(\{0,1\})$. This has two nonidentity elements which we write $\iota\colon 0 \to 1$ and $\iota^{-1}\colon 1 \to 0$. This apparently 'trivial' groupoid will play a key role in the theory, since it determines homotopies. It is also called the 'unit interval groupoid'.

A directed graph X is called *connected* if the free groupoid $F(X)$ on X is connected, and is called a *forest* if every object group $F(X)(a)$ of $F(X)$, $a \in \mathrm{Ob}(X)$, is trivial. A connected forest is called a *tree*. If X is a tree, then the groupoid $F(X)$ is indiscrete; an indiscrete groupoid is also called a *tree groupoid*.

Let G be a connected groupoid and let a_0 be an object of G. For each $a \in \mathrm{Ob}(G)$ choose an arrow $\tau a\colon a_0 \to a$, with $\tau a_0 = 1_{a_0}$. Then an isomorphism

$$\phi\colon G \to G(a_0) \times I(\mathrm{Ob}(G)) \tag{1.7.1}$$

is given by $g \mapsto ((\tau a) g (\tau b)^{-1}, (a,b))$ when $g \in G(a,b)$ and $a, b \in \mathrm{Ob}(G)$. The composition of ϕ with the projection yields a morphism $\rho\colon G \to G(a_0)$ which we call a *deformation retraction* since it is the identity on $G(a_0)$ and is in fact homotopic to the identity morphism of G, though we do not elaborate on this fact here.

It is also standard [Bro06], 8.1.5, that a connected groupoid G is isomorphic to the free product groupoid $G(a_0) * T$ where $a_0 \in \mathrm{Ob}(G)$ and T is any wide, tree subgroupoid of G. The importance of this is as follows.

Suppose that X is a graph which generates the connected groupoid G. Then X is connected. Choose a maximal tree T in X. Then T determines for each a_0 in $\mathrm{Ob}(G)$ a retraction $\rho_T\colon G \to G(a_0)$ and the isomorphisms

$$G \cong G(a_0) * I(\mathrm{Ob}(G)) \cong G(a_0) * F(T)$$

show that a morphism $G \to K$ from G to a groupoid K is completely determined by a morphism of groupoids $G(a_0) \to K$ and a graph morphism $T \to K$ which agree on the object a_0.

We shall use later the following proposition, which is a special case of [Bro06], 6.7.3:

Proposition 1.7.1. *Let G, H be groupoids with the same set of objects, and let $\phi\colon G \to H$ be a morphism of groupoids which is the identity on objects. Suppose that G is connected and $a_0 \in \mathrm{Ob}(G)$.*

Choose a retraction $\rho\colon G \to G(a_0)$. Then there is a retraction $\sigma\colon H \to H(a_0)$

such that the following diagram, where ϕ' is the restriction of ϕ,

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[d, "\phi"'] & G(a_0) \arrow[d, "\phi'"] \\ H \arrow[r, "\sigma"'] & H(a_0) \end{tikzcd} \tag{1.7.2}$$

is commutative and is a pushout of groupoids.

This result can be combined with Theorem 1.6.1 to determine the fundamental group of the circle S^1.

Corollary 1.7.2. *The fundamental group of the circle S^1 is a free group on one generator.*

Proof. We represent S^1 as the union of two semicircles E^1_+, E^1_- with intersection $\{-1, 1\}$. Then both fundamental groupoids $\pi_1(E^1_+, \{-1, 1\})$ and $\pi_1(E^1_-, \{-1, 1\})$ are easily seen to be isomorphic to the connected groupoid $\mathcal{I}$ with object set $\{-1, 1\}$ and trivial object groups. In fact this groupoid is the free groupoid on one generator $\iota\colon -1 \to 1$.

Also, $\pi_1(\{-1, 1\}, \{-1, 1\})$ is the discrete groupoid on these objects $\{-1, 1\}$. Consider the following two squares of morphisms of groupoids:

$$\begin{tikzcd} \{-1,1\} \arrow[r] \arrow[d] & \mathcal{I} \arrow[r] \arrow[d] & \{1\} \arrow[d] \\ \mathcal{I} \arrow[r] & \pi_1(S^1, \{-1,1\}) \arrow[r] & \pi_1(S^1, 1). \end{tikzcd}$$

By an application of Theorem 1.6.1 the lefthand square is a pushout of groupoids; from the previous proposition, the right-hand square is a pushout of groupoids. It follows that the outside composite square is a pushout of groupoids, and the result follows by an easy universal argument. □

Note that S^1 may be regarded as a pushout in the category of topological spaces

$$\begin{tikzcd} \{-1,1\} \arrow[r] \arrow[d] & \{1\} \arrow[d] \\ {[-1,1]} \arrow[r] & S^1. \end{tikzcd} \tag{1.7.3}$$

Exercise 1.7.3. By regarding $S^1 \vee S^1$ as obtained from the interval $[0, 2]$ by identifying $\{0, 1, 2\}$ to a single point, use the SvKT to prove that $\pi_1(S^1 \vee S^1, x)$ is isomorphic to the free group on two generators. (Compare [Bro06], p. 343.) We will use this example as part of a higher dimensional result in Remark 8.3.16. □

The correspondence between this pushout of spaces and the previous composite pushout of groupoids was for R.Brown a major incentive to exploring the use of groupoids. Here we have a successful algebraic model of a space, but of a different type from that previously considered. An aspect of its success is that groupoids have structure in two dimensions, namely 0 and 1, and this is useful for modeling the way spaces are built up using identifications in dimensions 0 and 1.

Another interesting aspect is that the groupoid $\mathcal{I}$ is finite, and it is very easy to explore all its internal properties. By contrast, the integers form an infinite set, and discussion of its properties usually requires induction.

The problem suggested by these considerations was to find analogous methods in higher dimensions.

1.8 Higher order groupoids

The successes of the use of groupoids in group theory and then in 1-dimensional homotopy theory as exposed in the books [Bro06], [Hig71] suggested the potential interest in the use of groupoids in higher dimensional homotopy theory. In particular, it was suggested in [Bro67] that a Higher Homotopy Seifert–van Kampen Theorem (HHSvKT) could be proved, the intuition being that the 'right' higher homotopy groupoids could be constructed, with properties analogous to those which enabled the proof of this theorem in dimension 1.

Experiments by Brown to obtain such a construction in the years 1965–74 proved abortive. However in 1971 Chris Spencer came to Bangor as a Science Research Council Research Assistant, and in this and a subsequent period considerable progress was made on discovering the algebra of double groupoids. It was in this collaboration that the relation with crossed modules was found, so linking the notion of double groupoids with more classical ideas.

Crossed modules had been defined by J. H. C. Whitehead in 1946 [Whi46] in order to express the properties of the properties of the boundary map

$$\partial \colon \pi_2(X, X_1, x) \to \pi_1(X_1, x)$$

of the second relative homotopy group, a group which is in general nonabelian. As explained in more detail on p. 32, he gave the first nontrivial determination of this group in showing that when X is formed from X_1 by attaching 2-cells, then $\pi_2(X, X_1, x)$ is isomorphic to the free crossed $\pi_1(X_1, x)$-module on the characteristic maps of the 2-cells.

This result was a crucial clue to Brown and Higgins in 1974. On the one hand it showed that a universal property, namely freeness, did exist in 2-dimensional homotopy theory. Also, if our proposed theory was to be any good, it should have this theorem as a corollary. However, Whitehead's theorem was about *relative homotopy groups*, which suggested that we should look at a relative theory, i.e. a space X with a subspace X_1.

With the experience obtained by then, we quickly found a satisfactory, even simple, construction of a relative homotopy double groupoid $\rho_2(X, X_1, x)$ and a proof of a 2-dimensional Seifert–van Kampen Theorem, as envisaged.

The equivalence between these sorts of double groupoids and crossed modules proved earlier by Brown and Spencer, then gave the required Seifert–van Kampen type theorem for the second homotopy crossed module, and so new calculations of second relative homotopy groups.

So we have a pattern of proof:

A) construct a homotopically defined multiple groupoid;

B) prove it is equivalent to a more familiar homotopical construction;

C) prove a Seifert–van Kampen Theorem in the multiple groupoid context; and

D) interpret this theorem in the more familiar context.

These combined give new nonabelian, higher dimensional, local-to-global results. This pattern has been followed in the corresponding result for crossed complexes, which is dealt with in our Part II, and in results for the cat^n-groups of Loday, [Lod82]. We give a brief indication of results on crossed squares in Appendix B, Section B.4.

Crossed modules had occurred earlier in other places. They were used in work on the cohomology of groups, [ML49], and in extensive work of Dedecker on cohomology with coefficients in a crossed module, for which we refer to [Ded60].

In the mid 1960s the great school of Grothendieck in Paris had considered sets with two structures, that of group and of groupoid, and had proved these were equivalent to crossed modules. However this result was not published, and so was known only to a restricted group of people.

It is now clear that once one moves to higher version of groupoids, the presence of crossed modules is inevitable, and is an important part of the theory and applications. This is why Part I is devoted entirely to the area of crossed modules and double groupoids. It should be said that the double groupoids we need for this book are not the most general kind, but are the kind most closely related to classical aspects of homotopy theory, and the ones which for us lead to explicit calculations in homotopy theory.

Notes

6 p. 12 The papers of Whitehead [Whi41b], [Whi39], [Whi41a] show a thorough understanding of work of Reidemeister, [Rei34], and are very original. They were worked up in new language in his papers after the war [Whi49a], [Whi49b], [Whi50b]. At Oberwolfach in 1957, Brown met R. Baer, who on learning of Brown's supervisor immediately quipped: 'Ah! The English Heraclitus!'. Whitehead was very pleased to be invited to Paris in the 1950s to talk on his theorem on homotopy equivalences, and said he was even more pleased when he found

that they had got it wrong! Brown also met H. Cartan in 1982, and his remark on Whitehead was: ‘Il à une idée toute minute!’.

7 p. 16 If one considers paths as rail journeys and the set A as the set of stations, then one wants to list all journeys between stations. It is not common sense to put the emphasis on return journeys from each station, and then to add ways of changing from one set of return journeys to another, called in the literature ‘change of base point’. Complicated combinatorial situations with even infinitely many components arise in group theory, for example in proving the Kurosch Subgroup Theorem using covering spaces or covering groupoids, [Hig64], [Hig71], [Bro06]. Many authors get around the problem of really needing many base points by choosing a base point in a connected graph and then a maximal tree: again, the result is usually unaesthetic.

8 p. 18 This proof differs from that in [Bro06] in working directly with path classes in $\pi_1(X, A)$ instead of first doing the case $A = X$ and then using a retraction. That retraction argument is not so easy to extend to the case of an arbitrary open cover of X, and, more importantly, seemingly impossible to extend to higher dimensions. So the proof we give returns in essence to the argument in [Cro59]. The result for general covers with best possible connectivity conditions is given in [BRS84]. A quotation from [Rot08], p. 48, is also relevant:

> “What can you prove with exterior algebra that you cannot prove without it?” Whenever you hear this question raised about some new piece of mathematics, be assured that you are likely to be in the presence of something important. In my time, I have heard it repeated for random variables, Laurent Schwartz’ theory of distributions, ideles and Grothendieck’s schemes, to mention only a few. A proper retort might be: “You are right. There is nothing in yesterday’s mathematics that could not also be proved without it. Exterior algebra is not meant to prove old facts, it is meant to disclose a new world. Disclosing new worlds is as worthwhile a mathematical enterprise as proving old conjectures.

9 p. 21 These collapsing techniques were developed in [Whi41b] and [Whi50b] by J. H. C. Whitehead and have become an important tool in geometric topology.

10 p. 25 For some discussion of the philosophy of moving from sets to directed graphs, see [Bro94]. We refer to [Bro06], [Hig71] for the construction of a free groupoid over a directed graph.

11 p. 25 Sometimes the term ‘transitive groupoid’ is used for what we have called a connected groupoid. The former term is more convenient when dealing with, say, topological or Lie groupoids.

Chapter 2
Homotopy theory and crossed modules

In this chapter we explain how crossed modules over groups arose in topology in the first half of the last century, and give some of the later developments.

The topologist J. H. C. Whitehead (1904–1960) was steeped in the combinatorial group theory of the 1930s, and much of his work can be seen as trying to extend the methods of group theory to higher dimensions, keeping the interplay with geometry and topology. These attempts led to greatly significant work, such as the theory of simple homotopy types [Whi50b], the algebraic background for which started the subject of algebraic K-theory. His ideas on crossed modules have taken longer to come into wide use, but they can be regarded as equally significant.

One of his starting points was the Seifert–van Kampen Theorem for the fundamental group. This tells us in particular how the fundamental group is affected by the attaching of a 2-cell, or of a family of 2-cells, to a space. Namely, if $X = A \cup \{e_i^2\}_{i \in I}$, where the 2-cell e_i^2 is attached by a map which for convenience we suppose is $f_i : (S^1, 1) \to (A, x)$, then each f_i determines an element ϕ_i in $\pi_1(A, x)$, and a consequence of the Seifert–van Kampen Theorem for the fundamental group is that the group $\pi_1(X, x)$ is obtained from the group $\pi_1(A, x)$ by adding the relations ϕ_i, $i \in I$.

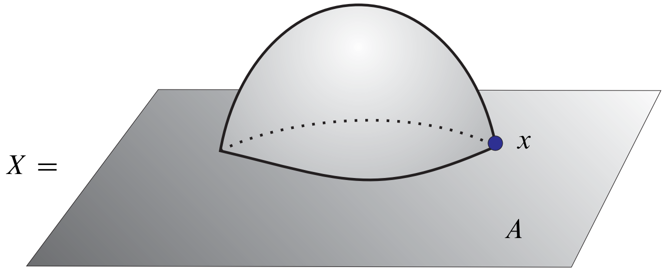


Figure 2.1. Picture of an attached 2-cell.

The next problem was clearly to determine the effect on the higher homotopy groups of adding cells to a space. So Whitehead's 1941 paper [Whi41a] was entitled 'On adding relations to homotopy groups'. If we could solve this problem in general then we would in particular be able to calculate all homotopy groups of spheres. Work since 1935 has shown the enormous difficulty of this task.

In this paper he gave important results in higher dimensions, but he was also able to obtain information on the second homotopy groups of $X = A \cup \{e_i^2\}_{i \in I}$. His results were clarified by him in two subsequent papers using the notion of *crossed* *module* [Whi46], and then *free crossed module,* [Whi49b]. Whitehead's highly original method of proof (an exposition is given in [Bro80]) uses knot theory and what is now called transversality, and has become the foundation of a technique called 'pictures', for which references are [BH82], [HAMS93] for computing second homotopy groups of 2-complexes. Free crossed modules are exploited rather differently and in a more algorithmic way in [BRS99] to compute second homotopy modules, see Section 10.3.ii.[12]

We begin this chapter by giving a definition of the fundamental crossed module

$$\Pi_2(X, A, x) = (\partial\colon \pi_2(X, A, x) \to \pi_1(A, x))$$

of a based pair of spaces and explaining some of Whitehead's work. Then we state two central results:

- the 2-dimensional Seifert–van Kampen Theorem, in Section 2.3;
- the notion of classifying space of a crossed module, in Section 2.4.

It is these two combined which give many of the important homotopical applications of crossed modules (including Whitehead's results). However the construction of the classifying space, and the proof of its properties, needs the methods of crossed complexes of Part II, and is given in Chapter 11. We give applications of the 2-dimensional Seifert–van Kampen Theorem in Chapters 4 and 5 and prove it in Chapter 6. This sets the scene for the corresponding higher dimensional results of Part II, and the substantial proofs of Part III.

Section 2.5 shows that crossed modules are equivalent to another algebraic structure, that of cat^1-groups. This is used in Section 2.6 to obtain the cat^1-group of a fibration, which yields an alternative way of obtaining the fundamental crossed module.

Section 2.7 shows that crossed modules are also equivalent to 'categories internal to groups', or, equivalently, to groupoids internal to groups. This is important philosophically, because groupoids are a generalisation of equivalence relations, and equivalence relations give an expression of the idea of quotienting, a fundamental process in mathematics and science, because it is concerned with classification. We can think of groupoids as giving ways of saying not only that two objects *are equivalent*, but also *how they are equivalent*: the arrows between two objects give different 'equivalences' between them, which can sometimes be regarded as 'proofs' that the objects are equivalent.

Moving now to the case of groups, to obtain a quotient of a group P we need not just an equivalence relation, but this equivalence relation needs to be a *congruence*, i.e. not just a subset but also a subgroup of $P \times P$. An elementary result in group theory is that a congruence on a group P is determined completely by a normal subgroup of P. The corresponding result for groupoids is that a groupoid with a group structure is equivalent to a crossed module $M \to P$ where P is the group of objects of the groupoid.

This family of equivalent structures – crossed modules, cat^1-groups, group objects in groupoids – gives added power to each of these structures. In fact in Chapter 6 we will use crucially another related structure, that of *double groupoids with connection*. This is equivalent to an important generalisation of a crossed module, that of *crossed module over a groupoid*, which copes with the varied base points of second relative homotopy groups.

2.1 Homotopy groups and relative homotopy groups

Recall that two maps $f, g\colon X \to Y$ between two topological spaces are said to be homotopic if f can be continuously deformed to g. Formally, they are *homotopic*, and this is denoted by $f \simeq g$, if there is a continuous map

$$F\colon I \times X \to Y$$

such that for all $x \in X$

$$F(0,x) = f(x) \text{ and } F(1,x) = g(x).$$

The map F is called a *homotopy* from f to g.

This definition gives an equivalence relation among the set of maps from X to Y. The quotient set is denoted $[X, Y]$ and the equivalence class of a map f is denoted by $[f]$.

Sometimes we are interested in considering only deformations that keep some subset fixed. If $A \subseteq X$, we say that two maps as above are *homotopic relative to* A, and denote this by $f \simeq g \text{ rel } A$, if there is a homotopy F from f to g satisfying $F(t,a) = f(a)$ for all $a \in A$, $t \in I$. This definition gives another equivalence relation among the set of maps from X to Y. The quotient set is written $[X, Y]_A$ and the equivalence class of a map f is again denoted by $[f]$.

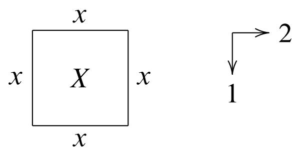
Since all maps homotopic relative to, or rel to, A must agree with a map $u\colon A \to Y$, this set for a fixed u is written $[X, Y; u]$. Thus $[X, Y]_A$ is the union of the disjoint sets $[X, Y; u]$ for all $u\colon A \to Y$.

A particular case of this definition is when we study maps sending a fixed subset A of X to a given point $y \in Y$. Then the quotient set corresponding to maps from X to Y sending all A to y with respect to homotopy rel A, is written as $[(X, A), (Y, y)]$ or, when $A = \{x\}$, as $[X, Y]_*$.

To define the *homotopy groups of a space*, we consider homotopy classes of maps from particular spaces. Namely if $x \in X$, the n-th homotopy group of X based at x is defined as

$$\pi_n(X, x) = [(I^n, \partial I^n), (X, x)]$$

where ∂I^n is the boundary of I^n. The elements of $\pi_n(X, x)$ are classes of maps that can be pictured for $n = 2$ as in the following diagram:



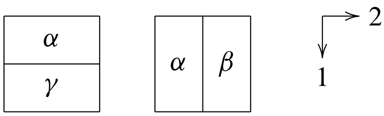
(2.1.1)

where we use throughout all the book a matrix like convention for directions. This is convenient for handling multiple compositions in Chapter 6, see Remark 6.1.4, and in Part III.

In the case $n = 1$ we obtain the fundamental group $\pi_1(X, x)$. For all $n \geqslant 1$ there initially seem to be n group structures on this set induced by the composition of representatives given for $1 \leqslant i \leqslant n$ by

$$(\alpha +_i \beta)(t_1, t_2, \dots, t_n) = \begin{cases} \alpha(t_1, t_2, \dots, 2t_i, \dots, t_n) & \text{if } 0 \leqslant t_i \leqslant 1/2, \\ \beta(t_1, t_2, \dots, 2t_i - 1, \dots, t_n) & \text{if } 1/2 \leqslant t_i \leqslant 1. \end{cases}$$

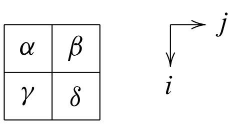
Remark 2.1.1. For the case $n = 2$ the following diagrams picture the two compositions:



$\alpha +_1 \gamma$ $\qquad$ $\alpha +_2 \beta$ □

Theorem 2.1.2. *If $n \geqslant 2$, then any of the multiplications $+_i$, $i = 1, \dots, n$ on $\pi_n(X, x)$ induce the same group structure, and all these group structures are abelian.*

Proof. By Theorem 1.3.1, we need only to verify the interchange law for the compositions $+_i, +_j$, $1 \leqslant i < j \leqslant n$. It is easily seen that if $\alpha, \beta, \gamma, \delta \colon (I^n, \partial I^n) \to (X, x)$ are representatives of elements of $\pi_n(X, x)$, then the two compositions obtained by evaluating the following matrix in two ways



in fact coincide. The rest of the argument is as in Theorem 1.3.1. □

We shall need later that π_n is functorial in the sense that to any map $\phi \colon X \to Y$ there is associated a homomorphism of groups

$$\phi_* \colon \pi_n(X, x) \to \pi_n(Y, \phi(x))$$

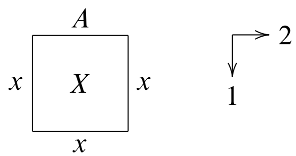
defined by $\phi_*[f] = [\phi f]$ satisfying the usual functorial properties $(\phi\psi)_* = \phi_*\psi_*$, $1_* = 1$.

Now we may repeat everything for maps of based pairs and homotopies among them. By a *based pair of spaces* (X, A, x) is meant a space X, a subspace A of X and a base point $x \in A$. The n^{th} *relative homotopy group* $\pi_n(X, A, x)$ of the based pair (X, A, x) is defined as the homotopy classes of maps of triples

$$\pi_n(X, A, x) = [(I^n, \partial I^n, J^{n-1}), (X, A, x)]$$

where $J^{n-1} = \{1\} \times I^{n-1} \cup I \times \partial I^{n-1}$; that is we consider maps $\alpha \colon I^n \to X$ such that $\alpha(\partial I^n) \subseteq A$ and $\alpha(J^{n-1}) = \{x\}$ and homotopies through maps of this kind.

The picture we shall have in mind as representing elements of $\pi_n(X, A, x)$ is



(2.1.2)

As before, a multiplication on $\pi_n(X, A, x)$ is defined by the compositions $+_i$ in any of the last $(n-1)$ directions. It is not difficult to check that any of these multiplications gives a group structure and analogously to Theorem 2.1.2 these all agree and are abelian if $n \geqslant 3$. Also, for any maps of based pairs $\phi \colon (X, A, x) \to (Y, B, y)$, there is a homomorphism of groups

$$\phi_* \colon \pi_n(X, A, x) \to \pi_n(Y, B, y)$$

as before .

The homotopy groups defined above fit nicely in an exact sequence called the *homotopy exact sequence of the pair* as follows:

$$\begin{aligned}
\cdots \to \pi_n(X, A, x) \xrightarrow{\partial_n} \pi_{n-1}(A, x) \xrightarrow{i_*} \pi_{n-1}(X, x) \xrightarrow{j_*} \cdots \\
\cdots \to \pi_2(X, A, x) \xrightarrow{\partial_2} \pi_1(A, x) \xrightarrow{i_*} \pi_1(X, x) \\
\xrightarrow{j_*} \pi_1(X, A, x) \xrightarrow{\partial_1} \pi_0(A) \xrightarrow{i_*} \pi_0(X),
\end{aligned} \tag{2.1.3}$$

where i_* and j_* are the homomorphisms induced by the inclusions, and the *boundary maps* ∂ are given by restriction, i.e. for any $[\alpha] \in \pi_n(X, A, x)$ represented by a map $\alpha \colon (I^n, \partial I^n, J^{n-1}) \to (X, A, x)$, we define $\partial[\alpha] = [\alpha']$ where α' is the restriction of α to the face $\{0\} \times I^{n-1}$, which we identify with I^{n-1}.

This exact sequence is of: abelian groups and homomorphisms until $\pi_2(X, x)$; of groups and homomorphisms until $\pi_1(X, x)$; and of based sets for the last three terms. The amount of exactness for the last terms is the same as for the exact sequence of a fibration of groupoids, see [Bro06], 7.2.9, which we use again in Section 12.1.ii (Theorem 12.1.15).

There is a structure on relative homotopy groups which is deduced from the above and which occurs when we have a triple (X_3, X_2, X_1) of pointed spaces, with base point x, say. That is, we have

$$x \in X_1 \subseteq X_2 \subseteq X_3.$$

Then we have for $n \geqslant 3$ a morphism

$$\delta\colon \pi_n(X_3, X_2, x) \to \pi_{n-1}(X_2, X_1, x)$$

which is defined as the composition

$$\pi_n(X_3, X_2, x) \xrightarrow{\partial} \pi_{n-1}(X_2, x) \xrightarrow{i_*} \pi_{n-1}(X_2, X_1, x).$$

We use this morphism δ, with (X_3, X_2, X_1) replaced by (X_n, X_{n-1}, X_{n-2}), in Section 7.1.v in constructing the fundamental crossed complex of a filtered space.

The final interesting piece of structure is the existence of a $\pi_1(A, x)$-*action* on all the terms of the above exact sequence which are groups. Let us define this action. For any $[\alpha] \in \pi_n(X, A, x)$ and any $[\omega] \in \pi_1(A, x)$, we define the map

$$F = F(\alpha, \omega)\colon I^n \times \{0\} \cup J^{n-1} \times I \to X$$

given by α on $I^n \times \{0\}$ and by ω on $\{t\} \times I$, for any $t \in J^{n-1}$. Then we have defined F on the subset of I^{n+1} indicated in Figure 2.2.

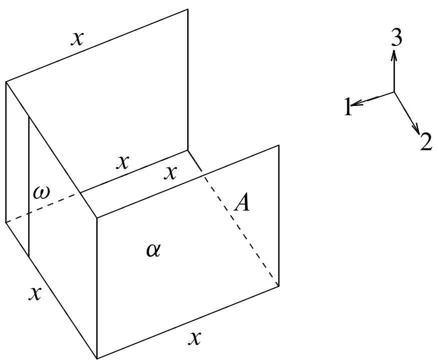


Figure 2.2. Action of $\pi_1(A, x)$.

We then compose with the retraction

$$r\colon I^{n+1} \to I^n \times \{0\} \cup J^{n-1} \times I$$

given by projecting from a point $P = (0, \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, 2)$ and indicated in Figure 2.3, getting a map $Fr\colon I^{n+1} \to X$ extending F.

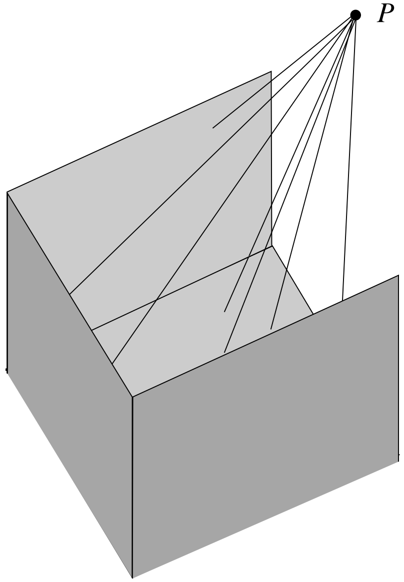


Figure 2.3. Retraction from above-lateral.

The 'restriction' map

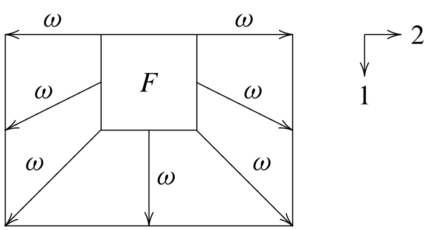
$$I^n \cong I^n \times \{1\} \hookrightarrow I^{n+1} \xrightarrow{Fr} X$$

represents an element $[\alpha]^{[\omega]} \in \pi_n(X, A, x)$.

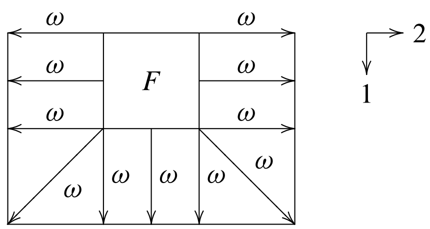
We leave the reader to develop proofs that the action is an action of a group on a group, that is that various axioms are satisfied. However all this will follow in a more algebraic fashion using the theory given in Chapter 14.

Notice that in this definition we use another of the filling arguments that we have started using in the proof of Theorem 1.6.1 in Section 1.6. Arguments of the same kind prove that the assignment just defined is independent of the several choices involved (α, ω and the extension of F), and that it defines an action.

Remark 2.1.3. When $n = 2$ the map representing $[\alpha]^{[\omega]}$ could be drawn



or, equivalently we could have chosen the one described as follows:



∎

In a similar way, we may define an action of $\pi_1(X,x)$ on $\pi_n(X,x)$. In our case, this gives an action of $\pi_1(A,x)$ on both $\pi_n(A,x)$ and $\pi_n(X,x)$. Moreover, all maps in the homotopy exact sequence are maps of $\pi_1(A,x)$-groups.

These constructions can be repeated for *based* r*-ads* $X_* = (X; X_1, X_2, \ldots, X_r, x)$, where all X_i are subspaces of X. Homotopy groups $\pi_n X_*$ are defined for $n \geqslant r+1$ and are abelian for $n \geqslant r+2$. There are various long exact sequences relating the homotopy groups of $(r+1)$-ads and r-ads. An account of these is in [Hu59]. The homotopy groups of an $(r+1)$-ad are also a special case of the homotopy groups of an r-cube of spaces [Lod82], [BL87], [Gil87]. All these groups are important for discussing the failure of excision for relative homotopy groups, to which we have referred earlier, and whose analysis in some cases using nonabelian methods will be an important feature of this book, see Section 5.4, Section 8.3.iii, and Section B.4.

2.2 Whitehead's work on crossed modules

We start with the basic definition of crossed module. (From Chapter 6 onwards we will need crossed modules over groupoids, but until then we stick to the group case).

Definition 2.2.1. A *crossed module* (*over a group*) $\mathcal{M} = (\mu \colon M \to P)$ is a morphism of groups $\mu \colon M \to P$ called the boundary of $\mathcal{M}$ together with an action[13] $(m, p) \mapsto m^p$ of the group P on the group M satisfying the two axioms[14]

CM1) $\mu(m^p) = p^{-1}\mu(m)p$,

CM2) $n^{-1}mn = m^{\mu n}$,

for all $m, n \in M$, $p \in P$. ∎

When we wish to emphasise the codomain P, we call $\mathcal{M}$ a *crossed* P*-module*. Basic algebraic examples of crossed modules are:

- A *conjugation crossed module* is an inclusion of a normal subgroup $N \trianglelefteq G$, with action of G on N given by conjugation. In particular, for any group P the identity map $1_P \colon P \to P$ is a crossed module with the action of P on itself by conjugation. Thus the concept of crossed module can be seen as an 'externalisation' of the concept of normal subgroup. That is, an inclusion is replaced by a homomorphism with special properties. This process occurs in other algebraic situations, and is relevant to the process of forming quotients, see Appendix C, Section C.8.

- If M is a group, its *automorphism crossed module* is given by $(\chi\colon M \to \mathrm{Aut}(M))$ where for $m \in M$, χm is the inner automorphism of M mapping n to $m^{-1}nm$.
- The trivial crossed module $0\colon M \to P$ whenever M is a P-module.
- A *central extension crossed module*, i.e. a surjective boundary $(\mu\colon M \to P)$ with kernel contained in the centre of M and $p \in P$ acts on $m \in M$ by conjugation with any element of $\mu^{-1}p$.
- Any homomorphism $(\mu\colon M \to P)$, with M abelian and $\mathrm{Im}\,\mu$ in the centre of P, provides a crossed module with P acting trivially on M.

The category $\mathsf{XMod}/\mathsf{Groups}$ has as objects all crossed modules over groups. Morphisms in $\mathsf{XMod}/\mathsf{Groups}$ from $\mathcal{M}$ to $\mathcal{N}$ are pairs of group homomorphisms (g, f) forming commutative diagrams with the two boundaries,

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "\mu"'] & N \arrow[d, "\nu"] \\ P \arrow[r, "f"'] & Q, \end{tikzcd}$$

and preserving the action in the sense that for all $m \in M$, $p \in P$ we have $g(m^p) = (gm)^{fp}$. If P is a group, then the category XMod/P of *crossed P-modules* is the subcategory of $\mathsf{XMod}/\mathsf{Groups}$ whose objects are the crossed P-modules and whose morphisms are the group homomorphisms $g\colon M \to N$ such that g preserves the action (i.e. $g(m^p) = (gm)^p$, for all $m \in M$, $p \in P$), and $\nu g = \mu$.

Here are some elementary general properties of crossed modules which we will often use.

Proposition 2.2.2. *For any crossed module $\mu\colon M \to P$, μM is a normal subgroup of P, in symbols $\mu M \trianglelefteq P$.*

Proof. This is immediate from CM1). □

The *centraliser* $C(S)$ of a subset S of a group M is the set of elements of M which commute with all elements of S. In particular, $C(M)$ is written ZM and called the *centre* of M and is abelian. Any subset of ZM is called *central* in M .

Proposition 2.2.3. *Let $\mu\colon M \to P$ be a crossed module. Then*

(i) $\mathrm{Ker}\,\mu$ *is central in M.*
(ii) $\mu(M)$ *acts trivially on ZM.*
(iii) ZM *and* $\mathrm{Ker}\,\mu$ *inherit an action of* $\mathrm{Cok}\,\mu$ *to become* $(\mathrm{Cok}\,\mu)$*-modules.*

Proof. Axiom CM2) shows that if $m, n \in M$ and $\mu n = 1$ then $mn = nm$. This proves (i). On the other hand, and by CM2) and CM1), $mn = nm$ implies $m^{\mu n} = m$, and this proves (ii). Then (iii) follows using these and Proposition 2.2.2, which implies $\operatorname{Cok} \mu = P/\mu(M)$. □

The *commutator* of elements m, n of a group M is the element

$$[m, n] = m^{-1}n^{-1}mn.$$

The *commutator subgroup* $[M, M]$ of M, is the subgroup of M generated by all commutators, and it is a standard fact that this is normal in M. We write M^{ab} for the abelian group $M/[M, M]$, the *abelianisation* of M.

Proposition 2.2.4. *Let $\mu\colon M \to P$ be a crossed module. Then P acts on M^{ab} and $\mu(M)$ acts trivially on M^{ab} which inherits an action of* $\operatorname{Cok}\mu$ *to become a* $(\operatorname{Cok}\mu)$*-module.*

Proof. Since $[m, n]^p = [m^p, n^p]$ for $m, n \in M$, $p \in P$, we have $[M, M]$ is P-invariant, so that P acts on M^{ab}. However the equation

$$m^{\mu n} = n^{-1}mn = m \bmod [M, M].$$

for $m, n \in M$ show that in this action $\mu(M)$ acts trivially. □

Remark 2.2.5. Thus for any crossed module $(\mu\colon M \to P)$ we have an exact sequence of $(\operatorname{Cok}\mu)$-modules

$$\operatorname{Ker}\mu \longrightarrow M^{\mathrm{ab}} \longrightarrow (\mu M)^{\mathrm{ab}} \longrightarrow 1.$$

The first map is not injective in general. To see this, consider the crossed module $\chi\colon M \to \operatorname{Aut}(M)$ associated to a group M. Then $\operatorname{Ker}\chi = ZM$, the centre of M. There are groups M for which

$$1 \neq ZM \subseteq [M, M],$$

for example the quaternion group, the dihedral groups and many others. For all these the composite map $\operatorname{Ker}\mu \to ZM \to M^{\mathrm{ab}}$ is trivial and so not injective. These examples give point to the following useful result. □

Proposition 2.2.6. *If there is a section $s\colon \mu M \to M$ of μ which is a group homomorphism (but not necessarily a P-map) then M is isomorphic as a group to* $\operatorname{Ker}\mu \times \mu M$. *Further* $[M, M] \cap \operatorname{Ker}\mu = 1$, *and the map* $\operatorname{Ker}\mu \to M^{\mathrm{ab}}$ *is injective.*

Proof. Because s is a section (i.e. μs is the identity on μM) we have that $M = (\operatorname{Ker}\mu)(\operatorname{Im}s)$ and $(\operatorname{Ker}\mu) \cap (\operatorname{Im}s) = \{1\}$. Because the action of $\operatorname{Im}s$ on $\operatorname{Ker}\mu$ is trivial, we have an internal product decomposition $M = (\operatorname{Ker}\mu)\times(\operatorname{Im}s)$. Furthermore, by Proposition 2.2.3 we know that $\operatorname{Ker}\mu$ is abelian so $[M, M] = [\operatorname{Im}s, \operatorname{Im}s]$.

So, $[M, M] \cap \operatorname{Ker}\mu = \{1\}$ and $\operatorname{Ker}\mu \to M^{\mathrm{ab}}$ is injective. □

An important example where the section s exists is when $\mu(M)$ is a free group. The well-known Schreier Subgroup Theorem of combinatorial group theory,[15] that a subgroup of a free group is itself free, assures us that such a section exists when P is free. The previous proposition is developed in Definition 7.4.23.

The result of the following exercise will be used in Example 10.3.10.

Exercise 2.2.7. Let $\mu\colon M \to P$ be a crossed module. If M has a single generator as P-group, and P is abelian, then M is abelian. □

Example 2.2.8. Crossed modules $\mu\colon M \to P$ in which both M and P are abelian form an interesting subcategory of that of crossed modules. As an example, let C_n be the cyclic group of order n, let $\mu\colon \mathsf{C}_2 \times \mathsf{C}_2 \to \mathsf{C}_4$ be the morphism which maps each C_2 summand injectively, and where C_4 operates by switching the two summands. This gives a crossed module which as we shall see in Example 12.7.12 is in a key sense nontrivial. □

Other algebraic examples of crossed modules arise from two important constructions with homotopical applications: *coproducts* of crossed P-modules developed in Chapter 4 and *induced* crossed modules in Chapter 5.

The major geometric example of a crossed module is the following, where the basic definitions were given in the last section. Let (X, A, x) be a based pair of spaces, that is X is a topological space and $x \in A \subseteq X$. Whitehead showed that the boundary map

$$\partial\colon \pi_2(X, A, x) \to \pi_1(A, x), \tag{2.2.1}$$

from the second relative homotopy group of (X, A, x) to the fundamental group $\pi_1(A, x)$, together with the standard action of $\pi_1(A, x)$ on $\pi_2(X, A, x)$,[16] has the structure of crossed module. We shall denote this crossed module by $\Pi_2(X, A, x)$ and it will be called *the fundamental crossed module of the based pair* (X, A, x). This result and its proof will be seen in various lights in this book. Because of this example it is convenient and sensible to regard crossed modules $\mu\colon M \to P$ as 2-*dimensional versions of groups*, with P, M being respectively the 1- and 2-dimensional parts. This analogy also will be pursued in more detail later. At this stage we only note that the full description of the 2-dimensional part requires specification of its 1-dimensional foundation and of the way the two parts fit together: that is, we need the whole structure of crossed module.

We have a functor from based pairs of topological spaces to crossed modules

$$\Pi_2\colon \mathsf{Top}^2_* \to \mathsf{XMod}/\mathsf{Groups} \tag{2.2.2}$$

which sends the based pair (X, A, x) to the crossed module given in (2.2.1) above. (In Chapter 6 we will formulate a groupoid version of this functor, allowing the base point to vary, but it is best to get familiar with this special case). Since all pairs are based in Chapters 2–5, we drop the base point from the notation from now on.

The work of Whitehead on crossed modules over the years 1941–1949 contained in [Whi41a], [Whi46], [Whi49b] and mentioned in the introduction to this chapter can be summarised as follows.

He started trying to obtain information on how the higher homotopy groups of a space are affected by adding cells. For the fundamental group, the answer is a direct consequence of the Seifert–van Kampen Theorem:

> adding a 2-cell corresponds to adding a relation to the fundamental group,
> adding an n-cell for $n \geqslant 3$ does not change the fundamental group.

So the next question is:

> how is the second homotopy group affected by adding 2-cells?, i.e. if $X = A \cup \{e_i^2\}$, what is the relation between $\pi_2(A)$ and $\pi_2(X)$?

In the first paper ([Whi41a]), he formulated a geometric proof of a theorem in this direction. In the second paper ([Whi46]) he gave the definition of crossed module and showed that the second relative homotopy group $\pi_2(X, A, x)$ of a pair of spaces could be regarded as a crossed module over the fundamental group $\pi_1(A, x)$. In the third paper ([Whi49b]) he introduced the notion of free crossed module and showed that his previous work could be reformulated as showing that the second relative homotopy group $\pi_2(X, A, x)$ was isomorphic to the *free crossed module*[17] on a set of generators corresponding to the 2-cells. This concept of free crossed module will be studied in detail in Section 3.4.

Whitehead was not able to obtain any detailed computations of second homotopy groups from this result, but he used it in his work on the classification of homotopy 2-types, and on a range of realisation problems.[18] We develop some of his results in Part II, particularly the relation with chain complexes with operators, and homotopy classification results.

The proof he gave was difficult to read, since it was spread over three papers, with some notation changes, and that is why a repackaged version of the proof by Brown was accepted for publication as [Bro80]. The main ideas of the proof included knot theory, and also transversality, techniques which became fashionable only in the 1960s (see also [HAMS93]).[19] A number of other proofs have been given, including the one we give in this book, see Corollary 5.4.8, in which the result is seen as one of the applications of a 2-dimensional Seifert–van Kampen type theorem.

The way this work was developed by Whitehead seems a very good example of what Grothendieck has called ‘struggling to bring new concepts out of the dark’ through the search for the underlying structural features of a geometric situation.

2.3 The 2-dimensional Seifert–van Kampen Theorem

Whitehead’s theorem on free crossed modules referred to in Section 2.2 demonstrated that a particular universal property was available for homotopy theory in dimension 2.

This suggested that there was scope for some broader kind of universal property at this level.

It also gave a clue to a reasonable approach. Such a universal property, in order to be broader, would clearly have to include Whitehead's theorem. Now this theorem is about the fundamental crossed module of a particular pair of spaces. So the broader principle should be about the fundamental crossed modules of *pairs* of spaces. The simplest property would seem to be, in analogy to the Seifert–van Kampen Theorem, that the functor

$$\Pi_2 \colon \mathsf{Top}^2_* \to \mathsf{XMod}/\mathsf{Groups}$$

described in (2.2.2) preserves certain pushouts. This led to the formulation of the next theorem. Also there had been a long period of experimentation by Brown and Spencer on the relations between crossed modules and double groupoids [BS76b], [BS76a], and by Higgins on calculation with crossed modules, so that the proof of the theorem, and the deduction of interesting calculations, came fairly quickly in 1974.

The next two theorems correspond to Theorem C of this Brown and Higgins paper ([BH78a]). We separate the statement into two theorems for an easier understanding. The first one is about coverings by two (open) subspaces, the second one about adjunction spaces.

First, we say the based pair (X, A) is *connected* if A and X are path connected and for $x \in A$ the induced map of fundamental groups $\pi_1(A, x) \to \pi_1(X, x)$ is surjective, or, equivalently, using the homotopy exact sequence, when $\pi_1(X, A, x) = 0$.

Having in mind that all pairs are based but not including the base point in the statement, we have:

Theorem 2.3.1 (2-dimensional Seifert–van Kampen Theorem 1). *Let A, U_1, and U_2 be subspaces of X such that the total space X is covered by the interiors of U_1 and U_2. We define $U_{12} = U_1 \cap U_2$, and $A_\nu = A \cap U_\nu$ for $\nu = 1, 2, 12$. If the pairs (U_ν, A_ν) are connected for $\nu = 1, 2, 12$, then:*

(Con) *The pair (X, A) is connected.*

(Iso) *The following diagram induced by inclusions*

$$\begin{tikzcd} \Pi_2(U_{12}, A_{12}) \arrow[r] \arrow[d] & \Pi_2(U_2, A_2) \arrow[d] \\ \Pi_2(U_1, A_1) \arrow[r] & \Pi_2(X, A) \end{tikzcd} \tag{2.3.1}$$

is a pushout of crossed modules.

Remark 2.3.2. Recall that this statement means that the above mentioned diagram is commutative and has the following universal property: For any crossed module $\mathcal{M}$ and morphisms of crossed modules $\phi_\nu \colon \Pi_2(U_\nu, A_\nu) \to \mathcal{M}$ for $\nu = 1, 2$ making the external square commutative, there is a unique morphism of crossed modules

$\phi\colon \Pi_2(X, A) \to \mathcal{M}$ such that the diagram

$$\begin{tikzcd}
\Pi_2(U_{12}, A_{12}) \arrow[r] \arrow[d] & \Pi_2(U_1, A_1) \arrow[d] \arrow[ddr, bend left, "\phi_1"] & \\
\Pi_2(U_2, A_2) \arrow[r] \arrow[drr, bend right, "\phi_2"'] & \Pi_2(U, A) \arrow[dr, dotted, "\phi"] & \\
& & \mathcal{M}
\end{tikzcd}$$

commutes. □

There is a slightly more general version of the theorem for adjunction spaces that can be deduced from the preceding theorem by using general mapping cylinder arguments.

Theorem 2.3.3 (2-dimensional Seifert–van Kampen Theorem 2). *Let X and Y be spaces, A a subset of X and $f : A \to Y$ a map. We consider subspaces $X_1 \subseteq X$ and $Y_1 \subseteq Y$ and define $A_1 = X_1 \cup A$ and let $f_1 = f| : A_1 \to Y_1$ be the restriction of f. If the inclusions $A \subseteq X$ and $A_1 \subseteq X_1$ are closed cofibrations and the pairs (Y, Y_1), (X, X_1), (A, A_1) are connected, then:*

(Con) *The pair $(X \cup_f Y, X_1 \cup_{f_1} Y_1)$ is connected.*

(Iso) *The following diagram induced by inclusions*

$$\begin{tikzcd}
\Pi_2(A, A_1) \arrow[r] \arrow[d] & \Pi_2(Y, Y_1) \arrow[d] \\
\Pi_2(X, X_1) \arrow[r] & \Pi_2(X \cup_f Y, X_1 \cup_{f_1} Y_1)
\end{tikzcd} \tag{2.3.2}$$

is a pushout of crossed modules.

Remark 2.3.4. The term closed cofibration included in the hypothesis of the theorem can be intuitively interpreted as saying that the placing of A in X and of A_1 in X_1 are 'locally not wild'. The condition is satisfied in a great number of useful cases, see Section 7.3 in [Bro06]. □

The interest in these theorems is at least seven fold:

- The theorem does have Whitehead's theorem on free crossed modules as a consequence (see Corollary 5.4.8).
- The theorem is a very useful computational tool and gives information unobtainable so far by other means.[20]
- The theorem is an example of a local-to-global theorem. Such theorems play an important rôle in mathematics and its applications.
- The theorem deals with nonabelian objects, and so cannot be proved by traditional means of algebraic topology.

- The two available proofs use groupoid notions in an essential way.
- The existence of the theorem confirms the value of the crossed module concept, and of the methods used in its proof. We should be interested in algebraic structures for which this kind of result is true.
- It shows the difficulty of homotopy theory since one has, it appears, to go through all this just to determine, as we explain in Section 5.8, the second homotopy groups of certain mapping cones.

A further point is that the proof we shall give later does not assume the general existence of pushouts of crossed modules. Instead, it verifies directly the required universal property for this case, and so that this pushout does exits.

We conclude this section by stating an analogue of Theorem 2.3.1, but for general covers of a space X; this also will be deduced from Theorem 6.8.2.

Let Λ be an indexing set and suppose we are given a family $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ of subsets of X such that the interiors of the sets of $\mathcal{U}$ cover X. For each $\nu = \{\nu_1, \dots, \nu_n\} \in \Lambda^n$, we write

$$U_\nu = U_{\nu_1} \cap \dots \cap U_{\nu_n}.$$

Let A be a subspace of X, and define $A_\nu = U_\nu \cap A$, for any $\nu \in \Lambda^n$. Suppose also given a base point $x \in A$ which is contained in every A_λ and so also in every A_ν.

Theorem 2.3.5. *Assume that for every $\nu \in \Lambda^n$, $n \geqslant 1$, the pair (U_ν, A_ν) is connected. Then*

(Con) *the pair (X, A) is connected, and*

(Iso) *the crossed module $\Pi_2(X, A)$ satisfies the following universal property: for any crossed module $\mathcal{M}$ and any family of morphisms of crossed modules*

$$\{\phi_\lambda\colon \Pi_2(U_\lambda, A_\lambda) \to \mathcal{M} \mid \lambda \in \Lambda\}$$

such that for any $\lambda, \mu \in \Lambda$ the diagram

$$\begin{tikzcd}
\Pi_2(U_{\lambda\mu}, A_{\lambda\mu}) \arrow[r] \arrow[d] & \Pi_2(U_\lambda, A_\lambda) \arrow[d, "\phi_\lambda"] \\
\Pi_2(U_\mu, A_\mu) \arrow[r, "\phi_\mu"] & \mathcal{M}
\end{tikzcd}$$

commutes, there is a unique morphism of crossed modules $\phi\colon \Pi_2(X, A) \to \mathcal{M}$ such that all triangles of the form

$$\begin{tikzcd}
\Pi_2(U_\lambda, A_\lambda) \arrow[r] \arrow[dr] & \Pi_2(X, A) \arrow[d, "\phi"] \\
& \mathcal{M}
\end{tikzcd}$$

commute.

The universal property of the theorem can be expressed as what is called a 'co-equaliser condition' (see Appendix A, Example A.4.4). It is in this way we rephrase the theorem in its double groupoid version in Theorem 6.8.2. We will also follow the groupoid philosophy and so have a many base point version, using crossed modules of groupoids.

Remark 2.3.6. It can be easily seen from the proof that the conditions on n-fold intersections for all $n \geqslant 1$ can be relaxed to path connectivity of all 4-fold intersections, and 1-connectivity of all pairs given by 8-fold intersections. More refinements of the arguments, using Lebesgue covering dimension, reduce these numbers to 3 and 4 respectively.[21] □

The proof of Theorem 2.3.5 will be given in Chapter 6 via another algebraic structure, that of double groupoids, since these have properties which are more appropriate than are those of crossed modules for expressing the geometry of the proof, which is analogous to that of the 1-dimensional theorem. Indeed the proof of the theorem was first conceived in terms of double groupoids, and the crossed module application appeared later.

2.4 The classifying spaces of a group and of a crossed module

We are going to construct in the second part of this book, in Chapter 11, a 'classifying space' of a crossed complex; this construction includes as particular cases (cubical versions of) the classifying space of a group and of a crossed module.

Nevertheless, this is a good point to state some of the properties of this classifying spaces for the special cases which classify the weak pointed homotopy 1-type and the 2-type of a space.

The *classifying space of a group* P is a functorial construction

$$B\colon \mathsf{Groups} \to \mathsf{Top}_*$$

assigning a reduced CW-complex BP to each group P so that

Proposition 2.4.1. *The homotopy groups of the classifying space BP of the group P are given by*

$$\pi_i(BP) \cong \begin{cases} P & \text{if } i = 1, \\ 0 & \text{if } i \geqslant 2. \end{cases}$$

This gives a natural equivalence from $\pi_1 B$ to the identity. There is also a relation between $B\pi_1$ and the identity, given by:

Proposition 2.4.2. *Let X be a reduced CW-complex and let $\phi\colon \pi_1(X) \to P$ be a homomorphism of groups. Then there is a map*

$$X \to BP$$

inducing the homomorphism ϕ on fundamental groups.

As a consequence we find that $B\pi_1$ captures all information on fundamental groups.

Theorem 2.4.3. *Let X be a reduced CW-complex and let $P = \pi_1(X)$. Then there is a map*

$$X \to BP$$

inducing an isomorphism of fundamental groups.

It is because of these results that groups are said to model pointed, connected homotopy 1-types.

Next, we state some properties of the *classifying space of a crossed module*. It is a functor

$$B\colon \mathsf{XMod} \to \mathsf{Top}_*$$

assigning to a crossed module $\mathcal{M} = (\mu\colon M \to P)$ a pointed CW-space $B\mathcal{M}$ with the following properties:

Proposition 2.4.4. *The homotopy groups of the classifying space of the crossed module $\mathcal{M}$ are given by*

$$\pi_i(B\mathcal{M}) \cong \begin{cases} \operatorname{Cok}\mu & \text{for } i = 1, \\ \operatorname{Ker}\ \mu & \text{for } i = 2, \\ 0 & \text{for } i > 2. \end{cases}$$

There is a twofold relation with the classifying space of a group defined before. On the one hand, the classifying space of a crossed module generalises that for groups, i.e.

Proposition 2.4.5. *If P is a group then the classifying space $B(1 \to P)$ is exactly the classifying space BP discussed before.*

On the other hand

Proposition 2.4.6. *Let $M \lhd P$ be a normal subgroup of the group P. Then the morphism of crossed modules $(M \to P) \to (1 \to P/M)$ induces a homotopy equivalence of classifying spaces*

$$B(M \to P) \to B(P/M). \qquad \square$$

This follows from another famous theorem of Whitehead that a map of CW-complexes inducing an isomorphism of all homotopy groups is a homotopy equivalence.[22]

Proposition 2.4.7. *The classifying space BP is a subcomplex of $B\mathcal{M}$, and there is a natural isomorphism of crossed modules*

$$\Pi_2(B\mathcal{M}, BP) \cong \mathcal{M}. \tag{2.4.1}$$

Theorem 2.4.8. *Let X be a reduced CW-complex, and let $\Pi_2(X, X^1)$ be the crossed module $\pi_2(X, X^1) \to \pi_1(X^1)$, where X^1 is the 1-skeleton of X. Then there is a map*

$$X \to B(\Pi_2(X, X^1)) \tag{2.4.2}$$

inducing an isomorphism of π_1 and π_2.

It is because of these results that it is reasonable to say that crossed modules model all pointed connected homotopy 2-types.[23]

We shall use cubical sets and crossed complexes to give in Definition 11.4.18 an elegant description of the cells of the classifying space $B(M \to P)$.[24] The existence and properties of the classifying space show that calculations of pushouts of crossed modules, such as those required by the 2-dimensional Seifert–van Kampen Theorem, may also be regarded as calculations of homotopy 2-types. This is evidence that the fundamental crossed module of a pair is an appropriate candidate for a 2-dimensional version of the fundamental group, as sought by an earlier generation of topologists (see Section 1.3), and indeed that crossed modules may be regarded as one kind of 2-dimensional version of groups.[25]

The situation we have for crossed modules and pairs of spaces comes under a format very similar to part of the Main Diagram on page xxxii of the Introduction:

$$\begin{tikzcd} \mathsf{Top}^2 \arrow[rr, shift left, "\Pi"] \arrow[dr, "U"'] & & \mathsf{XMod} \arrow[ll, shift left, "\mathscr{B}"] \arrow[dl, "B"] \\ & \mathsf{Top} & \end{tikzcd} \tag{2.4.3}$$

We suppose the following properties:

(i) The functor Π preserves certain colimits.

(ii) There is a natural equivalence $\Pi\mathscr{B} \simeq 1$.

(iii) $B = U\mathscr{B}$.

(iv) There is a convenient natural transformation $1 \simeq \mathscr{B}\Pi$ preserving some homotopy properties.

Property (i) is a form of the Seifert–van Kampen Theorem. This enables some computations to get started.

Property (ii) shows that the algebraic data forms a reasonable mirror of the topological data.

Property (iii) allows the classifying space to be defined: U is some kind of forgetful functor.

Property (iv) is difficult to state precisely in general terms. The intention is to show that the structure $\mathscr{B}\Pi$ captures some slice of the homotopy properties of the original topological data.

We shall not use any general format of or deduction from these properties, but it should be realised that the material we give on groups and on crossed modules forms part of a much more general pattern.

Let us finish this section by giving also some indications of how to go up one dimension further. First we give a theorem about the behaviour of the classifying space functor on crossed modules when applied to a short exact sequence. This theorem will be deduced from a more general result, Corollary 12.1.14, on the classifying space of crossed complexes, where more machinery is available for the proof.

Theorem 2.4.9. *Suppose the commutative diagram*

$$\begin{tikzcd} 1 \arrow[r] & L \arrow[r, "i"] \arrow[d, "\lambda"] & M \arrow[r, "p"] \arrow[d, "\mu"] & N \arrow[r] \arrow[d, "\nu"] & 1 \\ 1 \arrow[r] & K \arrow[r, "j"'] & P \arrow[r, "f"'] & Q \arrow[r] & 1 \end{tikzcd} \tag{2.4.4}$$

is such that the vertical arrows are crossed modules, the squares are morphisms of crossed modules, and the rows are exact sequences of groups. Then the diagram of induced maps of classifying spaces

$$B(L \to K) \to B(M \to P) \to B(N \to Q)$$

is a fibration sequence.

In the above situation we say that the crossed module $L \to K$ is a kernel of the morphism (p, f) of crossed modules. Note that the groups L, K may be considered as normal subgroups of M, P respectively. There is an additional property: if $k \in K$, $m \in M$, then $p(m^{-1}m^{j(k)}) = 1$, so that $m^{-1}m^{j(k)} \in \operatorname{Im} i$. This gives rise to a function $h\colon K \times M \to L$. The properties are summarised by saying that the first square of diagram (2.4.4) is a *crossed square*. These structures give the next stage after crossed modules for modeling homotopy types, that is they model homotopy 3-types. There seem to be good reasons why the analysis of kernels should give rise to a higher order structure modeling a further level of homotopy types. Such structures are quite subtle and cannot be pursued in this book, though we give a brief indication on crossed squares in Appendix B, Section B.4.[26]

2.5 Cat^1-groups

There are several algebraic and combinatorial categories that are equivalent to the category of crossed modules.[27]

Of the categories equivalent to XMod/Groups, perhaps the most used is the category Cat^1-Groups of cat^1-groups. It is also useful in some cases when describing the colimits used in the 2-dimensional Seifert–van Kampen Theorem.

In this section, we explain this equivalence and some of the applications. Let us begin by expressing the basic properties of a crossed module $\mathcal{M} = (\mu\colon M \to P)$ in an alternative way.

The action of P on M can be encoded using the semidirect product $P \ltimes M$, and the projection $s\colon P \ltimes M \to P$, $(p, m) \mapsto p$. The map μ gives a homomorphism $t\colon P \ltimes M \to P \ltimes M$ by the rule $(p, m) \mapsto (p\mu(m), 1)$; that t is a homomorphism of groups follows from CM1).

It is a bit more difficult to find the way CM2) can be translated, but after playing for a while it can be seen that it gives that the elements of $\operatorname{Ker} s$ and those of $\operatorname{Ker} t$ commute in the semidirect product. This is the kind of algebraic object we need.

A *cat*^{1}*-group* is a triple $\mathcal{G} = (G, s, t)$ such that G is a group and $s, t\colon G \to G$ are group homomorphisms satisfying

CG1) $st = t$ and $ts = s$,

CG2) $[\operatorname{Ker} s, \operatorname{Ker} t] = 1$.

A *homomorphism of cat*^{1}*-groups* between (G, s, t) and (G', s', t') is a homomorphism of groups $f : G \to G'$ preserving the structure, i.e. such that $s'f = fs$ and $t'f = ft$. These objects and morphisms define the category $\mathsf{Cat}^1\text{-}\mathsf{Groups}$ of cat^1-groups.

Example 2.5.1. The category of groups, Groups, can be considered a full subcategory of $\mathsf{Cat}^1\text{-}\mathsf{Groups}$ using the inclusion functor

$$I : \mathsf{Groups} \to \mathsf{Cat}^1\text{-}\mathsf{Groups}$$

given by $I(G) = (G, 1_G, 1_G)$. □

Having in mind the discussion at the beginning of this section, we define a functor

$$\lambda\colon \mathsf{XMod}/\mathsf{Groups} \to \mathsf{Cat}^1\text{-}\mathsf{Groups}$$

by $\lambda(\mu\colon M \to P) = (P \ltimes M, s, t)$, where $s(g, m) = (g, 1)$ and $t(g, m) = (g(\mu m), 1)$.

Proposition 2.5.2. *If $\mu\colon M \to P$ is a crossed module, then $\lambda(\mu\colon M \to P)$ is a cat*^{1}*-group.*

Proof. It is clear that s is a homomorphism. To check that t is also a homomorphism , let us consider elements $(g, m), (g', m') \in P \ltimes M$. Then, we have

$$\begin{aligned} t((g, m)(g', m')) &= t(gg', m^{g'} m') \\ &= (gg'\mu(m^{g'})\mu m', 1) = (gg'g'^{-1}\mu m g'\mu m', 1) \quad \text{by CM1)} \\ &= (g\mu m g'\mu m', 1) = t(g, m)t(g', m'). \end{aligned}$$

It is also easy to prove that s, t satisfy CG1).

To check CG2), let us consider elements $(1,m) \in \operatorname{Ker} s$ and $(\mu m', m'^{-1}) \in \operatorname{Ker} t$. Then, we have

$$\begin{aligned}(1,m)(\mu m', m'^{-1}) &= (\mu m', m^{\mu m'} m'^{-1}) = (\mu m', m'^{-1} m m' m'^{-1}) \quad \text{by CM2)}\\ &= (\mu m', m'^{-1} m) = (\mu m', m'^{-1})(1,m).\end{aligned}$$ □

Example 2.5.3. Thus, associated to any normal subgroup M of G, we have a cat^1-group $M \ltimes G$, where G acts on M by conjugation. □

To define the functor back, let us check that all cat^1-groups have a semidirect product decomposition.

Proposition 2.5.4. *For any cat^1-group (G,s,t):*

i) *The maps s,t have the same range, i.e. $s(G) = t(G) = N$, and are the identity on N.*

ii) *The morphisms s and t are 'projections', i.e. $t^2 = t$ and $s^2 = s$.*

Proof. i) As $st = t$, we have $\operatorname{Im} t \subseteq \operatorname{Im} s$ and as $ts = s$, we have $\operatorname{Im} s \subseteq \operatorname{Im} t$.

ii) We have $ss = sts = ts = s$. Similarly, $tt = t$. □

As an easy consequence, we have:

Corollary 2.5.5. *There are two split short exact sequences*

$$1 \longrightarrow \operatorname{Ker} s \longrightarrow G \xrightarrow{s} N \longrightarrow 1,$$
$$1 \longrightarrow \operatorname{Ker} t \longrightarrow G \xrightarrow{t} N \longrightarrow 1.$$ □

Remark 2.5.6. Thus G is isomorphic to both semidirect products $N \ltimes \operatorname{Ker} s$ and $N \ltimes \operatorname{Ker} t$, where N acts on each of the kernels by conjugation. The map $N \ltimes \operatorname{Ker} s \to G$ is just the product and the inverse isomorphism $G \to N \ltimes \operatorname{Ker} s$ is given by $g \mapsto (s(g), s(g^{-1})g)$. □

We can also define an inverse functor

$$\gamma \colon \mathsf{Cat^1\text{-}Groups} \to \mathsf{XMod/Groups}$$

given by $\gamma(G,s,t) = (t| \colon \operatorname{Ker} s \to \operatorname{Im} s)$ where $\operatorname{Im} s$ acts on $\operatorname{Ker} s$ by conjugation.

Proposition 2.5.7. *If (G,s,t) is a cat^1-group, then $\gamma(G,s,t)$ is a crossed module.*

Proof. With respect to CM1), for all $g \in \operatorname{Im} s$ and $m \in \operatorname{Ker} s$, we have

$$t(m^g) = t(g^{-1}mg) = (tg)^{-1}(tm)(tg).$$

Since $g \in \operatorname{Im} s = \operatorname{Im} t$, by Proposition 2.5.4, we have $tg = g$. Thus, $t(m^g) = g^{-1}(tm)g$.

On the other hand, with respect to CM2) for all $m, m' \in \operatorname{Ker} s$, we have

$$m'^{(tm)} = (tm^{-1})m'(tm) = (tm^{-1})m'(tm)m^{-1}m.$$

But $(tm)m^{-1}$, m' commute since $(tm)m^{-1} \in \operatorname{Ker} s$ and $m' \in \operatorname{Ker} s$, and so

$$m'^{(tm)} = (tm^{-1})(tm)m^{-1}m'm = m^{-1}m'm$$

as required. □

Proposition 2.5.8. *The functors λ and γ give an equivalence of categories.*

Proof. On the one hand we have $\lambda\gamma(G, s, t) = (\operatorname{Im} t \ltimes \operatorname{Ker} s, s', t')$ where $s'(g, m) = (g, 1)$ and $t'(g, m) = (gt(m), 1)$. Clearly there is a natural isomorphism of groups $\phi \colon G \to \operatorname{Im} t \ltimes \operatorname{Ker} s$ given by $\phi(g) = (s(g), s(g)^{-1}g)$ that is an isomorphism of cat^1-groups.

On the other hand, $\gamma\lambda(\mu \colon M \to P) = (\operatorname{Ker} \xrightarrow{t} \operatorname{Im} s)$ where $s \colon P \ltimes M \to P \ltimes M$ is given by $s(g, m) = (g, 1)$. There are obvious natural isomorphisms $\operatorname{Ker} s \cong M$ and $\operatorname{Im} s \cong P$ giving a natural isomorphism of crossed modules. □

2.6 The fundamental crossed module of a fibration

In this section the proofs will be omitted or be sketchy, since background in fibrations of spaces is needed. Throughout we assume that 'space' means 'pointed space'.

We are going to show that for any fibration $\mathcal{F} = (F \xrightarrow{i} E \xrightarrow{p} X)$ the induced map

$$i_* \colon \pi_1(F) \to \pi_1(E)$$

is a crossed module $\Pi_2(\mathcal{F})$ which we call the *fundamental crossed module* of the fibration $\mathcal{F}$. This is an observation first made by Quillen and from it can be deduced the fundamental crossed module of a pair of spaces.

Perhaps it is better first to recall in some detail the action of $\pi_1(E)$ on $\pi_1(F)$ for any fibration $\mathcal{F}$.

Let us consider $[\mu] \in \pi_1(F)$ and $[\alpha] \in \pi_1(E)$. The projection to X of the loop $\alpha^{-1}\mu\alpha$ is homotopic to the constant through a homotopy of loops $H \colon I \times I \to X$. Since p is a fibration, using the homotopy lifting property, we get a homotopy of loops $\bar{H} \colon I \times I \to E$ from $\alpha^{-1}\mu\alpha$ to a loop projecting to the constant, i.e. $\operatorname{Im} \bar{H}_1 \subseteq F$. We define

$$[\mu]^{[\alpha]} = [\bar{H}_1] \in \pi_1(E).$$

We omit the proof that this action is well defined. This is a good exercise on fibration theory.

To prove that i_* is a crossed module, we proceed in a roundabout way. Clearly, it is equivalent to prove that the semidirect product $\pi_1(E) \ltimes \pi_1(F)$ given by the action just defined is a cat^1-group. Again, this is not done directly, but instead we prove that there is a natural isomorphism of groups

$$\pi_1(E \times_X E) \cong \pi_1(E) \ltimes \pi_1(F)$$

and that the former is a cat^1-group, where $E \times_X E$ is the pullback of p along p, i.e.

$$E \times_X E = \{(e, e') \in E \times E \colon p(e) = p(e')\}.$$

First, let us prove that $\pi_1(E \times_X E)$ decomposes to the expected semidirect product.

Proposition 2.6.1. *For any fibration $\mathcal{F} = (F \xrightarrow{i} E \xrightarrow{p} X)$, there are two splitting short exact sequences*

$$1 \to \pi_1(F) \xrightarrow{i_{l*}} \pi_1(E \times_X E) \xrightarrow{p_{l*}} \pi_1(E) \to 1 \quad \textit{for } l = 1, 2$$

where i_l is the inclusion of F in the l^{th} factor. Moreover both sequences are natural with respect to maps of fibrations.

Proof. Recall that the projection in the first factor $E \times_X E \to E$ is a fibration with fibre F since it is the pullback of p along itself. Also, the diagonal map gives a section of this fibration. Thus, its homotopy exact sequence decomposes into a sequence of splitting short exact sequences. In particular,

$$1 \longrightarrow \pi_1(F) \xrightarrow{i_{1*}} \pi_1(E \times_X E) \xrightarrow{p_{1*}} \pi_1(E) \longrightarrow 1$$

is a splitting short exact sequence. The same is true in the second case. □

Now, we are able to prove that $(\pi_1(E \times_X E), s, t)$ where s (resp. t) is the homomorphism induced on the fundamental groups by the composition of the projection in the first (resp. second) factor and the diagonal is a cat^1-group for any fibration $\mathcal{F}$. We shall call it the *fundamental cat^1-group of the fibration $\mathcal{F}$*.[28]

Proposition 2.6.2. *Let $\mathcal{F} = (F \xrightarrow{i} E \xrightarrow{p} X)$ be a fibration. Then $(\pi_1(E \times_X E), s, t)$ is a cat^1-group.*

Proof. It clearly satisfies CG1) since the maps s, t are in essence projections.

To prove CG2), using the exact sequence of Proposition 2.6.1, we have $\operatorname{Ker} s = \operatorname{Im} i_{1*}$ and $\operatorname{Ker} t = \operatorname{Im} i_{2*}$

Also by Proposition 2.6.1 the elements of $\operatorname{Ker} s$ are of the form $[(ct, \mu)]$ where μ is a loop in the fibre and the elements of $\operatorname{Im} s$ are of the form $[(\alpha, \alpha)]$ where α is a loop in E.

We choose elements $[(ct,\mu)] \in \operatorname{Ker} s$ and $[(\nu,ct)] \in \operatorname{Ker} t$ where μ and ν are loops in the fibre. Then

$$[(\nu,ct)][(ct,\mu)] = [(\nu,\mu)] = [(ct,\mu)][(\nu,ct)]$$

which gives the commutativity of these elements. □

Now, we proceed to identify the crossed module associated with $(\pi_1(E \times_X E), s, t)$.

Proposition 2.6.3. *The crossed module* $(t| \colon \operatorname{Ker} s \to \operatorname{Im} s)$ *associated to the cat*^{1}*-group* $\pi_1(E \times_X E)$ *is naturally isomorphic to* $\Pi_2\mathcal{F} = (\pi_1(F) \to \pi_1(E))$.

Proof. There are natural isomorphisms $\pi_1(F) \cong \operatorname{Ker} s$ and $\pi_1(E) \cong \operatorname{Im} s$, given by $[\mu] \mapsto [(ct,\mu)]$ and $[\alpha] \mapsto [(\alpha,\alpha)]$ respectively. It remains only to check that these isomorphisms preserve actions.

The action of $\operatorname{Ker} s$ on $\operatorname{Im} s$ is given by conjugation in $\pi_1(E \times_X E)$. Under these isomorphisms the result of the action of $[\alpha] \in \pi_1(E)$ on $[\mu] \in \pi_1(F)$, is the homotopy class of any loop ν in F satisfying

$$[(ct,\nu)] = [(\alpha^{-1}\alpha, \alpha^{-1}\mu\alpha)].$$

Recalling the definition of the $\pi_1(E)$ action on $\pi_1(F)$ at the beginning of the section, we see that $[\mu]^{[\alpha]}$ is represented by just this same element. □

To define the fundamental cat^1-group functor on maps of general topological spaces we need some more homotopy theory. There is no space to develop this here in full, and so we just sketch the ideas.[29]

A standard procedure in homotopy theory is to factor any map $f : Y \to X$ through a homotopy equivalence i and a fibration $\bar{f} : \bar{Y} \to X$ where $\bar{Y} = \{(\lambda, y) \in X^I \times Y : \lambda(1) = f(y)\}$ and $\bar{f}(\lambda, y) = \lambda(0)$.

This gives a functor $\mathrm{Fib} \colon f \mapsto \bar{f}$ from maps to fibrations. We define the cat^1-group functor on maps of general topological spaces by composition with the cat^1-group of fibrations functor.

Let us sketch a direct description of the composite functor[30]

$$\mathsf{Maps} \to \mathsf{Cat}^1\text{-}\mathsf{Groups}.$$

The functor is defined by

$$(f : Y \to X) \mapsto (\pi_1(\bar{Y} \times_X \bar{Y}), p_{1*}, p_{2*}).$$

Using the homeomorphism

$$\bar{Y} \times_X \bar{Y} \equiv \{(y_1, \lambda, y_2) \in Y \times X^I \times Y \mid \lambda(0) = f(y_1) \text{ and } \lambda(1) = f(y_2)\}$$

the projections in the factors correspond to the maps

$$p_1(y_1, \lambda, y_2) = (y_1, \lambda_1), \quad \text{where } \lambda_1(t) = \lambda(t/2),$$
$$p_2(y_1, \lambda, y_2) = (y_2, \lambda_2), \quad \text{where } \lambda_2(t) = \lambda(1 - (t/2)).$$

Via the same homeomorphism, the elements of $\pi_1(\bar{Y} \times_X \bar{Y})$ correspond to homotopy classes of triples, $[(\alpha, \mu, \beta)]$, where $\mu\colon I \times I \to X$ maps $I \times \{0, 1\}$ to the base point and $\alpha, \beta\colon I \to Y$ are loops on Y lifting $\mu(0, -)$ and $\mu(1, -)$ respectively. The homotopies correspond to triples, (F, H, G), the map $H : I \times I \times I \to X$ sending $I \times \{0, 1\} \times I$ to the base point, and $F, G : I \times I \to Y$ being homotopies of loops, relative to the end points, lifting $H(0, -, -)$ and $H(1, -, -)$, respectively.

The description of p_{1*} and p_{2*} follows easily.

To assure consistency, let us point out that if f is already a fibration, both definitions of the fundamental cat^1-group produce the same group up to isomorphism.

If f is a fibration, f and $\bar{f}$ are fibre homotopy equivalent. It can be checked directly that $Y \times_X Y$ and $\bar{Y} \times_X \bar{Y}$ are also homotopy equivalent, but it is also a consequence of the following 'cogluing theorem'.[31]

Theorem 2.6.4. *Suppose given maps over X,*

$$\begin{tikzcd} Z \arrow[rr, "i"] \arrow[dr, "f"'] & & \bar{Y} \arrow[dl, "\bar{f}"] \\ & X, & \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[rr, "j"] \arrow[dr, "g"'] & & \bar{Z} \arrow[dl, "\bar{g}"] \\ & X, & \end{tikzcd}$$

such that f, $\bar{f}$, g, $\bar{g}$ are fibrations, and i, j are homotopy equivalences. Then the induced map on pullbacks

$$i \times_X j : Y \times_X Z \to \bar{Y} \times_X \bar{Z}$$

is also a homotopy equivalence, and in fact a fibre homotopy equivalence.

In the particular case in which we are mostly interested, we consider a pair of topological spaces (X, A). Associated to the inclusion $i : A \to X$ there is the fibration $\bar{A} \to X$ where $\bar{A}$ is the space of paths in X starting at some point of A and the map sends each path to its end point. The fibre of this fibration is the space

$$F_i = \{\lambda \in X^I \mid \lambda(0) \in A, \lambda(1) = *\}$$

whose homotopy groups are, by definition, those of the pair (X, A), i.e.

$$\pi_n(F_i) = \pi_{n+1}(X, A).$$

In particular, the *fundamental crossed module of a pair functor*

$$\Pi_2 : \mathsf{Top}^2_* \longrightarrow \mathsf{Fib} \longrightarrow \mathsf{XMod/Groups}$$

is given by

$$\Pi_2(X, A) = (\partial\colon \pi_2(X, A) \to \pi_1(A))$$

with the usual action already known and used by Whitehead.

Finally in this section, we mention some relations of crossed modules with algebraic K-theory, for those familiar with that area.

Let R be a ring. A basic structure for algebraic K-theory is the homotopy fibration

$$F(R) \to B\mathrm{GL}(R) \to B\mathrm{GL}(R)^+.$$

This yields the crossed module

$$\pi_1(F(R)) \to \pi_1(B\mathrm{GL}(R))$$

which is equivalent to

$$\mathrm{St}(R) \to \mathrm{GL}(R)$$

which has cokernel $K_1(R)$ and kernel $K_2(R)$. The group $\mathrm{St}(R)$ is called the *Steinberg group* of the ring R, and is usually given a direct definition in terms of generators and relations.

Now let I be an ideal of R, and let $\mathrm{GL}(R, I)$, the *congruence subgroup*, be the kernel of $\mathrm{GL}(R) \to \mathrm{GL}(R/I)$. By the same trick, we get a crossed module

$$\mathrm{St}(R, I) \to \mathrm{GL}(R, I)$$

which has cokernel $K_1(R, I)$ and kernel $K_2(R, I)$.[32]

2.7 The category of categories internal to groups

In this section, we study another category equivalent to $\mathsf{XMod}/\mathsf{Groups}$, namely the category of categories internal to groups, written $\mathsf{Cat}[\mathsf{Groups}]$. This category has easy generalisations both to higher dimensions and to other algebraic settings.

This category has two features that make it very interesting. On the one hand it can be used as an intermediate step to get a simplicial equivalent of crossed modules which can be generalised to crossed n-cubes.[33]

On the other hand, we shall see that the category $\mathsf{Cat}[\mathsf{Groups}]$ is formed by groupoids, being also the category of group-groupoids. This will be generalised in Chapter 6 to an equivalence from the category XMod of crossed modules over groupoids to a category of double groupoids, and then to higher dimensions in Chapter 13.

First, let us recall that the definition of a small category C is given by two sets, the object set, $\mathrm{Ob}\,\mathsf{C}$, and the morphism set, $\mathrm{Mor}\,\mathsf{C}$, and four maps, the identity i, the source and target s, t, and the composition of morphisms $\circ$, satisfying several axioms. Note that $\circ$ is considered as a function on its domain.

We say that C is a *category internal to* Groups, if both $\mathrm{Ob}\,\mathsf{C}$ and $\mathrm{Mor}\,\mathsf{C}$ have a group structure and the maps s, t, i and $\circ$ are homomorphisms of groups. Thus, a category internal to Groups is also a *group in the category of all small categories*. This principle for algebraic structure that 'an $\mathcal{A}$ in a $\mathcal{B}$ is also a $\mathcal{B}$ in an $\mathcal{A}$' is of wide applicability.

Similarly, a functor $f : \mathsf{C} \to \mathsf{C}'$ between two categories is a pair of maps $\mathrm{Ob}\, f$ and $\mathrm{Mor}\, f$ commuting with the structure maps (source, target, identity and composition).

A functor between categories internal to Groups is a *functor internal to* Groups if both maps are homomorphisms of groups.

Then, $\mathsf{Cat}[\mathsf{Groups}]$ is the category whose objects and morphisms are categories and functors internal to Groups.

For any object C in $\mathsf{Cat}[\mathsf{Groups}]$, we will write the product in $\mathrm{Mor}\,\mathsf{C}$ additively and the product in $\mathrm{Ob}\,\mathsf{C}$ multiplicatively. Then, if 1 and 0 are the identities in $\mathrm{Ob}\,\mathsf{C}$ and $\mathrm{Mor}\,\mathsf{C}$, we have $i(1) = 0$, $s(0) = 1$ and $t(0) = 1$. So, the elements of $\mathrm{Ker}\, s$ (resp. $\mathrm{Ker}\, t$) are the morphisms with source 1 (target 1).

The next property shows that, for any category internal to Groups, we can define the composition of morphisms in terms of the other structure maps.

Proposition 2.7.1. *For any two composable morphisms u and v we have*

(i) $v \circ u = v - itu + u = v - isv + u,$

(ii) $v \circ u = u - itu + v = u - isv + v.$

Proof. (i) We have

$$v \circ u = (v + 0) \circ (itu + (-itu + u)) = (v \circ itu) + (0 \circ (-itu + u)) = v - itu + u.$$

The second equality is immediate: $itu = isv$ because the morphisms are composable.

The proof of (ii) is similar. $\square$

Remark 2.7.2. Thus, to prove that a category where the objects and morphisms are groups, and the source, target and identity are homomorphisms, is internal to groups, all we need to check is that the composition defined using Proposition 2.7.1 is a homomorphism. $\square$

Proposition 2.7.3. *A category internal to groups is a groupoid, with the inverse of a morphism u given by*

$$u^{-1} = isu - u + itu.$$

Proof. You can easily check that this definition gives the appropriate source and target and that both compositions $u \circ u^{-1}$, $u^{-1} \circ u$ are identities. $\square$

Remark 2.7.4. As a consequence of this property, a category internal to groups is a groupoid internal to groups, or, equivalently, is a group in the category of groupoids.[34]

Considering that a group is just a groupoid with only one object, it is reasonable to study the category of 'groupoids of groupoids', or 'double groupoids'. We shall do this in Chapter 6. $\square$

To end this section, we state the relation of Cat[Groups] to the previous categories. The equivalence with Cat^1- Groups is easily defined.

In one direction, we assign to the cat^1-group (G, s, t) the category having $\operatorname{Im} s = \operatorname{Im} t$ as set of objects, G as set of morphisms, s and t as source and target, identity the inclusion $\operatorname{Im} s \subseteq G$ and composition defined by $g' \circ g = g' - itg + g$, for any $g, g' \in G$ with $tg = sg'$. It can be easily checked that this gives a category internal to Groups.

In the other direction, to any category C internal to Groups we assign the cat^1-group $(\operatorname{Mor} \mathsf{C}, i \circ s, i \circ t)$.

Thus, the categories XMod/Groups and Cat[Groups] are equivalent, since both are equivalent to Cat^1- Groups.[35] However, it is convenient to record for further use the functors giving this equivalence.

The functor one way is defined as $\mathsf{C} \mapsto (s| \colon \operatorname{Ker} t \to \operatorname{Ob} \mathsf{C})$, where C is a cat^1-group. The reverse functor assigns to any crossed module $\mathcal{M} = (\mu \colon M \to P)$ the category having P as set of objects, $P \ltimes M$ as set of morphisms; identity map given by the inclusion; source and target maps given by $s(g, m) = g$ and $t(g, m) = g(\mu m)$ and composition given by any of the formulae in Proposition 2.7.1.

Nevertheless, there is a simpler expression for the composition in this case. Notice first that two morphisms $(g', m'), (g, m) \in P \ltimes M$ are composable when $g\mu m = g'$.

Proposition 2.7.5. *The composition of morphisms in $P \ltimes M$,*

$$\circ \colon P \ltimes M \ {}_s\!\times_t P \ltimes M \to P \ltimes M$$

is given by $(g(\mu m), m') \circ (g, m) = (g, mm')$.

Proof. This is not difficult to prove using the definition of composition given in Proposition 2.7.1 (i). □

With this property we can get another model of the category internal to Groups associated to a crossed module.

Proposition 2.7.6. *The map $A \colon \operatorname{Mor} \mathsf{C} \ {}_s\!\times_t \operatorname{Mor} \mathsf{C} \to P \ltimes M \ltimes M$ defined by*

$$A((g', m'), (g, m)) = (g, m, m')$$

is an isomorphism carrying the composition to the map

$$\circ' \colon P \ltimes M \ltimes M \to P \ltimes M$$

sending (g, m, m') to (g, mm').

Proof. Clearly A is bijective and transforms the composition to the afore mentioned map. It remains to check that A is a homomorphism and that is left as an exercise. □

Let us consider now the composite functor

$$\mathsf{Fib} \to \mathsf{Cat}^1\text{-}\mathsf{Groups} \to \mathsf{Cat}[\mathsf{Groups}]$$

i.e., mapping $\mathcal{F}$ first to the cat^1-group $\pi_1(E \times_X E)$ and then to the category internal to Groups associated to that.

Using the isomorphism $\operatorname{Im} p_{i*} \cong \pi_1(E)$, this category is isomorphic to the category that has $\pi_1(E)$ as objects, $\pi_1(E \times_X E)$ as morphisms, source and target given by projections, identity given by the diagonal and composition the only one possible to make this a category internal to groups.

As seen before, this category is also isomorphic to the one associated to $\pi_1(E) \ltimes \pi_1(F)$, that has $\pi_1(E)$ as objects, $\pi_1(E) \ltimes \pi_1(F)$ as morphisms, $([\alpha], [\mu]) \mapsto [\alpha]$ and $([\alpha], [\mu]) \mapsto [\alpha] * i_*([\mu])$ as source and target maps and composition given by

$$([\alpha] * i_*[\mu], [\mu']) \circ ([\alpha], [\mu]) = ([\alpha][\mu * \mu']).$$

We finish by stating a description of the composition in $\pi_1(E \times_X E)$.

Proposition 2.7.7. *Let $[(\alpha, \beta)], [(\beta', \gamma')] \in \pi_1(E \times_X E)$ be such that $[\beta] = [\beta']$,i.e. there is a homotopy $G\colon \beta' \cong \beta$. Since p is a fibration there is a homotopy H lifting pG and starting with γ'. Then*

$$[(\beta', \gamma')] \circ [(\alpha, \beta)] = [(\alpha, H_1)].$$

Proof. [36] It is clear that $[(\beta', \gamma')]$ and $[(\beta, H_1)]$ are homotopic using the homotopy (G, H). Then, $[(\beta', \gamma')] \circ [(\alpha, \beta)] = [(\beta, H_1)] \circ [(\alpha, \beta)]$. So, we only have to consider the composition in the case $[(\beta, \gamma)] \circ [(\alpha, \beta)]$. Using that $\mathcal{F}$ is a fibration there are unique $[\mu], [\mu'] \in \pi_1(F)$ with

$$[(\alpha, \beta)] = A([\alpha], [\mu]) = [(\alpha * ct, \alpha * \mu)]$$

and

$$[(\beta, \gamma)] = A([\beta], [\mu']) = [(\beta * ct, \beta * \mu')].$$

Clearly, $[\beta] = [\alpha] * i_*([\mu])$, and

$$\begin{aligned}
[(\beta, \gamma)] \circ [(\alpha, \beta)] &= A([\beta], [\mu']) \circ A([\alpha], [\mu]) \\
&= [(\beta * ct, \beta * \mu')] \circ [(\alpha * ct, \alpha * \mu)] \\
&= [(\alpha * ct, \alpha * \mu' * \mu)] \\
&= [((\alpha * ct) * ct, \alpha * \mu' * \mu)] \\
&= [(\alpha * ct, \beta * \mu')] \\
&= [(\alpha, \gamma)].
\end{aligned}$$

□

We can also describe easily the functor

$$\mathsf{Maps} \to \mathsf{Cat}[\mathsf{Groups}].$$

Notice that $\pi_1(\overline{Y})$ is isomorphic to $\pi_1(Y)$ under the projection. So the associated category internal to groups is equivalent to the one having $\pi_1(Y)$ as objects, $\pi_1(\overline{Y} \times_X \overline{Y})$ as morphisms, source and target given by $[(\alpha, \mu, \beta)] \to [\alpha]$ and $[(\alpha, \mu, \beta)] \to [\beta]$, and composition given by $[(\beta, \mu', \gamma)] \circ [(\alpha, \mu, \beta)] = [(\alpha, \mu' * \mu, \gamma)]$.

Note that if ν is a homotopy from β to β', the composition of $[(\alpha, \mu, \beta)]$ with $[(\beta', \mu', \gamma)]$ is given by $[(\alpha, \mu' * \nu * \mu, \gamma)]$ since $[(\beta', \mu', \gamma)] = [(\beta, \mu' * \nu, \gamma)]$.

Notes

12 p. 32 The method in Section 10.3.ii of constructing inductively a universal cover and its contracting homotopy has been developed by Ellis into a substantial GAP package for computing the homology of groups, [Ell04].

13 p. 38 The standard axioms for such an action are:

$$m^1 = m, \quad (mn)^p = m^p n^p, \quad m^{pq} = (m^p)^q$$

for all $p, q \in P$, $m, n \in M$. In such case M is also called a P-group. Note that P itself is also a P-group with action given by conjugation. Later we will need actions of a groupoid on a family of groups.

14 p. 38 The earliest statement of the axiom CM2 known to us is as footnote 25 on page 422 of [Whi41a], and therefore we call this the Whitehead axiom. The definition of crossed module appeared in [Whi46]. Related ideas were in [Pei49], submitted in 1944.

15 p. 41 See books on combinatorial group theory, for example [LS01], [Joh97] and also [Hig71], or [Bro06], 10.8.2, for a groupoid proof.

16 p. 41 Note that if $A = \{x\}$ then this action is trivial, and so this action does not include the action of $\pi_1(X, x)$ on $\pi_2(X, x)$. Whitehead proposed a more general operation to include both cases in [Whi48]. This area was then developed in [Hu48]. The former paper also includes important results on automorphisms of crossed modules which have been developed in [Lue79], [Nor90], [BG89a], [Bİ03a].

17 p. 42 Whitehead's work on free crossed modules parallelled independent work by Reidemeister and his student Renee Peiffer at about the same time on the closely related notion of identities among relations [Rei49], [Pei49], which we deal with in Section 3.1. Whitehead also acknowledged in [Whi46] that some of his results on second homotopy groups were also obtainable from work of Reidemeister, [Rei34], on universal coverings and chain complexes with operators,

now recognised as given by the complex of cellular chains of the universal cover of the space, and which has been extensively used for example in simple homotopy theory, [Coh73]. Relations between crossed modules and chain complexes are discussed in our Section 7.4, developing work of Whitehead in [Whi49b], and related to the cellular chains of universal covers in Section 8.4.

18 p. 42 There is a general statement of such kind of problem as follows. Let $T : \mathsf{A} \to \mathsf{B}$ be a functor. One realisation problem is that for objects: characterise the objects b of B are isomorphic in B to some $T(a)$ where a is an object of A? Another realisation problem is for morphisms: let $f : T(a) \to T(a')$ be a morphism in B where a, a' are objects of A. Determine whether or not $f = T(g)$ for some morphism $g : a \to a'$ in A. These problems are of interest in the case when T is a functor from a category of topological data to one of algebraic data. For further discussion of these and related problems see the expository article [Ste72].

19 p. 42 Developments of these ideas are seen in [BH82], [Hue09] and the references there. The result is used in [Pap63] in relation to the Poincaré Conjecture. Most texts on algebraic topology and homotopy theory opt out of giving a proof. Among other proofs of Whitehead's theorem we mention [Rat80], [GH86]. The fact that this result does not fit into the usual approach to algebraic topology should be seen as an anomaly, needing correction.

20 p. 44 Example applications are given in [KFM08], [FM09].

21 p. 46 These improvements were originally shown by Razak Salleh in his thesis [RS76].

22 p. 47 This is an important standard theorem in algebraic topology, first proved in [Whi49a], and is found in many texts on homotopy theory or algebraic topology. The proof is outlined in Exercises 7–10 of [Bro06], Section 7.6. The theorem is put in the general context of model categories for homotopy theory as Theorem B.8.1 in Section B.8.

23 p. 48 This result is originally due to Mac Lane and Whitehead [MLW50] (they use the term 3-type for what later came to be called 2-type), and with a different proof which uses essentially the notion of free crossed module. It is generalised by Eilenberg–Mac Lane [EML50], Ando [And57] who consider the cohomological invariant, often called Postnikov invariant, or k-invariant, for a space X with $\pi_i(X) = 0$ for $1 < i < n$, $n \geqslant 2$. See also Section 12.7.

24 p. 48 The simplicial construction is given in [BH91], see also [Bro99]. Two recent papers using crossed modules are [FM09], [KFM08]. A general background to crossed modules is in [Jan03].

25 p. 48 See [Bro82], [Bro96], and the use of the term ‘2-group’ in for example [BL04] for related and also more general constructions of 2-dimensional versions of groups.

26 p. 49 For a discussion of normal sub crossed module, see [Nor90], and a web search will give further references.

27 p. 49 Some of these equivalences were already known to Verdier in the late 60s, but the first published account seems to have been by Brown and Spencer in 1976 [BS76b]. Later, these equivalences have been generalised by Porter, [Por87], to a more categorical setting. A paper on the general setting for this kind of result is [Jan03]. One of the advantages is the naturality of the generalisation to higher dimensions, namely to cat^n-groups, and in this way was used by Loday in [Lod82]. These objects have been shown by Ellis and Steiner in [ES87] to be equivalent to what they call ‘crossed n-cubes of groups’, a rich algebraic structure.

28 p. 53 This fundamental cat^1-group of a fibration has been generalised by several authors, and the most general is the ‘fundamental double groupoid of a map’, [BJ04]. This process is the first step in constructing the *fundamental cat^n-group* of an n-cube of maps, [BL87].

29 p. 54 These ideas are well covered in books on abstract homotopy theory, for example [KP97].

30 p. 54 We follow ideas of Gilbert in [Gil87]. A general construction of a double groupoid on a map of spaces (without base point) is given in [BJ04], which also takes care to show the precise relation between the description of the composition given by the algebra and that given by the geometry.

31 p. 55 This is a special case of the results of [BH70]. The dual of this result, namely a ‘gluing theorem’, is proved in [Bro06], 7.5.7, and in an abstract setting in [KP97], Theorem 7.1.

32 p. 56 This is Loday’s definition of the relative K_2 [Lod78]. It differs from that of Milnor’s in [Mil71] by relations corresponding to those of the second rule CM2) for a crossed module. One advantage of this procedure is the generalisation to multirelative groups $K_2(R; I_1, \dots, I_n)$ [GWL81], [Ell88a]. The relevant algebra is that of crossed n-cubes of groups. All this was one of the motivations for the Seifert–van Kampen Theorem for n-cubes of spaces [BL87], based on Loday’s subtle notion of the fundamental cat^n-group of an n-cube of spaces, described in [Lod82] and elucidated further in [BL87], [Gil87].

33 p. 56 A way of deriving crossed n-cubes of groups from simplicial groups has been given by T. Porter in [Por93].

34 p. 57 This has led such structures being called 2-*groups*, but this term is also used for related structures but with a weakening of the axioms. See for example [BL04]. The notions of internal category and groupoid have proved important. See for example [BJ01], [MLM94].

35 p. 58 An example of groupoid object in groups is the fundamental groupoid of a topological group. This example is considered in [BS76b], and is also used in [BM94] to analyse in the nonconnected case topological groups which are also covering spaces of a given topological group. The theory here turns out to be related to the extension theory of groups given in Chapter 12. This equivalence between crossed modules and category objects in groups should also be considered in the more general light of the papers [Jan03], [MM10].

36 p. 59 This proof is related to a proof in [BJ04] which verifies that in the construction of a homotopy double groupoid of a map of spaces, a composition defined geometrically agrees with that derived from Generalised Galois Theory, [BJ01]. Such a verification is also necessary in the notion of the fundamental cat^n-group of an n-cube of maps, compare [BL87], [Gil87].

Chapter 3
Basic algebra of crossed modules

In this chapter we analyse what historically was the second source of crossed modules over groups: identities among relations in presentations of groups. This also leads to the area of *crossed resolutions of groups, and groupoids* with which we deal in Chapter 10.

A central problem in mathematics is the representation of infinite objects in manipulable, and preferably finite, terms. One method of doing this is by what is called a *resolution*. There is not a formal definition of this, but we can see several examples.

This notion first arose in the 19th century study of invariants. *Invariant theory* deals with subalgebras of polynomial algebras $\Lambda = k[x_1, \dots, x_n]$, where k is a ring. Consider for example, the subalgebra A of $\mathbb{Z}[a, b, c, d]$ generated by

$$a^2 + b^2, \quad c^2 + d^2, \quad ac + bd, \quad ad - bc.$$

It is called an *invariant subalgebra* since it is invariant under the action of $\mathbb{Z}_2$ which switches the variables a, b and c, d. As pointed out in [Gar80], p. 247, 'these generators satisfy the relation

$$(ac + bd)^2 + (ad - bc)^2 = (a^2 + b^2)(c^2 + d^2)$$

which is classically called a *syzygy*, and the algebra A of invariant polynomials turns out to be the homomorphic image of the polynomial algebra in four variables given by the quotient algebra

$$\mathbb{Z}[x, y, z, w]/(z^2 + w^2 - xy).$$

In particular, the algebra is finitely generated by four explicit polynomials, and the ideal of relations is finitely generated by a single explicit relation.'

Hilbert solved also the so-called second main problem of invariant theory, in showing that the ideal of relations among the invariants was also finitely generated.[37] In [Gar80], p. 253–254, we have:

> Since the second main problem had succumbed so easily, it was natural to turn to chains of syzygies, studying relations among the generating set of relations and so on. More precisely, this work involved the sequence of finitely generated $k[x_1, \dots, x_n]$-modules

$$\begin{array}{l}
0 \longrightarrow J_1 \longrightarrow F_1 \longrightarrow B_1 \longrightarrow 0, \\
0 \longrightarrow J_2 \longrightarrow F_2 \longrightarrow J_1 \longrightarrow 0, \\
\qquad \dots \qquad \dots \qquad \dots \\
0 \longrightarrow J_q \longrightarrow F_q \longrightarrow J_{q-1} \longrightarrow 0,
\end{array}$$

where the F_i are free with rank equal to the minimal number of generators of the i'th syzygy J_i. Hilbert's main theorem on the chains of syzygies says that if k is a field then $J_q = 0$ if $q > n$. In effect, this launched the theory of homological dimension of rings.

It was also natural to splice the morphisms $F_q \to J_{q-1} \to F_{q-1}$ together to get a sequence

$$\cdots \xrightarrow{\partial_{q+1}} F_q \xrightarrow{\partial_q} F_{q-1} \xrightarrow{\partial_{q-1}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} B_1$$

which was *exact* in the sense that

$$\operatorname{Ker} \partial_q = \operatorname{Im} \partial_{q+1}$$

for all q. This sequence was called a *free resolution* of the module B_1.

A basic question was the dependence of this sequence on the choices made. It was found that given any two such free resolutions $F_* \to B_1$, $F'_* \to B_1$, then there was a morphism $F_* \to F'_*$ and any two such morphisms were *homotopic*. It was also later found that the condition 'free' could conveniently be replaced by the condition *projective*.

Another source for homological algebra was the homology and cohomology theory of groups. As pointed out in [ML78], the starting point for this was the 1942 paper of Hopf [Hop42]. Let X be an aspherical space (i.e. connected and with $\pi_i X = 0$ for $i > 1$), and let

$$1 \to R \to F \to \pi_1 X \to 1$$

be an exact sequence of groups with F free. Hopf proved the formula

$$H_2 X \cong \frac{R \cap [F, F]}{[F, R]}.$$

We shall see in Section 5.5 that this formula follows from our 2-dimensional Seifert–van Kampen Theorem for crossed modules. Thus we see the advantage for homotopy theory of having a 2-dimensional algebraic model of homotopy types. (A higher dimensional version of Hopf's result appears in Proposition 8.3.21).

Later work of Eilenberg–Mac Lane [EML47] found an algebraic formula for $H_n X$, $n \geqslant 2$ as follows. Produce sequences of $\mathbb{Z}G$-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J_1 & \longrightarrow & F_1 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0, \\
0 & \longrightarrow & J_2 & \longrightarrow & F_2 & \longrightarrow & J_1 & \longrightarrow & 0, \\
 & & \cdots & & \cdots & & \cdots & & \\
0 & \longrightarrow & J_q & \longrightarrow & F_q & \longrightarrow & J_{q-1} & \longrightarrow & 0,
\end{array}$$

in which $\mathbb{Z}$ is the trivial $\mathbb{Z}G$-module, and each F_n is a free $\mathbb{Z}G$-module. Splice these together to give a *free resolution of* $\mathbb{Z}$:

$$F_*\colon \cdots \to F_n \to F_{n-1} \to \cdots \to F_2 \to F_1 \to \mathbb{Z}.$$

Form the chain complex $C = F \otimes_{\mathbb{Z}G} \mathbb{Z}$. Then $H_n C \cong H_n X$. Using particular choices of the F_n, the Hopf formula may be deduced [Bro94], p. 46.

Thus we see an input from the homotopy and homology theory of spaces into the development of homological algebra. The use of homological methods across vast areas of mathematics is a feature of 20th century mathematics. It seems the solution of Fermat's last theorem depended on it, but it has also been applied in differential equations, coding theory and theoretical physics.

In its 20th century form, homological algebra is primarily an abelian theory. There is considerable work on nonabelian homological algebra, but this is only beginning to link with work in homotopical algebra, differential topology, and related areas. This book has an aim of showing one kind of start to a more systematic background to such an area.

Now the elementary, computational and example-oriented approach to groups considers presentations $\langle X; R\rangle$ of a group P: that is X is a subset of P and there is an exact sequence

$$1 \longrightarrow N \longrightarrow F(X) \xrightarrow{\ \phi\ } P \longrightarrow 1, \tag{$*$}$$

where $F(X)$ is the free group on generators $[x]$, $x \in X$; p is defined by $p[x] = x$, $x \in X$; and R is a set of generators of N as normal subgroup of $F(X)$. Thus, each element of N is a *consequence*

$$c = (r_1^{\varepsilon_1})^{u_1} \dots (r_n^{\varepsilon_n})^{u_n},$$

$r_i \in R$, $\varepsilon_i = \pm 1$, $u_i \in F(X)$ and $a^b = b^{-1}ab$. However, this representation of elements of N, and the persistent use of N and $F(X)$ as nonabelian groups, rather than of modules derived from them, plays a small role in the homological algebra of groups. One would expect, *a priori*, that the sequence $(*)$ would be the beginning of a 'nonabelian resolution' of the group P. We will show that this is so in Chapter 10.

Another curiosity is that there are a number of results in homotopy theory which are satisfactory for 1-connected spaces, but for which no formulation has been given when this assumption has been dropped, particularly when some nonabelian group has to be described. As long as interest was focussed on high-dimensional, or stable, problems, this restriction seemed not to matter. In many problems of current interest (for example low-dimensional topology and homology of groups, algebraic K-theory) this restriction has proved irksome, but few appropriate constructions have been generally seen to be available. This is one of the reasons for promoting the subject matter of this book.

In Section 3.1 we recall what is a presentation $\langle X \mid \omega\rangle$ of a group P, and show that the 'identities among the relations' can be seen as the elements of the kernel of a morphism $\theta\colon F(R \times P) \to P$ which satisfies CM1) in the definition of crossed modules.

This gives good reason to relax the concept of crossed module. In Section 3.3 we define precrossed modules in terms of axiom CM1) and also the functor that associates to every precrossed module a crossed module. This construction $(-)^{\mathrm{cr}}$ is adjoint to the inclusion of categories $\mathsf{XMod}/\mathsf{Groups} \hookrightarrow \mathsf{PXMod}/\mathsf{Groups}$.

The morphism $\theta\colon F(R \times P) \to P$ has some extra freeness properties, making it what is called a 'free precrossed module'. These are studied in Section 3.4.

The chapter ends with the definition of a category of algebraic objects equivalent to that of precrossed modules and generalising the equivalence defined in Section 2.5.

3.1 The context of presentation of groups and identities among relations

3.1.i Introduction

We now show how crossed modules arise in combinatorial group theory.[38]

A group G is of course defined as a set with a multiplication satisfying certain axioms. In some cases this multiplication can be specified by a formula involving the elements: notable examples are certain matrix groups, such as the Heisenberg group H of matrices of the form

$$\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$$

for $x, y, z \in \mathbb{Z}$. Thus the elements of H are given by triples (x, y, z) of integers with multiplication

$$(x, y, z)(u, v, w) = (x + u, y + v + xw, z + w).$$

This is known as a 'polynomial group law'. So we have a formula for the elements of the group H and for the multiplication.

The reader should not be surprised that this could raise difficulties in other cases. Part of the problem may be to give a useful formula for the elements of the group, let alone a formula for the multiplication. In mathematics as a whole, the question of 'presenting' information on a structure is often a key part of a problem.

An often useful way of representing the elements of a group is by giving generators for the group.

Example 3.1.1. Let D_8 be the dihedral group of order 8, i.e. the group of symmetries of the square. This group is generated by the elements x, y where x is rotation anticlockwise through 90° and y is reflection in a vertical bisector of the square. The elements of D_4 can then be written as

$$1, \quad x, \quad x^2, \quad x^3, \quad y, \quad yx, \quad yx^2, \quad yx^3$$

and this is quite a convenient labeling of the elements. However if you try to work out the multiplication table in terms of this labeling you find you need more information, namely *relations* among the generators, for example

$$x^4 = 1, \quad y^2 = 1, \quad xyxy = 1.$$

If you are not already familiar with these, particularly the last one, then you are expected to verify them using some kind of model of a square. It turns out that every relation you might need in working out the multiplication table is a consequence only of these three. Thus we can specify the group completely also in terms of what we call a 'presentation'

$$\mathcal{P} = \langle x, y \mid x^4,\ y^2,\ xyxy \rangle.$$

We shall write $D_8 = \operatorname{gp}\mathcal{P}$. □

We need a definition of the concept of a presentation. The first thing to note is that the term x^4 in the presentation $\mathcal{P}$ is not an element of the group D_8, since the 4th power of the element x in D_8 is 1. Rather, as is common with the mathematical use of $=$, one side of the $=$ sign in $x^4 = 1$ is in fact an instruction: 'multiply x by itself 4 times', while the other side tells you what will be the result. A convenient language to express both an 'instruction for a procedure' and the result of the procedure is that of a morphism defined on a free group.

A *free group* $F(X)$ on a set X is intuitively a group $F(X)$ together with an inclusion mapping $i: X \to F(X)$ such that X generates the group $F(X)$ and 'there are no relations among these generators'. There are two useful ways of expressing this precisely.

One of them is to give what is called a 'universal property': this is that a morphism $g: F(X) \to G$ to a group G is entirely determined by its values on the set X. Put in another way, given any function $f: X \to G$, there is a unique morphism $g: F(X) \to G$ such that $gi = f$. This 'external' definition thus defines a free group by its relation to all other groups, and is a model for the notion of 'freeness' in other algebraic situations. A set X generating a free group plays a rôle similar to that of a basis for a vector space, and we also talk about X as a basis for the free group $F(X)$. However, unlike vector spaces, not every group is free. The simplest example is the group $\mathbb{Z}_2$ with two elements: it is not free because there is only one morphism $\mathbb{Z}_2 \to \mathbb{Z}$, the zero morphism.

The other 'internal' way of specifying a free group is to specify its elements and the multiplication, and this can be done in terms of 'reduced words': every nonidentity element of $F(X)$ is uniquely expressible in the form

$$x_1^{r_1} x_2^{r_2} \dots x_n^{r_n} \tag{3.1.1}$$

where $n \geqslant 1$, $x_i \in X$, $r_i \in \mathbb{Z}$, $r_i \neq 0$, and for no i is $x_i = x_{i+1}$, i.e. no cancelation in the expression (3.1.1) is possible. In this specification, work is needed to give the multiplication since adjoining two reduced words often yields a nonreduced word, and the reduction process has to be given.[39] Reduced words are commonly used to store elements of a free group in computer implementations of combinatorial group theory.

We assume now that we have free groups, and this allows us to give our first definition of a presentation of a group.

Definition 3.1.2. A *presentation* $\mathcal{P} = \langle X \mid R \rangle$ of a group P consists of a set X and a subset R of the free group $F(X)$ together with a surjective morphism $\phi\colon F(X) \to P$ such that $\operatorname{Ker}\phi$ is the normal closure in $F(X)$ of the set R.

We also write $P = \operatorname{gp}\mathcal{P}$. □

We explain in more detail the notion of normal closure, since this gives a useful model of an important general process. First recall that for any normal subgroup $K \lhd P$, the group P acts on the group K by conjugation. A basic aspect of group theory is that a normal subgroup is a kernel of a morphism (in this case, for example, of the quotient morphism $P \to P/K$), and that the kernel of any morphism from P to a group is normal in P.

If R is a subset of the group P then the *normal closure* $\langle R \rangle^P$ of R in P is the smallest normal subgroup of P containing R. The elements of $\langle R \rangle^P$ are all *consequences* of R in P, namely all products

$$c = (r_1^{\varepsilon_1})^{p_1} \dots (r_m^{\varepsilon_m})^{p_m} \tag{3.1.2}$$

where $r_i \in R$, $\varepsilon_i = \pm 1$, $p_i \in P$ and $m \geqslant 1$. An important point is that if $\phi\colon P \to Q$ is any morphism to a group Q such that $\phi(R) = \{1\}$, then $\phi(\langle R \rangle^P) = \{1\}$, since $\operatorname{Ker}\phi$ is normal. Thus ϕ factors as $P \to P/\langle R \rangle^P \to Q$ where the first morphism is the quotient morphism.[40]

Now we can see that there might be what are called *identities among relations*.[41] Intuitively, such an identity is a 'formal' product such as (3.1.2) which is 1 when evaluated in the group P. This is formalised in Definition 3.2.1. Here we consider some examples.

Example 3.1.3. For any elements r, s of R, we have the identities

$$\begin{aligned} r^{-1}s^{-1}rs^r &= 1, \\ rs^{-1}r^{-1}s^{(r^{-1})} &= 1. \end{aligned}$$

These identities hold always, whatever R. □

Example 3.1.4. Suppose $r \in R$, $p \in P$ and $r = p^m$, $m \in \mathbb{Z}$. Then $rp = pr$, i.e. p belongs to the centraliser $C(r)$ of r in P. We have the identity

$$r^{-1}r^p = 1. \tag{3.1.3}$$

It is known that if the group P is free and $r \in R$ then there is a unique element p of P such that $r = p^m$ with $m \in \mathbb{N}$ maximal and then $C(r)$ is the infinite cyclic group generated by p. This element p is called the *root* of r and if $m > 1$ then r is called a *proper power*. □

Example 3.1.5. Suppose the commutators $[p, q] = p^{-1}q^{-1}pq$, $[q, r]$, $[r, p]$ lie in R. Then the well-known rule

$$[p, q][p, r]^q\,[q, r][q, p]^r\,[r, p][r, q]^p = 1 \tag{3.1.4}$$

is an identity among the relations of the presentation, since $[q, p] = [p, q]^{-1}$. □

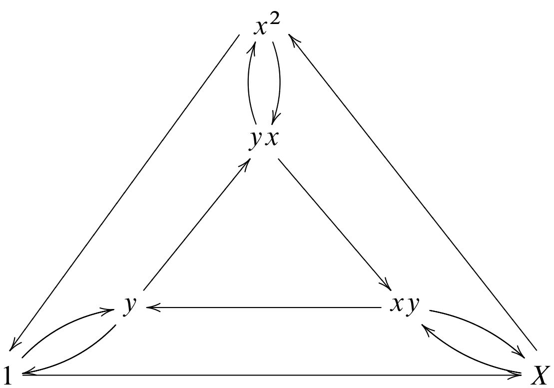
Example 3.1.6. Let S_3 be the symmetric group on three letters with presentation $\langle x, y \mid r, s, t\rangle$ where $r = x^3$, $s = y^2$, $t = xyxy$. The fact that each relation is a proper power gives rise to three identities among relations, namely

$$r^{-1}r^x, \quad s^{-1}s^y, \quad t^{-1}t^{xy}.$$

However there is also a fourth identity namely

$$(s^{-1})^{x^{-1}}ts^{-1}(r^{-1})^{y^{-1}}t^x(s^{-1})^x r^{-1}t^{x^{-1}}.$$

We leave it to you to verify that this is an identity among relations by writing out the formula in the free group on x, y. This identity can also be interpreted as a kind of composition of 2-cells in the following picture:



We shall discuss this a bit more in the next section in terms of van Kampen diagrams. The general notion of ‘composition of 2-cells’ makes more sense with our discussion of computing identities among relations in Section 10.3.ii. □

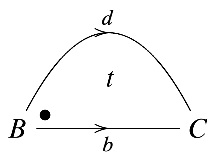
3.1.ii Van Kampen diagrams

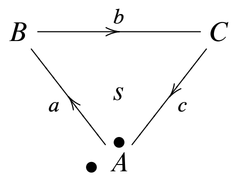
These diagrams give a geometric method of deducing consequences of relations, and can, as we shall see, be used to show exactly how to write a word as a consequence of the relations. We do not give a general definition or description, but illustrate it with examples. The idea has been used extensively in some sophisticated theorems in combinatorial group theory. For our purposes, the idea illustrates geometric aspects of the use of crossed modules.

The idea of these diagrams come from the fact that a relation in a presentation can be represented by a based cell whose sides are labeled by the letters of the relation in such way that when they are read clockwise from the base point we get the relation.

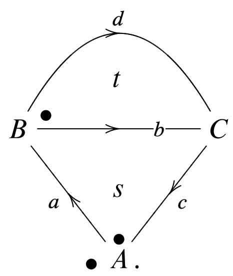
Then, we can get new relations by gluing two or more of these cell along some common sides. Let us consider a simple case.

Example 3.1.7. Suppose for a given presentation we have the relations $t = db^{-1}$ and $s = abc$. They can be represented as based cells as follows:





We write $\delta s = abc$, $\delta t = db^{-1}$. Now, we glue t and s alongside b getting



The boundary of this new cell is

$$adc = abc \cdot c^{-1}b^{-1} \cdot db^{-1} \cdot bc = (\delta s)(\delta(t^{bc})).$$

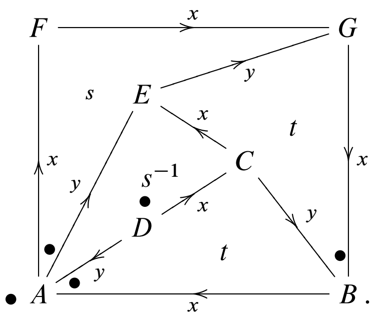
Of course t^{bc} makes sense in the context of crossed modules over groupoids, since t is based at B whereas t^{bc} is based at A. $\square$

Here is a more complex example.

Example 3.1.8. The quaternion group of order 8 may be presented in the form

$$Q_8 = \mathrm{gp}\langle x, y \mid x^4, x^2y^{-2}, y^{-1}xyx\rangle.$$

However the following diagram shows that the relation x^4 is a consequence of the other two relations. Set $r = x^4$, $s = x^2y^{-2}$, $t = y^{-1}xyx$ and consider the drawing



In this diagram, each cell has a base point, represented by a $\bullet$, which is where the reading of the boundary starts in clockwise direction. This explains why we have an s and s^{-1}, since the latter is s read counterclockwise.

Now we have to show how we can deduce from this diagram the expression we want.

We take the outside loop starting from A (which has a base point for the outside 'cell') and then change it to traverse the boundary of each internal cell, obtaining the rule which you can easily verify:

$$xxy^{-1}y^{-1} \cdot yx^{-1}x^{-1}y \cdot y^{-1}xyx \cdot x^{-1} \cdot y^{-1}xyx \cdot x = x^4.$$

This can be reread as

$$s \cdot yx^{-1}x^{-1}y \cdot t \cdot x^{-1} \cdot t \cdot x = x^4.$$

But $yx^{-1}x^{-1}y = yx^{-1}x^{-1} \cdot yyx^{-1}x^{-1} \cdot xxy^{-1} = (s^{-1})^{xxy^{-1}}$. So our final result is that

$$s \cdot (s^{-1})^{xxy^{-1}} \cdot t \cdot t^x \cdot r^{-1}$$

is an identity among relations, or, alternatively, shows in a precise way how x^4 is a consequence of the other relations. $\square$

One context for van Kampen diagrams is clarified by the notion of *shelling* of such a diagram.[42] It is a sequence of 2-dimensional subcomplexes $K_0, K_1, \dots, K_n$ each of which is formed of 2-dimensional cells, with K_0 consisting of a chosen basepoint $*$, K_1 being a 2-cell s_1 with $*$ on its boundary, and such that for $i = 2, \dots, n$, K_i is obtained from K_{i-1} by adding a 2-cell s_i such that $s_i \cap K_{i-1}$ is a nonempty union of 1-cells which form a connected and 1-connected set, i.e. a path. Such a shelling will yield a formula for the boundary of K_n in terms of the boundaries of each individual cell, provided each cell is given a base point and orientation.

Here is a clear way of getting the formula (explained to us by Chris Wensley):

Choose $* = K_0$ as base point for all the K_i. The relation for K_0 is the trivial word. If B_1 is the base point for s_1 and P_1 is the anticlockwise path around s_1 from B_1 to $*$ and w_1 is the word in the generators read off along P_1, then the relation for K_1 is $\delta({s_1}^{w_1})$. For $i \geqslant 2$, let B_i be the base point for s_i, and let U_i, V_i be the first and last vertices in the intersection $s_i \cap K_{i-1}$ met when traversing the boundary of K_{i-1} in a clockwise direction (so that the intersection is a path $U_i \ldots V_i$). Then if B_i lies on $U_i \ldots V_i$ let P_i be the path $B_i \ldots U_i \ldots *$, otherwise let P_i be the path $B_i \ldots V_i \ldots U_i \ldots *$ (traversing the boundary of s_i in an anticlockwise direction and the boundary of K_{i-1} clockwise). If w_i is the word in the generators read off along P_i then

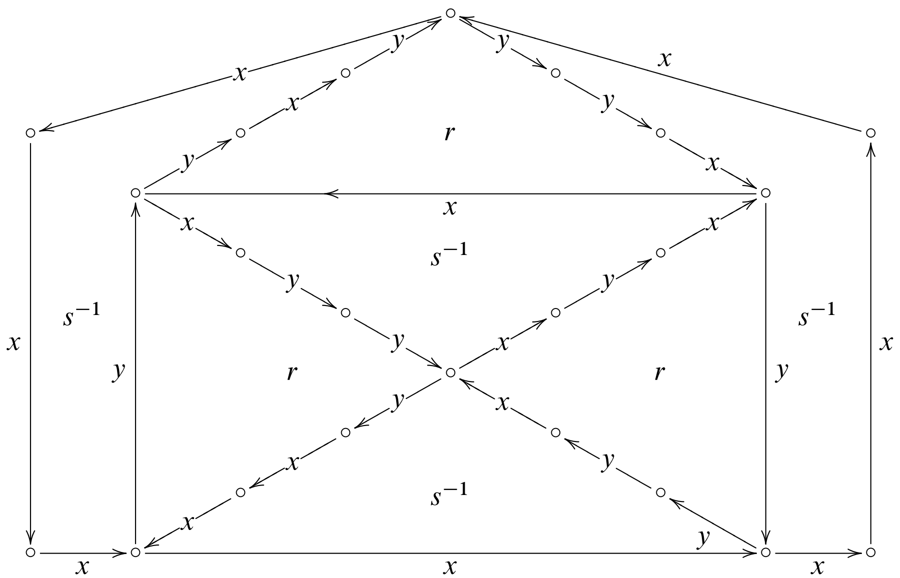
$$(\text{relation for } K_i) = (\text{relation for } K_{i-1}) \cdot \delta({s_i}^{w_i}).$$

We finish this short section with a more involved example.

Example 3.1.9. These methods can be used to prove the not obvious fact that the relations

$$r = x^2yxy^3, \quad s = y^2xyx^3$$

have x^7 as a consequence. We leave it as an exercise to prove this by inserting base points and orientations to the cells in the next picture, and then, which is harder, to give x^7 as an explicit consequence of r, s.[43]



□

Here is a more formal definition of a van Kampen diagram.

A *complete Higher Homotopy van Kampen diagram* is a finite regular CW-complex K on a compact subset of the sphere S^2. *Regularity* here means that each attaching map $f_s\colon (S^1,1)\to(K^1,K^0)$ of a 2-cell s is a homeomorphism into. By omitting one 2-cell s_∞ from K and using stereographic projection we can also regard $K\setminus s_\infty$ as a subset of the plane $\mathbb{R}^2$. The projection of $K\setminus s_\infty$ gives a planar van Kampen diagram.

Whitehead's theorem (Corollary 5.4.8) says essentially that $\Pi(K,K^1,K^0)$ is the free crossed $\pi_1(K^1,K^0)$-module on the characteristic maps of the 2-cells of K.

3.2 Presentations and identities: 2

In Section 3.1 we have discussed examples of identities among relations for a presentation of a group. Note that in all these examples conjugation is crucial. This is related to the fact that the kernel K of a morphism from a group P should be thought of not just as a subgroup K of the group P but also as a subgroup with an action of P on K. This principle, that a kernel in nonabelian situations has more structure than just that of subobject, is of general applicability; it is of direct applicability to the definition of an identity among the relations for a presentation of a group P.

Definition 3.2.1. Let R be a set and P a group. We define the *free P-group on R*, to be a P-group $F_P(R)$ and function $i\colon R\to F_P(R)$ with the universal property that for any function $f\colon R\to M$ where M is a P-group, there is a unique morphism of P-groups $f'\colon F_P(R)\to M$ such that $f'i=f$. □

Proposition 3.2.2. *The free P-group on R exists and may be constructed as the free group on $R\times P$, with action defined by $(r,p)^q=(r,pq)$ for $r\in R$, $p,q\in P$ and with $i\colon R\to F_P(R)$ defined by $r\mapsto(r,1)$.* □

The proof is easy.[44]

In discussing identities among relations and the corresponding geometric situation of adding 2-cells, it is convenient to allow for repeated relations. Another reason for this is that we may have some difficulty in recognising algorithmically that two elements of P specified say by words in generators are in fact the same. So we consider functions say $\omega\colon R\to P$. By the definition of $F_P(R)$, ω determines a unique morphism of P-groups

$$\theta\colon F_P(R)\to P$$

such that $\theta i=\omega$. Note that as a morphism of P-groups, θ satisfies $\theta(h^p)=p^{-1}\theta(h)p$ for all $h\in F_P(R)$, $p\in P$. We recognise this as rule CM1) for a crossed module. The image of θ is the normal closure $\langle\omega R\rangle^P$ of ωR in P, i.e. the group of consequences of ωR in P. The elements of $F_P(R)$ will be called the *formal consequences* of $\omega\colon R\to P$ (as against the actual consequences, which lie in P itself).

If $P=F(X)$ is the free group on X, then imposing the relations R on $F(X)$ gives a group G, say, and we have a *presentation* $\mathcal{P}=\langle X\mid\omega\rangle$ of G. Nevertheless, we keep the notation $\langle X\mid R\rangle$ whenever $R\subseteq F(X)$ and ω is the inclusion.

Our next definition is that an *identity among the relations* of $\omega\colon R \to P$ is an element of $E = \operatorname{Ker}\theta$. Equivalently, an identity among relations is a formal consequence which gives 1 when evaluated as an actual consequence in P.

The idea of specifying an identity among relations is thus analogous to that of specifying a relation as an element of the free group $F(X)$, but here we must take into account the action of $F(X)$. This leads to an appropriate concept of 'free' (see Section 3.4). However, we are not yet at our final position.

Note that because $\theta\colon F_P(R) \to P$ is a P-morphism, if $h, k \in F_P(R)$ we have

$$\theta(h^{\theta k}) = \theta(k)^{-1}\theta(h)\theta(k)$$

and so

$$k^{-1}h^{-1}kh^{\theta k} \in \operatorname{Ker}\theta.$$

We call such an element a *basic Peiffer commutator* and write[45]

$$[\![k, h]\!] = k^{-1}h^{-1}kh^{\theta(k)}.$$

These should be thought of as 'twisted commutators'.[46] The wider context for these is that of the precrossed modules of the next section.

3.3 Precrossed and crossed modules

Following the concepts introduced in Section 3.2, we need to study morphisms having the same formal properties as $\theta\colon F_P(R) \to P$. One way of describing the distinctive feature of θ is to say that θ is a morphism of P-groups, where P acts on itself by conjugation.

Definition 3.3.1. Let P be a group and M a P-group. Then a morphism of P-groups of the form $\mu\colon M \to P$ is called a *precrossed module*[47]. A *morphism* between two precrossed modules $\mathcal{M} = (\mu\colon M \to P)$ and $\mathcal{N} = (\nu\colon N \to Q)$ is a pair (g, f) of homomorphisms of groups $g\colon M \to N$ and $f\colon P \to Q$ such that

i) the diagram

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "\mu"'] & N \arrow[d, "\nu"] \\ P \arrow[r, "f"'] & Q \end{tikzcd}$$

commutes, i.e. $f\mu = \nu g$, and

ii) the actions are preserved, i.e. $g(m^p) = (gm)^{fp}$ for any $p \in P$ and $m \in M$.

These objects and morphisms define the category $\mathsf{PXMod/Groups}$ of precrossed modules and morphisms. □

Example 3.3.2. Given a map $\omega\colon R \to P$ the associated map $\theta\colon F_P(R) \to P$ defined in Section 3.2 is a precrossed module.

Remark 3.3.3. We affix the term Groups to the name for this category as later we will need an analogous category with groups replaced by groupoids, and this we will denote simply by PXMod. □

Analogously to our method in the example $\theta\colon F_P(R) \to P$ determined by $\omega\colon R \to P$, we can define Peiffer elements in any precrossed module. Let $\mathcal{M} = (\mu\colon M \to P)$ be a precrossed module and let m, m' be elements of M. Their *Peiffer commutator* is defined as[48]

$$[\![m, m']\!] = m^{-1}m'^{-1}m m'^{\mu m}.$$

The precrossed modules in which all Peiffer commutators are trivial are precisely the crossed modules. Thus the category of crossed modules is the full subcategory of the category of precrossed modules whose objects are crossed modules.

Since the Peiffer elements are always defined in a precrossed module, it is a natural idea to factor out by the normal subgroup that they generate and consider the induced map from the quotient. Let us check that this produces a crossed module.

The *Peiffer subgroup* $[\![M, M]\!]$ of M is the subgroup of M generated by all Peiffer commutators.[49] We now prove that this subgroup inherits the P-action and, as for the usual commutator subgroup, is a normal subgroup.

Theorem 3.3.4. *For any precrossed module $\mu\colon M \to P$, the Peiffer subgroup $[\![M, M]\!]$ of M is a P-invariant normal subgroup.*

Proof. The Peiffer subgroup is P-invariant since for any $m, m' \in M$ and $p \in P$, we have

$$\begin{aligned}
[\![m, m']\!]^p &= (m^{-1}m'^{-1}m m'^{\mu m})^p \\
&= (m^p)^{-1}(m'^p)^{-1}m^p m'^{(\mu m)p} \\
&= (m^p)^{-1}(m'^p)^{-1}m^p m'^{p(\mu m)^p} \\
&= (m^p)^{-1}(m'^p)^{-1}m^p m'^{p(\mu m^p)} \\
&= [\![m^p, m'^p]\!].
\end{aligned}$$

It is also normal since for any $m, m', n \in M$ we have

$$\begin{aligned}
n^{-1}[\![m, m']\!]n &= n^{-1}m^{-1}m'^{-1}m m'^{\mu m}n \\
&= n^{-1}m^{-1}m'^{-1}m(n m'^{\mu m n}(m'^{-1})^{\mu m n}n^{-1})m'^{\mu m}n \\
&= ((mn)^{-1}m'^{-1}m n m'^{\mu m n})(((m'^{\mu m})^{\mu n})^{-1}n^{-1}m'^{\mu m}n) \\
&= [\![mn, m']\!][\![n, m'^{\mu m}]\!]^{-1}.
\end{aligned}$$ □

For future computations it is interesting to have as small a set of generators of the Peiffer subgroup as possible. The following property is useful for this.

Proposition 3.3.5. *Let $\mu\colon M \to P$ be a precrossed module and let V be a subset of M which generates M as a group and is also P-invariant. Then the Peiffer subgroup $[\![M, M]\!]$ of M is the normal closure in M of the set of Peiffer commutators*

$$\{[\![a, b]\!] \mid a, b \in V\}.$$

Proof. Let Z be the normal closure of $W = \{[\![a, b]\!] \mid a, b \in V\}$. Since $[\![M, M]\!]$ is normal and contains W, it is clear that $Z \subseteq [\![M, M]\!] \subseteq \operatorname{Ker} \mu$. On the other hand W is P-invariant since $[\![a, b]\!]^p = [\![a^p, b^p]\!]$ as was proved in Theorem 3.3.4. So Z is also P-invariant. Thus μ induces a homomorphism of groups $\bar{\mu}\colon M/Z \to P$ which is P-invariant, so that we have a precrossed module. Let us check that it is also a crossed module.

Let $\bar{V}$ be the image of V in M/Z, i.e. $\bar{V}$ is the set of cosets of all elements in V. Notice that we have

$$y^{\bar{\mu}x} = x^{-1}yx = y^x \qquad (**)$$

for any x and y lying in $\bar{V}$, which is a set of generators of M/Z. It is easy to see that for a fixed x in M/Z the set P_x of y's satisfying this equation $(**)$ is a subgroup containing $\bar{V}$ so P_x has to be all of M/Z.

Consider now the set Q of x in M/Z satisfying $(**)$ for all y in M/Z. It is closed under multiplication (since

$$y^{\bar{\mu}(xx')} = (y^{\bar{\mu}x})^{\bar{\mu}x'} = (y^x)^{\bar{\mu}x'} = (y^x)^{x'} = y^{xx'}$$

and also under inversion (since if $w = y^{x^{-1}}$, we have $w^x = y$ and $w^x = x^{-1}wx$, so that $x^{-1}wx = y$ and $w = xyx^{-1}$). So $Q = M/Z$ and thus $\bar{\mu}\colon M/Z \to P$ is a crossed module. It follows that $[\![M, M]\!] \subseteq Z$. □

Corollary 3.3.6. *Let $\omega\colon R \to P$ be a function to the group P and let $\theta\colon F_P(R) \to P$ be the associated precrossed module. Then the Peiffer subgroup $[\![F_P(R), F_P(R)]\!]$ of $F_P(R)$ is the normal closure in $F_P(R)$ of the basic Peiffer elements $[\![a, b]\!] = a^{-1}b^{-1}ab^{\theta a}$ where $a, b \in R \times P$.*

Now for any precrossed module $\mu\colon M \to P$ we define

$$M^{\mathrm{cr}} = \frac{M}{[\![M, M]\!]},$$

and let $\phi\colon M \to M^{\mathrm{cr}}$ denote the quotient morphism By the previous property, M^{cr} is a P-group. Let us see that the homomorphism μ induces a crossed module.

Proposition 3.3.7. *Let $\mu\colon M \to P$ be a precrossed module. Then*

(i) *the induced map gives a crossed module*

$$\mathcal{M}^{\mathrm{cr}} = (\mu^{\mathrm{cr}}\colon M^{\mathrm{cr}} \to P),$$

(ii) *if $\nu\colon N \to P$ is a crossed module and $\alpha\colon M \to N$ is a morphism of precrossed modules over P, then α determines a unique morphism $\alpha'\colon M^{\mathrm{cr}} \to N$ such that $\alpha'\phi = \alpha$.*

Proof. (i) It is easy to see that for each $m, m' \in M$, $\mu[\![m, m']\!] = 1$, so μ induces a homomorphism of groups μ^{cr}. Clearly μ^{cr} satisfies CM1) because it was already satisfied by μ. It also satisfies CM2) because all Peiffer commutators have been quotiented out.

(ii) This follows because in a crossed module, all Peiffer commutators vanish. □

The association of the crossed module $M^{\mathrm{cr}} \to P$ to a precrossed module $M \to P$ gives a functor

$$(-)^{\mathrm{cr}} \colon \mathsf{PXMod/Groups} \to \mathsf{XMod/Groups}.$$

That is, a morphism (g, f) of precrossed modules yields a morphism (g^{cr}, f) of crossed modules, and this association satisfies the usual functorial rules. Clearly $(-)^{\mathrm{cr}}$ is left adjoint to the inclusion of categories $\mathsf{XMod/Groups} \to \mathsf{PXMod/Groups}$.

3.4 Free precrossed and crossed modules

In any algebraic structure, the notion of free structure usually plays an important role. This also for holds for the structures of precrossed modules and crossed modules. They will also be seen later in Chapter 5 as arising as induced precrossed and crossed modules, so they give an introduction to that topic.

We first note:

Proposition 3.4.1. *If P is a group, and $\omega \colon R \to P$ is a function from a set R, then the precrossed P-module $\theta \colon F_P(R) \to P$ determined by ω has with the function $i \colon R \to F_P(R)$ the property that for any precrossed P-module $\nu \colon N \to P$ and function $f \colon R \to N$ such that $\nu f = \omega$, there is a unique morphism $\alpha \colon F_P(R) \to N$ of P-groups such that $\nu\alpha = \theta$.*

Proof. The morphism α is defined by its values on the P-generators by $\alpha(r, 1) = f(r), r \in R$. The equation $\nu\alpha = \theta$ holds on $F_P(R)$ because it holds on generators. □

Because of this result we call $\theta \colon F_P(R) \to P$ with i the *free precrossed P-module on ω*.

Definition 3.4.2. Let $\omega \colon R \to P$ be a function from the set R to the group P. The *free crossed P-module on ω* is a crossed module $\partial \colon \mathrm{FX}(\omega) \to P$ with a function $i \colon R \to \mathrm{FX}(\omega)$ such that $\partial i = \omega$ and with the universal property that for any crossed module $\nu \colon N \to P$ and function $f \colon R \to N$ such that $\nu f = \omega$, there is a unique morphism $\alpha \colon \mathrm{FX}(\omega) \to N$ of crossed P-modules such that $\alpha i = f$. □

Proposition 3.4.3. *The free crossed P-module on ω exists and may be constructed as*

$$\mathrm{FX}(\omega) = F_P(R)^{\mathrm{cr}} \to P$$

together with the composite function $R \xrightarrow{i} F_P(R) \xrightarrow{\phi} \mathrm{FX}(\omega)$.

Proof. This follows easily from the universal properties already discussed. □

Example 3.4.4. Suppose $\omega\colon R \to P$ has image only the identity of P. Then $\partial\colon \mathrm{FX}(\omega) \to P$ is trivial, and $\mathrm{FX}(\omega)$, which we may write in this case as $\mathrm{FX}(R)$, is the free P-module on the set R. Its elements may be written as formal sums $\sum_{i=1}^{n} x_i (r_i)^{p_i}$ where $x_i \in \mathbb{Z}$, $r_i \in R$, $p_i \in P$, $n \geqslant 0$. The function $i\colon R \to \mathrm{FX}(R)$ is injective. □

Remark 3.4.5. This construction will be generalised to free crossed modules over groupoids in Section 7.3.ii.[50] □

From the construction of the free precrossed module as a free group, it is clear that $R \to F_P(R)$ is injective. It is not so clear that $R \to \mathrm{FX}(\omega) = F_P(R)^{\mathrm{cr}}$ is also injective. This is a consequence of the following property:

Proposition 3.4.6. *Given a free crossed P-module $\mathcal{M} = (\mu\colon M \to P)$ on $\omega\colon R \to P$ with basis $i\colon R \to M$, then M^{ab} is a free* $(\mathrm{Cok}\,\mu)$*-module with basis the composition* $R \xrightarrow{i} M \to M^{\mathrm{ab}}$.

Proof. Let $G = \mathrm{Cok}\,\mu$. We know by Proposition 2.2.4 that M^{ab} may be given the structure a G-module, derived from the crossed P-module structure on M. To see that M^{ab} is free on the given basis we will verify the appropriate universal property.

Let N be a G-module. The projection $P \times N \to P$ becomes a crossed P-module when P acts on $P \times N$ by conjugation on P and via the G-action on N. For any map $f\colon R \to N$ we define $f' = (\mu\omega, f)\colon R \to P \times N$. Since $\mu\colon M \to P$ is a free crossed P-module we get a unique morphism of P-crossed modules $\phi\colon M \to P \times N$ such that $f' = \phi\omega$. The composite $M \to N$ factors through a G-morphism $\bar{\phi}\colon M^{\mathrm{ab}} \to N$ which is the only morphism of G-modules satisfying $\bar{\phi}\omega^{\mathrm{ab}} = f$. □

Remark 3.4.7. This result can be seen later in the context of fibrations of categories and induced crossed modules. (See Section 7.3.ii.) □

3.4.i Free crossed module as an adjoint functor

As indicated in Example A.6.2 of Appendix A, a free construction in a category of algebraic structures is, in most cases the left adjoint of some forgetful functor. The appropriate forgetful functor for crossed modules goes to the category of *sets over a group* forgetting the algebra of the top group and considering only the underlying boundary map. We recall the appropriate categories.

Let P be a group. We have defined the category XMod/P of crossed P-modules in Section 2.2. In a similar way, we define the category PXMod/P by restricting to precrossed modules over P.

Let P be a set or group. We define the category Set/P of *sets over* P to be the category whose objects are functions $\omega\colon R \to P$ and whose morphisms $\alpha\colon \omega \to \xi$

where $\xi\colon S \to P$ are commutative diagrams

$$\begin{tikzcd} R \arrow[rr, "\alpha"] \arrow[dr, "\omega"'] & & S \arrow[dl, "\xi"] \\ & P. & \end{tikzcd}$$

We have a forgetful functor

$$U : \mathsf{XMod}/P \to \mathsf{Set}/P$$

and we define the *free crossed module functor*

$$\mathrm{FX}\colon \mathsf{Set}/P \to \mathsf{XMod}/P$$

to be the left adjoint of this functor U. Thus for any function $\omega\colon R \to P$ we have a P-crossed module $\partial\colon \mathrm{FX}(\omega) \to P$ such that for any crossed P-module $\mathcal{M} = (\mu\colon M \to P)$ there is a natural bijection

$$\phi\colon (\mathsf{Set}/P)(\omega, U\mathcal{M}) \cong (\mathsf{XMod}/P)(\mathrm{FX}(\omega), \mathcal{M}).$$

This is equivalent to saying that there is a map $i\colon R \to U\,\mathrm{FX}(\omega)$ in Set/P, corresponding to the morphism $1_{\mathrm{FX}(\omega)}$ of crossed P-modules such that $\partial i = \omega$ and for any map over $f\colon R \to M$ over P there exists a unique morphism $\bar{f}\colon \mathrm{FX}(\omega) \to M$ of crossed P-modules such that $\bar{f} i = f$. (For more information on the relation between adjoint functors and universal properties, see Appendix A, Section A.6).

We define in a similar way the *free precrossed module* using the forgetful functor $\mathsf{PXMod}/P \to \mathsf{Set}/P$.

As always in universal constructions, the free crossed and free precrossed P-modules on ω are unique up to isomorphism. For this reason we often talk about *the* free crossed P-module on ω. The previous sections prove the existence of these free constructions.

We now give an example and a proposition which illustrate some of the difficulties of working with free crossed modules.[51]

Let $(\partial\colon C(R) \to F(X))$ be the free crossed module on the subset R of $F(X)$ and suppose that Y is a subset of X, and S a subset of R. Let M be the subgroup of $C(R)$ generated by $F(Y)$ operating on S, and assume that $\partial(M) \subseteq F(Y)$. Let $\mathcal{M}' = (\partial'\colon M \to F(Y))$ be the crossed module given by restricting ∂ to M. Then $\mathcal{M}'$ is not necessarily a free crossed module.

Example 3.4.8. Let $X = Y = \{x\}$, $R = \{a, b\}, S = \{b\}$ be such that $\partial a = x$, $\partial b = 1$. Since $\partial b = 1$, we have $ab = ba$, whence $b^x b^{-1} = a^{-1}bab^{-1} = 1$. Therefore $\mathcal{M}'$ is not a free crossed module. □

Proposition 3.4.9. *Let ∂, ∂' be as before and $\eta\colon \mathrm{Cok}\,\partial \to \mathrm{Cok}\,\partial'$ be the morphism induced by the inclusion $i\colon F(Y) \to F(X)$. If η is injective, then $\mathcal{M}'$ is the free crossed $F(Y)$-module on S.*

Proof. Let $d: C(S) \to F(Y)$ be the free crossed $F(Y)$-module on S. It is clear that $d(C(S)) = \partial(M)$. Let $j: C(S) \to M$ be the morphism of crossed $F(Y)$-modules. Clearly j is surjective, and the result is proved when we have shown that j is injective.

Suppose that $u \in C(S)$ and $j(u) = 1$. Then $d(u) = 1$. Let $k: C(S)^{\mathrm{ab}} \to C(R)^{\mathrm{ab}}$ be the induced morphism of the abelianised groups. These abelianised groups are in fact free modules over Cok ∂, Cok ∂' respectively on the bases S, R respectively. Since η is injective, it follows that k is injective. Let $\bar{u}$ denote the class of u in $C(S)^{\mathrm{ab}}$. Then $k\bar{u} = 0$, and hence $\bar{u} = 0$. But the morphism $C(S) \to C(S)^{\mathrm{ab}}$ is injective on $\operatorname{Ker} d$. It follows that $u = 1$. □

3.5 Precat^1-groups and the existence of colimits

In the two previous section we have seen that when working with crossed modules it is sometimes convenient to consider the weaker structure of precrossed modules and see the category $\mathsf{XMod/Groups}$ as a full subcategory of $\mathsf{PXMod/Groups}$.

In Section 2.5 we have seen that the category $\mathsf{Cat^1\text{-}Groups}$ of cat^1-groups is equivalent to the category $\mathsf{XMod/Groups}$. It is an easy exercise to put both together and construct a category bigger than $\mathsf{Cat^1\text{-}Groups}$ and equivalent to $\mathsf{PXMod/Groups}$.

So, a *precat^1-group* is a triple (G, s, t) where G is a group and $s, t: G \to G$ are endomorphisms satisfying $st = t$ and $ts = s$. Thus we are omitting CG2) from the axioms of a cat^1-group, i.e. we do not impose commutativity between elements of $\operatorname{Ker} s$ and $\operatorname{Ker} t$.

As before, a *morphism* between pre-cat^1-groups is just a homomorphism of groups commuting with the s's and t's. These objects and morphisms define the category $\mathsf{PCat^1\text{-}Groups}$. It contains $\mathsf{Cat^1\text{-}Groups}$ as a full subcategory.

Proposition 3.5.1. *The categories* $\mathsf{PCat^1\text{-}Groups}$ *and* $\mathsf{PXMod/Groups}$ *are equivalent, by an equivalence extending that between* $\mathsf{Cat^1\text{-}Groups}$ *and* $\mathsf{XMod/Groups}$.

Proof. The definitions of both functors are the same as in Section 2.5, namely

$$\lambda: \mathsf{PXMod/Groups} \to \mathsf{PCat^1\text{-}Groups}$$

is given by $\lambda(\mu: M \to P) = (P \ltimes M, s, t)$, s and t being defined as before, and

$$\gamma: \mathsf{PCat^1\text{-}Groups} \to \mathsf{PXMod/Groups}$$

is defined by $\gamma(G, s, t) = (t|: \operatorname{Ker} s \to \operatorname{Im} s)$.

It is easily checked that both functors are well defined and both compositions are naturally equivalent to the identity. □

As in the Section 3.3, we may define a functor associating to each pre-cat^1-group a cat^1-group

$$(-)^{\mathrm{cat}}: \mathsf{PCat^1\text{-}Groups} \to \mathsf{Cat^1\text{-}Groups}$$

defined by $(G,s,t)^{\mathrm{cat}} = (G/N, s', t')$, where $N = [\operatorname{Ker} s, \operatorname{Ker} t]$.

It is easy to see that the functor $(-)^{\mathrm{cat}}$ corresponds through the equivalences of categories to

$$(-)^{\mathrm{cr}} \colon \mathsf{PXMod/Groups} \to \mathsf{XMod/Groups}.$$

Then, it follows

Proposition 3.5.2. *The functor $(-)^{\mathrm{cat}}$ is a left adjoint of the inclusion.*

Using this last property we can prove the existence of colimits in $\mathsf{Cat}^1\text{-}\mathsf{Groups}$.

Since left adjoint functors preserve colimits (see [ML71] or Section A.7 of Appendix A, for any indexed family $\mathcal{G}_\lambda = (G_\lambda, s_\lambda, t_\lambda)$ of cat^1-groups and morphisms between them, we have

$$\operatorname{colim}_{\mathrm{cat}}\{\mathcal{G}_\lambda\} = (\operatorname{colim}_{\mathrm{pre}}\{\mathcal{G}_\lambda\})^{\mathrm{cat}}.$$

So, the existence of colimits in $\mathsf{Cat}^1\text{-}\mathsf{Groups}$ has been reduced to the existence of colimits in $\mathsf{PCat}^1\text{-}\mathsf{Groups}$.

It is not difficult now to check that in $\mathsf{PCat}^1\text{-}\mathsf{Groups}$ the colimits are as expected, i.e. for an indexed family $\{\mathcal{G}_\lambda \mid \lambda \in \Lambda\}$ of pre-cat^1-groups $\mathcal{G}_\lambda = (G_\lambda, s_\lambda, t_\lambda)$ and morphisms between them,

$$\operatorname{colim}_{\mathrm{pre}}\{\mathcal{G}_\lambda\} = (\operatorname{colim}_{\mathrm{gr}}\{G_\lambda\}, \operatorname{colim}_{\mathrm{gr}}\{s_\lambda\}, \operatorname{colim}_{\mathrm{gr}}\{t_\lambda\}).$$

From the existence of colimits in $\mathsf{Cat}^1\text{-}\mathsf{Groups}$ follows from the existence of colimits in $\mathsf{XMod/Groups}$ using the equivalence between these categories.

Remark 3.5.3. We have just added another way of computing colimits of crossed modules. So, if we have an indexed family of crossed modules $\{\mu_\lambda \colon M_\lambda \to P_\lambda\}$, we construct the associated family of cat^1-groups $\{(M_\lambda \ltimes P_\lambda, s_\lambda, t_\lambda)\}$ getting their colimit (G,s,t) and the colimit crossed module is $t| \colon \operatorname{Ker} s \to \operatorname{Im} t$.

Even if it seems a long way around, it is worthwhile because for example $M_\lambda \ltimes P_\lambda$ may be finitely generated, even if M_λ and P_λ are not. Also, there are some efficient computer-assisted ways of getting colimits, kernels and images of finitely generated groups and homomorphisms. □

3.6 Implementation of crossed modules in GAP

To make any serious computational work in group theory it is usually necessary to use a computational group theory package. Some of these packages have evolved to accommodate more structures becoming veritable computational discrete algebra packages.

Work at Bangor (in particular by M. Alp and C. D. Wensley) has produced the GAP module XMOD which includes a number of constructions on crossed modules,

cat^1-groups and their morphisms. In particular: derivations, kernels and images; the Whitehead group; cat^1-groups and their relation with crossed modules; induced crossed modules.

This package has already been in use for some time, and has been incorporated into GAP4.[52]

In Section 5.9 we will show how XMOD has been used to determine explicitly some induced crossed modules whose computation do not follow from general theorems and seem too hard to compute by hand.

Notes

37 p. 64 This finiteness no longer holds for groups. A group may be finitely generated, but not finitely presented; finitely presented but not finitely identified; and so on. For more information see Wikipedia on finitely presented groups.

38 p. 67 The subject of identities among relations for groups was founded by Peiffer and Reidemeister in the papers [Pei49], [Rei49], with the intention of working towards normal forms for 3-manifolds. This work was related to, but independent of, work by Whitehead on crossed modules, to which we have already referred. Interest in the work of Peiffer and Reidemeister was widened by the exposition in [BH82]; the historical investigation, as reported there, is due to J. Huebschmann. The exposition here, and some of the results, e.g. Proposition 3.3.5, follows that article to some extent. Reidemeister, as one of the founders of knot theory, was aware of the intuitive relations between the identities for crossed modules and those for knots and links.

39 p. 69 Accounts of this are in many books on combinatorial group theory, for example [Joh97], [LS01], [Coh89]. The notion of free groupoid on a directed graph is described in [Bro06] and [Hig71]: the construction in the former is in terms of reduced words, and in the latter the reduced words come after the formal definition in terms of the universal property. Free groupoids are important to us in Section 10.3.ii.

40 p. 69 In Chapter 5 we construct for a subgroup R of a group P a crossed module $i_*(R) \to P$ whose image is $\langle R\rangle^P$ and which has an important universal property as a so called ‘induced crossed module’.

41 p. 69 This term was introduced by Peiffer and Reidemeister in [Pei49], [Rei49] and given an exposition in [BH82]. A better term might be ‘identity among consequences’, but the term we use is embedded in the literature. Another term used is ‘homotopical syzygy’, [Lod00]. Because of the relation of this concept to the

construction of a CW-complex, identities among relations are used in [Hue99] in relation to a construction of free simplicial groups from a CW-complex, and this is applied to lattice gauge theory to give a solution of a question of Atiyah as to whether there is a finite-dimensional construction of the Chern–Simons function on a 3-manifold.

42 p. 72 The concept of shelling and shellable is well known in combinatorics, and the article [Bjö92] gives more information on this. The concept is also used in the thesis [Jon88] to control the kinds of cell decompositions of a sphere to be allowed. The contrast between the 2-dimensional van Kampen diagram and the formula for a consequence can be viewed as that between a 2-dimensional and a 1-dimensional form of algebra. The two views illuminate each other, but the 1-dimensional view is more susceptible to calculation and computer implementation.

43 p. 73 These examples are from the book [Joh97]. Other examples on van Kampen diagrams may be found there, and by a web search. The geometric and metric analysis of van Kampen diagrams has proved important in aspects of combinatorial group theory.

44 p. 74 There is an alternative description of $F_P(R)$ which we give for those familiar with the group theoretic background; it is sometimes useful but does not immediately give the free property. *The group $F_P(R)$ is isomorphic to the normal closure of R in the free product $P * F(R)$.* This result is [Pei49], Satz 3 on p. 69, see also [Met79]. This result is a simple consequence of the Kurosch Subgroup Theorem for free products, which may be found in books on combinatorial group theory and in [Hig71], [Bro06] from a groupoid viewpoint.

45 p. 75 These elements were introduced in [Pei49], p. 70, and in [Rei49]. The term ‘Peiffer element’ was first used in [BH82]. This work is related to but independent of that of Whitehead on free crossed modules; the war ensured a lack of communication. The goal of the program of Reidemeister in [Rei49] and of Peiffer was to develop normal forms for 3-manifolds. A major use of Whitehead’s work was in [Pap63], which reduced the 3-dimensional Poincaré Conjecture to problems in combinatorial group theory, which were however difficult to solve.

46 p. 75 In this spirit, there is a ‘Peiffer commutator calculus’ whose study has been advanced considerably by Baues and Conduché [BC90].

47 p. 75 The term ‘precrossed module’ was introduced in [BH82], following a suggestion of P. J. Higgins.

48 p. 76 There is a substantial theory of Peiffer commutator calculus, see [CE89], [BC90].

49 p. 76 The term Peiffer group was used in [Met79].

50 p. 79 A more combinatorial description of free crossed modules is given in essence in [Pei49]; this is explained in [BH82], Section 5, in terms of 'Peiffer transformations on Y-sequences', and related to an important 'identity property'. The reader may like to develop an analogous account for the groupoid case.

51 p. 80 These are due to Whitehead in [Whi50b].

52 p. 83 We note that Alp and Wensley have in [AW00] used this programme to list many finite cat^1-groups.

Chapter 4

Coproducts of crossed P-modules

In this chapter we start to show how the combination of the 2-dimensional Seifert–van Kampen Theorem and the algebra of crossed modules allows specific nonabelian calculations in homotopy theory in dimension 2. To this end, we study the coproduct of crossed modules (mainly of two crossed modules) over the same group P. We construct this coproduct of crossed P-modules, check some properties and, using the 2-dimensional Seifert–van Kampen Theorem, we apply these general results to some topological cases.

Section 4.1 gives the construction of the coproduct of crossed P-modules. First, we see what the definition of coproduct in a general category means in this case, and then we prove its existence in two steps. As a first step, we prove that the free product of groups gives the coproduct in the category of precrossed P-modules. Then, using the fact that the functor $(-)^{\mathrm{cr}}$ preserves coproducts, we see that its associated crossed P-module is the coproduct in the category of P-modules.

This procedure is not immediately suitable for computations, because the free product of groups is always a very big group (it is normally infinite even if all the groups concerned are finite). So in Section 4.2 we give an alternative description of the coproduct of two crossed P-modules. This is obtained by dividing the construction of the associated crossed module for this case into two steps, of which the first gives a semidirect product. Thus the coproduct of two crossed P-modules is a quotient of a semidirect product. Hence we can get presentations of the coproduct using the known presentations of the semidirect product.

This has topological applications as explained in Section 4.3. First, we know that the coproduct of two crossed P-modules is just the pushout of these two crossed modules with respect to the trivial crossed module $1 \to P$. Thus if the topological space X is the union of two open subsets U_1, U_2 such that both pairs (U_1, U_{12}), (U_2, U_{12}) are 1-connected, then the crossed module $\Pi_2(X, U_{12})$ is the coproduct $\Pi_2(U_1, U_{12}) \circ \Pi_2(U_2, U_{12})$ (Theorem 4.3.3) and we can use the previous results to get information on the second homotopy group of X. We end this section by studying some consequences in this case.

In the last section (4.4) we study the coproduct in a particular case used later. We begin with two crossed P-modules $\mathcal{M} = (\mu\colon M \to P)$ and $\mathcal{N} = (\nu\colon N \to P)$ satisfying the condition

$$\nu(N) \subseteq \mu(M) \text{ and there is an equivariant section of } \mu. \tag{$*$}$$

In this case we get a description of their coproduct using the displacement subgroup N_M (Theorem 4.4.8). This case is not uncommon and we get some topological applications when the space X is got from Y by attaching a cone CA, that is, X is a mapping cone.

We finish this last section with a description of the coproduct for an arbitrary set of indices satisfying the above condition ($*$). This result will be used at the end of the next chapter (see Section 5.8).

4.1 The coproduct of crossed P-modules

We give a construction of coproducts in the category XMod/P of crossed modules over the group P. We do this for a general family of indices since this causes no more difficulty than the case of two crossed modules.

From the general definition of the coproduct in a category given in Example A.4.4 of Appendix A, we see that the coproduct of a family $\{\mathcal{M}_t \mid t \in T\}$ of crossed modules over P is given by a crossed module $\mathcal{M}$ and a family of morphisms of crossed P-modules $\{i_t : \mathcal{M}_t \to \mathcal{M} \mid t \in T\}$ satisfying the following universal property: for any family $\{u_t : \mathcal{M}_t \to \mathcal{M}' \mid t \in T\}$ of morphisms of crossed modules over P, there is a unique morphism $u : \mathcal{M} \to \mathcal{M}'$ of crossed modules over P such that $u_t = ui_t$ for each $t \in T$. Diagrammatically, there exists a unique dashed arrow such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{M}_t \arrow[r, "i_t"] \arrow[dr, "u_t"'] & \mathcal{M} \arrow[d, dashed, "u"] \\ & \mathcal{M}. \end{tikzcd}$$

As with any universal construction, the coproduct is unique up to isomorphism.

As we have seen in Section 3.3, the functor $(-)^{\mathrm{cr}}$ from precrossed modules to crossed modules, obtained by factoring out the Peiffer subgroup, is left adjoint to the inclusion of crossed modules into precrossed modules, and so takes coproducts into coproducts. Thus to construct the coproduct of crossed P-modules we construct the coproduct in PXMod/P, the category of precrossed modules over the group P and apply the functor $(-)^{\mathrm{cr}}$ to it. The coproduct in PXMod/P is simply obtained using the coproduct in the category Groups of groups, and this is the well-known free product $*_{t\in T}\, G_t$ of a family $\{G_t\}$ of groups.[53]

Proposition 4.1.1. *Let T be an indexing set, and for each $t \in T$ let $\mathcal{M}_t = (\mu_t : M_t \to P)$ be a precrossed P-module. We define $*_{t\in T}\, M_t$ to be the free product of the groups M_t, $t \in T$. There is an action of P on $*_{t\in T}\, M_t$ defined by the action of P on each M_t. Consider the morphism*

$$*_{t\in T}\, \mathcal{M}_t = (\partial' : *_{t\in T}\, M_t \to P),$$

together with the homomorphisms $i_t : M_t \to *_t M_t$ given by the inclusion in the free product, and where $\partial' = *_{t\in T}\, \mu_t$ is the homomorphism of groups induced from the homomorphisms μ_t using the universal property of the coproduct of groups. Then *the above defined* $*_{t\in T}\mathcal{M}_t$ *is a precrossed* P*-module and the homomorphisms* i_t *are morphisms of precrossed modules over* P *giving the coproduct in the category* PXMod/P.

Proof. Let $\mathcal{M} = *_{t\in T}\mathcal{M}_t$. If we represent by $p_\#$ the action by $p \in P$, then the action $p_\#\colon \mathcal{M} \to \mathcal{M}$ of p is defined by the composite morphisms $\mathcal{M}_t \xrightarrow{p_\#} \mathcal{M}_t \xrightarrow{i_t} \mathcal{M}$.

In terms of the normal form of an element of the free product, this means that the action is given by the formula

$$(m_{t_1}\dots m_{t_n})^p = (m_{t_1})^p \dots (m_{t_n})^p, \quad m_{t_i} \in M_{t_i}.$$

As already pointed out, the homomorphisms μ_t extend uniquely to a homomorphism $*_t\mu_t$. So

$$\begin{aligned}(*_t\mu_t)((m_{t_1}\dots m_{t_n})^p) &= (*_t\mu_t)(m_{t_1}^p\dots m_{t_n}^p)\\ &= (\mu_{t_1}(m_{t_1}^p))\dots(\mu_{t_n}(m_{t_n}^p))\\ &= p^{-1}(\mu_{t_1}m_{t_1})p\dots p^{-1}(\mu_{t_n}(m_{t_n}))p\\ &= p^{-1}((\mu_{t_1}m_{t_1})\dots(\mu_{t_n}(m_{t_n}))p\end{aligned}$$

and $*_t\mu_t$ is a precrossed module.

The verification of the universal property is straightforward. □

We now easily obtain:

Corollary 4.1.2. *If* $\mathcal{M}_t = (\mu_t\colon M_t \to P), t \in T$ *is a family of crossed* P*-modules, then applying the functor* $(-)^{\mathrm{cr}}$ *to* $*_{t\in T}\mathcal{M}_t$ *to give*

$$\partial'^{\mathrm{cr}}\colon (*_{t\in T}\mathcal{M}_t)^{\mathrm{cr}} \to P$$

with the morphisms $j_t\colon M_t \xrightarrow{i_t} *_{t\in T}\mathcal{M}_t \to (*_{t\in T}\mathcal{M}_t)^{\mathrm{cr}}$*, where the second morphism is the quotient homomorphism, gives the coproduct of crossed* P*-modules.*

We denote this coproduct by

$$\bigcirc_{t\in T}\mathcal{M}_t = (\partial\colon \bigcirc_{t\in T} M_t \to P)$$

where the morphisms $j_t\colon \mathcal{M}_t \to \bigcirc_{t\in T}\mathcal{M}_t$ are understood to be part of the structure. These morphisms need not be injective. In the case $T = \{1, 2, \dots, n\}$, this coproduct will be written $M_1 \circ \dots \circ M_n \to P$. As is standard for coproducts in any category, the coproduct in XMod/P is associative and commutative up to natural isomorphisms.

Remark 4.1.3. 1. There is a generalisation of this universal property given in Exercise B.1.10 in Appendix B. The proof uses a pullback construction of crossed modules given in Section 5.1.

2. If all the crossed modules $M_t \to P$, $t \in T$ are the same then their coproduct is called the *copower* construction

$$\mathcal{M}^{\circ T} = \bigcirc_{t\in T}\mathcal{M}. \qquad \square$$

4.2 The coproduct of two crossed P-modules

Throughout this section we suppose given two crossed P-modules $\mathcal{M} = (\mu\colon M \to P)$ and $\mathcal{N} = (\nu\colon N \to P)$, and we develop at some length the study of their coproduct in XMod/P

$$\mathcal{M} \circ \mathcal{N} = (\mu \circ \nu\colon M \circ N \to P)$$

and the canonical morphisms from M, N into $M \circ N$.[54]

The basic observation is that $M \circ N$ may be obtained as a quotient of the semidirect product group $M \ltimes N$ where M operates on N *via* P. This result makes the coproduct of two crossed modules computable and from this we get some topological computations.

For convenience, we assume M, N are disjoint. To study $M \circ N = (M * N)^{\mathrm{cr}}$ in some detail we should have a closer look at $[\![M * N, M * N]\!]$, the Peiffer subgroup of $M * N$. As seen in Section 3.3, $[\![M * N, M * N]\!]$ is the subgroup of $M * N$ generated by all Peiffer commutators

$$[\![k, k']\!] = k^{-1}k'^{-1}kk'^{(\mu*\nu)k}$$

for all $k, k' \in M * N$.

Notice that by Proposition 3.3.5, $[\![M * N, M * N]\!]$ is also the normal subgroup generated by the Peiffer commutators of any given P-invariant set of generators. Now $M \cup N$ generates $M * N$ and is P-invariant. Since $\mathcal{M}$ and $\mathcal{N}$ are crossed modules, we have $[\![m, m']\!] = 1$ and $[\![n, n']\!] = 1$, for all $m, m' \in M$ and $n, n' \in N$. Thus $[\![M * N, M * N]\!]$ is the normal subgroup of $M * N$ generated by the elements

$$r(m, n) = n^{-1}m^{-1}nm^n \quad \text{and} \quad s(m, n) = m^{-1}n^{-1}mn^m$$

for all $m \in M$, $n \in N$.

The process of quotienting out by the Peiffer subgroup may be divided into two steps. First, we consider the quotient of $M * N$ by the group U generated by all Peiffer commutators of the form $\{s(m, n) \mid m \in M, n \in N\}$, and show that this quotient is the well-known semidirect product.

Proposition 4.2.1. *Let U be the normal P-invariant subgroup of $M * N$ generated by the set $\{m^{-1}n^{-1}mn^m \mid m \in M, n \in N\}$. Then the precrossed P-module*

$$(\mathcal{M} * \mathcal{N})/U = (\mu * \nu\colon (M * N)/U \to P)$$

is isomorphic to

$$\mathcal{M} \ltimes \mathcal{N} = (\mu \ltimes \nu\colon M \ltimes N \to P)$$

where the semidirect product is associated to the action of M on N via μ and the P-action.

Proof. The inclusions $M \to M \ltimes N$ and $N \to M \ltimes N$ extend to a homomorphism of groups

$$\varphi\colon M * N \to M \ltimes N.$$

Let us check that $\varphi(U) = 1$ by computing φ on all generators:

$$\begin{aligned}\varphi(m^{-1}n^{-1}mn^m) &= (m^{-1},1)(1,n^{-1})(m,1)(1,n^m)\\ &= (m^{-1},n^{-1})(m,n^m)\\ &= (m^{-1}m,(n^{-1})^m n^m)\\ &= (1,1).\end{aligned}$$

So we have an induced homomorphism of P-groups

$$\overline{\varphi}\colon (M * N)/U \to M \ltimes N.$$

We define a homomorphism in the other direction

$$\psi\colon M \ltimes N \to (M * N)/U$$

by $\psi(m,n) = [mn]$ the equivalence class of the element $mn \in M * N$. To check the homomorphism property, we compute

$$\begin{aligned}\psi(m',n')^{-1}\psi(m,n)^{-1}\psi((m,n)(m',n')) &= [n'^{-1}m'^{-1}][n^{-1}m^{-1}]\psi(mm',n^{m'}n')\\ &= [n'^{-1}m'^{-1}n^{-1}m^{-1}mm'n^{m'}n']\\ &= [n'^{-1}(m'^{-1}n^{-1}m'n^{m'})n']\\ &= [1]\end{aligned}$$

since $m'^{-1}n^{-1}m'n^{m'} \in U$.

Clearly $\overline{\varphi}\psi = 1$. Since $\psi\overline{\varphi}$ is a homomorphism, to prove that it is 1 it is enough to check this on the generators $[mn]$, $m \in M$, $n \in N$, and this is clear.

It now follows, as may be proved directly, that $\mu \ltimes \nu\colon M \ltimes N \to P$, $(m,n) \mapsto (\mu m)(\nu n)$ is a homomorphism which with the action of P given by $(m,n)^p = (m^p,n^p)$ is a precrossed P-module. $\square$

So $\mathcal{M} \ltimes \mathcal{N}$ is a precrossed module containing $\mathcal{M}$ and $\mathcal{N}$ as submodules. Let us see that it satisfies a universal property with respect to maps of the crossed modules $\mathcal{M}$ and $\mathcal{N}$ to any given crossed P-module $\mathcal{M}'$.

Proposition 4.2.2. *Let $\mathcal{M}' = (\mu'\colon M' \to P)$ be a crossed P-module and let $f\colon M \to M'$ and $g\colon N \to M'$ be morphisms of crossed P-modules. Then there is a unique map of precrossed P-modules extending f and g, namely $f \ltimes g\colon M \ltimes N \to M'$, $(m,n) \mapsto (fm)(gn)$.*

Proof. Uniqueness is obvious.

To prove existence we check that the morphism of precrossed P-modules

$$f * g\colon M * N \to M'$$

sends all elements of U to 1, where U is the subgroup specified in Proposition 4.2.1. On generators of U we have

$$(f * g)(m^{-1}n^{-1}mn^m) = f(m^{-1})g(n^{-1})f(m)g(n^m) = g(n^{-1})^{\mu' fm}g(n)^{\mu m} = 1$$

since $\mu' \colon M' \to P$ is a crossed module and $\mu' f = \mu$. □

Therefore it is clear that the coproduct of two crossed P-modules $\mu \colon M \to P$ and $\nu \colon N \to P$ is the crossed module associated to the precrossed module $\mu \ltimes \nu \colon M \ltimes N \to P$, i.e.

$$\mathcal{M} \circ \mathcal{N} = ((\mu \ltimes \nu)^{\mathrm{cr}} \colon (M \ltimes N)^{\mathrm{cr}} \to P) = (M \circ N \to P).$$

This has some striking consequences.

Remark 4.2.3. If $M \to P$, $N \to P$ are crossed P-modules such that M and N are finite groups (resp. finite p-groups), then so also is the semidirect product $M \ltimes N$ and hence their coproduct as crossed modules $M \circ N$ is also a finite group (resp. a finite p-group). This result was not so clear from previous descriptions of the coproduct of crossed P-modules. □

Remark 4.2.4. If $(\mu \colon M \to P)$, $(\nu \colon N \to P)$ are crossed P-modules such that each of M, N act trivially on the other via P, then $M \ltimes N = M \times N$, the product of the groups, with action of P given by $(m,n)^p = (m^p, n^p)$ and with boundary $\partial \colon M \times N \to P$ given by $(m,n) \mapsto (\mu m)(\nu n)$. □

The Peiffer subgroup $[\![M \ltimes N, M \ltimes N]\!]$ of $M \ltimes N$ is of course the subgroup of $M \ltimes N$ generated by the Peiffer commutators of all elements of $M \ltimes N$, and we write it as $\{M, N\}$. Alternatively, $\{M, N\}$ is generated by the images by φ of $r(m,n) = n^{-1}m^{-1}nm^n$, i.e. by

$$\{\{n,m\} \mid m \in M, n \in N\}.$$

Lemma 4.2.5. *The elements* $\{n,m\}$ *satisfy*

$$\{n,m\} = ([m,n],[n,m]),$$

where $[m,n] = m^{-1}m^n$ *and* $[n,m] = n^{-1}n^m$.

Proof. Notice that any $m, m' \in M$ and $n \in N$ satisfy the relation

$$n'^{(m^n)} = ((n'^{n^{-1}})^m)^n = n^{-1}(nn'n^{-1})^m n = n^{-1}n^m n'^m (n^{-1})^m n. \qquad (*)$$

Thus,

$$\begin{aligned}
\{n,m\} &= n^{-1}m^{-1}nm^n \\
&= (1,n^{-1})(m^{-1},1)(1,n)(m^n,1) \\
&= (m^{-1},(n^{-1})^{m^{-1}})(m^n,(n^m)^n) \\
&= (m^{-1}m^n,((n^{-1})^{m^{-1}})^{m^n}(n^m)^n) \\
&= (m^{-1}m^n, n^{-1}n^m) \qquad \text{using } (*) \\
&= ([m,n],[n,m])
\end{aligned}$$

as required. □

Using the previous result and some well-known facts on the semidirect product, we get a presentation of the coproduct of two crossed modules as follows. First, recall that the semidirect product has a presentation with generators the elements $(m,n) \in M \times N$ and relations

$$(m,n)(m',n') = (mm', n^{m'}n')$$

for all $m, m' \in M$ and $n, n' \in N$. The set of relations may equivalently be expressed as

$$(m, n^{m'^{-1}})(m',n') = (mm', nn').$$

To get a presentation of $M \circ N$ we add the relations corresponding to the Peiffer subgroup $\{M, N\}$. By the preceding property the relation $\{m,n\} = 1$ is equivalent to $[m,n] = [n,m]^{-1}$, giving $(m^n)^{-1}m = n^{-1}n^m$, or $n(m^{-1})^n = (n^{m^{-1}})^{-1}m^{-1}$. This may be expressed, taking $m' = m^{-1}$, as

$$nm'^n = (n^{m'})^{-1}m'$$

suggesting the next theorem.

Theorem 4.2.6. *If $M \to P$, $N \to P$ are crossed P-modules, then the group $M \circ N$ has a presentation with generators $\{m \circ n \mid m \in M, n \in N\}$, and relations*

$$mm' \circ nn' = (m \circ n^{m'^{-1}})(m' \circ n') = (m \circ n)(m'^n \circ n'),$$

for all $m, m' \in M$ and $n, n' \in N$.

Proof. Let K be the group with this presentation. Then P acts on K by $(m \circ n)^p = m^p \circ n^p$, and the map

$$\xi \colon K \to P, \quad m \circ n \mapsto (\mu m)(\nu n),$$

is a well-defined homomorphism. It is routine to verify the crossed module rules for this structure.

It is also not difficult to check that this crossed module together with the morphisms $i : M \to K, m \mapsto m \circ 1$, and $j : N \to K, n \mapsto 1 \circ n$, satisfy the universal property of the coproduct. We omit further details. □

We describe some extra facts about $\{M, N\}$, for example the expression of the products and inverses of the elements $\{n, m\}$.

Proposition 4.2.7. *For any $m, m' \in M$ and $n, n' \in N$ we have*

$$\{n, m\}\{n', m'\} = ([m, n][m', n'], [n', m'][n, m]).$$

Proof.

$$\begin{aligned}\{n, m\}\{n', m'\} &= ([m, n], [n, m])([m', n'], [n', m']) \\ &= ([m, n][m', n'], [n, m]^{[m', n']}[n', m'],\end{aligned}$$

and, using $(*)$ in Lemma 4.2.5 for the fourth equality in the following:

$$\begin{aligned}[n, m]^{[m', n']}[n', m'] &= (n^{-1}n^m)^{m'^{-1}m'^{n'}} n'^{-1}n'^{m'} \\ &= ((n^{-1})^{m'^{-1}} (n^m)^{m'^{-1}})^{m'^{n'}} n'^{-1}n'^{m'} \\ &= n'^{-1}n'^{m'} n^{-1}n^m (n'^{-1})^{m'} n'n'^{-1}n'^{m'} \\ &= n'^{-1}n'^{m'} n^{-1}n^m \\ &= [n', m'][n, m].\end{aligned}$$

Thus

$$\{n, m\}\{n', m'\} = ([m, n][m', n'], [n', m'][n, m])$$

as required. □

Remark 4.2.8. This result extends to any finite product of elements $\{n_i, m_i\}$ with $m_i \in M, n_i \in N$. □

Corollary 4.2.9. *For any $m \in M$ and $n \in N$ we have*

$$\{n, m\}^{-1} = \{n^{-1}, m^n\}.$$ □

The proof is left to the reader.

4.3 The coproduct and the 2-dimensional Seifert–van Kampen Theorem

The 2-dimensional Seifert–van Kampen Theorem as stated in Theorem 2.3.1 allows for topological applications of the coproduct of crossed P-modules, since the coproduct of two crossed P-modules may also be interpreted as a pushout, as follows.

Proposition 4.3.1. *If $(\mu\colon M \to P)$, $(\nu\colon N \to P)$ are crossed P-modules then the following diagram*

$$\begin{tikzcd} (1 \to P) \arrow[r] \arrow[d] & (N \to P) \arrow[d] \\ (M \to P) \arrow[r] & (M \circ N \to P) \end{tikzcd} \tag{4.3.1}$$

is a pushout in the category XMod/P *and also in the category* $\mathsf{XMod}/\mathsf{Groups}$.

Proof. The equivalence of the pushout property in the category XMod/P with the universal property of the coproduct is easy to verify. We defer the proof of the pushout property in the category $\mathsf{XMod}/\mathsf{Groups}$ until we have introduced in Section 5.2 the pullback functor $f^*\colon \mathsf{XMod}/Q \to \mathsf{XMod}/P$ for a morphism $f\colon P \to Q$ of groups. □

Remark 4.3.2. The relation between these two pushouts also follows from a general result in Appendix B, see Remark B.1.9. □

One of the simpler cases of the 2-dimensional Seifert–van Kampen Theorem is the following.

Theorem 4.3.3. *Suppose that the connected space X is the union of the interior of two connected subspaces U_1, U_2, with connected intersection U_{12}. Suppose that the pairs (U_1, U_{12}) and (U_2, U_{12}) are 1-connected. Then the pair (X, U_{12}) is 1-connected and the morphism*

$$\Pi_2(U_1, U_{12}) \circ \Pi_2(U_2, U_{12}) \to \Pi_2(X, U_{12})$$

induced by inclusions is an isomorphism of crossed $\pi_1(U_{12})$-modules.

Proof. We apply Theorem 2.3.1 to the cover of X given by U_1 and U_2 with $A = U_{12}$. The connectivity result is immediate. Also by the same theorem the following diagram is a pushout of crossed modules:

$$\begin{tikzcd} \Pi_2(U_{12}, U_{12}) \arrow[r] \arrow[d] & \Pi_2(U_1, U_{12}) \arrow[d] \\ \Pi_2(U_2, U_{12}) \arrow[r] & \Pi_2(X, U_{12}). \end{tikzcd}$$

Since $\Pi_2(U_{12}, U_{12}) = (1 \to \pi_1(U_{12}))$, the result follows from Proposition 4.3.1. □

We would like to extract from this result some information on the absolute homotopy group $\pi_2(X)$. Consider the following part of the homotopy exact sequence of the pair (X, U_{12}) stated in page 35 (see (2.1.3)),

$$\cdots \to \pi_2(U_{12}) \xrightarrow{i_*} \pi_2(X) \xrightarrow{j_*} \pi_2(X, U_{12}) \xrightarrow{\partial} \pi_1(U_{12}) \to \cdots.$$

It is clear that we have an isomorphism

$$\frac{\pi_2(X)}{i_*(\pi_2(U_{12}))} \cong \operatorname{Ker}\partial = \operatorname{Ker}(\partial_1 \circ \partial_2). \tag{4.3.2}$$

Notice than, in particular, this result gives complete information on $\pi_2(X)$ when $\pi_2(U_{12}) = 0$.

To identify the kernel of the coproduct of two crossed P-modules in a more workable way, we introduce the pullback of crossed P-modules. Given two crossed modules $\mathcal{M} = (\mu\colon M \to P)$, $\mathcal{N} = (\nu\colon N \to P)$ we form the pullback square

$$\begin{tikzcd} M \times_P N \arrow[r, "p_1"] \arrow[d, "p_2"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd} \tag{4.3.3}$$

where $M \times_P N = \{(m,n) \in M \times N \mid \mu(m) = \nu(n)\}$, p_1 and p_2 are the projections. Obviously $M \times_P N$ is a P-group, with P acting diagonally.

Proposition 4.3.4. *The pullback $M \times_P N$ is isomorphic as P-group to* $\operatorname{Ker}(\mu \ltimes \nu)$.

Proof. We check that

$$\phi\colon M \times_P N \to M \ltimes N$$

defined as $\phi(m,n) = (m, n^{-1})$ is a morphism of groups. Let $m, m' \in M$ and $n, n' \in N$; then

$$\begin{aligned} \phi(m,n)\phi(m',n') &= (m,n^{-1})(m',n'^{-1}) \\ &= (mm', (n^{-1})^{m'}n'^{-1}) \\ &= (mm', (n^{-1})^{n'}n'^{-1}) \\ &= (mm', n'^{-1}n^{-1}n'n'^{-1}) \\ &= (mm', (nn')^{-1}) \\ &= \phi(mm', nn'). \end{aligned}$$

Clearly, ϕ is a bijection onto $\operatorname{Ker}(\mu \ltimes \nu)$ which preserves the P-actions. □

Now to any $m \in M$ and $n \in N$ we associate the element of $M \times_P N$ defined by

$$\langle m, n\rangle = (m^{-1}m^n, (n^{-1})^m n). \tag{4.3.4}$$

Correspondingly, we write $\langle M, N\rangle$ for the normal subgroup of $M \times_P N$ generated by $\{\langle m,n\rangle \mid m \in M, n \in N\}$. It follows that $\phi(\langle M, N\rangle) = \{M, N\}$.

Thus, there is an induced map

$$\bar{\phi}\colon \frac{M \times_P N}{\langle M, N\rangle} \longrightarrow \frac{M \ltimes N}{\{M, N\}} = M \circ N.$$

We deduce immediately from the proposition:

Corollary 4.3.5. *The map $\bar{\phi}$ gives an isomorphism of P-modules*

$$\bar{\phi}\colon \frac{M \times_P N}{\langle M, N\rangle} \cong \operatorname{Ker}(\mu \circ \nu).$$

Remark 4.3.6. Notice that this result has some purely algebraic consequences. Since $\mathcal{M} \circ \mathcal{N}$ is a crossed module, Ker $(\mu \circ \nu)$ is abelian; so $\langle M, N\rangle$ contains the commutator subgroup of $M \times_P N$. □

Now we can translate this algebraic result into a topological one.

Theorem 4.3.7. *If (U_1, U_{12}) and (U_2, U_{12}) are 1-connected and $\pi_2(U_{12}) = 0$, we have,*

$$\pi_2(X) \cong \frac{\pi_2(U_1, U_{12}) \times_{\pi_1(U_{12})} \pi_2(U_2, U_{12})}{\langle \pi_2(U_1, U_{12}), \pi_2(U_2, U_{12})\rangle}.$$

Proof. Since $\pi_2(U_{12}) = 0$, from (4.3.2), we have $\pi_2(X) \cong \operatorname{Ker}(\partial_1 \circ \partial_2)$ and the result follows from Corollary 4.3.5. □

Let us study some other algebraic ways of computing $\operatorname{Ker}(\mu \circ \nu)$ or, equivalently, the quotient

$$\frac{M \times_P N}{\langle M, N\rangle}.$$

We may also define a homomorphism of groups $k\colon M \times_P N \to P$ by the formula $k(m,n) = \mu(m) = \nu(n)$. This gives the following result.

Proposition 4.3.8. *There is an exact sequence of P-groups*

$$0 \to \operatorname{Ker}\mu \oplus \operatorname{Ker}\nu \to M \times_P N \xrightarrow{k} \mu(M) \cap \nu(N) \to 1.$$

Proof. It is immediate that $k(M \times_P N) = \mu(M) \cap \nu(N)$. It remains to check that

$$\operatorname{Ker} k = \operatorname{Ker}\mu \oplus \operatorname{Ker}\nu;$$

but this is clear since

$$\operatorname{Ker} k = \{(m,n) \mid \mu(m) = \nu(n) = 0\}$$

and all $m \in \operatorname{Ker}\mu$ and $n \in \operatorname{Ker}\nu$ commute. □

Bringing the subgroup $\langle M, N\rangle$ into the picture, it is immediate that $k(\langle m,n\rangle) = [\mu(m), \nu(n)]$. Then we have $k(\langle M, N\rangle) = [\mu(M), \nu(N)]$ giving a homomorphism $\bar{k}$ onto the quotient. This gives directly the next result.

Corollary 4.3.9. *There is an exact sequence of P-modules*

$$\begin{aligned} 0 \to (\operatorname{Ker}\mu \oplus \operatorname{Ker}\nu) \cap (\langle M, N\rangle) &\to \operatorname{Ker}\mu \oplus \operatorname{Ker}\nu \\ &\to \frac{M \times_P N}{\langle M, N\rangle} = \operatorname{Ker}(\mu \circ \nu) \xrightarrow{\bar{k}} \frac{\mu(M) \cap \nu(N)}{[\mu(M), \nu(N)]} \to 0. \end{aligned}$$

Remark 4.3.10. An easy consequence is that $\mu \circ \nu \colon M \circ N \to P$ is injective if and only if

i) $\operatorname{Ker}\mu \oplus \operatorname{Ker}\nu \subseteq \langle M, N\rangle$, and

ii) $[\mu(M), \nu(N)] = \mu(M) \cap \nu(N)$. □

As above, we can apply this result to the topological case, getting a way to compute the second homotopy group of a space in some cases.

Theorem 4.3.11. *If (U_1, U_{12}) and (U_2, U_{12}) are 1-connected and $\pi_2(U_{12}) = 0$, the following sequence of groups and homomorphisms is exact:*

$$\begin{aligned} 0 \to (\pi_2(U_1) \oplus \pi_2(U_2)) \cap \langle \pi_2(U_1, U_{12}), \pi_2(U_2, U_{12})\rangle \\ \to \pi_2(U_1) \oplus \pi_2(U_2) \to \pi_2(X) \longrightarrow \frac{R_1 \cap R_2}{[R_1, R_2]} \to 1, \end{aligned}$$

where $R_l = \operatorname{Ker}(\pi_1(U_{12}) \to \pi_1(U_l))$ for $l = 1, 2$.

If further $\pi_2(U_1) = \pi_2(U_2) = 0$, then there is an isomorphism

$$\pi_2(X) \cong \frac{R_1 \cap R_2}{[R_1, R_2]}.$$

Proof. Let us consider the crossed modules $\partial_l : \pi_2(U_l, U_{12}) \to \pi_1(U_{12})$. Recall from (2.1.3) that the homotopy exact sequence of the pair (U_l, U_{12}) is

$$\cdots \to \pi_2(U_{12}) \xrightarrow{i_{l*}} \pi_2(U_l) \xrightarrow{j_{l*}} \pi_2(U_l, U_{12}) \xrightarrow{\partial_l} \pi_1(U_{12}) \to \cdots .$$

Directly from this exact sequence, we have

$$\operatorname{Im} \partial_l = R_l.$$

On the other hand,

$$\operatorname{Ker} \partial_l = \pi_2(U_l)$$

using the same homotopy exact sequence and $\pi_2(U_{12}) = 0$.

Thus the result is a translation of Corollary 4.3.9. □

Remark 4.3.12. Whenever U_1, U_2 are based subspaces of X with intersection U_{12} there is always a natural map

$$\sigma : \pi_2(U_1, U_{12}) \circ \pi_2(U_2, U_{12}) \to \pi_2(X, U_{12})$$

determined by the inclusions, but in general σ is not an isomorphism.[55] □

4.4 Some special cases of the coproduct

We end this chapter by giving a careful description of the coproduct of crossed P-modules $\mu\colon M \to P$, $\nu\colon N \to P$ in the useful special case when $\nu(N) \subseteq \mu(M)$ and there is a P-equivariant section $\sigma\colon \mu M \to M$ of μ. Notice that this includes the case when $M = P$ and μ is the identity.

This case is important because of the topological applications and also because it is useful in Section 5.6 for describing as a coproduct the crossed module induced by a monomorphism.

We start with some general results that will be used several times in this book.

Definition 4.4.1. If the group M acts on the group N we define $[N, M]$ to be the subgroup of N generated by the elements, often called *pseudo-commutators*, $n^{-1}n^m$ for all $n \in N$, $m \in M$. This subgroup is called the *displacement subgroup displacement subgroup* y and measures how much N is moved under the M-action. □

The following result is analogous to a standard result on the commutator subgroup.

Proposition 4.4.2. *The displacement subgroup $[N, M]$ is a normal subgroup of N.*

Proof. It is enough to prove that the conjugate of any generator of $[N, M]$ lies also in $[N, M]$.

Let $m \in M$, $n, n_1 \in N$. We easily check that

$$n_1{}^{-1}(n^{-1}n^m)n_1 = ((nn_1)^{-1}(nn_1)^m)(n_1{}^{-1}n_1{}^m)^{-1}$$

and the product on the right-hand side belongs to $[N, M]$ since both factors are generators. So we have proved $n_1{}^{-1}[N, M]n_1 \subseteq [N, M]$, whence $[N, M]$ is a normal subgroup of N. □

Definition 4.4.3. We write $N_M = N/[N, M]$ for the quotient of N by the displacement subgroup. The class in N_M of an element $n \in N$ is written $[n]$. It is clear that N_M is a trivial M-module since $[n^m] = [n]$. □

Proposition 4.4.4. *Let $\mu\colon M \to P$, $\nu\colon N \to P$ be crossed P-modules, so that M acts on N via μ. Then P acts on N_M by $[n]^p = [n^p]$. Moreover this action is trivial when restricted to μM.*

Proof. To see that the P-action on N induces one on N_M, we have to check that $[N, M]$ is P-invariant. This follows from

$$(n^{-1}n^m)^p = (n^{-1})^p(n^m)^p = (n^p)^{-1}(n^p)^{m^p}$$

for all $n \in N$, $m \in M$, $p \in P$.

The action of μM is trivial since $[n]^{\mu m} = [n^{\mu m}] = [n^m] = [n]$. □

Now we study the homomorphism

$$\xi \colon M \times N_M \to P, \quad (m, [n]) \mapsto \mu m.$$

We have just seen that N_M is a P-group.

Proposition 4.4.5. *With P acting on $M \times N_M$ by the diagonal action, $\xi \colon M \times N_M \to P$ is a precrossed P-module.*

Proof. If $m \in M$, $n \in N$, $p \in P$ then

$$\begin{aligned} \xi((m,[n])^p) &= \xi(m^p, [n^p]) \\ &= \mu(m^p) \\ &= p^{-1}(\mu m)p \\ &= p^{-1}(\xi(m,[n])^p))p. \end{aligned}$$ □

Remark 4.4.6. In general ξ is not a crossed module. Nevertheless when N_M is abelian, the actions of both factors on each other are trivial. In this case it follows from Remark 4.2.4 that $\xi \colon M \times N_M \to P$ is a crossed module. It is also an easy exercise to prove this directly. □

Proposition 4.4.7. *Let $\mu \colon M \to P$, $\nu \colon N \to P$ be crossed P-modules such that $\nu N \subseteq \mu M$. Then N_M is abelian and therefore $\xi \colon M \times N_M \to P$ is a crossed P-module.*

Proof. Let $n, n_1 \in N$. Choose $m \in M$ such that $\nu n_1 = \mu m$. Then by the crossed module rule CM2)

$$n_1{}^{-1} n n_1 = n^{\nu n_1} = n^{\mu m}$$

and so in the quotient $[n_1]^{-1}[n][n_1] = [n^{\mu m}] = [n]$. □

We now study the case where there is also a P-equivariant section $\sigma \colon \mu M \to M$ of μ defined on μM. We will see that in this case $\xi \colon M \times N_M \to P$ is isomorphic to the coproduct $\mathcal{M} \circ \mathcal{N}$ of crossed P-modules.[56]

Theorem 4.4.8. *Let $\mu \colon M \to P$, $\nu \colon N \to P$ be crossed P-modules with $\nu N \subseteq \mu M$ and let $\sigma \colon \mu M \to M$ be a P-equivariant section of μ. Then the morphisms of crossed P-modules*

$$\begin{aligned} i \colon M &\to M \times N_M, & j \colon N &\to M \times N_M, \\ m &\mapsto (m, 1), & n &\mapsto (\sigma \nu n, [n]), \end{aligned}$$

give a coproduct of crossed P-modules. Hence the canonical morphism of crossed P-modules

$$M \circ N \to M \times N_M$$

given by $m \circ n \mapsto (m(\sigma \nu n), [n])$ is an isomorphism.

Proof. We need to verify that the pair (i, j) satisfies the universal property of the coproduct of crossed P-modules. Consider an arbitrary crossed P-module $\chi\colon C \to P$ and morphisms of crossed P-modules $\beta\colon M \to C$, and $\gamma\colon N \to C$. We have the following diagram:

$$\begin{tikzcd}
M \arrow[dr, "i"] \arrow[ddr, bend right, "\beta"'] & & N \arrow[dl, "j"'] \arrow[ddl, bend left, "\gamma"] \\
 & M \times N_M \arrow[d, dotted, "\phi"] & \\
 & C &
\end{tikzcd}$$

and we want to prove that there is a unique $\phi\colon M \times N_M \to C$ determining a morphism of crossed P-modules closing the diagram i.e. such that $\phi i = \beta$, and $\phi j = \gamma$.

Let us consider uniqueness. For any $m \in M, n \in N$, since ϕ has to be a homomorphism, we have

$$\begin{aligned}
\phi(m, [n]) &= \phi((m, 0)(\sigma\nu n, 0)^{-1}(\sigma\nu n, [n])) \\
&= (\beta m)(\beta\sigma\nu n)^{-1}(\gamma n).
\end{aligned}$$

This proves uniqueness of any such a ϕ. We now prove that this formula gives a well-defined morphism.

It is immediate from the formula that $\phi\colon M \times N_M \to C$ has to be β on the first factor and is defined on the second one by the map $[n] \mapsto (\beta\sigma\nu n)^{-1}(\gamma n)$. We have to check that this latter map is a well-defined homomorphism.

We define the function

$$\psi\colon N \to C$$

by $n \mapsto (\beta\sigma\nu n^{-1})(\gamma n)$ and prove in turn the following statements.

4.4.9. $\psi(N) \subseteq Z(C)$, the centre of C, and $\chi(C)$ acts trivially on $\psi(N)$.

Proof of 4.4.9. Since $\chi\beta = \mu$ and $\chi\gamma = \nu$, it follows that $\chi\psi = 0$ and $\psi(N) \subseteq \operatorname{Ker}\chi$. Since C is a crossed module, $\chi(C)$ acts trivially on $\operatorname{Ker}\chi$ and $\operatorname{Ker}\chi \subseteq Z(C)$. □

4.4.10. ψ is a morphism of crossed P-modules.

Proof of 4.4.10. We have to prove that ψ is a morphism and is P-equivariant. The latter is clear, since $\beta, \gamma, \sigma, \nu$ are P-equivariant. So let $n, n_1 \in N$. Then

$$\begin{aligned}
\psi(nn_1) &= (\beta\sigma\nu n_1^{-1})(\beta\sigma\nu n^{-1})(\gamma n)(\gamma n_1) \\
&= (\beta\sigma\nu n_1^{-1})(\psi n)(\gamma n_1) \\
&= (\psi n)(\beta\sigma\nu n_1^{-1})(\gamma n_1) && \text{by (4.4.9)} \\
&= (\psi n)(\psi n_1).
\end{aligned}$$

□

Note that even if σ is not P-equivariant, ψ is still a group homomorphism.

4.4.11. M acts trivially on $\psi(N)$.

Proof of 4.4.11. Let $m \in M$, $n \in N$. Note that $(\beta\sigma\mu m)(\beta m^{-1})$ lies in Ker χ, and so belongs to $Z(C)$. Hence

$$\begin{aligned}
(\psi n)^m &= (\beta\sigma\nu n^m)^{-1}(\gamma n)^{\mu m} \\
&= \beta\sigma((\mu m^{-1})(\nu n)(\mu m))^{-1}(\gamma n)^{\chi\beta m} \\
&= (\beta\sigma\mu m^{-1})(\beta\sigma\nu n^{-1})(\beta\sigma\mu m)(\beta m^{-1})(\gamma n)(\beta m) \\
&= (\beta\sigma\mu m^{-1})(\beta\sigma\mu m)(\beta m^{-1})(\beta\sigma\nu n^{-1})(\gamma n)(\beta m) \\
&= (\beta m^{-1})(\psi n)(\beta m) \\
&= \psi n \qquad \text{by (4.4.9).}
\end{aligned}$$
□

It follows that ψ induces a morphism $\psi' \colon N_M \to C$, $[n] \mapsto \psi n$, and so we define

$$\phi = (\beta, \psi') \colon M \times N_M \to C$$

by $(m, [n]) \mapsto (\beta m)(\psi n)$. Since ψn commutes with βm we easily verify that ϕ is a homomorphism, $\phi i = \beta$, $\phi j = \gamma$ and $\chi\phi = \xi$. Thus the pair of morphisms $i \colon M \to M \times N_M$, $j \colon N \to M \times N_M$ satisfies the universal property of a coproduct. This completes the proof of the theorem. □

A standard consequence of the existence of a homomorphism $\sigma \colon \mu M \to M$ which is a section of μ on μM is that M is isomorphic to the semidirect product $\mu M \ltimes \operatorname{Ker} \mu$, where μM acts on Ker μ by conjugation, i.e. $m'^{\mu m} = m^{-1} m' m$. Moreover, in the case when μ is a crossed module and σ is P-equivariant, the isomorphism is as crossed P-modules. Thus we have a third expression for the coproduct.

Proposition 4.4.12. *Let $\mu \colon M \to P$ and $\nu \colon N \to P$ be crossed P-modules with $\nu N \subseteq \mu M$ and let $\sigma \colon \mu M \to M$ be a P-equivariant section of μ. There is an isomorphism of crossed P-modules*

$$M \circ N \cong (\mu M \ltimes \operatorname{Ker} \mu) \times N_M$$

given by $m \circ n \mapsto (m(\sigma\mu m)^{-1}, (\mu m)(\nu n), [n])$.

We now give a topological application.

Corollary 4.4.13. *Let (Y, A) be a connected based pair of spaces, and let $X = Y \cup CA$ be obtained from Y by attaching a cone on A. Then there is an isomorphism of crossed $\pi_1(A)$-modules*

$$\pi_2(X, A) \cong \pi_1(A) \times \pi_2(Y, A)_{\pi_1(A)}.$$

Proof. We apply Theorem 4.3.3 with $U_1 = CA$, $U_2 = Y$, so that $U_{12} = A$. Then $\pi_2(CA, A) \cong \pi_1(A)$, by the exact sequence of the pair (CA, A), so that we have

$$\pi_2(X, A) \cong \pi_1(A) \circ \pi_2(Y, A).$$

The result now follows from Theorem 4.4.8. $\square$

As another application of Theorem 4.4.8, we analyse the symmetry of the coproduct in a special case.

The symmetry morphism $\tau\colon M \circ N \to N \circ M$ is, as usual for a coproduct, given by the pair of canonical morphisms $M \to N \circ M$, $N \to N \circ M$. Hence τ is given by $m \circ n \mapsto (1 \circ m)(n \circ 1) = n \circ m^n$.

Proposition 4.4.14. *Let $\mu\colon M \to P$ be the crossed module given by the inclusion of the normal subgroup M of the group P. Then the isomorphism of crossed P-modules*

$$\theta\colon M \circ M \to M \times M^{\mathrm{ab}},$$
$$\theta(m \circ n) = (mn, [n]),$$

transforms the twist isomorphism $\tau\colon M \circ M \to M \circ M$ to the isomorphism

$$\theta^{-1}\tau\theta\colon M \times M^{\mathrm{ab}} \to M \times M^{\mathrm{ab}},$$
$$(m, [n]) \mapsto (m, [n^{-1}m]).$$

Proof. Notice that in this case $M^{\mathrm{ab}} = M_M$. The isomorphism $\theta\colon M \circ M \to M \times M^{\mathrm{ab}}$ is given in Theorem 4.4.8. The twist isomorphism is transformed into the composition

$$(m, [n]) \mapsto mn^{-1} \circ n \mapsto n \circ (mn^{-1})^n = n \circ n^{-1}m \mapsto (m, [n^{-1}m]). \qquad \square$$

For an application in Section 5.6, we now extend the previous results to more general coproducts. We first prove:

Proposition 4.4.15. *Let T be an indexing set, and let $\mu\colon M \to P$ and $\nu_t\colon N_t \to P$, $t \in T$, be crossed P-modules. Let*

$$N = \bigcirc_{t \in T} N_t.$$

Suppose that $\nu_t N_t \subseteq \mu M$ for all $t \in T$. Then there is an isomorphism of P-modules

$$N_M \cong \bigoplus_{t \in T} : (N_t)_M.$$

Proof. Since $N = \bigcirc_{t \in T} N_t$ is the quotient of the free product $*N_t$ by the Peiffer relations, N_M can be presented as the same free product with the Peiffer relations

$$n_s^{-1} n_t^{-1} n_s n_t^{\nu_s n_s} = 1$$

and the relations $n_t^{\mu m} = n_t$ for all $n_s \in N_s$, $n_t \in N_t$, $m \in M$.

These relations are equivalent to the commutator relations $[n_s, n_t] = 1$ together with $n_t^{\mu m} = n_t$ for all $n_s \in N_s$, $n_t \in N_t$, $m \in M$. $\square$

Corollary 4.4.16. *Suppose in addition that the restriction $\mu| \colon M \to \mu M$ of μ has a P-equivariant section σ. Then there are isomorphisms of crossed P-modules between*

(i) $M \circ (\bigcirc_{t\in T} N_t)$,
(ii) $\xi \colon M \times \bigoplus_{t\in T} (N_t)_M \to P$, $\xi(m,n) = \mu m$, *and*
(iii) $\xi\eta^{-1} \colon \mu M \times \operatorname{Ker}\mu \times \bigoplus_{t\in T} (N_t)_M \to P$.

Under the first isomorphism, the coproduct injections $i \colon M \to M \circ (\bigcirc_{t\in T} N_t)$, $j_t \colon N_t \to M \circ (\bigcirc_{t\in T} N_t)$ are given by $m \mapsto (m,0)$, $n_t \mapsto (\sigma\nu_t n_t, [n_t])$. □

When T is well-ordered, we may also obtain explicit isomorphisms by writing a typical element of $\bigcirc_{t\in T} N_t$ as $\bigcirc_{t\in T} n_t$, and by writing a finite product of elements $\nu_t n_t \in P$ as $\prod_{t\in T} \nu_t n_t$.

Corollary 4.4.17. *Under the same assumptions, and also when T is well-ordered, there are isomorphisms*

$$\begin{aligned} M \circ (\bigcirc_{t\in T} N_t) &\cong M \times \bigoplus_{t\in T} (N_t)_M, \\ m \circ (\bigcirc_{t\in T} n_t) &\mapsto \Big(m\big(\prod_{t\in T}(\sigma\nu_t n_t)\big), \bigoplus_{t\in T}[n_t]\Big), \end{aligned}$$

and

$$\begin{aligned} M \circ (\bigcirc_{t\in T} N_t) &\cong \mu M \times \operatorname{Ker}\mu \times \bigoplus_{t\in T} (N_t)_M, \\ m \circ (\bigcirc_{t\in T} n_t) &\mapsto \Big(m(\sigma\mu m^{-1}),\ (\mu m)\big(\prod_{t\in T} \nu_t n_t\big), \bigoplus_{t\in T}[n_t]\Big). \end{aligned}$$ □

Notes

53 p. 87 The free product of groups is well studied in books on combinatorial group theory such as [LS01], and is also a special case of a groupoid construction, namely the universal group of a disjoint union of groups, viewed as a groupoid, as in [Hig71], [Bro06].

54 p. 89 The coproduct of two crossed P-modules is the case that has been analysed more deeply in the literature. Most of the results of this section were in print for the first time in [Bro84a]. The basic observation in [Bro84a] is that $M \circ N$ may be obtained as a quotient of the semidirect product group $M \ltimes N$. Further results were obtained in [GH89], and some more applications and results are also given in [HAMS93]. An analogous construction for two groups M, N which act on each other is also called the *Peiffer product* of M and N and is often written

$M \bowtie N$, see [GH89], [BG00]. The construction of this product as a quotient of the free product really goes back to Whitehead [Whi41a], p. 428, but his topological application is only to 2-dimensional cell complexes.

55 p. 97 Bogley and Gutiérrez in [BG92] have had some success in describing Ker σ and Coker σ in the case when all the above spaces are connected. It would also be interesting to know if these methods can be applied to the study of unions of Cockcroft complexes, compare [Bog94].

56 p. 99 The condition $\nu N \subseteq \mu M$ was used in this context in [GH89]. Theorem 4.4.8 contains the main result of [GH89] in the sense that it determines explicitly the coproduct structure. We follow the later proof given in [BW96].

Chapter 5
Induced crossed modules

In the previous chapter we used coproducts of crossed modules to describe certain relative homotopy groups. Essentially, we first obtain information on homotopy 2-types, in terms of crossed modules, and then try to deduce from this information on second homotopy groups as modules over the fundamental group. Traditional methods of getting at this operation are usually via covering spaces, and are not so convenient or direct.

Induced crossed modules, which are the subject of this chapter, give another construction which allows detailed computations of nonabelian information on some second relative homotopy groups; they arise topologically on applying the 2-dimensional Seifert–van Kampen Theorem 2.3.3 to a pushout of pairs of spaces of the form of the following left-hand square

$$\begin{array}{ccc} (A,A) & \xrightarrow{(f,1)} & (X,A) \\ {\scriptstyle (i,i)}\downarrow & & \downarrow \\ (Y,Y) & \longrightarrow & (X\cup_f Y, Y), \end{array} \qquad \begin{array}{ccc} (1\to\pi_1(A)) & \longrightarrow & \Pi_2(X,A) \\ \downarrow & & \downarrow \\ (1\to\pi_1(Y)) & \longrightarrow & \Pi_2(X\cup_f Y, Y) \end{array}$$

to give the right-hand pushout square of crossed modules. The left-hand square gives a format for what is known topologically as *excision*, since if both maps $i: A \to Y$, $f: A \to X$ are closed inclusions then $X \cup_f Y$ with Y cut out, or excised, is the same as X with A excised.

For homology, if (X, A) is closed and cofibred, we end up with isomorphisms $H_n(X, A) \to H_n(X \cup_f Y, Y)$.

This is by no means so for relative homotopy groups. The induced construction illustrates a feature of homotopy theory:

> identifications in low dimensions can strongly influence
> high dimensional homotopy.

The Higher Homotopy Seifert–van Kampen Theorems give information on how this influence is controlled, with some results not obtainable otherwise.

With these methods we obtain some standard result, for example: the Relative Hurewicz Theorem in dimension 2; the theorem of Whitehead on free crossed modules; and a formula of Hopf for the second homology of an aspherical space, which was one of the starting points of the important theory of the homology and cohomology of groups. These applications gave a model for the higher dimensional results in Chapter 8, and for the stronger results of [BL87a].

The constructions in this chapter can be elaborate and in places technical. This illustrates the complications of 2-dimensional homotopy theory, and also the aspects with which the algebra can cope.

The crossed module 'induced' by a homomorphism of groups $f\colon P \to Q$ may be seen as one of the family of 'change of base' functors of algebraic categories that have proved useful in many fields from algebraic geometry to homological algebra. A general account of induced constructions in the context of cofibred categories is given in Section B.2 of Appendix B.

The construction of the induced crossed module follows a natural pattern. Given the morphism f as above and a crossed P-module $\mu\colon M \to P$, we need a new group N, depending on M and f, on which Q acts so that N is a candidate for a crossed Q-module. Therefore we need new elements of the form m^q for $m \in M, q \in Q$. Since these do not for the moment exist, we form the free group on pairs (m, q) and then add appropriate relations. This is done in detail in Section 5.3.

In Section 5.1 we describe the pullback crossed module $f^*(M)$ of a crossed P-module M. This is quite easy to construct and the existence of the induced crossed module $f_*(M)$ defined in Section 5.2 is essentially the construction of a left adjoint to the pullback construction. We prove by the universal property that the free crossed module of Section 3.4 is a particular case of the induced crossed module and that an induced crossed module is the pushout of M and the trivial crossed module $1 \to Q$ over the trivial crossed module $1 \to P$.

That leaves the induced crossed module ready to be used in applications of the 2-dimensional Seifert–van Kampen Theorem. In Section 5.4 we prove that when the topological space X is the union of two path connected sets U_1, U_2 such that the pair (U_2, U_{12}) is 1-connected, then the pair (X, U_1) is 1-connected and the fundamental crossed module $\Pi_2(X, U_1)$ is the crossed module induced from $\Pi_2(U_2, U_{12})$ by the homomorphism $i_*\colon \pi_1(U_{12}) \to \pi_1(U_2)$ induced by the inclusion (Theorem 5.4.1). As a consequence we get some homotopical results, in particular Whitehead's theorem on free crossed modules.

The second part of the chapter analyses in more detail the construction of the induced crossed module. In particular we study in the next two sections the cases when f is surjective or injective.

The surjective case (Section 5.5) is quite direct and we prove that $f_*(M)$ is the quotient of M by the displacement subgroup $[M, \operatorname{Ker} f]$ defined in 4.4.1. This case has some interesting topological applications, in particular the Relative Hurewicz Theorem in dimension 2 and Hopf's formula for the second homology group of a group.

The case when f is injective, i.e. a monomorphism (Section 5.6), is essentially the inclusion of a subgroup. This case is much more intricate and we need the concept of the copower construction $M^{\circ T}$ where T is a transversal of P in Q. We get a description of the induced crossed module as a quotient of the copower (Corollary 5.6.6). Both the group and the action have alternative descriptions that can be used to develop some examples, so obtaining in particular a bound for the number of generators and relations for an induced crossed module.

It is also proved (in Section 5.7) that the induced crossed module is finite when both M and the index $[P : Q]$ are finite. This suggests the problem of explicit computation, and in the last section of the chapter we explain some computer calculations in the finite case obtained using the package GAP.

The next section (5.8) contains a detailed description of the induced crossed module in a useful special case, with many interesting examples, namely when P and M are both normal subgroups of Q. We start by studying the induced crossed module when P is a normal subgroup of Q, getting a description in terms of the coproduct $M^{\circ T}$. Then we use the description of the coproduct given in the last section of the preceding chapter to derive just from the universal property both the action (Theorem 5.8.6) and the map (Theorem 5.8.7). When M is itself another normal subgroup included in P, we get some more concrete formulas.

This leaves many finite examples not covered by the previous theorems: the last section gives some computer calculations.[57]

In Section 8.3.iii we will also indicate ways of generalising methods of this chapter to the case of many base points.

The general categorical background to these methods is given in Appendix B.

5.1 Pullbacks of precrossed and crossed modules

The work of this section can be done both for crossed and for precrossed modules. We shall state only the crossed case but, if nothing is said, it is understood that a similar result is true for precrossed modules. We shall not repeat the statement, but we only give indications of the differences.

Let us start by defining the functor that is going to be the right adjoint of the induced crossed module, the 'pullback'. This is an important construction which, given a morphism of groups $f : P \to Q$, enables us to move from crossed Q-modules to crossed P-modules.

Definition 5.1.1. Let $f : P \to Q$ be a homomorphism of groups and let $\mathcal{N} = (\nu : N \to Q)$ be a crossed module. We define the subgroup of $N \times P$

$$f^*N = N \times_Q P = \{(n, p) \in N \times P \mid \nu n = fp\}.$$

This is the usual pullback in the category Groups. There is a commutative diagram

$$\begin{tikzcd} f^*N \arrow[r, "\bar f"] \arrow[d, "\bar\nu"'] & N \arrow[d, "\nu"] \\ P \arrow[r, "f"'] & Q \end{tikzcd}$$

where $\bar{\nu}\colon (n, p) \mapsto p$, $\bar{f}\colon (n, p) \mapsto n$. Then P acts on f^*N via f and the diagonal, i.e. $(n, p)^{p'} = (n^{fp'}, p'^{-1}pp')$. It is easy to see that this gives a P-action. The *pullback crossed module* is

$$f^*\mathcal{N} = (\bar{\nu}\colon f^*N \to P).$$

It is also called the pullback of $\mathcal{N}$ along f and it is easy to see that $f^*\mathcal{N}$ is a crossed module. □

This construction satisfies a crucial universal property, analogous to that of the pullback of groups. To state it, we use also the morphism of crossed modules

$$(\bar{f}, f)\colon f^*\mathcal{N} \longrightarrow \mathcal{N}.$$

Theorem 5.1.2. *For any crossed module $\mathcal{M} = (\mu\colon M \to P)$ and any morphism of crossed modules*

$$(h, f)\colon \mathcal{M} \longrightarrow \mathcal{N}$$

there is a unique morphism of crossed P-modules $(h', 1)\colon \mathcal{M} \to f^*\mathcal{N}$ such that the following diagram commutes:

$$\begin{tikzcd}
M \arrow[dr, dashed, "h'"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "\mu"'] & & \\
& f^*N \arrow[r, "\bar{f}"'] \arrow[d, "\bar{\nu}"] & N \arrow[d, "\nu"'] \\
& P \arrow[r, "f"] & Q.
\end{tikzcd}$$

Proof. The existence and uniqueness of the homomorphism h' follows from the fact that f^*N is the pullback in the category of groups. It is defined by $h'(m) = (h(m), \mu(m))$. So we have only to prove that h' is a morphism of crossed P-modules. This can be checked directly. □

Using this universal property, it is not difficult to see that this construction gives a functor

$$f^*\colon \mathsf{XMod}/Q \to \mathsf{XMod}/P.$$

Remark 5.1.3. These functors have the property that for any homomorphisms $f\colon P \to Q$ and $f'\colon Q \to R$ there are natural equivalences $f^*f'^* \simeq (f'f)^*$. We refer to Appendix B and in particular to Theorem B.1.7 for general background and on the use of this pullback construction. □

5.2 Induced precrossed and crossed modules

Now we define a functor f_* left adjoint to the pullback f^* of the previous section. In particular we prove that the free crossed module is a particular case of an induced crossed module. Then we apply this to the topological case to get Whitehead's theorem (Corollary 5.4.8).

The 'induced crossed module' functor is defined by the following universal property, adjoint to that of pullback.

Definition 5.2.1. For any crossed P-module $\mathcal{M} = (\mu\colon M \to P)$ and any homomorphism $f: P \to Q$ the crossed module *induced* by f from $\mathcal{M}$ should be given by:

i) a crossed Q-module $f_*\mathcal{M} = (f_*\mu\colon f_*M \to Q)$;

ii) a morphism of crossed modules $(\phi, f)\colon \mathcal{M} \to f_*\mathcal{M}$, satisfying the dual universal property that for any morphism of crossed modules

$$(h, f)\colon \mathcal{M} \longrightarrow \mathcal{N}$$

there is a unique morphism of Q-crossed modules $h'\colon f_*M \to N$ such that the diagram

$$\begin{tikzcd}
 & & N \arrow[dd, bend left, "\nu"] \\
M \arrow[r, "\phi"'] \arrow[urr, bend left, "h"] \arrow[d, "\mu"] & f_*M \arrow[ur, dotted, "h'"] \arrow[d, "f_*\mu"'] & \\
P \arrow[r, "f"] & Q &
\end{tikzcd}$$

commutes. □

Now we prove that this functor if it exists, forms an adjoint pair with the pullback functor. Using general categorical considerations, we can deduce the existence of the induced crossed module functor

$$f_*\colon \mathsf{XMod}/P \to \mathsf{XMod}/Q$$

and, also, that they satisfy the 'naturality condition' that there is a natural equivalence of functors $f'_*f_* \simeq (f'f)_*$.

Theorem 5.2.2. *For any homomorphism of groups $f: P \to Q$, f_* is the left adjoint of f^*.*

Proof. From the naturality conditions expressed earlier, it is immediate that for any crossed modules $\mathcal{M} = (\mu\colon M \to P)$ and $\mathcal{N} = (\nu\colon N \to Q)$ there are bijections

$$(\mathsf{XMod}/P)(\mathcal{N}, f^*\mathcal{N}) \cong \{h\colon M \to N \mid (h, f)\colon \mathcal{M} \to \mathcal{N} \text{ is a morphism in } \mathsf{XMod}\},$$

as proved in Proposition 5.1.2 , and

$$(\mathsf{XMod}/Q)(f_*\mathcal{M},\mathcal{N}) \cong \{h\colon M \to N \mid (h,f)\colon \mathcal{M} \to \mathcal{N} \text{ is a morphism in } \mathsf{XMod}\}$$

as given in the definition.

Their composition gives the bijection needed for adjointness. □

We end this section by comparing the universal properties defining the induced crossed module and two other constructions. The first one is the free crossed module on a map. Using the induced crossed module, we get an alternative description of the free crossed module.

Proposition 5.2.3. *Let P be a group and $\{\omega_r \mid r \in R\}$ be an indexed family of elements of P, or, equivalently, a map $\omega\colon R \to P$. Let F be the free group generated by R and $f : F \to P$ the homomorphism of groups such that $f(r) = \omega_r \in P$. Then the crossed module $f_*(1_F)\colon f_*F \to P$ induced from $1_F = (1_F : F \to F)$ by f is the free crossed P-module on $\{(1,r) \in f_*F \mid r \in R\}$.*

Proof. Both universal properties assert the existence of morphisms of crossed P-modules commuting the appropriate diagrams. Let us check that the data in both constructions are equivalent.

The data in the induced crossed module are a crossed module $\mathcal{N}$ and a morphism of crossed modules $(h,f)\colon 1_F \to \mathcal{N}$. The data in the free crossed module are a crossed module $\mathcal{N}$ and a map $\omega'\colon R \to N$ lifting ω. Since F is the free group on R, the map ω' is equivalent to a homomorphism of groups $h\colon F \to N$ lifting ω (i.e. $h(r) = \omega'(r)$). Moreover, h satisfies

$$h(r^{r'}) = h(r'^{-1}rr') = h(r')^{-1}h(r)h(r') = (hr)^{\nu h(r')} = (hr)^{f(r')} \tag{5.2.1}$$

for all $r, r' \in R$. So h preserves the action and (h,f) is a morphism of crossed modules.

Thus the data in both cases are equivalent. □

Remark 5.2.4. It is clear that the proof in Proposition 5.2.3 does not work for precrossed modules since in proving the equality (5.2.1) we have used axiom CM2). It is easy to see that the precrossed module induced from $1_F : F \to F$ is not the free precrossed module but its quotient with respect to the normal subgroup generated by all relations

$$(p, r^{r'}) = (p\omega(r), r')$$

for $p \in P$ and $r, r' \in R$.

It is a nice exercise to find a precrossed module $L(R) \to F(R)$ such that any free precrossed Q-module on generators R is induced from $L(R)$ by a morphism $F(R) \to Q$. □

We now give an important re-interpretation of induced crossed modules in terms of pushouts of crossed modules. This is how we can show that induced crossed modules arise from a 2-dimensional Seifert–van Kampen Theorem. The proof is obtained by relating the two universal properties. The general situation of which this proof is an example is given using notions of fibrations of categories in Proposition B.2.8 and Theorem B.3.2 of Appendix B.

Proposition 5.2.5. *For any crossed module $\mathcal{M} = (\mu\colon M \to P)$ and any homomorphism $f\colon P \to Q$, the induced crossed module $f_*\mathcal{M}$ is exactly given by the condition that the commutative diagram of crossed modules*

$$\begin{tikzcd}
(1 \to P) \arrow[r, "{(0,f)}"] \arrow[d, "{(0,1)}"'] & (1 \to Q) \arrow[d, "{(0,1)}"] \\
(M \to P) \arrow[r, "{(\phi,f)}"'] & (f_*M \to Q)
\end{tikzcd}$$

is a pushout of crossed modules.

Proof. To check that the diagram satisfies the universal property of the pushout, let $\mathcal{N} = (\nu\colon N \to R)$ be a crossed module, and $(h, f')\colon \mathcal{M} \to \mathcal{N}$ and $(0, g)\colon 1_Q \to \mathcal{N}$ morphisms of crossed modules, such that the diagram of full arrows commutes. We have to construct the dotted morphism of crossed modules (k, g):

$$\begin{tikzcd}
(1 \to P) \arrow[r, "{(0,f)}"'] \arrow[d, "{(0,1)}"'] & (1 \to Q) \arrow[d, "{(0,1)}"'] \arrow[ddr, bend left, "{(0,g)}"] & \\
(M \to P) \arrow[r, "{(\phi,f)}"] \arrow[drr, bend right, "{(h,f')}"'] & (f_*M \to Q) \arrow[dr, dotted, "{(k,g)}"] & \\
& & (N \to R).
\end{tikzcd}$$

It is immediate that $f' = gf$, $k\phi = h$. So we can transform morphisms in turn

$$(M \to P) \xrightarrow{(k\phi,\, gf)} (N \to R)$$

$$(M \to P) \xrightarrow{(\overline{k\phi},\, 1)} ((gf)^*N \to P)$$

$$(M \to P) \xrightarrow{(\,\overline{k\phi},\, 1)} (f^*g^*N \to P)$$

$$(f_*M \to Q) \xrightarrow{(\bar{\phi},\, 1)} (g^*N \to Q)$$

$$(f_*M \to Q) \xrightarrow{(k,\, g)} (N \to R)$$

as required. $\square$

Remark 5.2.6. If $f: P \to Q$ is a morphism of groups and M is a P-module, then there is a well-known definition of an *induced Q-module* f_*M as a tensor product $M \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ where P acts on Q on the left via f. If $f: P \to Q$ is the inclusion of a subgroup, and T is a transversal of P in Q, then there is an alternative description of f_*M as the direct sum of copies M_t of M for all $t \in T$, with action $(m,t)^q = (m^p, t')$ where $t, t' \in T$, $p \in P$, $q \in Q$ satisfy $pt' = tq$. Accounts of this will be found in books on the representation theory of groups, e.g. [CR06]. These constructions are also part of the general categorical setting of Appendix B, and Proposition 5.2.5 is a special case of Proposition B.3.2. □

5.3 Construction of induced crossed modules

We now give a direct construction of the induced crossed module, thus showing its existence. The problem of evaluating particular examples is dealt with in later sections of this chapter.

We are going to construct the induced crossed module in two steps, producing first the induced precrossed module and then from this the associated crossed module by quotienting out by its Peiffer subgroup.

Let us start with a homomorphism of groups $f: P \to Q$ and a crossed module $\mathcal{M} = (\mu: M \to P)$.

First we consider the free precrossed Q-module generated by $f\mu$. As seen in Section 3.4 it is

$$\theta: F_Q(M) \to Q$$

where $F_Q(M)$ is the free group generated by the elements of $M \times Q$ (to make things easier to remember, we think of (m,q) as m^q, i.e. m operated on by q), the Q-action on $F_Q(M)$ is given on generators by

$$(m,q)^{q'} = (m, qq')$$

for any $q, q' \in Q$ and $m \in M$ and the map θ is given on generators by

$$\theta(m,q) = q^{-1} f\mu(m) q.$$

It is a precrossed Q-module as seen in Section 3.4.

To get the *induced precrossed module* from this map θ, we take into the picture both the P-action and the multiplication on M, making a quotient by the appropriate normal subgroup. Let S be the normal subgroup generated by all the relations of the two following types:

$$(m,q)(m',q) = (mm',q), \tag{5.3.1}$$

$$(m^p,q) = (m, f(p)q) \tag{5.3.2}$$

for any $m, m' \in M$, $p \in P$, $q \in Q$. We define $E = F_Q(M)/S$. It is easy to see that the action of Q on $F_Q(M)$ induces one on E. Also, θ induces a precrossed module

$$\hat{\mu}\colon E \to Q.$$

There is a map

$$\phi\colon M \to E$$

got by projecting the map on F defined as $\phi(m) = (m, 1)$. This map is a morphism of groups thanks to the relations of type (5.3.1), while (ϕ, f) is a morphism of precrossed modules thanks to the relations of type (5.3.2).

Theorem 5.3.1. *The precrossed module $\hat{\mu}\colon E \to Q$ is that induced from μ by the homomorphism f.*

Proof. We have only to check the universal property.

For any morphism of precrossed modules

$$(h, f)\colon (\mu\colon M \to P) \longrightarrow (\nu\colon N \to Q)$$

there is a unique morphism of precrossed Q-modules $h'\colon E \to N$ such that $h = h'\phi$ because the only way to define this homomorphism is by $h'(m, q) = (hm)^q$ on generators. It is a very easy exercise to check that this definition maps S to 1, and that the induced homomorphism gives a morphism of crossed modules. □

Remark 5.3.2. If $\mathcal{M} = (\mu\colon M \to P)$ is a crossed module, there are two equivalent ways to obtain the induced crossed module $f_*\mathcal{M} = (f_*M \to Q)$. One way is to get the crossed module associated to the precrossed one in the theorem. The second way is to quotient out $F_Q(M)$, not only by the relations of the above two kinds, but also adding the Peiffer relations

$$(m_1, q_1)^{-1}(m_2, q_2)(m_1, q_1) = (m_2, q_2 q_1^{-1} f\mu(m_1) q_1) \tag{5.3.3}$$

for any $q_1, q_2 \in Q$ and $m_1, m_2 \in M$. □

Every morphism of groups decomposes as the composition of a monomorphism and an epimorphism. We give later in Sections 5.5 and 5.6 direct descriptions of induced crossed modules in these two cases.[58]

5.4 Induced crossed modules and the Seifert–van Kampen Theorem in dimension 2

The relation between induced crossed modules and pushouts of crossed modules suggests that the induced crossed module may appear in some cases when using the 2-dimensional Seifert–van Kampen Theorem 2.3.1. After looking to the statement of the theorem for general subspaces $A, U_1, U_2 \subseteq X$ it is easy to see that this case occurs when $A = U_1$; this situation ia also known as 'excision'. We give some background to this idea.

In the situation where the space X is the union $U_1 \cup U_2$, the inclusion of pairs

$$E\colon (U_1, U_1 \cap U_2) \to (X, U_2)$$

is known as the 'excision map' because the smaller pair is obtained by cutting out or 'excising' $X \setminus U_2$ from the larger pair. It is a theorem of homology (the Excision Theorem) that if U_1, U_2 are open in X then the excision map induces an isomorphism of relative homology groups. This is one of the basic results which make homology groups readily computable.

Here we get a result that can be interpreted as a limited form of Excision Theorem for homotopy, but it shows that the excision map is in general not an isomorphism even for second relative homotopy groups. Lack of excision is one of the reasons for the difficulty of computing homotopy groups of spaces.

Theorem 5.4.1 (Homotopical excision in dimension 2). *Let X be a space which is the union of the interior of two subspaces U_1 and U_2 and define $U_{12} = U_1 \cap U_2$. If all spaces are connected and (U_2, U_{12}) is 1-connected, then (X, U_1) is also 1-connected and the morphism of crossed modules*

$$\Pi_2(U_2, U_{12}) \to \Pi_2(X, U_1)$$

realises the crossed module $\Pi_2(X, U_1)$ as induced from $\Pi_2(U_2, U_{12})$ by the homomorphism $\pi_1(U_{12}) \to \pi_1(U_1)$ induced by the inclusion.

Proof. Following the notation of Theorem 2.3.1 with $A = U_1$ we have

$$A_1 = A \cap U_1 = U_1, \quad A_2 = A \cap U_2 = U_{12} \quad \text{and} \quad A_{12} = A \cap U_{12} = U_{12}.$$

It is clear that the hypothesis of Theorem 2.3.1 are satisfied since $(U_1, A_1) = (U_1, U_1)$, $(U_2, A_2) = (U_2, U_{12})$ and $(U_{12}, A_{12}) = (U_{12}, U_{12})$ are 1-connected. The consequence is that the diagram of crossed modules

$$\begin{tikzcd}
\Pi_2(U_{12}, U_{12}) \arrow[r] \arrow[d] & \Pi_2(U_2, U_{12}) \arrow[d] \\
\Pi_2(U_1, U_1) \arrow[r] & \Pi_2(X, U_1)
\end{tikzcd} \tag{5.4.1}$$

is a pushout.

Proposition 5.2.5 now implies the result. □

As in the case of Theorem 2.3.1, using standard mapping cylinder arguments, we can prove a more general statement.[59]

Corollary 5.4.2. *Let (X, A) be a pair of spaces and $f: A \to Y$ a continuous map. If all spaces are connected, the inclusion $i: A \to X$ is a closed cofibration and the pair (X, A) is 1-connected, then the pair $(Y \cup_f X, Y)$ is also 1-connected and $\Pi_2(Y \cup_f X, Y)$ is the crossed module induced from $\Pi_2(X, A)$ by $f_*: \pi_1(A) \to \pi_1(Y)$.*

Proof. This can either be deduced from the proceeding theorem by use of mapping cylinder arguments, or can be seen as a particular case of Theorem 2.3.3 when $U_1 = A$ and $Y_1 = Y$. □

This last corollary has as a consequence a Homotopical Excision Theorem for closed subsets under weak conditions.

Corollary 5.4.3. *Let X be a space that is the union of two closed subspaces U_1 and U_2 and let $U_{12} = U_1 \cap U_2$. If all spaces are connected, the inclusion $U_1 \to X$ is a cofibration, and the pair (U_2, U_{12}) is connected, then the pair (U_1, X) is also connected and the crossed module $(\pi_2(X, U_1) \to \pi_1(U_1))$ is the one induced from $(\pi_2(U_2, U_{12}) \to \pi_1(U_{12}))$ by the morphism $\pi_1(U_{12}) \to \pi_1(U_1)$ induced by the inclusion.*

Before proceeding any further, we consider the case of a space X given as the homotopy pushout of classifying spaces.

Theorem 5.4.4. *Let $\mathcal{M} = (\mu: M \to P)$ be a crossed module, and let $f: P \to Q$ be a morphism of groups. Let $\beta: BP \to B\mathcal{M}$ be the inclusion. Consider the pushout diagram*

$$\begin{tikzcd} BP \arrow[r, "Bf"] \arrow[d, "\beta"'] & BQ \arrow[d, "\beta'"] \\ B\mathcal{M} \arrow[r] & X, \end{tikzcd}$$

i.e. $X = BQ \cup_{Bf} B\mathcal{M}$. Then the fundamental crossed module $\Pi_2(X, BQ)$ is isomorphic to the induced crossed module $f_*\mathcal{M}$.

Further, there is a map of spaces $X \to Bf_*\mathcal{M}$ inducing an isomorphism of the corresponding π_1, π_2.

Proof. This first part is immediate from Corollary 5.4.2.

The last statement requires a generalisation of Proposition 2.4.8, in which the 1-skeleton is replaced by a subcomplex Z with the property that $\pi_2(Z) = 0$ and the induced map $\pi_1(Z) \to \pi_1(X)$ is surjective. (In our case $Z = BQ$.) This result is proved in Proposition 12.3.4. □

Remark 5.4.5. The most striking consequence of the last theorem is that we have determined completely a nontrivial homotopy 2-type of a space. That is, we have replaced geometric constructions by corresponding algebraic ones. As we shall see, induced crossed modules are computable in many cases, and so we can obtain many explicit computations of homotopy 2-types. The further surprise is that all this theory is needed for just this example. This shows the difficulty of homotopy theory, in that new ranges of algebraic structures are required to explain what is going on. □

In the next sections, we will be able to obtain some explicit calculations as a consequence of the last results.

Remark 5.4.6. An interesting special case of the last theorem is when $\mathcal{M}$ is an inclusion $M \trianglelefteq P$ of a normal subgroup, since then $B\mathcal{M}$ has the homotopy type of $B(P/M)$ by Proposition 2.4.6. So we have determined the fundamental crossed module of (X, BR) when X is the homotopy pushout

$$\begin{tikzcd} BP \arrow[r, "Bp"] \arrow[d, "Bf"'] & BR \arrow[d] \\ BQ \arrow[r, "p'"'] & X \end{tikzcd}$$

in which $p \colon P \to R$ is surjective. In this case $\mathcal{M}$ is the crossed module given by the normal inclusion $(\mathrm{Ker}\ p) \to P$. □

To end this section, we consider the useful case when the space we are attaching is a cone.

Theorem 5.4.7. *Let $f: A \to Y$ be a continuous map between connected spaces. Then the pair $(CA \cup_f Y, Y)$ is 1-connected and $\Pi_2(CA \cup_f Y, Y)$ is the crossed module induced from the identity crossed module $1_{\pi_1(A)}$ by $f_* \colon \pi_1(A) \to \pi_1(Y)$.*

Proof. Using part of the homotopy exact sequence of the pair (CA, A),

$$\pi_2(CA, x) = 0 \to \pi_2(CA, A, x) \to \pi_1(A, x) \to \pi_1(CA, x) = 0$$

we get an isomorphism of $\pi_1(A, x)$ groups transforming the fundamental crossed module $\Pi_2(CA, A)$ to the identity crossed module $1_{\pi_1(A,x)}$.

Now, we can use Corollary 5.4.2 and identify the induced crossed module with the free module by Proposition 5.2.3. □

As a consequence we get a proof [60] of Whitehead's theorem on free crossed modules [Whi49b].

Corollary 5.4.8 (Whitehead's theorem on free crossed modules). *Let Y be a space constructed from the path-connected space X by attaching cells of dimension two. Then the map $\pi_1(X) \to \pi_1(Y)$ is surjective and $\Pi_2(Y, X)$ is isomorphic to the free crossed module on the characteristic maps of the 2-cells.*

As before, we apply the results just obtained to the case of a space X which is a pushout of classifying spaces.

Theorem 5.4.9. *Let $f: P \to Q$ be a morphism of groups. Then the crossed module $\Pi_2(BQ \cup_{Bf} CBP, BQ)$ is isomorphic to the induced crossed module $f_*(1_P)$.*

Proof. Taking $R = 1$ in Remark 5.4.6, its classifying space is contractible. Thus, we can take CBP as equivalent to the classifying space BR. □

5.5 Calculation of induced crossed modules: the epimorphism case

Let us consider now the case where $f: P \to Q$ is an epimorphism. Then $\operatorname{Ker} f$ acts on M via the map f and the induced crossed module f_*M may be seen as M quotiented out by the normal subgroup appropriate for trivialising the action of $\operatorname{Ker} f$ (since Q is isomorphic to $P/\operatorname{Ker} f$), i.e. by quotienting out the displacement subgroup studied in Section 4.4.

Proposition 5.5.1. *If $f: P \to Q$ is an epimorphism and $\mu: M \to P$ is a crossed module, then*

$$f_*M \cong \frac{M}{[M, \operatorname{Ker} f]}$$

where $[M, \operatorname{Ker} f]$ is the displacement subgroup of Definition 4.4.1, i.e. the subgroup of M generated by $\{m^{-1}m^k \mid m \in M, k \in \operatorname{Ker} f\}$.

Proof. Let us recall that by Proposition 4.4.7 the quotient $M/[M, \operatorname{Ker} f]$ is a Q-crossed module with the Q-action on $M/[M, \operatorname{Ker} f]$ given by $[m]^q = [m^p]$ for $m \in M, q \in Q, q = f(p), p \in P$, and the homomorphism

$$\overline{f\mu}: \frac{M}{[M, \operatorname{Ker} f]} \to Q,$$

is induced by the composition $f\mu: M \to Q$.

It remains only to prove that this $\overline{f\mu}$ satisfies the universal property. Let

$$(h, f): (\mu: M \to P) \longrightarrow (\nu: N \to Q)$$

be a morphism of crossed modules. We have to prove that there exists a unique homomorphism of groups

$$h': \frac{M}{[M, \operatorname{Ker} f]} \longrightarrow N$$

such that

$$(h', f): \left(\overline{f\mu}: \frac{M}{[M, \operatorname{Ker} f]} \to P\right) \longrightarrow (\nu: N \to Q)$$

is a morphism of crossed modules and $h'\phi = h$ where ϕ is the natural projection. Equivalently, we have to prove that h induces a homomorphism of groups h' and that (h', f) is a morphism of crossed modules.

Since $h(m^p) = (hm)^{f(p)}$ for any $m \in M$ and $p \in P$, we have $h[M, \operatorname{Ker} f] = 1$. Then h induces a homomorphism of groups h' as above such that $h'\phi = h$.

We have only to check that h' is a map of Q-crossed modules. But

$$\nu h'[m] = \nu h(m) = f\mu(m) = \overline{f\mu}[m],$$

so the square commutes, and

$$h'([m]^q) = h'[m^p] = h(m^p) = (hm)^{f(p)} = (h'[m])^q$$

so h' preserves the actions. $\square$

This description gives as a topological consequence a version of the Relative Hurewicz Theorem in dimension 2. A version for all dimensions $\geqslant 3$ is given in Theorem 8.3.19, and a version in dimension 1 is given in Theorem 14.7.7.[61]

Theorem 5.5.2 (Relative Hurewicz Theorem: dim 2). *Consider a 1-connected pair of spaces (Y, A) such that the inclusion $i: A \to Y$ is a closed cofibration. Then the space $Y \cup CA$ is simply connected and its second homotopy group $\pi_2(Y \cup CA)$ and the singular homology group $H_2(Y \cup CA)$ are each isomorphic to $\pi_2(Y, A)$ factored by the action of $\pi_1(A)$.*

Proof. It is clear from the classical Seifert–van Kampen Theorem that the space $Y \cup CA$ is 1-connected.

Using the homotopy exact sequence of the pair $(Y \cup CA, CA)$,

$$\cdots \to 0 = \pi_2(CA) \to \pi_2(Y \cup CA) \to \pi_2(Y \cup CA, CA) \to 0 = \pi_1(CA) \to \cdots$$

we have

$$\pi_2(Y \cup CA) \cong \pi_2(Y \cup CA, CA).$$

Now we can apply Corollary 5.4.2 to show that the crossed module

$$\pi_2(Y \cup CA, CA) \to \pi_1(CA) = 1$$

is induced from $\pi_2(Y, A) \to \pi_1(A)$ by the map given by the morphism $\pi_1(A) \to 1$ induced by the inclusion $A \to CA$.

Moreover, since the map $i_*: \pi_1(A) \to \pi_1(Y)$ is onto, by Proposition 5.5.1 we have

$$\pi_2(Y \cup CA, CA) \cong \frac{\pi_2(Y, A)}{[\pi_2(Y, A), \pi_1(A)]}.$$

This yields the result on the second homotopy group.

The Absolute Hurewicz Theorem for $Y \cup CA$ (which we prove in Theorem 14.7.8) yields the result on the second homology group. $\square$

Corollary 5.5.3. *The first two homotopy groups of S^2 are $\pi_1(S^2) = 0, \pi_2(S^2) \cong \mathbb{Z}$.*

Proof. This is the case of Theorem 5.5.2 when $A = S^1, Y = E^2_+$, where E^2_+ denotes the top hemisphere of the 2-sphere S^2. Then $\pi_2(Y, A) \cong \mathbb{Z}$ with trivial action by $\pi_1(A) \cong \mathbb{Z}$. □

Actually we have a more general result.[62]

Corollary 5.5.4. *If A is a path connected space, and $SA = CA \cup_A CA$ denotes the suspension of A, then SA is simply connected and*

$$\pi_2(SA) \cong \pi_1(A)^{\mathrm{ab}}.$$

Proof. This is simply the result that $\pi_1(A)^{\mathrm{ab}} = \pi_1(A)/[\pi_1(A), \pi_1(A)]$. □

Example 5.5.5. Let $f : A \to Y$ be as in Theorem 5.4.7, let $Z = Y \cup_f CA$, and suppose that $f_* : \pi_1(A) \to \pi_1(Y)$ is surjective with kernel K. An application of Proposition 5.5.1 to the conclusion of Theorem 5.4.7 gives $\pi_2(Z) = \pi_1(A)/[\pi_1(A), K]$, and it follows from the homotopy exact sequence of the pair (Z, Y) that there is an exact sequence

$$\pi_2(Y) \to \pi_2(Z) \to \frac{K}{[\pi_1(A), K]} \to 0. \tag{5.5.1}$$

It follows from this exact sequence that if $A = BP$ and $Y = BQ$, so that we have an exact sequence $1 \to K \to P \to Q \to 1$ of groups, then $\pi_2(Z) \cong K/[P, K]$. Now we assume some knowledge of homology of spaces. In particular, the homology $H_i(P)$ of a group P is defined to be the homology $H_i(BP)$ of the space BP, $i \geqslant 0$. Since Z is simply connected, we get the same value for $H_2(Z)$, by the Absolute Hurewicz Theorem, which we prove in all dimensions in Part III, Theorem 14.7.8. Now the homology exact sequence of the cofibre sequence $A \to Y \to Z$ gives an exact sequence[63]

$$H_2(P) \to H_2(Q) \to \frac{K}{[P, K]} \to H_1(P) \to H_1(Q) \to 0.$$

In particular if $P = F$ is a free group, or one with $H_2(F) = 0$, then we obtain an exact sequence

$$0 \to H_2(Q) \to \frac{K}{[F, K]} \to F^{\mathrm{ab}} \to Q^{\mathrm{ab}} \to 0.$$

This gives the famous Hopf formula

$$H_2(Q) \cong \frac{K \cap [F, F]}{[K, F]}$$

which was one of the starting points of homological algebra. This is generalised to all dimensions in Proposition 8.3.21.[64] □

5.6 The monomorphism case: inducing from crossed modules over a subgroup

In Section 5.3 we have considered the construction of an induced crossed module for a general homomorphism, and in Section 5.5 we have given a simpler expression for the case when f is an epimorphism. Now we study the case of a monomorphism, which is essentially the same as studying the case of an inclusion in a subgroup. So in all this section we shall consider the inclusion $\iota\colon P \to Q$ of a subgroup P of Q.

As we shall see this case is rather involved and we get an expression of the induced crossed module that is quite complicated and in some cases very much related to the coproduct. Let us introduce some concepts that shall be helpful.

Definition 5.6.1. Let M be a group and let T be a set, we define the *copower* M^{*T} to be the free product of the groups $M_t = M \times \{t\}$ for all $t \in T$. Notice that all M_t are naturally isomorphic to M under the map $(m,t) \mapsto m$. So M^{*T} can be seen as the free product of copies of M indexed by T. □

The copower construction satisfies the adjointness condition that for any group N there is a bijection

$$\mathsf{Set}(T, \mathsf{Groups}(M, N)) \cong \mathsf{Groups}(M^{*T}, N)$$

natural in M, N, T. Notice also that the precrossed module induced from $\mathcal{M}\colon (\mu\colon M \to P)$ by $f\colon P \to Q$ is a quotient of M^{*UQ} where UQ is the underlying set of Q.

In the case where we have the inclusion of a subgroup $\iota\colon P \to Q$, we choose T to be a *right transversal* of P in Q, by which is meant a subset of Q including the identity 1 and such that any $q \in Q$ has a unique representation as $q = pt$ where $p \in P, t \in T$. For any crossed P-module $\mathcal{M} = (\mu\colon M \to P)$, the precrossed Q-module induced by ι will have the form $\hat{\mu}\colon M^{*T} \to Q$. Let us describe the Q-action.

Proposition 5.6.2. *Let $\iota\colon P \to Q$, $\mathcal{M}$, and T be as above. Then there is a Q-action on M^{*T} defined on generators using the coset decomposition by*

$$(m,t)^q = (m^p, u)$$

for any $q \in Q$, $m \in M$, $t \in T$, where p, u are the unique $p \in P$, $u \in T$, such that $tq = pu$.

Proof. Let $m \in M$, $t, u, u' \in T$, $p, p' \in P$ and $q, q' \in Q$ be elements such that $tq = pu$ and $uq' = p'u'$. We have $t(qq') = puq' = pp'u'$. Therefore,

$$((m,t)^q)^{q'} = (m^p, u)^{q'} = (m^{pp'}, u') = (m,t)^{qq'}$$

and Q acts on M^{*T}. □

Remark 5.6.3. We can think of (m,t) as m^t, so the action is $(m^t)^q = (m^p)^u$ where $tq = pu$. Notice that if P is normal in Q then the Q-action induces an action of P on M_t given by $(m,t)^p = (m^{tpt^{-1}}, t)$. We shall exploit this later, in Corollary 5.6.9. □

Now we define the boundary homomorphism by specifying the images of the generators

$$\hat{\mu}\colon M^{*T} \to Q, \quad (m,t) \mapsto t^{-1}\mu(m)t.$$

Proposition 5.6.4. *Let $\iota\colon P \to Q$, $\mathcal{M}$ and T be as above. Then $(\hat{\mu}\colon M^{*T} \to Q)$ with the above action is a precrossed Q-module.*

Proof. We verify axiom CM1). For any $m \in M$, $t \in T$, and $q \in Q$, we have

$$\begin{aligned}
\hat{\mu}((m,t)^q) &= \hat{\mu}(m^p,u) && \text{when } tq = pu\\
&= u^{-1}\mu(m^p)u && \text{by definition of } \hat{\mu}\\
&= u^{-1}(p)^{-1}\mu(m)pu && \text{because } \mu \text{ is a crossed module}\\
&= q^{-1}(t)^{-1}\mu(m)tq && \text{because } tq = pu\\
&= q^{-1}\hat{\mu}(m,t)q && \text{by definition of } \hat{\mu}.
\end{aligned}$$

□

To complete the characterisation we now prove that this precrossed module is induced.

Theorem 5.6.5. *If $\iota\colon P \to Q$ is a monomorphism, and $\mathcal{M} = (\mu\colon M \to P)$ is a crossed P-module then $\hat{\mu}\colon M^{*T} \to Q$ is the precrossed module induced by ι from μ.*

Proof. We check the universal property. There is a homomorphism of groups $\phi\colon M \to M^{*T}$ defined by $\phi(m) = (m,1)$ that makes commutative the square

$$\begin{tikzcd}
M \arrow[r, "\phi"] \arrow[d, "\mu"'] & M^{*T} \arrow[d, "\hat{\mu}"] \\
P \arrow[r, "\iota"'] & Q
\end{tikzcd}$$

and so that (ϕ,ι) is a morphism of precrossed modules.

For any morphism of precrossed modules

$$(h,\iota)\colon (\mu\colon M \to P) \longrightarrow (\nu\colon N \to Q)$$

the only possible definition of a homomorphism of groups $h'\colon M^{*T} \to N$ such that $h'\phi = h$ is the one given by $h'(m,t) = (hm)^t$ on generators. It is easy to see that it is a morphism of Q-precrossed modules. □

It is immediate that the induced crossed module is the one associated to the precrossed module $\hat{\mu}$, i.e. is the quotient with respect to the Peiffer subgroup.

Corollary 5.6.6. *If $\iota\colon P \to Q$ is a monomorphism, and $(\mu\colon M \to P)$ is a crossed P-module, then the crossed module induced by ι from μ is the homomorphism induced by $\hat{\mu}$ on the quotient*

$$\hat{\mu}\colon \frac{M^{*T}}{[\![M^{*T}, M^{*T}]\!]} \to Q$$

together with the induced action of Q.

Corollary 5.6.7. *If $\iota\colon P \to Q$ is the inclusion of a subgroup, and $(\mu\colon M \to P)$ is a crossed P-module, then the boundary $\partial\colon \iota_* M \to Q$ has image the normaliser of $\iota\mu(M)$ in Q.*

In particular, taking $\mu\colon P \to P$ to be the identity, we see that the induced crossed module $\iota_* P \to Q$ expands the standard group theoretic normaliser of P in Q to a construction with a universal property.

It is advantageous to have a smaller number of generators of the Peiffer subgroup $[\![M^{*T}, M^{*T}]\!]$.

Proposition 5.6.8. *Let $\mathcal{M} = (\mu\colon M \to P)$ be a crossed P-module and let $\iota\colon P \to Q$ be an inclusion of a subgroup. Let T be a transversal of P in Q, and M^{*T} the copower of M as before. Let S be a set of generators of M as a group, and let us define $S^P = \{s^p \mid s \in S,\ p \in P\}$. Then there is an isomorphism*

$$\iota_* M \cong \frac{(M^{*T})}{R}$$

of the induced crossed module $\iota_*\mathcal{M} = (\iota_* M \to Q)$ to a quotient of the copower, where R is the normal closure in M^{*T} of the elements

$$[\![(r,t),(s,u)]\!] = (r,t)^{-1}(s,u)^{-1}(r,t)(s,u)^{\hat{\mu}(r,t)} \tag{5.6.1}$$

for all $r, s \in S^P$ and $t, u \in T$.

Proof. By Corollary 5.6.6 we just have to prove that R is exactly the Peiffer subgroup $[\![M^{*T}, M^{*T}]\!]$ of M^{*T}.

Now, M^{*T} is generated by the set

$$(S^P, T) = \{(s^p, t) \mid s \in S,\ p \in P,\ t \in T\}$$

and this set is Q-invariant since $(s^p, t)^q = (s^{pp'}, u)$ where $u \in T$, $p' \in P$ satisfy $tq = p'u$. Then by Proposition 3.3.5 $\{M^{*T}, M^{*T}\}$ is the normal closure of the set $\{(S^P, T), (S^P, T)\}$ of basic Peiffer commutators and this is just R. □

Corollary 5.6.9. *If, further to the proposition, M is abelian and $\iota\mu(M)$ is normal in Q, then $\iota_* M$ is abelian and is isomorphic as module to the cosum $M^{\oplus T}$.*

Proof. Note that if $u, t \in T$ and $r \in S$ then

$$u\hat{\mu}(r,t) = ut^{-1}\mu(r)t = \iota\mu(m)ut^{-1}t = \mu(m)u$$

for some $m \in M$, by the normality condition. The Peiffer commutator given in Equation (5.6.1) can therefore be rewritten as

$$(r,t)^{-1}(s,u)^{-1}(r,t)(s,u)^{\hat{\mu}(r,t)} = (r,t)^{-1}(s,u)^{-1}(r,t)(s^m, u).$$

Since M is abelian, $s^m = s$. Thus the basic Peiffer commutators reduce to ordinary commutators. Hence $\iota_* M$ is the cosum $M^{\oplus T}$. □

Exercise 5.6.10. In the last corollary, set $G = \operatorname{Cok} \partial$, and assume $M = P$ and μ is the identity crossed module. Show using the canonical bijection $T \to G$ how to represent $\iota_* P$ as a direct sum $\bigoplus_{g \in G} P^g$ and describe the operation of G on this direct sum in detail. □

Example 5.6.11. The last corollary and exercise are easiest to interpret and apply when Q is abelian. Let $\partial \colon \iota_* M \to Q$ be the boundary morphism, and let $\gamma \colon Q \to G = \operatorname{Cok} \partial$ be the quotient morphism. Then γ induces a bijection $T \to G$, $t \mapsto g_t$. Under this bijection, we can write $\iota_* M \cong \bigoplus_{g \in G} M^{g_t}$ where the operation is $(m, g_t)^{h_s} = (m^p, g_r)$ where $pr = ts$ in Q.

A simple case is for cyclic groups and the inclusion $\iota \colon \mathsf{C}_a \to \mathsf{C}_{ab}$ which maps a generator x of C_a to y^b in C_{ab} where y generates C_{ab}, the crossed C_a module is the identity. Then the cokernel of $\partial \colon \iota_* \mathsf{C}_a \to \mathsf{C}_{ab}$ is C_b. We can identify $\iota_* \mathsf{C}_a$ with $\mathbb{Z}_a[\mathsf{C}_b]$. □

The following example of the dihedral crossed module exhibits a number of typical features.

Example 5.6.12. Recall that the dihedral group D_{2n} of order $2n$ has presentation

$$\langle x, y \mid x^n,\ y^2,\ xyxy \rangle.$$

We consider another copy $\widetilde{D}_{2n}$ of D_{2n} with presentation $\langle u, v \mid u^n,\ v^2,\ uvuv \rangle$ and the homomorphism

$$\partial \colon \widetilde{D}_{2n} \to D_{2n}, \quad u \mapsto x^2,\ v \mapsto y.$$

With this boundary and action of D_{2n} on $\widetilde{D}_{2n}$ given on generators by the equations

$$u^y = vuv^{-1}, \quad v^y = v, \quad u^x = u, \quad v^x = vu,$$

this becomes the *dihedral crossed module*. As an exercise, check this result and also that $\partial \colon \widetilde{D}_{2n} \to D_{2n}$ is an isomorphism if n is odd, and has kernel and cokernel isomorphic to C_2 if n is even. □

Example 5.6.13. We let $Q = D_{2n}$ be the dihedral group as in the last example and let $M = P = C_2$ be the cyclic subgroup of order 2 generated by y. Let us denote by $\iota \colon C_2 \hookrightarrow D_{2n}$ the inclusion.

We have that $1 \colon C_2 \to C_2$ is a crossed module and we are going to identify the induced crossed module

$$\hat{\mu} = \iota_*(1) \colon \iota_*(C_2) \longrightarrow D_{2n}.$$

A right transversal of C_2 in D_{2n} is given by the elements

$$T = \{x^i \mid i = 0, 1, 2, \ldots, n-1\}.$$

By Proposition 5.6.8, $\iota_* C_2$ has a presentation with generators $a_i = (y, x^i)$, $i = 0, 1, 2, \dots, n-1$ and relations $a_i^2 = 1$, $i = 0, 1, 2, \dots, n-1$, together with the Peiffer relations associated to these generators.

Since the D_{2n}-action on C_2^{*T} is given by

$$a_i^x = a_{i+1} \quad \text{and} \quad a_i^y = a_{n-i},$$

and

$$\hat{\mu}(a_i) = x^{-i} y x^i = y x^{2i},$$

we have $(a_i)^{\hat{\mu} a_j} = a_{2j-i}$, so that the Peiffer relations become

$$a_j^{-1} a_i a_j = a_{2j-i}.$$

In this group C_2^{*T} we define $u = a_0 a_1$, $v = a_0$. As a consequence, we have $u = a_i a_{i+1}$ and $u^i = a_0 a_i$ and it is now easy to check that $(C_2^{*T})^{\mathrm{cr}} \cong \widetilde{D}_{2n}$. Also the map $\hat{\mu}$ satisfies

$$\hat{\mu} u = \hat{\mu}(a_0 a_1) = y y x^2 = x^2, \quad \hat{\mu} v = y.$$

Thus y acts on $\iota_* C_2$ by conjugation by v. However x acts by $u^x = u$, $v^x = vu$.

This crossed module is the dihedral crossed module of the previous Example 5.6.12.

It is worth pointing out that this induced crossed module is finite while the corresponding precrossed module M^{*T} is clearly infinite. We will develop these points in the next section. □

5.7 On the finiteness of induced crossed modules

With the results of the previous section, we have an alternative way of constructing the induced crossed module associated to a homomorphism f. We can factor f in an epimorphism and a monomorphism and then apply the constructions. As pointed out before it is always a good thing to have as many equivalent ways as possible since then we can choose the most appropriate to some particular situation.

As we have seen in the previous section, if we have a (pre)crossed module $\mathcal{M} = (M \to P)$ in which M is generated by a finite P-set of a generators, and a group homomorphism $P \to Q$ with finite cokernel, then the induced (pre)crossed module is also generated by a finite set. In this section we give an algebraic proof that a crossed module induced from a finite crossed module by a morphism with finite cokernel is also finite. The result is false for precrossed modules.

Theorem 5.7.1. *Let $\mu\colon M \to P$ be a crossed module and let $f\colon P \to Q$ be a morphism of groups. Suppose that M and the index of $f(P)$ in Q are finite. Then the induced crossed module $f_* M$ is finite.*

Proof. Factor the morphism $f\colon P \to Q$ as $\tau\sigma$ where τ is injective and σ is surjective. Then f_*M is isomorphic to $\tau_*\sigma_*M$. It is immediate from Proposition 5.5.1 that if M is finite then so also is σ_*M. So it is enough to assume that f is injective.

Let T be a right transversal of $f(P)$ in Q. Then there are maps

$$(\xi, \eta)\colon T \times Q \to f(P) \times T$$

defined by $(\xi, \eta)(t, q) = (p, u)$ where $p \in P$, $u \in T$ are elements such that $tq = f(p)u$. With this notation, the form of a basic Peiffer relation got in Corollary 5.6.6 is then of the form

$$(m, t)(n, u) = (n, u)(m^{\xi(t,u^{-1} f\mu(n)u)}, \eta(t, u^{-1} f\mu(n)u)) \tag{5.7.1}$$

where $m, n \in M$, $t, u \in T$.

We now assume that the finite set T has l elements and has been given the total order $t_1 < t_2 < \cdots < t_l$. An element of M^{*T} may be represented as a word

$$(m_1, u_1)(m_2, u_2)\ldots(m_e, u_e). \tag{5.7.2}$$

Such a word is said to be *reduced* when $u_i \neq u_{i+1}$, $1 \leqslant i < e$, and to be *ordered* if $u_1 < u_2 < \cdots < u_e$ in the given order on T. This yields a partial ordering of $M * T$ where $(m_i, u_i) \leqslant (m_j, u_j)$ whenever $u_i \leqslant u_j$.

A *twist* replaces a reduced word $w = w_1(m, t)(n, v)w_2$, with $v < t$, by $w' = w_1(n, v)(m^p, u)w_2$ using the Peiffer relation (5.7.1). If the resulting word is not reduced, multiplication in M_v and M_u may be used to reduce it. In order to show that any word may be ordered by a finite sequence of twists and reductions, we define an integer weight function on the set W_n of nonempty words of length at most n by

$$\Omega_n\colon W_n \to \mathbb{Z}^+,$$
$$(m_1, t_{j_1})(m_2, t_{j_2})\ldots(m_e, t_{j_e}) \mapsto l^e \sum_{i=1}^{e} l^{n-i} j_i.$$

It is easy to see that $\Omega_n(w') < \Omega_n(w)$ when $w \to w'$ is a reduction. Similarly, for a twist

$$w = w_1(m_i, t_{j_i})(m_{i+1}, t_{j_{i+1}})w_2 \to w' = w_1(m_{i+1}, t_{j_{i+1}})(n, t_k)w_2$$

the weight reduction is

$$\Omega_n(w) - \Omega_n(w') = l^{n+e-i-1}(\, l(j_i - j_{i+1}) + j_{i+1} - j_k\,) \geqslant l^{n+e-i-1},$$

so the process terminates in a finite number of moves.

We now specify an algorithm for converting a reduced word to an ordered word. Various algorithms are possible, some presumably more efficient than others, but we are not interested in efficiency here. We call a reduced word k-*ordered* if the subword consisting of the first k elements is ordered and the remaining elements are greater than these. Every reduced word is at least 0-ordered. Given a k-ordered, reduced word, find the rightmost minimal element to the right of the k-th position. Move this element one place to the left with a twist, and reduce if necessary. The resulting word may only be j-ordered, with $j < k$, but its weight will be less than that of the original word. Repeat until an ordered word is obtained.

Let $Z = M_{t_1} \times M_{t_2} \times \cdots \times M_{t_l}$ be the product of the sets $M_{t_i} = M \times \{t_i\}$. Then the algorithm yields a function $\phi \colon Y \to Z$ such that the quotient morphism $Y \to f_* M$ factors through ϕ. Since Z is finite, it follows that $f_* M$ is finite. □

Remark 5.7.2. In this last proof, it is in general not possible to give a group structure on the set Z such that the quotient morphism $Y \to f_* M$ factors through a morphism to Z. For example, in the dihedral crossed module of Example 5.6.12, with $n = 3$, the set Z will have 8 elements, and so has no group structure admitting a morphism onto D_6.

So the proof of the main theorem of this section does not extend to a proof that the induced crossed module construction is closed also in the category of p-groups. Nevertheless, the result is true and there is a topological proof in [BW95]. □

5.8 Inducing crossed modules by a normal inclusion

We continue the study of Section 5.6 of the crossed modules induced by the inclusion $P \to Q$ of a subgroup, by considering the case when P is normal in Q. We shall show in Theorem 5.8.4 that the coproduct of crossed P-modules described in Section 4.1 may be used to give a presentation of crossed Q-modules induced by the inclusion $\iota \colon P \to Q$ analogous to known presentations of induced modules over groups.[65]

Let us start by digressing a bit about crossed modules constructed from a given one using an isomorphism.

Definition 5.8.1. Let $\mu \colon M \to P$ be a crossed P-module and let α be an automorphism of P. The crossed module $\mu_\alpha \colon M_\alpha \to P$ *associated to* α is defined as follows. The group M_α is just $M \times \{\alpha\}$, the morphism μ_α is given by $(m, \alpha) \mapsto \alpha\mu m$ and the action of P is given by $(m, \alpha)^p = (m^{\alpha^{-1}p}, \alpha)$. □

Proposition 5.8.2. *The map $\mu_\alpha \colon M_\alpha \to P$ is a crossed module. Moreover this crossed module is isomorphic to μ since the map $k_\alpha \colon M \to M_\alpha$ given by $k_\alpha m = (m, \alpha)$ produces an isomorphism over α.*

Proof. Let us check both properties of a crossed module:

$$\begin{aligned} \mu_\alpha(m^{\alpha^{-1}p}, \alpha) = \alpha(\mu m^{\alpha^{-1}p}) &= \alpha(\alpha^{-1}(p)^{-1}\mu(m)\alpha^{-1}(p)) \\ &= p^{-1}\alpha\mu(m)p = p^{-1}\mu^\alpha(m)p \end{aligned}$$

and

$$\begin{aligned}(m,\alpha)^{\mu_\alpha(m',\alpha)} &= (m,\alpha)^{\alpha\mu(m')} = (m^{\alpha^{-1}\alpha\mu(m')},\alpha)\\ &= (m^{\mu(m')},\alpha) \quad = (m',\alpha)^{-1}(m,\alpha)(m',\alpha).\end{aligned}$$

It is immediate that the map $k_\alpha\colon M \to M_\alpha$ is an isomorphism. Also, the diagram

$$\begin{tikzcd} M \arrow[r, "k_\alpha"] \arrow[d, "\mu"'] & M_\alpha \arrow[d, "\mu_\alpha"] \\ P \arrow[r, "\alpha"'] & P \end{tikzcd}$$

commutes and the map k_α preserves the P-action over α. □

Remark 5.8.3. Notice that if $\alpha = 1$, there is a natural identification $M_1 = M$. □

We continue to assume that P is a normal subgroup of Q. In this case, for any $t \in Q$, there is an inner automorphism $\alpha_t\colon P \to P$ defined by $\alpha_t(p) = t^{-1}pt$. Let us write $(\mu_t\colon M_t \to P)$ instead of $(\mu_{\alpha_t}\colon M_{\alpha_t} \to P)$.

Recall that this crossed P-module is the same $(\mu_t\colon M_t \to P)$ that we have used to construct $\iota_* M$ in Section 5.6, namely $M_t = M \times \{t\}$, the P-action was given by $(m,t)^p = (m^{tpt^{-1}}, t)$ and the homomorphism μ_t was defined by $\mu_t(m,t) = t^{-1}\mu m t$. We have just seen that it is a crossed P-module isomorphic to $\mathcal{M}$.

Now let T be a right transversal of P in Q. We can form the precrossed Q-module $\mathcal{M}' = (\partial'\colon M^{*T} \to Q)$ as in Proposition 5.6.2. Recall that the Q-action is defined on generators as follows. For any $q \in Q$, $m \in M$, $t \in T$ we define

$$(m,t)^q = (m^p, u),$$

where $p \in P$ and $u \in T$ are the only ones satisfying $tq = pu$. Also the homomorphism ∂' is defined by $\partial'(m,t) = t^{-1}pt$

We had seen in Theorem 5.6.5 that the induced crossed Q-module $\iota_*\mathcal{M}$ is the quotient of M^{*T} by the Peiffer subgroup associated to the Q-action. On the other hand, we have seen in Corollary 4.1.2 that the coproduct as crossed P-modules

$$\partial\colon M^{\circ T} \to P$$

is the quotient of M^{*T} with respect to the Peiffer subgroup associated to the P-action. We are going to check that they are the same.

Theorem 5.8.4. *In the situation we have just described, the homomorphism*

$$M^{\circ T} \xrightarrow{\partial} P \xhookrightarrow{\iota} Q$$

with the morphism of crossed modules

$$(i_1,\iota)\ :\ \mathcal{M} \to (\iota\partial\colon M^{\circ T} \to Q)$$

is the induced crossed Q-module.

Proof. It is immediately checked in this case that the Peiffer subgroup is the same whether M^{*T} is considered as a precrossed P-module $M^{*T} \to P$ or as a precrossed Q-module $M^{*T} \to Q$. It can also be directly checked. We leave that as an exercise. □

We remark that the result of Theorem 5.8.4 is analogous to well-known descriptions of induced modules, except that here we have replaced the direct sum which is used in the module case by the coproduct of crossed modules. Corresponding descriptions in the non-normal case look to be considerably harder.

As a consequence we obtain easily a result on p-finiteness.[66]

Proposition 5.8.5. *If M is a finite p-group and P is a normal subgroup of finite index in Q, then the induced crossed module $\iota_* M$ is a finite p-group.*

Proof. This follows immediately from the discussion in Section 4.1. □

Now the induced module $(\iota\partial\colon M^{\circ T} \to Q)$ in Theorem 5.8.4 may be described using Corollary 4.4.16, if the hypotheses there are satisfied. So let P be a normal subgroup of Q and T a transversal as before, and let $(\mu\colon M \to P)$ be a crossed P-module.

We can divide the construction of the group $M^{\circ T}$ into two steps. We define $W = M^{\circ T'}$ the coproduct of all but $M_1 = M$. Then there is an isomorphism of crossed Q-modules
$$\iota_* \mathcal{M} \cong M \circ W.$$

To apply Corollary 4.4.16 we have to assume that for all $t \in T$ we have $\mu_t(M) \subseteq \mu(M)$, i.e. that for all $t \in T$ we have $t^{-1}\mu(M)t \subseteq \mu(M)$ (notice that this is immediately satisfied if μM is normal in Q), and that there is a section $\sigma\colon \mu M \to M$ of μ defined on μM. Most of the time we shall require also that σ is P-equivariant.

Then there is an isomorphism
$$\iota_* \mathcal{M} \cong M \times \bigoplus_{t \in T'} (M_t)_M$$
through which the morphisms giving the coproduct structure become
$$(i, \iota)\colon (\mu\colon M \to P) \longrightarrow (\xi = \iota\mu\,\mathrm{pr}_1\colon M \times \bigoplus_{t \in T'} (M_t)_M \to Q)$$
where $i = i_1\colon (m, 1) \mapsto (m, 0)$ and
$$(i_t, \iota)\colon (\mu\colon M_t \to P) \longrightarrow (\xi = \iota\mu\,\mathrm{pr}_1\colon M \times \bigoplus_{t \in T'} (M_t)_M \to Q)$$
where for $t \neq 1$, $i_t(m, t) = (\sigma((\mu m)^t), [m, t])$.

Let us describe first how the Q-action is defined on this last crossed Q-module. Later we shall check the universal property.

The result we next give is quite complicated and technical. It is given principally because it illustrates the method, and also shows that these methods give an understanding of actions of the fundamental group in a way which seems unobtainable by traditional methods. Those methods go via universal covers and homology, whereas ours go via direct descriptions of crossed modules, i.e. nonabelian structures in dimensions 2.

Theorem 5.8.6. *The Q-action on the group $M \times \bigoplus_{t \in T'} (M_t)_M$ is given as follows.*

(i) *For any $m \in M$, $q \in Q$*

$$(m,0)^q = \begin{cases} (m^q, 0) & \text{if } v = 1, \\ (\sigma((\mu m)^q), [m^r, v]) & \text{if } v \neq 1, \end{cases}$$

where $r \in P$ and $v \in T$ satisfy $q = rv$ and $[m,v]$ denotes the class of (m,v) in $(M_v)_M$.

(ii) *If $m \in M$, $t \in T'$, $q \in Q$ then*

$$(1,[m,t])^q = \begin{cases} (1,[m^p,t]) & \text{if } v = 1, \\ (\sigma(\mu m^p)^{-1} m^p, -[\sigma((\mu m^p)^{v^{-1}}), v]) & \text{if } v \neq 1, u = 1, \\ (1, -[\sigma((\mu m^p)^{uv^{-1}}), v] + [m^p, u]) & \text{if } v \neq 1, u \neq 1, \end{cases}$$

where $p \in P$, $u \in T$ are the unique elements satisfying $tq = pu$.

Proof. We use the description of the morphisms associated to the coproduct structure given above to calculate the action given by Theorem 5.8.4.

The formulae (i) and (ii) for the case $v = 1$ follow from the description of the action of P on M_t given at the beginning of this section.

The remaining cases will be deduced from the formula for the action of Q given in Theorem 5.8.4, namely if $m \in M$, $t \in T$, $q \in Q$ then

$$(i_t(m,t))^q = \begin{cases} i_1(m^p, 1) = (m^p, 0) & \text{if } tq = p \in P, \\ i_u(m^p, u) = (\sigma((\mu m^p)^u), [m^p, u]) & \text{if } tq = pu, \ p \in P, \ u \in T'. \end{cases}$$

We first prove (i) for $v \neq 1$. We have since $q = rv$, $v \in T'$,

$$\begin{aligned} (m,0)^q &= (i_1(m,1))^{rv} \\ &= i_v(m^r, v) \\ &= (\sigma((\mu m^r)^v), [m^r, v]). \end{aligned}$$

To prove (ii) with $v \neq 1$, first note that

$$\begin{aligned} (1,[m,t]) &= (\sigma((\mu m)^t), 0)^{-1} (\sigma((\mu m)^t), [m,t]) \\ &= (\sigma((\mu m)^t), 0)^{-1} i_t(m,t). \end{aligned}$$

But

$$\begin{aligned}(\sigma((\mu m)^t),0)^q &= (\sigma((\mu\sigma((\mu m)^t))^q),[(\sigma((\mu m)^t))^r,v]) && \text{by (i)}\\ &= (\sigma((\mu m)^{tq}),[\sigma((\mu m)^{tr}),v]) && \text{since } \mu\sigma = 1,\end{aligned}$$

and, from the definition of the Q-action,

$$(i_t(m,t))^q = \begin{cases}(m^p,0) & \text{if } u = 1,\\ (\sigma((\mu m)^{tq}),[m^p,u]) & \text{if } u \neq 1.\end{cases}$$

It follows that

$$(1,[m,t])^q = \begin{cases}(\sigma(\mu m^p)^{-1}m^p,-[\sigma((\mu m^p)^{v^{-1}}),v]) & \text{if } u = 1,\\ (1,-[\sigma((\mu m^p)^{uv^{-1}}),v]+[m^p,u]) & \text{if } u \neq 1.\end{cases} \qquad \square$$

Now we check that the universal property is satisfied.

Theorem 5.8.7. *For any crossed module $\mathcal{N} = (\nu\colon N \to Q)$ and any morphism of crossed modules $(\beta,\iota)\colon \mathcal{M} \to \mathcal{N}$, the induced morphism $\phi\colon M \times \bigoplus_{t\in T'}(M_t)_M \to N$ is given by*

$$\phi(m,0) = \beta m, \quad \phi(m,[n,v]) = (\beta m)\,\beta(\sigma((\mu n)^v))^{-1}\,(\beta n)^v.$$

Proof. The formula for ϕ is obtained as follows:

$$\begin{aligned}\phi(m,[n,v]) &= \phi(m,0)\,\phi(\sigma((\mu n)^v),0)^{-1}\,\phi(i_v(n,v))\\ &= (\beta m)\,(\beta(\sigma((\mu n)^v))^{-1})\,(\beta n)^v\end{aligned}$$

where the definition of ϕ is taken from Theorem 5.8.4 $\square$

We now include an example for Theorem 5.8.6 showing the action in the case $v \neq 1$, $u = 1$.

Example 5.8.8. Let n be an odd integer and let $Q = D_{8n}$ be the dihedral group of order $8n$ generated by elements $\{t, y\}$ with relators $\{t^{4n}, y^2, (ty)^2\}$. Let $P = D_{4n}$ be generated by $\{x, y\}$, and let $\iota\colon P \to Q$ be the monomorphism given by $x \mapsto t^2$, $y \mapsto y$. Then let $M = C_{2n}$ be generated by $\{m\}$. Define $\mathcal{M} = (\mu\colon M \to P)$ where $\mu m = x^2$, $m^x = m$ and $m^y = m^{-1}$. This crossed module is isomorphic to a sub-crossed module of $(D_{4n} \to \mathrm{Aut}(D_{4n}))$ and has kernel $\{1, m^n\}$.

The image μM is the cyclic group of order n generated by x^2, and there is an equivariant section $\sigma\colon \mu M \to M$, $x^2 \mapsto m^{n+1}$ because $(x^2)^{(n+1)} = x^2$ and $\gcd(n+1, 2n) = 2$. Then $Q = P \cup Pt$, $T = \{1, t\}$ is a transversal, M_t is generated by (m,t) and $\mu_t(m,t) = x^2$. The action of P on M_t is given by

$$(m,t)^x = (m,t), \quad (m,t)^y = (m^{-1},t).$$

Since M acts trivially on M_t,

$$\iota_* M \cong M \times M_t \cong C_{2n} \times C_{2n}.$$

Using the section σ given above, Q acts on $\iota_* M$ by

$$\begin{aligned}(m,0)^t &= (m^{n+1},[m,t]),\\ (m,0)^y &= (m^{-1},0),\\ (1,[m,t])^t &= (m^n,(n-1)[m,t]),\\ (1,[m,t])^y &= (1,-[m,t]).\end{aligned}$$ □

It is worth recalling that our objective was not only to get an easier expression of the induced crossed module, but also to have some information about the kernel of its boundary map. We can obtain some information on the latter in the case where P is of index 2 in Q, even without the assumption that μM is normal in Q as follows.

Suppose then that $T = \{1, t\}$ is a right transversal of P in Q. Let the morphism $M \ltimes M_t \to P$ be given as usual by $(m,(n,t)) \mapsto (\mu m)(\mu_t(n,t)) = mt^{-1}nt$.

Write $\langle M, M_t\rangle$ for the subgroup of $M \times_P M_t$ generated by the elements

$$\langle m,(n,t)\rangle = (m^{-1}m^{t^{-1}(\mu n)t}, ((n,t)^{-1})^m(n,t)),$$

for all $m \in M$, $(n,t) \in M_t$.

Proposition 5.8.9. *Let $\mu\colon M \to P$ and $\iota\colon P \to Q$ be inclusions of normal subgroups. Suppose that P is of index 2 in Q, and $t \in Q \setminus P$. Then the kernel of the induced crossed module $(\partial\colon \iota_* M \to Q)$ is isomorphic to*

$$\frac{M \cap t^{-1}Mt}{[M, t^{-1}Mt]}.$$

In particular, if M is also normal in Q, then this kernel is isomorphic to $M/[M,M]$, i.e. to M made abelian.

Proof. By previous results $\iota_* M$ is isomorphic to the coproduct crossed P-module $M \circ M_t$ with a further action of Q. The result follows from Corollary 4.3.9. □

We now give some homotopical applications of the last result.

Example 5.8.10. Let $\iota\colon P = D_{4n} \to Q = D_{8n}$ be as in Example 5.8.8, and let $M = D_{2n}$ be the subgroup of P generated by $\{x^2, y\}$, so that $\iota M \lhd \iota P \lhd Q$ and $t^{-1}Mt$ is isomorphic to a second D_{2n} generated by $\{x^2, yx\}$. Then

$$M \cap t^{-1}Mt = [M, t^{-1}Mt]$$

(since $[y, yx] = x^2$), and both are isomorphic to C_n generated by $\{x^2\}$.

It follows from Proposition 5.8.9 that if X is the homotopy pushout of the maps

$$\begin{tikzcd} BD_{4n} \arrow[r] \arrow[d] & BC_2 \arrow[d] \\ BD_{8n} \arrow[r] & X \end{tikzcd}$$

where the horizontal map is induced by $D_{4n} \to D_{4n}/D_{2n} \cong C_2$, then $\pi_2(X) = 0$. □

Example 5.8.11. Let M, N be normal subgroups of the group G, and let Q be the wreath product

$$Q = G \wr C_2 = (G \times G) \rtimes C_2.$$

Take $P = G \times G$, and consider the crossed module $(\partial \colon Z \to Q)$ induced from $M \times N \to P$ by the inclusion $P \to Q$. If t is the generator of C_2 which interchanges the two factors of $G \times G$, then $Q = P \cup Pt$ and $t^{-1}(M \times N)t = N \times M$. So

$$(M \times N) \cap t^{-1}(M \times N)t = (M \cap N) \times (N \cap M)$$

and

$$[M \times N, N \times M] = [M, N] \times [N, M].$$

It follows that if X is the homotopy pushout of

$$\begin{tikzcd} BG \times BG \arrow[r] \arrow[d] & B(G/M) \times B(G/N) \arrow[d] \\ B(G \wr C_2) \arrow[r] & X \end{tikzcd}$$

then

$$\pi_2(X) \cong ((M \cap N)/[M, N])^2.$$

If $([m], [n])$ denotes the class of $(m, n) \in (M \cap N)^2$ in $\pi_2(X)$, the action of Q is determined by

$$([m], [n])^{(g,h)} = ([m^g], [n^h]), \quad (g, h) \in P, \qquad ([m], [n])^t = ([n], [m]). \qquad \square$$

We end this section by giving an explicit description of the induced crossed module in the case that M is a normal subgroup of P and P is a normal subgroups of Q.

There are two construction used in the description. The first one is the abelianisation M^{ab} of a group M. If $n \in M$, then the class of n in M^{ab} is written $[n]$.

The second construction is the augmentation ideal IQ of a group Q, which we develop for groupoids in Section 7.4.i. For groups it is defined as the kernel of the augmentation map $\varepsilon \colon \mathbb{Z}[Q] \to \mathbb{Z}$ which takes $\Sigma_i n_i q_i, n_i \in \mathbb{Z}, q_i \in Q$ to $\Sigma_i n_i$; this ideal has an additive basis of elements $q - 1, q \in Q, q \neq 1$, so that the augmentation ideal $I(Q/P)$ of a quotient group Q/P has basis $\{\bar{t} - 1 \mid t \in T'\}$ where T is a transversal of P in Q, $T' = T \setminus \{1\}$ and $\bar{q}$ denotes the image of q in Q/P.

Theorem 5.8.12. *Let $M \subseteq P$ be normal subgroups of Q, so that Q acts on both P and M by conjugation. Let $\mu\colon M \to P$, $\iota\colon P \to Q$ be the inclusions and let $\mathcal{M} = (\mu\colon M \to P)$. Then the induced crossed Q-module $\iota_*\mathcal{M}$ is isomorphic as a crossed Q-module to*

$$(\zeta\colon M \times (M^{\mathrm{ab}} \otimes I(Q/P)) \to Q)$$

where for $m, n \in M$, $x \in I(Q/P)$:

(i) $\zeta(m, [n] \otimes x) = m$;

(ii) *the action of Q is given by*

$$(m, [n] \otimes x)^q = (m^q, [m^q] \otimes (\bar{q} - 1) + [n^q] \otimes x\bar{q}).$$

The universal map $i\colon M \to M \times (M^{\mathrm{ab}} \otimes I(Q/P))$ is given by $m \mapsto (m, 0)$.

Proof. We deduce this from Theorem 5.8.6.[67] Specialising this theorem to the current situation, in which $\sigma\mu = 1$ and $i_t(m, t) = (m^t, [m, t])$, yields an isomorphism of crossed Q-modules

$$\iota_*\mathcal{M} \to \mathcal{X} = (\xi = \iota\mu\, \mathrm{pr}_1\colon M \times \bigoplus_{t \in T'} (M^{\mathrm{ab}}) \to Q).$$

In $\mathcal{X}$ the action of Q is given as follows, where $m \in M$, $r \in P$, $q = rv$ and $v \in T$:

(i)

$$(m, 0)^q = \begin{cases} (m^q, 0) & \text{if } v = 1, \\ (m^q, [m^r, v]) & \text{if } v \neq 1; \end{cases}$$

(ii) if $tq = pu$, $t \in T'$, $p \in P$ and $u \in T$, then

$$(1, [m, t])^q = \begin{cases} (1, [m^p, t]) & \text{if } v = 1, \\ (1, -[m^{pv^{-1}}, v]) & \text{if } v \neq 1,\ u = 1, \\ (1, -[m^{puv^{-1}}, v] + [m^p, u]) & \text{if } v \neq 1,\ u \neq 1. \end{cases}$$

Now we construct an isomorphism

$$\omega\colon M \times \bigoplus_{t \in T'} (M^{\mathrm{ab}}) \to M \times (M^{\mathrm{ab}} \otimes I(Q/P))$$

where for $m, n \in M$, $t \in T'$,

$$\omega(m, 0) = (m, 0), \quad \omega(m, [n, t]) = (m, [n^t] \otimes (\bar{t} - 1)).$$

Clearly ω is an isomorphism of groups, since it is an isomorphism on the part determined by a fixed $t \in T'$, and $I(Q/P)$ has a basis $\{\bar{t} - 1\colon t \in T'\}$ when considered as an abelian group. Now we prove that ω preserves the action of Q. Let $m, n \in M$, $t \in T'$,

$q \in Q$. Let $q = rv$, $tq = pu$, $p, r \in P$, $u, v \in T$. When $v = 1$ we have $tqt^{-1} \in P$ and so $u = t$. Then

$$\begin{aligned}
\omega((m,0)^q) &= \begin{cases} \omega(m^q, 0) & \text{if } v = 1, \\ \omega(m^q, [m^r, v]) & \text{if } v \neq 1, \end{cases} \\
&= \begin{cases} (m^q, 0) & \text{if } v = 1, \\ (m^q, [m^q] \otimes (\bar{v} - 1) & \text{if } v \neq 1, \end{cases} \\
&= (\omega(m,0))^q.
\end{aligned}$$

Further,

$$\begin{aligned}
\omega((1,[m,t])^q) &= \begin{cases} \omega(1, [m^p, t]) & \text{if } v = 1, \\ \omega(1, -[m^{pv^{-1}}, v]) & \text{if } v \neq 1,\ u = 1, \\ \omega(1, -[m^{puv^{-1}}, v] + [m^p, u]) & \text{if } v \neq 1,\ u \neq 1, \end{cases} \\
&= \begin{cases} (1, [m^{pt}] \otimes (\bar{t} - 1)) & \text{if } v = 1, \\ (1, -[m^p] \otimes (\bar{v} - 1)) & \text{if } v \neq 1,\ u = 1, \\ (1, -[m^{pu}] \otimes (\bar{v} - 1) + [m^{pu}] \otimes (\bar{u} - 1)) & \text{if } v \neq 1,\ u \neq 1, \end{cases} \\
&= (1, -[m^{pu}] \otimes (\bar{v} - 1) + [m^{pu}] \otimes (\bar{u} - 1) \qquad \text{in every case}, \\
&= (1, [m^{tq}] \otimes (\bar{t} - 1)\bar{q}), \\
&= (\omega(1,[m,t]))^q
\end{aligned}$$

since, in $I(Q/P)$,

$$(\bar{t} - 1)\bar{q} = \overline{pu} - \overline{rv} = \bar{u} - \bar{v} = (\bar{u} - 1) - (\bar{v} - 1).$$

Finally, we have to compute the universal extension ϕ of β. For this, it is sufficient to determine

$$\begin{aligned}
\phi(1, [n] \otimes (\bar{q} - 1)) &= \phi\omega(1, [n^{v^{-1}}, v]) \\
&= \phi\omega((n^{-1}, 0)\, i_v(n^{v^{-1}}, v)) \\
&= \beta(n^{-1})\, \beta(n^{v^{-1}})^v \\
&= \beta(n^{-1})\, \beta(n^{q^{-1}})^q
\end{aligned}$$

since β is a P-morphism and $\bar{q} = \overline{rv} = \bar{v}$. □

With this description, we can get new results on the fundamental crossed module of a space which is the pushout of classifying spaces. The following corollary is immediate.

Corollary 5.8.13. *Under the assumptions of the theorem, let us consider the space* $X = BQ \cup_{BP} B(P/M)$. *Its fundamental crossed module* $\Pi_2(X, BQ)$ *is isomorphic to the above crossed* Q*-module*

$$(\zeta\colon M \times (M^{\mathrm{ab}} \otimes I(Q/P)) \to Q).$$

In particular, the second homotopy group $\pi_2(X)$ *is isomorphic to* $M^{\mathrm{ab}} \otimes I(Q/P)$ *as* Q/M*-module.*

Proof. The proof is immediate. □

Note again one of our major arguments: we compute an abelian second homotopy group by using a 2dSvKT to compute a crossed module, a nonabelian invariant, representing the homotopy 2-type.

Corollary 5.8.14. *In particular, if the index* $[Q : P]$ *is finite, and* $\mathcal{P}$ *is the crossed module* $(1\colon P \to P)$, *then* $\iota_*\mathcal{P}$ *is isomorphic to the projection crossed module*

$$(\mathrm{pr}_1\colon P \times (P^{\mathrm{ab}})^{[Q:P]-1} \to Q)$$

with action as above.

Remark 5.8.15. In this case, $X = BQ \cup_{BP} B(P/P)$ may be interpreted either as the space obtained from BQ by collapsing BP to a point, or, better, as

$$X = BQ \cup_{BP} CB(P),$$

the space got by attaching a cone: this is a consequence of the gluing theorem for homotopy equivalences, [Bro06], 7.5.7.

The crossed module of Corollary 5.8.14 is not equivalent to the trivial one. At first sight, it seems that the projection

$$\mathrm{pr}_2\colon P \times (P^{\mathrm{ab}} \otimes I(Q/P)) \to (P^{\mathrm{ab}} \otimes I(Q/P))$$

should determine a morphism of crossed modules to the trivial one

$$0\colon (P^{\mathrm{ab}} \otimes I(Q/P)) \to I(Q/P)),$$

but this is not so because the map pr_2 is not a Q-morphism.

We are going to show later (Theorem 12.7.10) that this crossed module is in a clear sense not equivalent to the projection crossed module. □

In the next section, we explain how the computer algebra system GAP has been used to give further computations of induced crossed modules, and of course these have topological applications according to the results of this chapter.

We have now completed the applications of the 2-dimensional Seifert–van Kampen Theorem which we will give in this book. In the next chapter we give the proof of the theorem, using the algebraic concepts of double groupoids.

5.9 Computation of induced crossed modules

This section is work of C. D. Wensley.

The following discusses significant aspects of the computation of induced crossed modules. Let us consider the description of the induced module from a computational point of view. It involves the copower, i.e. a free product of groups. This usually gives infinite groups, but let us consider how to get a finite presentation in the case $M \leqslant P \leqslant Q$.

If $M = \langle X \mid R \rangle$ is a finite presentation of M, and T is a right transversal of P in Q, there is a finite presentation of M^{*T} with $|X||T|$ generators and $|R||T|$ relations.

Let X^P be the closure of X under the action of P. Then $\iota_*(M) = (M^{*T})/N$ where N is the normal closure in M^{*T} of the elements

$$\langle (m,t),(n,u)\rangle = (m,t)^{-1}(n,u)^{-1}(m,t)(n,u)^{\delta(m,t)} \tag{5.9.1}$$

for all $m,n \in X^P$, $t,u \in T$. The homomorphism ι_* is induced by the projection $\mathrm{pr}_1 m = (m,1)$ onto the first factor, and the boundary δ of $\iota_*\mathcal{M}$ is induced from δ' as shown in the following diagram:

$$\begin{tikzcd} M \arrow[r, "\iota_*"] \arrow[d, "\mu"'] & (M^{*T})/N \arrow[d, "\delta"] \\ P \arrow[r, "\iota"'] & Q. \end{tikzcd}$$

When Σ is a set and $\sigma\colon \Sigma \to Q$ any map, take $M = P = F(\Sigma)$ to be the free group on Σ and let $\mathcal{F}_\Sigma = (\mathrm{id}_{F(\Sigma)}\colon F(\Sigma) \to F(\Sigma)\ )$. Then σ extends uniquely to a homomorphism $\sigma'\colon F(\Sigma) \to Q$ and $\sigma'_*\mathcal{F}_\Sigma$ is the free crossed module $\mathcal{F}_\sigma$ described in Section 3.4. However, computation in free crossed modules is in general difficult since the groups are usually infinite.

So, in order to compute the induced crossed module $\iota_*\mathcal{M}$ for $\mathcal{M} = (\mu\colon M \to P)$ a conjugation crossed module and $\iota\colon P \to Q$ an inclusion, we construct finitely presented groups FM, FP, FQ isomorphic to the permutation groups M, P, Q and monomorphisms $FM \to FP \to FQ$ mimicking the inclusions $M \to P \to Q$.

As well as returning an induced crossed module, the construction should return a morphism of crossed modules $(\iota_*,\iota)\colon \mathcal{M} \to \iota_*\mathcal{M}$.

A finitely presented form FC for the copower M^{*T} is constructed with $|X||T|$ generators. The relators of FC comprise $|T|$ copies of the relators of FM, the i-th copy containing words in the i-th set of generators.

The inclusion δ' maps the generators of FM to the first $|X|$ generators of FC. A finitely presented form FI for ι_*M is then obtained by adding to the relators of FC further relators corresponding to the list of elements in Equation (5.9.1).

Then we can apply some Tietze transformationsto the resulting presentation. During the resulting simplification, some of the first $|X|$ generators may be eliminated, so the projection pr_1 may be lost. In order to preserve this projection, and so obtain the morphism ι_*, it is necessary to record for each eliminated generator g a relator gw^{-1} where w is the word in the remaining generators by which g was eliminated.

Let us see how this process works in some examples, and note some of the limitations of the process.

Polycyclic groups, which are implemented in GAP4 as PcGroups (see [GAP08], Chapters 43, 44), have algorithms which can be more efficient than those for arbitrary finitely-presented groups, since they work with elements in normal form. Recall that a *polycyclic group* is a group G with power-conjugate (or power-commutator) presentation having generators $\{g_1, \ldots, g_n\}$ and relations

$$\{g_i^{o_i} = w_{ii}(g_{i+1}, \ldots, g_n),\ g_i^{g_j} = w'_{ij}(g_{j+1}, \ldots, g_n) \text{ for all } 1 \leqslant j < i \leqslant n\}. \tag{5.9.2}$$

Since subgroups $M \leqslant P \leqslant G$ have induced power-conjugate presentations, and T is a transversal for the right cosets of P in G, then the relators of M^{*T} are all of the form in (5.9.2).

Furthermore, all the Peiffer relations in Equation (5.9.1) are of the form $g_i^{g_j} = g_k^p$, so one might hope that a power conjugate presentation would result. Consideration of the cyclic-by-cyclic case in the following example shows that this does not happen in general.

Example 5.9.1. Let C_n be the cyclic group of order n and let $\alpha \colon x \mapsto x^a$ be an automorphism of C_n of order p. Take

$$G = \langle g, h \mid g^p,\ h^n,\ h^g h^{-a}\rangle \cong C_p \ltimes C_n.$$

It follows from these relators that $h^i g = gh^{ai}$, $0 < i < n$ and that $h^{-1}(gh^{i(1-a)})h = gh^{(i+1)(1-a)}$. So if we put $g_i = gh^{i(1-a)}$, $0 \leqslant i < n$ then $g_i^{g_j} = g_{[j+a(i-j)]}$. When $M = P = C_n \lhd G$ Theorem 5.8.12 apply, and $\iota_* P \cong C_n^m$. Now take $M = P = C_p$, with power-conjugate form $\langle g \mid g^p\rangle$, and $\iota \colon C_p \to G$. We may choose as transversal $T = \{\lambda, h, h^2, \ldots, h^{n-1}\}$, where λ is the empty word. Then M^{*T} has generators $\{(g, h^i) \mid 0 \leqslant i < n\}$, all of order p, and relators $\{(g, h^i)^p \mid 0 \leqslant i < n\}$. The additional Peiffer relators in Equation (5.9.1) have the form

$$(g, h^i)(g, h^j) = (g, h^j)(g^k, h^l) \quad \text{where } h^i h^{-j} g h^j = g^k h^l,$$

so $k = 1$ and $l = [j + a(i-j)]$. Hence $\theta \colon \iota_* M \to Q, (g, h^i) \mapsto g_i$ is an isomorphism, and $\iota_* \mathcal{M}$ is isomorphic to the identity crossed module on Q. Furthermore, if we take M to be a cyclic subgroup C_m of C_p then $\iota_* \mathcal{M}$ is the conjugation crossed module $(\partial \colon C_m \ltimes C_n \to C_p \ltimes C_n)$. □

A desirable outcome of a computation is to identify $\iota_* M$ up to isomorphism. Small examples show that many of the induced groups $\iota_* M$ are direct products. However the generating sets in the presentations that arise following the Tietze transformations do not in general split into generating sets for direct summands. This is clearly illustrated by the following simple example.

Example 5.9.2. Let $Q = S_4$ be the symmetric group of degree 4, and let $M = P = A_4$, the alternating subgroup of Q of index 2. Since the abelianisation of A_4 is cyclic of order 3, Theorem 5.8.12 shows that $\iota_* M \cong A_4 \times C_3$. However a typical presentation for $\iota_* M$ obtained from the program is

$$\langle x, y, z \mid x^3, y^3, z^3, (xy)^2, zy^{-1}z^{-1}x^{-1}, yzyx^{-1}z^{-1}, y^{-1}x^2y^2x^{-1} \rangle,$$

and one generator for the C_3 summand is yzx^2. Converting $\iota_* M$ to an isomorphic permutation group gives a degree 18 representation with generating set

$$\begin{aligned}\{\ &(1,2,3)(4,6,9)(5,7,10)(8,11,15)(12,16,17)(13,14,18),\\ &(1,4,5)(2,6,7)(3,9,10)(8,16,14)(11,17,18)(12,13,15),\\ &(2,8)(3,11)(4,12)(5,13)(7,14)(9,17)\ \}.\end{aligned}$$

Converting $\iota_* M$ to a PcGroup produces a group with generators $\{g_1, g_2, g_3, g_4\}$, composition series $A_4 \times C_3 > A_4 > C_2^2 > C_2 > I$, and $g_1 g_2$ is a generator for the normal C_3. In all these representations, the cyclic summand remains hidden, and an explicit search among the normal subgroups must be undertaken to find it. □

To illustrate the results obtained from our computations, we list all the induced crossed modules coming from subgroups of groups of order at most 24 (excluding 16) which are not covered by the special cases mentioned earlier. This enables us to exclude abelian and dihedral groups, cases $P \lhd Q$ and $Q \cong C_m \ltimes C_n$.

In the first table, we assume given an inclusion $\iota \colon P \to Q$ of a subgroup P of a group Q, and a normal subgroup M of P. We list the group $\iota_* M$ induced from

Table 1

$\lvert Q \rvert$	M	P	Q	$\iota_* M$	$\nu_2(\iota)$
12	C_2	C_2	A_4	H_8^+	C_4
	C_3	C_3	A_4	$\mathrm{SL}(2,3)$	C_2
18	C_2	C_2	$C_3^2 \rtimes C_2$	$(C_3^2 \rtimes C_3) \rtimes C_2$	C_3
	S_3	S_3	$C_3^2 \rtimes C_2$	$(C_3^2 \rtimes C_3) \rtimes C_2$	C_3
20	C_2	C_2	H_5	D_{10}	C_2
	C_2	C_2^2	D_{20}	D_{10}	I
	C_2^2	C_2^2	D_{20}	D_{20}	I
21	C_3	C_3	H_7^+	H_7+	I

$(\mu \colon M \to P)$ by the inclusion ι. The kernel of $\partial \colon \iota_* M \to Q$ is written $\nu_2(\iota)$. In the topological application this kernel is related to the second homotopy group, and in some cases such as Theorems 5.4.4 and 5.4.7 it is *exactly* the second homotopy group.

In this table the labels I, C_n, D_{2n}, A_n, S_n denote the identity, cyclic, dihedral, alternating and symmetric groups of order $1, n, 2n, n!/2$ and $n!$ respectively. The group H_n is the holomorph of C_n, and H_n^+ is its positive subgroup in degree n; $\mathrm{SL}(2,3)$ is the special linear group of order 24.

The second table contains the results of calculations with $Q = S_4$, where $C_2 = \langle(1,2)\rangle$, $C_2' = \langle(1,2)(3,4)\rangle$, and $C_2^2 = \langle(1,2),(3,4)\rangle$. $\mathrm{GL}(2,3)$ and $\mathrm{GL}(3,2)$ are the general linear groups of order 48 and 168 respectively. The final column contains the automorphism group $\mathrm{Aut}(\iota_* M)$, except for one case where the automorphism group has order 33,030,144.

Table 2

M	P	$\iota_* M$	$\nu_2(\iota)$	$\mathrm{Aut}(\iota_* M)$
C_2	C_2	$\mathrm{GL}(2,3)$	C_2	$S_4 C_2$
C_3	C_3	$C_3\ \mathrm{SL}(2,3)$	C_6	$S_4 S_3$
C_3	S_3	$\mathrm{SL}(2,3)$	C_2	S_4
S_3	S_3	$\mathrm{GL}(2,3)$	C_2	$S_4 C_2$
C_2'	C_2'	$C_2^3\ H_8^+$	$C_2^3 C_4$	
C_2'	$C_2^2,\ C_4$	H_8^+	C_4	$S_4 C_2$
C_2'	D_8	C_2^3	C_2	$\mathrm{SL}(3,2)$
C_2^2	C_2^2	$S_4 C_2$	C_2	$S_4 C_2$
$C_2^2,\ C_4$	D_8	S_4	I	S_4
C_4	C_4	$\mathrm{SL}(2,3) \rtimes C_4$	C_4	$S_4 C_2^2$
D_8	D_8	$S_4 C_2$	C_2	$S_4 C_2$

These computations confirm our point that the second homotopy group as a module over the fundamental group can be but a pale shadow of a crossed module representing a homotopy 2-type. It is not clear how to obtain what might be the 'simplest' crossed module of such a representation: see the discussion in Section 12.7. Also note again that our method is to obtain a computation of a second homotopy group from a computation of a crossed module obtained by a colimit process. Such calculations essentially of nonabelian second relative homotopy groups have not been obtained by other methods of algebraic topology.

Notes

57 p. 107 The results of this chapter are taken mainly from [BH78a, BW95, BW96, BW03].

58 p. 113 These results on presentations of induced crossed modules go back to [BH78a].

59 p. 114 This type of argument is dealt with for the fundamental groupoid in [Bro06], Section 9.1.

60 p. 116 A modern, kind of repackaged, account of Whitehead's proof is given in [Bro80]. The use of knot theory in this proof is developed in [Hue09]. There are a number of papers which give different proofs of Whitehead's Theorem 5.4.8, e.g. [Rat80], [EP86], [GH86], but it is not generally acknowledged that the theorem is a consequence of a 2-dimensional Seifert–van Kampen type Theorem, a result not mentioned in, say, [HAMS93], although other deductions from that theorem are given.

Whitehead's theorem was seen by Brown and Higgins as a significant example of a universal property in 2-dimensional homotopy theory. It was the desire to obtain this theorem as a consequence of a 2-dimensional Seifert–van Kampen Theorem which made Brown and Higgins realise in 1974 the advantages of using a relative theory.

61 p. 118 Note that in Theorem 5.5.2 we obtain immediately a result on the second absolute homotopy group of $Y \cup C(A)$ without using any homology arguments. This is significant because the setting up of singular homology, proving all its basic properties, and proving the Absolute Hurewicz Theorem takes a considerable time. An exposition of the Hurewicz theorems occurs on pages 166–180 of G. Whitehead's text [Whi78], assuming the properties of singular homology. The cubical account of singular homology in [Mas80] fits best with our story.

Again, one of the reason for emphasising these kinds of results is that they arise from a uniform procedure, which involves first establishing a Higher Homotopy Seifert–van Kampen Theorem. This theorem has analogues for algebraic models of homotopy types which are more elaborate than just groups or crossed modules; these analogues have led to new results, such as a higher order Hopf formula [BE88], which is deduced from an $(n+1)$-adic Hurewicz Theorem [BL87a]. The only proof known of the last result is as a deduction from a Seifert–van Kampen Theorem for n-cubes of spaces [BL87], which has also stimulated research into related areas.

62 p. 119 One interest in these results is the method, which extends to other situations where the notion of abelianisation is not so clear, [BL87a].

63 p. 119 This sequence was originally due to Stallings .

64 p. 119 It was only in 1989 that a generalisation of Hopf's formula to all dimensions was published in [BE88]. This involved the notion of 'double' and 'n-fold presentation', which eventually was reformulated into that of n-fold extension, in work of Janelidze, Everaert, [EGVdL08], and others. This suggests that the reason for the limitations of the crossed complex methods dealt with in Part II is that crossed complexes form a 'linear theory', as is shown by the linear form of a crossed complex. For further work, we would seem to need double and higher crossed complexes. The natural conjecture is that the quadratic theory needs double crossed complexes, and so on! This is a matter for considerable further exploration.

65 p. 126 See for example [CR06].

66 p. 128 Using topological methods, this result is generalised to the non-normal case, in [BW95], Corollary 4.3.

67 p. 133 A direct verification of the universal property is given in [BW96].

Chapter 6
Double groupoids and the 2-dimensional Seifert–van Kampen Theorem

In Chapter 2 we defined the important topological example of the fundamental crossed module of a based pair of spaces

$$\Pi_2(X, A)(x) = (\partial\colon \pi_2(X, A, x) \to \pi_1(A, x)),$$

and the applications of this using a 2-dimensional Seifert–van Kampen Theorem were developed in Chapters 3–5.

However the structure of crossed module is inadequate to yield for this 2-dimensional Seifert–van Kampen Theorem a proof modelling in higher dimensions the proof of the Seifert–van Kampen Theorem, as given in Section 1.6. For this kind of proof, we need a new structure which can express the ideas of multiple compositions of squares and of commutative cubes, and also enables the kinds of deformations used in the proof.[68]

Such a structure is provided by the homotopy double groupoid $\rho(X, A, x)$ of a based pair, which we construct in Section 6.3, and which is discussed briefly in the section on p. xxii in our Introduction to the book.

The proofs that $\rho_2(X, A, C)$ has the required properties are not entirely trivial. The generalisation to all dimensions which we give in Chapter 14 is quite tricky, and needs new ideas; this the main reason for introducing the methods first of all in dimension 2.

The key reason for conceiving of the homotopy double groupoid was to find an algebraic gadget appropriate for giving an

algebraic inverse to subdivision.

This is the slogan underlying the work on Higher Homotopy Seifert–van Kampen Theorems, and explains the emphasis in this book on cubical, rather than the traditional simplicial, methods. Subdividing a square into little squares has a convenient expression in terms of double groupoids, and much more inconvenient expressions, if they exist at all, simplicially or in terms of crossed modules. The 2-dimensional Seifert–van Kampen Theorem was conceived first in terms of double groupoids, and it was only gradually that the link with crossed modules was realised. In the end, the aim of obtaining Whitehead's theorem on free crossed modules (Corollary 5.4.8) as a corollary was an important impetus to forming a definition of a homotopy double groupoid for a pointed *pair* of spaces, since that theorem involved a crossed module defined for such a pair of spaces.

Further, the structure of double groupoids that we use was expressly sought for two reasons. One was to make them equivalent to crossed modules, see Section 6.6. The other was in order to make valid Lemma 6.8.4 in the proof of our 2-dimensional

Seifert–van Kampen Theorem in the final section of this chapter. This lemma, which shows that a construction of an element of a double groupoid is independent of all the choices made, uses crucially the notion of 'commutative cube'. The proof of the higher dimensional theorem in Chapter 14 has a similar structure.

This theory gives also in a sense an *algebraic formulation* of different ways which have been classically used and found necessary in considering properties of second and higher relative homotopy groups. We find that the multiple groupoid viewpoint is useful both for understanding the theory and for proving theorems, while the crossed module, and later crossed complex, viewpoint is useful both for specific calculations, and because of its closer relation to the more traditional chain complexes, see Section 7.4. One important consequence of the algebraic formulation of the equivalence given in Section 6.6 between crossed modules and our particular kind of double groupoids is the equivalence between colimits, and in particular pushouts, in the two categories.

Since this is a longish chapter, it seems a good idea to include a more detailed sketch of the way that all this material is presented here.

The first part describes the step up one dimension from groupoids to double groupoids. Since these are double categories where all structures are groupoids and have either a connection pair or a thin structure the first few sections are devoted to defining first double categories and then connections. In parallel another algebraic category is described, that of crossed modules over groupoids, which is equivalent to that of double groupoids. The equivalence is finally proved in Section 6.6

The first section gives the definition and properties of double categories. Some notions to be used later are also presented here, e.g. the double category of commutative squares or 2-shells in a category or groupoid.

With this model in mind, we can think of the elements of a double category D as squares. Also, we can restrict our attention to the subspace γD of 'squares' having all faces trivial but the top one.

The double categories G which have all three structures groupoids have a substructure γG which is algebraically a crossed module over a groupoid. These algebraic structures, which are an easy step away from that of a crossed module over a group, are studied in Section 6.2.

A direct topological example of such a structure is the fundamental crossed module of a triple of topological spaces (X, A, C) formed by all the crossed modules $\partial\colon \pi_2(X, A, x) \to \pi_1(A, x)$ for varying $x \in C$. We denote this crossed module by $\Pi_2(X, A)$ and we shall prove that it is a crossed module in an indirect way by showing in Proposition 6.3.8 that $\Pi_2(X, A)$ is the crossed module associated to the fundamental double groupoid of a triple $\rho(X, A, C)$ defined in Section 6.3.

Both the fundamental crossed module of a triple and the double category of commutative 2-shells on a groupoid have some extra structure that can be defined in two equivalent ways: as a *thin structure* (as in Section 6.4) and as a *connection pair* (in Section 6.5). In this way we define the objects in the category of double groupoids.

Using 2-shells that 'commute up to some element', in Section 6.6 we associate to each crossed module $\mathcal{M}$ a double groupoid $\lambda\mathcal{M}$ in such a way that it is clear that $\gamma\lambda\mathcal{M}$

is naturally isomorphic to $\mathcal{M}$. It is a bit more challenging to prove that for any double groupoid G, $\lambda\gamma G$ is also naturally isomorphic to G. In order to do this we use the folding operation $\Phi\colon G_2 \to G_2$ which has the effect of folding all faces of an element of G_2 into the top face.

With all the algebra in place, we turn to the topological part. As seen in Chapter 1, the proof of the 1-dimensional Seifert–van Kampen Theorem uses the homotopy commutativity of squares. Thanks to the algebraic machinery developed earlier, we can talk about commutative 3-cubes and prove that any composition of commutative cubes is commutative (Theorem 6.7.4). This commutativity of the boundary of a cube in $\rho(X, A, C)$ has a homotopy meaning which is analogous to the 1-dimensional case and which stated in Section 6.7.

We finish this chapter by giving in Section 6.8 a proof of the 2-dimensional Seifert–van Kampen Theorem for the fundamental double groupoid and so of the main consequences elaborated in the previous chapters.

The whole chapter can be seen as an introduction to the generalisation to all dimensions which is carried out in Part III. Chapter 13 generalises the algebraic part by giving an equivalence between crossed complexes and cubical ω-groupoids with connections, while Chapter 14 covers the topological part, including the statement and proof of the HHSvKT for the fundamental ω-groupoid ρX_* of a filtered space X_*. This result is the basis for the applications to crossed complexes given in Part II.

6.1 Double categories

Let us start by pointing out that there are several candidates for the term 'double groupoid'. We are going to keep that name for the structures which are defined in Section 6.4 and are then used to prove the 2-dimensional Seifert–van Kampen Theorems. We start by investigating what a double category should be.

It is useful, particulary for the generalisations to higher dimensions, to think of a category as having a composition imposed on the underlying geometry of a directed graph, C. This is to have two sets C_0, C_1 called the sets of objects and of morphisms respectively, and three maps: the source $\partial^-\colon C_1 \to C_0$, the target $\partial^+\colon C_1 \to C_0$ and the identity $1 = \varepsilon\colon C_0 \to C_1$, satisfying

$$\partial^-\varepsilon = \partial^+\varepsilon = \mathrm{id}.$$

For a category we also have a partial composition

$$C_1 \times_{C_0} C_1 \to C_1$$

which is associative and has $1_x = \varepsilon(x), x \in C_0$ as giving right and left identities. We think of this composition as a partial binary operation[69] whose domain of definition is determined by geometric conditions.

Thus we can think of the elements of C_0 as 0-dimensional, called points, and the elements of C_1 as 1-dimensional and oriented, called arrows. An element $a \in C_1$ is represented by

$$\underset{\partial^- a}{\bullet} \overset{a}{\text{———}} \underset{\partial^+ a}{\bullet}$$

and for any $x \in C_0$ its identity $1_x = \varepsilon(x)$ is

$$\underset{x}{\bullet} \overset{1_x}{\text{———}} \underset{x}{\bullet}$$

The composition ab of two elements $a, b \in C_1$ is described by juxtaposition:

$$\underset{\partial^- a}{\bullet} \overset{a}{\text{———}} \underset{\partial^+ a = \partial^- b}{\bullet} \overset{b}{\text{———}} \underset{\partial^+ b}{\bullet} \quad = \quad \underset{\partial^-(ab)}{\bullet} \overset{\text{ab}}{\text{————}} \underset{\partial^+(ab)}{\bullet}$$

This gives a well-known 1-dimensional pictorial description of a category.

Remark 6.1.1. In general the notation for categories in this book follows the usual notation for functions in which the composition of $f : X \to Y$ and $g : Y \to Z$ is written $gf : X \to Z$ according to the usual formula $gf(x) = g(f(x))$. However this convention turns out inconvenient when categories are viewed as algebraic objects, with a partial composition, and becomes even more troublesome in higher dimensions. So, when we use categories and groupoids as algebraic structures we will write the composition of $a : x \to y$ and $b : y \to z$ as one of ab, $a \circ b$, $a + b$, all from $x \to z$, but will keep the usual notation for composition in categories of structures, such as the category of groupoids.[70] $\square$

For a 2-dimensional generalisation, namely a double category D, apart from the sets of 'points', D_0 and of 'arrows', D_1, we need a set of 'squares', D_2. We shall also have two categories associated to the 'horizontal' and 'vertical' structures on squares, with their faces and compositions. Also, we should have all the appropriate compatibility conditions dictated by the geometry. The double categories we use are special since the objects of the horizontal and the vertical category structures on squares are the same; in other words, the horizontal and vertical edges of the squares come from the same category. They are sometimes called 'edge-symmetric' double categories. This is the case we need in this book.

Thus we think of an element $u \in D_2$ as a square

$$\begin{array}{c|c|c} & c & \\ \hline a & u & d \\ \hline & b & \end{array} \qquad \begin{array}{l} \ulcorner\!\!\to 2 \\ \downarrow \\ 1 \end{array}$$

where the directions are labeled as indicated, and we call a, b, c, d the *edges, or faces* of u.

Definition 6.1.2. A *double category*[71] is given by three sets D_0, D_1 and D_2 and three structures of category. The first one on (D_1, D_0) has maps ∂^-, ∂^+ and ε and composition denoted as multiplication. The other two are defined on (D_2, D_1), a 'vertical' one with maps ∂_1^-, ∂_1^+ and ε_1 and composition denoted by $u +_1 w$ and the 'horizontal' one with maps ∂_2^-, ∂_2^+ and ε_2 and composition denoted by $u +_2 v$, satisfying some conditions.

Before describing the compatibility conditions it is worth getting used to the diagrammatic expression of the elements in a double category. Thus an element $u \in D_2$ is represented using a matrix like convention

$$\begin{array}{ccc} & \partial_1^- u & \\ \partial_2^- u & \boxed{\quad u \quad} & \partial_2^+ u \\ & \partial_1^+ u & \end{array} \qquad \begin{array}{l} \ulcorner\!\!\longrightarrow 2 \\ \downarrow \\ 1 \end{array}$$

where the labels on the sides are given as indicated.

From this representation it seems indicated, and we assume, that the sources and targets have to satisfy

$$\partial^\tau \partial_1^\sigma = \partial^\sigma \partial_2^\tau \quad \text{for } \sigma, \tau = \pm, \tag{DC 1}$$

since they represent the same vertex. We shall find it convenient to represent the horizontal identity in several ways, i.e.

$$\varepsilon_2(a) = \quad a \boxed{\qquad} a \quad = \boxed{\begin{array}{c} - \\ - \end{array}} = \begin{array}{c} - \\ - \end{array} .$$

In the first representation the unlabeled sides are identities:

$$\partial_1^\sigma \varepsilon_2 = \varepsilon \partial^\sigma \quad \text{for } \sigma = \pm. \tag{DC 2.1}$$

In the other two, the sides corresponding to those drawn in the middle are identities. Similarly, the vertical identity is represented by

$$\varepsilon_1(a) = \begin{array}{c} a \\ \boxed{\qquad} \\ a \end{array} = \boxed{\;|\;\;|\;} = \;|\;\;|$$

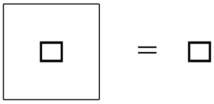
with the same conventions as before. It has also the expected faces in the horizontal direction:

$$\partial_2^\sigma \varepsilon_1 = \varepsilon \partial^\sigma \quad \text{for } \sigma = \pm. \tag{DC 2.2}$$

There are also some relations between the identities. The two double degenerate maps are the same and are denoted by 0:

$$\varepsilon_2\varepsilon = \varepsilon_1\varepsilon = 0. \tag{DC 3}$$

So $0_x = 0(x)$ is both a horizontal and a vertical identity and is represented as:

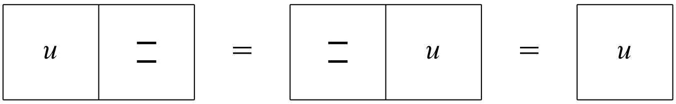


All elements $\varepsilon(x),\ \varepsilon_1(a),\ \varepsilon_2(a)$ are called *degeneracies*.

The vertical and horizontal compositions can be represented by ‘juxtaposition’ in the corresponding direction, and are indicated by:

$$u +_1 w = \begin{array}{|c|}\hline u \\ \hline w \\ \hline \end{array} \qquad u +_2 v = \begin{array}{|c|c|}\hline u & v \\ \hline \end{array}$$

They satisfy all the usual rules of a category, and may be given a diagrammatic representation. For example, the fact that ε_2 is the horizontal identity may be represented as:



The composition in one direction satisfies compatibility conditions with respect to the faces and degeneracies in the other direction, i.e. these functions are homomorphisms. This can be read from the representation. Thus the horizontal faces of a vertical composition are

$$\partial_2^\sigma(u +_1 w) = (\partial_2^\sigma u)(\partial_2^\sigma w) \quad \text{ for } \sigma = \pm, \tag{DC 4.1}$$

and the vertical faces of the horizontal composition are

$$\partial_1^\sigma(u +_2 v) = (\partial_1^\sigma u)(\partial_1^\sigma v) \quad \text{ for } \sigma = \pm. \tag{DC 4.2}$$

The same applies to the vertical and horizontal identities, i.e.

$$\varepsilon_2(ab) = \varepsilon_2(a) +_1 \varepsilon_2(b), \tag{DC 5.1}$$

$$\varepsilon_1(ab) = \varepsilon_1(a) +_2 \varepsilon_1(b). \tag{DC 5.2}$$

Our final compatibility condition is known as the 'interchange law' and says that, when composing 4 elements in a square, it is irrelevant if we compose first in the horizontal direction and then in the vertical one, or the other way around, i.e.

$$(u +_2 v) +_1 (w +_2 x) = (u +_1 w) +_2 (v +_1 x) \tag{DC 6}$$

when both sides are defined. This can be represented as giving only one way of evaluating the double composition:

$$\begin{array}{|c|c|}
\hline
u & v \\
\hline
w & x \\
\hline
\end{array}$$

To complete the description of the category of double categories, a *double functor* between two double categories D and D' is given by three maps $F_i : D_i \to D'_i$ for $i = 0, 1, 2$ which commute with all structure maps (faces, degeneracies, composition, etc.). In particular, the pair (F_1, F_0) gives a functor from (D_1, D_0) to (D'_1, D'_0).

With these objects and morphisms, we get the category DCat of double categories. □

Remark 6.1.3. Thus a double category has a structure which is called a 2-*truncated cubical set with compositions*. Properties (DC 1)–(DC 3) give the 2-truncated cube structure and (DC 4)–(DC 6) the compatibility with compositions. The corresponding definitions in all dimensions are given in Part III in Section 13.1. □

Remark 6.1.4. On matrix notation. There is also a matrix notation for the compositions which will be useful in this chapter and is applied often in Part III. We write:

$$u +_1 w = \begin{bmatrix} u \\ w \end{bmatrix}, \quad u +_2 v = [u, v].$$

With this notation we can represent all the rules in the definition of double categories. For instance, we have

$$\begin{bmatrix} u \\ |\,| \end{bmatrix} = [\ u, \ \ = \ ] = u.$$

Choosing the matrix description, the 'interchange law' (DC 6) may be written

$$\begin{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} & \begin{bmatrix} v \\ x \end{bmatrix} \end{bmatrix} = \begin{bmatrix} [u \quad v] \\ [w \quad x] \end{bmatrix},$$

and this his common value is also written

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix}.$$

Here is a caution about using this interchange law. Let u, v be squares in a double category such that

$$w = \begin{bmatrix} u & v \end{bmatrix} = u +_2 v$$

is defined. Suppose further that

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u_1 +_1 v_1, \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_2 +_1 v_2.$$

Then we can assert

$$w = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$$

only when $u_1 +_2 v_1$, and $u_2 +_2 v_2$ are defined. Thus care is needed in 2-dimensional rewriting, such as that given on p. 175.[72] □

This matrix notation has a generalisation which we use later.

Definition 6.1.5. Let D be a double category. A *composable array* (u_{ij}) in D is given by elements $u_{ij} \in D_2$ $(1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n)$ satisfying

$$\begin{cases} \partial_2^+ u_{i,j-1} = \partial_2^- u_{i,j} & (1 \leqslant i \leqslant m,\ 2 \leqslant j \leqslant n), \\ \partial_1^+ u_{i-1,j} = \partial_1^- u_{i,j} & (2 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n). \end{cases}$$

It follows from the interchange law that a composable array (u_{ij}) in D can be composed both ways, getting the same result which is denoted by $[u_{ij}]$.

If $u \in D_2$, and (u_{ij}) is a composable array in D satisfying $[u_{ij}] = u$, we say that the array (u_{ij}) is a *subdivision* of u. We also say that u is the *composite* of the array (u_{ij}). □

Remark 6.1.6. Subdivisions and their use. The interchange law implies that if in the composable array (u_{ij}) we partition the rows and columns into blocks which produce another composable array and compute the composite v_{kl} of each block, then $[u_{ij}] = [v_{kl}]$. We call the (u_{ij}) a *refinement* of (v_{kl}) in this case.

This observation is used in several ways to prove equalities. The method consists usually in starting from the definition of one side of the equation, then changing the array using this subdivision technique and finally composing the new array to get the other side of the equation. For quite elaborate examples of this method, see the proofs of Theorem 6.4.10 and Proposition 6.5.3.

Changes in a composable array that are clearly possible using this subdivision technique are:

(i) Select a block of an array and change it for another block having the same composition and the same boundary (see Proposition 6.6.8)

(ii) Substitute some adjacent columns by another set of adjacent columns having the same boundary and such that each row has the same horizontal composition in both cases. The same can done with rows (see Theorems 6.4.10 and 6.4.11). □

Example 6.1.7. Let us give a couple of examples of double categories associated to a category C. The first one is the double category of 'squares' or, better still, '2-shells' in a category C, denoted by $\Box' C$.

The category structure of $(\Box' C)_1$, i.e. points and arrows and composition, is the same as that of C. The squares of $(\Box' C)_2$ are defined to be the quadruples $\left(a \begin{smallmatrix} c \\ b \end{smallmatrix} d\right)$ of elements $a, b, c, d \in C_1$ such that

$$\partial^+ a = \partial^- b, \quad \partial^- d = \partial^+ c \quad \partial^+ b = \partial^+ d, \quad \partial^- a = \partial^- c$$

and the horizontal and vertical face and degeneracy maps are the ones clear from this representation. For example

$$\partial_1^- \left(a \begin{smallmatrix} c \\ b \end{smallmatrix} d\right) = c, \quad \partial_2^+ \left(a \begin{smallmatrix} c \\ b \end{smallmatrix} d\right) = d; \qquad \begin{array}{l} \longrightarrow 2 \\ \downarrow \\ 1 \end{array}$$

if $a \colon x \to y$ in C_1 then

$$\varepsilon_1(a) = \left(1_x \begin{smallmatrix} a \\ a \end{smallmatrix} 1_y\right), \quad \varepsilon_2(a) = \left(a \begin{smallmatrix} 1_x \\ 1_y \end{smallmatrix} a\right).$$

The compositions are defined by

$$\left(a \begin{smallmatrix} c \\ b \end{smallmatrix} d\right) +_1 \left(f \begin{smallmatrix} b \\ g \end{smallmatrix} h\right) = \left(af \begin{smallmatrix} c \\ g \end{smallmatrix} dh\right)$$

and

$$\left(a \begin{smallmatrix} c \\ b \end{smallmatrix} d\right) +_2 \left(d \begin{smallmatrix} u \\ v \end{smallmatrix} w\right) = \left(a \begin{smallmatrix} cu \\ bv \end{smallmatrix} w\right)$$

It is easy to see that $\Box' C$ is a double category and that $\Box'$ is a functorial construction. Moreover this functor is right adjoint to the truncation functor which sends each double category D to the category (D_1, D_0). We leave the proof of adjointness as an exercise.

We now define $\Box\, C$, the category of 'commutative squares' or 'commutative 2-shells'. Its squares are the quadruples $\left(a \begin{smallmatrix} c \\ b \end{smallmatrix} d\right)$ such that $ab = cd$.

The horizontal and vertical face and degeneracy maps and the compositions are the restriction of those in $\Box' C$. □

Example 6.1.8. There are quite a few categories that can be defined in a similar manner, but requiring that the compositions ab and cd differ in some way by the action of an element of some fixed subset of C. It is a good exercise to investigate which conditions C and the action have to satisfy to obtain a double category.[73] We shall come back to this in Example 6.1.10. □

As we have stated above, our main objects of interest are double groupoids.[74] These are in the first instance double categories in which all the categories involved are groupoids. We first study these and later impose further structure of 'connections' and the equivalent thin structure which are crucial for this work.

Definition 6.1.9. The category DCatG is the full subcategory of DCat which has as objects double categories in which all three structures are groupoids. □

By definition of a groupoid as a small category G in which all arrows are isomorphisms, there is a map $(\ )^{-1}\colon G_1 \to G_1$ such that

$$aa^{-1} = 1_{\partial^- a} \quad \text{and} \quad a^{-1}a = 1_{\partial^+ a}, \qquad a \in G.$$

Thus in a double category G where all three category structures are groupoids, there are three 'inverse' maps

$$(\ )^{-1}\colon G_1 \to G_1, \quad -_1\colon G_2 \to G_2 \quad \text{and} \quad -_2\colon G_2 \to G_2,$$

where

$$(\varepsilon_i a) +_j (\varepsilon_i a^{-1}) = 0_{\partial^- a}, \quad (\varepsilon_i a^{-1}) +_j (\varepsilon_i a) = 0_{\partial^+ a}, \qquad \text{for } i \neq j.$$

From the compatibility conditions (DC 4.1), (DC 4.2), we see that the boundary maps preserve inverses in the other direction since they are homomorphisms, i.e.

$$\partial_1^\sigma(-_2\alpha) = (\partial_1^\sigma(\alpha))^{-1}, \quad \partial_2^\sigma(-_1\alpha) = (\partial_2^\sigma(\alpha))^{-1}. \tag{DCG 4}$$

From the compatibility conditions (DC 5.1), (DC 5.2), we get that the degeneracy maps also preserve inverses, i.e.

$$\varepsilon_1(a^{-1}) = -_2(\varepsilon_1(a)), \quad \varepsilon_2(a^{-1}) = -_1(\varepsilon_2(a)). \tag{DCG 5}$$

We also easily check from the interchange law that for $u \in G_2$,

$$-_1 -_2 u = -_2 -_1 u; \tag{DCG 6}$$

we denote $-_1 -_2$ by $-_{12}$ and call it a 'rotation', an idea which will be developed in Definition 6.4.5.

Example 6.1.10. In the case G is a groupoid, the double categories $\square\, G$ of commutative 2-shells and $\square' G$ of 2-shells in G defined in Example 6.1.7 are all double groupoids, the inverses of $\alpha = \begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix}$ being as follows:

$$-_1\alpha = \begin{pmatrix} & b & \\ a^{-1} & & d^{-1} \\ & c & \end{pmatrix}, \quad -_2\alpha = \begin{pmatrix} & c^{-1} & \\ d & & a \\ & b^{-1} & \end{pmatrix}, \quad -_1 -_2\, \alpha = \begin{pmatrix} & b^{-1} & \\ d^{-1} & & a^{-1} \\ & c^{-1} & \end{pmatrix}.$$

□

There are interesting differences between the category and groupoid cases with regard to commutative 2-shells. If G is a groupoid, the commutativity condition of a 2-shell α as above can also be stated as $c = abd^{-1}$ or even as $b^{-1}a^{-1}cd = 1$.

Exercise 6.1.11. In homotopy theory we may have noncommutative shells, so it is interesting to see if we can modify the above construction of a double groupoid: so start with a group G and a subgroup S of G and consider 2-shells $\left(a \begin{smallmatrix} c \\ b \end{smallmatrix} d\right)$ of elements of G such that $b^{-1}a^{-1}cd \in S$. Prove that the compositions of these can be defined as above to make a double groupoid if and only if S is a normal subgroup of G. (This exercise is a preparation for the construction of a double groupoid from a crossed module in Section 6.6.) □

This leads to a possible extension of the notion of normal subgroups to 'normal subgroupoids'.[75] At a further stage, the concept of normal subgroupoid can be 'externalised' as a crossed module of groupoids, analogously to what has been done for groups, and this will be done in Section 6.2.[76]

6.2 The category of crossed modules over groupoids

Crossed modules of groupoids are an easy step away from crossed modules of groups and mimic the structure of the family of fundamental crossed modules $\Pi_2(X, A, x)$ when $x \in A \subseteq X$. Also we can construct crossed modules over groupoids associated to any double category which has all three structures of groupoid.

It is natural to define a crossed module of groupoids to be a morphism of groupoids $(\mu\colon M \to P)$ with an action of P on M such that axioms equivalent to CM1) and CM2) are satisfied. Thus, we start with a groupoid P with its set of vertices P_0, and its initial and final maps ∂^-, ∂^+. We write $P_1(p,q)$ for the set of arrows from p to q $(p, q \in P)$ and $P_1(p)$ for the group $P_1(p, p)$. Recall that the composition of $a\colon p \to q$ and $b\colon q \to r$ is written $ab\colon p \to r$.

Definition 6.2.1. A *crossed module* over the groupoid $P = (P_1, P_0)$ is given by a groupoid $M = (M_2, P_0)$ and a morphism of groupoids

$$M \xrightarrow{\mu} P$$

which is the identity on objects and satisfies:

• M is a totally disconnected groupoid with the same objects as P. Equivalently, M is a family of groups $\{M_2(p)\}_{p \in P_0}$.

We shall use additive notation for all groups $M_2(p)$ and we shall use the same symbol 0 for all their identity elements.

Also, μ is given by a family of homomorphisms $\{\mu_p\colon M_2(p) \to P_1(p)\}_{p \in P_0}$.

• The groupoid P operates on the right on M. The action is denoted $(x, a) \mapsto x^a$. If $x \in M_2(p)$ and $a \in P_1(p, q)$ then $x^a \in M_2(q)$. It satisfies the usual axioms of an action:

i) $x^1 = x$, $(x^{ab}) = (x^a)^b$,

ii) $(x + y)^a = x^a + y^a$.

(Thus $M_2(p) \cong M_2(q)$ if p and q lie in the same component of the groupoid P).

• These data are to satisfy two laws:

CM1) μ preserves the actions, i.e. $\mu(x^a) = (\mu x)^a$.

CM2) For all $c \in M_2(p)$, μc acts on M by conjugation by c, i.e. for any $x \in M_2(p)$,

$$x^{\mu c} = -c + x + c.$$

Notice that $M_2(p)$ is a crossed module over $P_1(p)$ for all $p \in P_0$. In the case when P_0 is a single point μ is a *crossed module over a group* and we also call it a *reduced* crossed module.

A *morphism* of crossed modules $f : (\mu: M \to P) \to (\nu: N \to Q)$ is a pair of morphisms of groupoids $f_2: M \to N$, $f_1: P \to Q$ inducing the same map of vertices and compatible with the boundary maps and the actions of both crossed modules. We denote by XMod the resulting category of crossed modules over groupoids. Notice that the category XMod/Groups studied in the preceding chapters can be seen as the full subcategory of XMod whose objects are reduced crossed modules of groupoids. □

Example 6.2.2. As we have pointed out, there is an immediate topological example. For any topological pair (X, A) and $C \subseteq A$, we consider $P = \pi_1(A, C)$, the fundamental groupoid of (A, C). Recall that the objects of $\pi_1(A, C)$ are the points of C and for any $x, y \in C$, the elements of $\pi_1(A, C)(x, y)$ are the homotopy classes rel $\{0, 1\}$ of maps

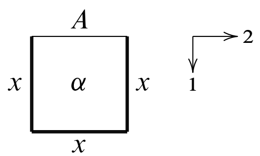
$$\omega: (I, 0, 1) \to (A, x, y).$$

The *fundamental crossed module* $\Pi_2(X, A, C)$ of the triple (X, A, C) includes the family of groups $\{\pi_2(X, A, x)\}_{x \in C}$ which we defined already in Section 2.1.

Recall that any $[\alpha] \in \pi_2(X, A, x)$ is a homotopy class rel J^1 of maps

$$\alpha: (I^2, \partial I^2, J^1) \to (X, A, x),$$

and so can be represented as a square



where we follow our 'matrix' convention for the directions 1 and 2, as shown by the arrows.

The action

$$\pi_2(X, A, x) \times \pi_1(A, C)(x, y) \to \pi_2(X, A, y)$$

was also described in Section 2.1.

The morphism of groupoids $\partial\colon \pi_2(X, A, C) \to \pi_1(A, C)$ is given, for each $x \in C$, by the restriction to the top face $0 \times I$, so giving

$$\partial(x)\colon \pi_2(X, A, x) \to \pi_1(A, x).$$

As before, it could be verified directly that these maps satisfy the properties of a crossed module over a groupoid, but we prefer the roundabout way of verifying this by proving that this crossed module is the one associated to a double groupoid called the *fundamental double groupoid* which will be defined in Section 6.3.[77] □

Let us go back to the general theory and see how to associate to any object $G \in \mathsf{DCatG}$ a crossed module of groupoids which we denote by

$$\gamma G = (\partial\colon \gamma G \to P).$$

We start by defining P to be the groupoid (G_1, G_0). Thus the objects of γG are $(\gamma G)_0 = G_0$ and as morphisms we choose all $u \in G_2$ that have all faces degenerate except $\partial_1^- u$, i.e.

$$(\gamma G)_2 = \{u \in G_2 \mid \partial_2^+ u = \partial_2^- u = \partial_1^+ u = \varepsilon\partial^-\partial_1^- u = \varepsilon\partial^+\partial_1^- u\}.$$

The reason we chose to use the subindex 2 in the set of morphisms M_2 of M is now apparent: because in this very important example they have 'dimension' 2. The elements in γG_2, when represented with a matrix like convention for the directions, are

$$\begin{array}{ccc} & \partial_1^- u & \\ 1 & \boxed{\quad u \quad} & 1 \\ & 1 & \end{array} \qquad \begin{tikzcd} \cdot \arrow[r] \arrow[d] & 2 \\ 1 & \end{tikzcd}$$

With the obvious source, target, and identity, and the composition $u + v$ defined to be $u +_2 v$, we get a totally disconnected groupoid $(\gamma G)_2$.

To make it a crossed module we need in addition to the groupoid structures on γG and P, a morphism of groupoids ∂ and an action satisfying CM1) and CM2).

The morphism of groupoids is defined by

$$\partial = \partial_1^-\colon \gamma G_2 \to P_1. \tag{6.2.1}$$

The final ingredient is an action

$$\gamma G_2(x) \times G_1(x, y) \to \gamma G_2(y)$$

for all $x, y \in G_0$. It is given by degeneration and conjugation: i.e. for any $u \in \gamma G_2(x)$ and $a \in G_1(x, y)$,

$$u^a = [-_2\varepsilon_1 a,\ u,\ \varepsilon_1 a], \tag{6.2.2}$$

or, in the usual representation,

$$\begin{array}{ccc} & (\partial_1^- u)^a & \\ 1 & \boxed{u^a} & 1 \\ & 1 & \end{array} \; = \; \begin{array}{|c|c|c|} \multicolumn{1}{c}{a^{-1}} & \multicolumn{1}{c}{\partial_1^- u} & \multicolumn{1}{c}{a} \\ \hline \mid\mid & u & \mid\mid \\ \hline \multicolumn{1}{c}{a^{-1}} & \multicolumn{1}{c}{1} & \multicolumn{1}{c}{a} \end{array}$$

Now we have to check that this gives an action which satisfies both properties in the definition of crossed module.

Proposition 6.2.3. *The equation* (6.2.2) *defines a right action of G_1 on γG_2.*

Proof. From the diagram, it is clear that $u^a \in \gamma G_2$. It is also not difficult to prove all properties of an action:

$$u^{ab} = (u^a)^b, \quad (u +_2 v)^a = u^a +_2 v^a \quad \text{and} \quad u^1 = u. \qquad \square$$

It remains to check the two axioms CM1) and CM2).

Proposition 6.2.4. *$\gamma G = (\partial_1^- : \gamma G_2 \to G_1)$ is a crossed module with the action defined by* (6.2.2).

Proof. The law CM1) is clear from the diagram, since the top face is the conjugate:

$$\partial(u^a) = \partial_1^-(u^a) = \partial_1^-(-_2\varepsilon_1 a)\partial_1^- u \partial_1^-(\varepsilon_1 a) = a^{-1}\partial_1^- u a = (\partial u)^a.$$

With respect to CM2), for any $a = \partial v$, $v \in \gamma G_2$, we may construct an array such that when computing both ways gives the equality. In this case the array is

$$\begin{array}{|c|c|c|} \multicolumn{1}{c}{a^{-1}} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{a} \\ \hline \mid\mid & u & \mid\mid \\ \hline -_2 v & \square & v \\ \hline \end{array}$$

Composing first in the horizontal direction and then in the vertical one, the first row gives u^a and the second one a degenerate square, so we get u^a.

On the other hand, composing first vertically, we get

$$[-_2 v, u, v] = u^v. \qquad \square$$

Definition 6.2.5. The previous construction of the *crossed module over groupoids associated to a double category in* DCatG gives a functor

$$\gamma : \mathsf{DCatG} \to \mathsf{XMod}. \qquad \square$$

Remark 6.2.6. We finish this section by pointing out that for a double category which has all three structures groupoids we have not only one associated crossed module over groupoids but in principle four, possibly with differing conventions, since we may chose any of the sides to be the unique one not equal to the identity. Let us call γG_j^σ the crossed module structure on the set of all elements of G_2 having all faces degenerate but the $-$-face in the j-direction defined by the map ∂_j^-. In general, γG_1^- and γG_2^- are not isomorphic but we shall see that they are isomorphic in the topological case of interest here, namely Example 6.2.2. The reason for this is that they are isomorphic for double groupoids with connections, using rotations (see Definition 6.4.5). $\square$

6.3 The fundamental double groupoid of a triple of spaces

We shall start by describing a space of maps and some structure over it before taking homotopy classes.

We consider a triple (X, A, C). We shall use the triple $(I^2, \partial I^2, \partial^2 I^2)$ given by the square, its boundary and the four vertices, respectively. We consider three sets

$$\begin{aligned} R_0(X, A, C) &= C, \\ R_1(X, A, C) &= \{\sigma : (I, \{0, 1\}) \to (A, C)\}, \\ R_2(X, A, C) &= \{\alpha : (I^2, \partial I^2, \partial^2 I^2) \to (X, A, C)\}. \end{aligned}$$

and call the elements of $R_2(X, A, C)$ *filtered maps*

$$\alpha : (I^2, \partial I^2, \partial^2 I^2) \to (X, A, C).$$

Remark 6.3.1. The elements of R_2 can be represented by squares as follows.

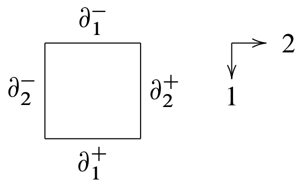
$$\begin{tikzcd} C \arrow[r, dash, "A"] \arrow[d, dash, "A"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, dash, "A"] \\ C \arrow[r, dash, "A"'] & C \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r] \arrow[d] & 2 \\ 1 & \end{tikzcd}$$

Note that, unlike the definition of the relative homotopy groups, no choice of direction or initial edge is made. We shall see that the 'good practice' of not making choices too early combines with the aesthetic requirement of symmetry to lead to a construction with more power! $\square$

There is an obvious definition of source and target maps on R_2, in two directions and given by restriction to the appropriate faces of I^2. More formally they are compositions with the maps given for $x \in I$ by

$$\partial_1^+(x) = (1, x), \quad \partial_1^-(x) = (0, x) \quad \text{and} \quad \partial_2^+(x) = (x, 1), \quad \partial_2^-(x) = (x, 0)$$

and they can be seen in the diagram



The identities are given by composing with the projection in the appropriate direction, i.e.

$$p_1(x, y) = x \quad \text{and} \quad p_2(x, y) = y$$

and we use the same notation for degenerate squares as in the previous section.

The set R_1 has the standard composition of paths used in defining the fundamental groupoid. The set R_2 has two similar compositions given for α, β, γ in $R_2(X, A, C)$ such that $\partial_1^+\alpha = \partial_1^-\beta$, $\partial_2^+\alpha = \partial_2^-\gamma$ by

$$(\alpha +_1 \beta)(x, y) = \begin{cases} \alpha(2x, y) & \text{if } 0 \leqslant x \leqslant 1/2, \\ \beta(2x - 1, y) & \text{if } 1/2 \leqslant x \leqslant 1, \end{cases}$$

and

$$(\alpha +_2 \gamma)(x, y) = \begin{cases} \alpha(x, 2y) & \text{if } 0 \leqslant y \leqslant 1/2, \\ \gamma(x, 2y - 1) & \text{if } 1/2 \leqslant y \leqslant 1. \end{cases}$$

We leave the reader to check that the interchange law holds for these two compositions. The reverse of an element $\alpha \in R_2$, with respect these two directions are written $-_1\alpha$, $-_2\alpha$ and are defined respectively by $(x, y) \mapsto \alpha(1 - x, y)$, $(x, y) \mapsto \alpha(x, 1 - y)$.

All this structure means in particular that $R(X, A, C)$ is a 2-truncated cubical set with compositions. (The general situation in all dimensions is discussed in Section 13.1). This structure does not give a double category, for the usual reasons of lack of associativity, identities, etc. Nevertheless, it is useful to fix the meaning of composition of arrays. We study this in the next remark.[78]

Remark 6.3.2. For positive integers m, n let $\varphi_{m,n} \colon I^2 \to [0, m] \times [0, n]$ be the map $(x, y) \mapsto (mx, ny)$. An $m \times n$ subdivision of a square $\alpha \colon I^2 \to X$ is a factorisation $\alpha = \alpha' \circ \varphi_{m,n}$; its *parts* are the squares $\alpha_{ij} \colon I^2 \to X$ defined by

$$\alpha_{ij}(x, y) = \alpha'(x + i - 1, y + j - 1).$$

We then say that α is the *composite* of the squares α_{ij}, and we write $\alpha = [\alpha_{ij}]$. Similar definitions apply to paths and cubes.

Such a subdivision determines a cell-structure on I^2 as follows. The intervals $[0, m]$, $[0, n]$ have cell-structures with integral points as 0-cells and the intervals $[i, i+1]$ as closed 1-cells. Then $[0, m] \times [0, n]$ has the product cell-structure which is transferred to I^2 by $\varphi_{m,n}^{-1}$. We call the 2-cell $\varphi_{m,n}^{-1}([i-1, i] \times [j-1, j])$ the *domain* of α_{ij}. □

Definition 6.3.3. To define the *fundamental double groupoid associated to a triple of spaces* (X, A, C) we shall use the three sets

$$\begin{aligned}
\rho_0(X, A, C) &= C,\\
\rho_1(X, A, C) &= R_1(X, A, C)/\equiv,\\
\rho_2(X, A, C) &= R_2(X, A, C)/\equiv,
\end{aligned}$$

where $\equiv$ is the relation of homotopy rel vertices on R_1 and of homotopy of pairs rel vertices on R_2. That is, for such a homotopy $H_t\colon I^2 \to X$, we have for all $t \in I$ that $H_t(c) = H_0(c)$ if $c \in \partial^2 I^2$, and also of course $H_t(b) \in A$ for all $b \in \partial I^2$. We call this type of homotopy a *thin homotopy* to distinguish it from the usual homotopy, written $\simeq$, of maps $I \to A$ or $I^2 \to X$. It is important that a thin homotopy is rel vertices, that is, the vertices of I and of I^2 are fixed in the homotopies. This allows us to obtain the groupoid structures on the thin homotopy classes without imposing any condition on the spaces.

The *thin homotopy class* of an element α is written $\langle\!\langle \alpha \rangle\!\rangle$. □

Remark 6.3.4. We will see later that the condition of 'rel vertices' is a key to obtaining the double groupoid structure.[79] □

We expect all the structure maps in $\rho(X, A, C)$ to be those induced by the corresponding structure maps of $R(X, A, C)$. So we have to prove that they are compatible with the homotopies. In the case of the structure maps for (ρ_1, ρ_0) this is clear, since they form the relative fundamental groupoid of the pair (A, C).

Let us try the maps for the horizontal and vertical structure on (ρ_2, ρ_1). There is no problem with the source and target since the homotopies are thin. Also a homotopy between elements of $R_1(X, A, C)$ gives easily a homotopy between the associated identities. The only problems appear to be with the compositions.

We develop only the horizontal case; the other follows by symmetry. So, let us consider two elements $\langle\!\langle \alpha \rangle\!\rangle, \langle\!\langle \beta \rangle\!\rangle \in \rho_2(X, A, C)$ such that $\langle\!\langle \partial_2^+ \alpha \rangle\!\rangle = \langle\!\langle \partial_2^- \beta \rangle\!\rangle$, i.e. we have continuous maps

$$\alpha, \beta\colon (I^2, \partial I^2, \partial^2 I^2) \to (X, A, C)$$

and a homotopy

$$h\colon (I, \partial(I)) \times I \to (A, C)$$

from $\alpha|_{\{1\}\times I}$ to $\beta|_{\{0\}\times I}$ rel vertices, i.e. $h(0 \times I) = y$ and $h(1 \times I) = x$. We define now the composition by

$$\langle\!\langle \alpha \rangle\!\rangle +_2 \langle\!\langle \beta \rangle\!\rangle = \langle\!\langle \alpha +_2 h +_2 \beta \rangle\!\rangle = \langle\!\langle [\alpha, h, \beta] \rangle\!\rangle.$$

This is given in a diagram in which the outer labels A, X denote the codomain for the edges and the inner labels A, X denote the codomain for the squares:

$$\begin{array}{c|c|c|c|c} \multicolumn{1}{c}{} & \multicolumn{1}{c}{A} & \multicolumn{1}{c}{x} & \multicolumn{1}{c}{A} & \multicolumn{1}{c}{} \\ \cline{2-4} & X & A & X & \\ A & \alpha & h & \beta & A\,. \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c}{A} & \multicolumn{1}{c}{y} & \multicolumn{1}{c}{A} & \multicolumn{1}{c}{} \end{array} \tag{6.3.1}$$

Our first important step is that these compositions are well defined.

Proposition 6.3.5. *The compositions $+_i$ in $\rho_2(X, A, C)$ are well defined.*

Proof. To prove this we chose two other representatives $\alpha' \in \langle\!\langle \alpha \rangle\!\rangle$ and $\beta' \in \langle\!\langle \beta \rangle\!\rangle$ and a homotopy h' from $\alpha'|_{\{1\}\times I}$ to $\beta'|_{\{0\}\times I}$. Using them, we get

$$\begin{array}{c|c|c|c|c} \multicolumn{1}{c}{} & \multicolumn{1}{c}{A} & \multicolumn{1}{c}{x} & \multicolumn{1}{c}{A} & \multicolumn{1}{c}{} \\ \cline{2-4} & X & A & X & \\ A & \alpha' & h' & \beta' & A\,. \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c}{A} & \multicolumn{1}{c}{y} & \multicolumn{1}{c}{A} & \multicolumn{1}{c}{} \end{array}$$

which should give the same composition in ρ_2 as (6.3.1).

Since $\langle\!\langle \alpha \rangle\!\rangle = \langle\!\langle \alpha' \rangle\!\rangle$, $\langle\!\langle \beta \rangle\!\rangle = \langle\!\langle \beta' \rangle\!\rangle$ there are thin homotopies $\phi \colon \alpha \equiv \alpha'$, $\psi \colon \beta \equiv \beta'$ which can be seen in the next figure, in which the thick lines denote edges on which the maps are constant.

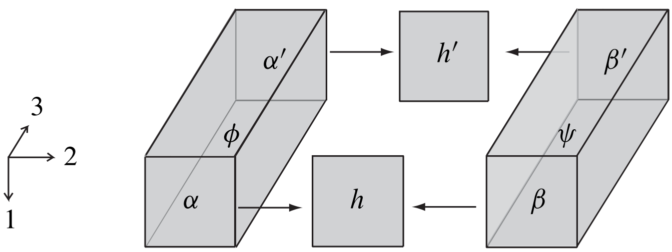


Figure 6.1. Filling a hole in the middle

To complete this to a thin homotopy

$$\alpha +_2 h +_2 \beta \equiv \alpha' +_2 h' +_2 \beta'$$

we need to ‘fill’ appropriately the hole in the middle (see Figure 6.3).

Let

$$k\colon I \times \partial I^2 \to A$$

be given by $(r,s,0) \mapsto h(r,s), (r,s,1) \mapsto h'(r,s)$, $(r,0,t) \mapsto \phi_t(r,1), (r,1,t) \mapsto \psi_t(r,0)$. In terms of Figure 6.3, k is the map defined on the four side faces of the central hole. But k is constant on the edges of the bottom face, since all the homotopies are rel vertices. So k extends over $\{1\} \times I^2 \to A$ extending k to five faces of I^3.

Now we can retract I^3 onto any five faces by projecting from a point above the centre of the remaining face. Composing with this retraction, we obtain a further extension $k'\colon I^3 \to A$. The composite cube $\phi +_2 k' +_2 \psi$ is a thin homotopy $\gamma \equiv \gamma'$ as required: the key point is that the extension k' of k maps the top face of the middle cube into A, since that is true for all the other faces of this middle cube.[80] □

Once we have proved that compositions in ρ_2 are well defined, we can easily prove that they are groupoids, with $\langle\!\langle -_i\alpha \rangle\!\rangle$ being the inverse of $\langle\!\langle \alpha \rangle\!\rangle$ for the composition $+_i$, $i = 1, 2$. We also need to prove the interchange law.

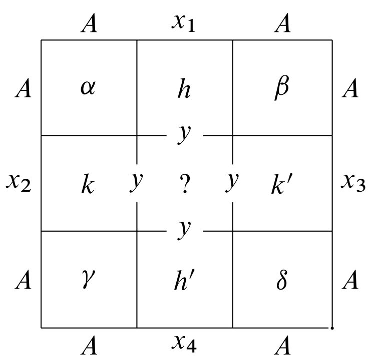
Proposition 6.3.6. *The compositions $+_1$, $+_2$ in $\rho_2(X, A, C)$ satisfy the interchange law.*

Proof. We start with four elements $\langle\!\langle \alpha \rangle\!\rangle, \langle\!\langle \beta \rangle\!\rangle, \langle\!\langle \gamma \rangle\!\rangle, \langle\!\langle \delta \rangle\!\rangle \in \rho_2(X, A, C)$ such that $\langle\!\langle \partial_2^+ \alpha \rangle\!\rangle = \langle\!\langle \partial_2^- \beta \rangle\!\rangle$, $\langle\!\langle \partial_2^+ \gamma \rangle\!\rangle = \langle\!\langle \partial_2^- \delta \rangle\!\rangle$, $\langle\!\langle \partial_1^+ \alpha \rangle\!\rangle = \langle\!\langle \partial_1^- \gamma \rangle\!\rangle$ and $\langle\!\langle \partial_1^+ \beta \rangle\!\rangle = \langle\!\langle \partial_1^- \delta \rangle\!\rangle$. To prove that

$$(\langle\!\langle \alpha \rangle\!\rangle +_2 \langle\!\langle \beta \rangle\!\rangle) +_1 (\langle\!\langle \gamma \rangle\!\rangle +_2 \langle\!\langle \delta \rangle\!\rangle) = (\langle\!\langle \alpha \rangle\!\rangle +_1 \langle\!\langle \gamma \rangle\!\rangle) +_2 (\langle\!\langle \beta \rangle\!\rangle +_1 \langle\!\langle \delta \rangle\!\rangle)$$

we construct an element of $R_2(X, A, C)$ that represents both compositions. The argument involves ‘filling a hole’.

Using thin homotopies $h\colon \partial_2^+ \alpha \equiv \partial_2^- \beta$, $h'\colon \partial_2^+ \gamma \equiv \partial_2^- \delta$, $k\colon \partial_1^+ \alpha \equiv \partial_1^- \gamma$ and $k'\colon \partial_1^+ \beta \equiv \partial_1^- \delta$ given because the compositions are defined we have a map defined on the whole square except on a hole in the middle:



However the edges of this ‘hole’ determine constant paths at y. So we extend with the constant map, and evaluate the resulting composition in two ways to prove the interchange law. □

Thus we have proved that $\rho(X, A, C)$ is a double category where all three structures are groupoids. We call this *the fundamental double groupoid of the triple* (X, A, C) and leave the study of its extra structure which justifies its name till Section 6.4.

A map $f : (X, A, C) \to (X', A', C')$ of triples clearly defines a morphism

$$\rho(f) : \rho(X, A, C) \to \rho(X', A', C')$$

of double categories.

Proposition 6.3.7. *If $f : (X, A, C) \to (X', A', C')$ is a map of triples such that each of $f : X \to X'$, $f_1 : A \to A'$ are homotopy equivalences, and $f_0 : C \to C'$ is a bijection, then $\rho(f) : \rho(X, A, C) \to \rho(X', A', C')$ is an isomorphism.*

Proof. This is a consequence of a cogluing theorem for homotopy equivalences. We give some details for the analogous result for filtered spaces in Proposition 14.1.10. □

Now let us check the not quite so straightforward fact that the crossed module associated to the fundamental double groupoid $\rho(X, A, C)$ is the fundamental crossed module $\Pi_2(X, A, C)$, i.e. $\gamma(\rho(X, A, C))_2 = \Pi_2(X, A, C)$. Recall that the elements of $\gamma(\rho(X, A, C))_2(x)$ are thin homotopy classes of filtered maps

$$\alpha : (I^2, \partial I^2, \partial^2 I^2) \to (X, A, x)$$

such that the restriction to all sides but the last vertical one are homotopically trivial. On the other hand, $\pi_2(X, A, x)$ consists of homotopy classes of maps

$$\alpha : (I^2, \partial I^2, J^+) \to (X, A, x).$$

Let us check that they are the same.

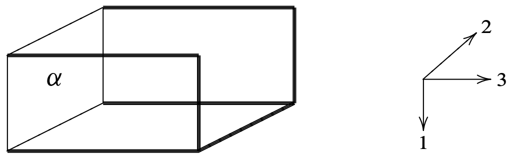
Proposition 6.3.8. *If $x \in C$, then the group $\gamma(\rho(X, A, C))_2(x)$ may be identified with the group $\pi_2(X, A, x)$.*

Proof. We have defined $R_2(X, A, C)$ as a set and we also consider it as a space with the usual compact-open topology. We also define $J_2(X, A, C)$ as the space of maps $I^2 \to X$ which take ∂_1^- into A and the other faces to a single element of C. Then we have a natural homeomorphism into $i : J_2(X, A, C) \to R_2(X, A, C)$. This homeomorphism induces a map

$$\phi : \pi_2(X, A, x) \to \gamma(\rho(X, A, C))_2(x),$$

$\phi([\alpha]) = \langle\langle i(\alpha) \rangle\rangle$, which is well defined, is a group homomorphism and preserves the actions. We only have to prove that ϕ is bijective. We shall use filling arguments which are developed for more general situations in Section 11.3.i. In the following we abbreviate $R_2(X, A, C)$, $J_2(X, A, C)$ to R_2, J_2.

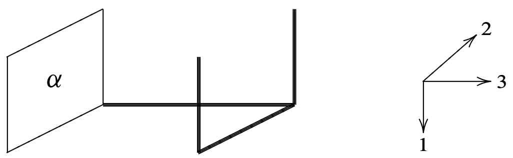
For surjectivity of ϕ we use the following diagram showing a subspace L of I^3, and in which we find it convenient to switch the axes:



in which edges which are eventually mapped constantly are shown in thick lines. The face ∂_3^- gives α such that $\langle\!\langle \alpha \rangle\!\rangle \in \gamma\rho(X, A, C)$. This implies that there are three homotopies of edges of α to constant maps, and these homotopies are shown as the three faces $\partial_2^\pm$, ∂_1^+ of L. We now construct a retraction $I^3 \to L$ in two steps. The first retraction is on the face ∂_1^- from the point $(0, \frac{1}{2}, 2)$. The second is on the whole cube from $(\frac{1}{2}, \frac{1}{2}, 2)$. These retractions define an extension of the given maps on L to a homotopy of α to a map in J_2. This proves surjectivity.

For injectivity, we construct a continuous map $r\colon R_2 \to J_2$. This will not be a retraction, but if $\alpha \in J_2$, then $r(\alpha)$ is homotopic in J_2 to α.

Consider the subspace K of I^3 pictured as follows:



in which again thick lines depict edges eventually to be mapped constantly. We construct in turn a sequence of retractions, the first four on the faces, to give a retraction $I^3 \to K$, namely retract: the face ∂_1^+ from $(1, -1, \frac{1}{2})$; the face ∂_2^- from $(-1, 0, \frac{1}{2})$; retract ∂_1^- from $(0, \frac{1}{2}, 2)$; finally retract the cube onto the above filled faces from $(\frac{1}{2}, \frac{1}{2}, 2)$. (You should draw the successive stages).

An element $\alpha \in R_2$ defines a map $\alpha'\colon K \to X$ in which the part shown by wavy lines is mapped constantly. Using the above retractions, α extends over I^3 to give a map $r'(\alpha)\colon I^3 \to X$ which is in J_2. This defines a continuous map $r\colon R_2 \to J_2$, $\alpha \mapsto \partial_3^+ r'(\alpha)$.

Now suppose $h_t\colon I^2 \to X$ is a homotopy rel vertices of α through elements of R_2 to a constant map. Then $r(h_t)\colon I^2 \to X$ is a homotopy rel vertices of $r(\alpha)$ to a constant map and through elements of J_2. But $r(\alpha)$ is easily seen to be homotopic in J_2 to α, so α is homotopic to 0 in J_2. This completes the proof. □

The reader will have noticed the wide use of filling arguments in the above proofs. These arguments are developed in higher dimensions in Section 11.3.i and in Chapter 14 and are the key to the proof of corresponding results, for example Theorem 14.4.1, which generalises Proposition 6.3.8 to all dimensions.

6.4 Thin structures on a double category: the category of double groupoids

We have examples of double categories coming from two sources: first, the 2-shells commutative up to an element of a crossed modules over groupoids hinted at the end of Section 6.1 and which will be properly developed in Section 6.6, and second, the fundamental double groupoid of a topological pair seen in Section 6.3. In both cases not only are all three structures groupoids but they have also some extra structure. Let us see one way of introducing this structure.

We have already introduced in Example 6.1.7 the double category $\square' C$ of 2-*shells* in the category C and its sub double category $\square\, C$ of *commuting* 2-*shells*.

For any double category D there is a morphism of double categories $D \to \square' D_1$ which is the identity in dimensions 0, 1 and in dimension 2 gives the bounding shell of any element. On the other hand, there is no natural morphism the other way, from either $\square' D_1$ or $\square\, D_1$, which is the identity on D_1.

In this section, we are going to study double categories endowed with such a morphism, i.e. for any given commuting shell in D_1, there is a chosen 'filler' in D_2. Next, in Section 6.5, we develop an alternative approach using some extra kind of degeneracies called *connections*.

Definition 6.4.1. We therefore define a *thin structure* on a double category D to be a morphism of double categories

$$\Theta \colon \square\, D_1 \to D$$

which is the identity on D_1, D_0. The 2-dimensional elements of the form $\Theta\alpha$ for $\alpha \in (\square\, D_1)_2$ will be called *thin squares* in (D, Θ) or simply in D if Θ is given. $\square$

Equivalently, a thin structure may be given by giving the thin squares and checking the following axioms:

T0) Any identity square in D is thin.

T1) Each commuting shell in D has a unique thin filler.

T2) Any composite of thin squares is thin.

By T0), particular thin squares represent the degenerate squares, namely those of the form

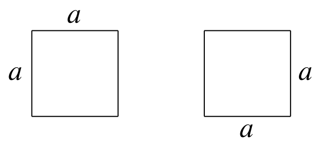
$$\begin{tikzcd} \cdot \arrow[r, dash, "1"] \arrow[d, dash, "1"'] & \cdot \arrow[d, dash, "1"] \\ \cdot \arrow[r, dash, "1"'] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, dash, "a"] \arrow[d, dash] & \cdot \arrow[d, dash] \\ \cdot \arrow[r, dash, "a"'] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, dash] \arrow[d, dash, "a"'] & \cdot \arrow[d, dash, "a"] \\ \cdot \arrow[r, dash] & \cdot \end{tikzcd} \tag{6.4.1}$$

which we write in short as

$$\square \qquad | \;\, | \qquad \overline{\underline{\quad}} \; .$$

Notice that identity edges are those drawn with a solid line. The notation is ambiguous, since for example the second element is the same as the first if $a = 1$. Also we have not named the vertices. Nevertheless, it is clear that they represent the degenerate squares since Θ is a morphism of double categories.

We also have two new kind of 'degenerate' squares, but called 'thin':



(6.4.2)

which we write in short as

$$\lrcorner \qquad\qquad \ulcorner \ .$$

The fact that Θ is a morphism of double categories leads immediately to some equations for compositions of such elements, i.e.

$$\begin{bmatrix} \ulcorner & \lrcorner \end{bmatrix} = \mid\,\mid \qquad \begin{bmatrix} \ulcorner \\ \lrcorner \end{bmatrix} = \, = \ . \tag{6.4.3}$$

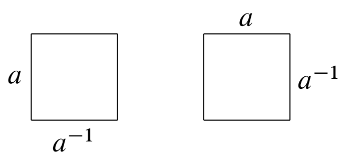
Remark 6.4.2. In writing such matrix compositions, of course we always assume that the compositions are defined. In order to understand these and other equations between composites of thin elements, and their combinations with elements of G_2 given later, you should expand our diagrams by filing in names for the edges of the squares to ensure the composites are well defined, and see how these edges are 'transferred' by the thin elements. Some such 'filling in' has to be done in the proof of Proposition 6.5.5 since that has no element $u \in G_2$ to 'fix' some edges. $\square$

The reason why these equations hold is that the composites are certainly thin, by T2), and since they are determined by their shell, by T1), they are by T0) of the form given.

Here are some more consequences of these equations, known as 'transport laws':[81]

$$\begin{bmatrix} \lrcorner & \mid\,\mid \\ = & \lrcorner \end{bmatrix} = \lrcorner\ , \qquad \begin{bmatrix} \ulcorner & = \\ \mid\,\mid & \ulcorner \end{bmatrix} = \ulcorner\ . \tag{6.4.4}$$

If in addition the category D_1 is a groupoid then we have two further thin elements namely



which we write again with ambiguity as

$$\urcorner \qquad\qquad\qquad \llcorner \; .$$

These elements give rise to new equations, for example

$$\begin{bmatrix} \ulcorner & \urcorner \\ | \; | & | \; | \end{bmatrix} = \square \; .$$

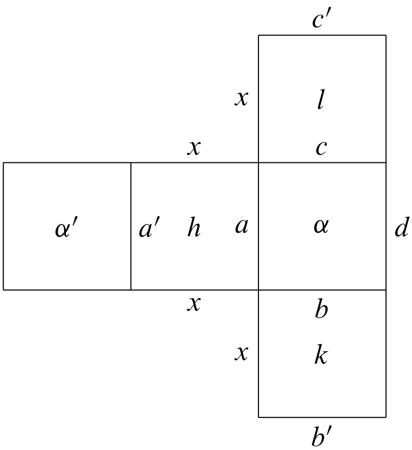
Note here that three of the sides of this last composition are identities, and hence so also is the fourth, by commutativity.

Now we apply these ideas to the fundamental double groupoid $\rho(X, A, C)$.

Proposition 6.4.3. *The fundamental double groupoid $\rho(X, A, C)$ has a natural thin structure in which a class $\langle\langle\alpha\rangle\rangle$ is thin if and only it has a representative α such that $\alpha(I^2) \subseteq A$.*

Proof. Let $a, b, c, d : I \to A$ be paths in A such that $ab \simeq cd$ in A. It is a standard property of the fundamental groupoid that the given paths can then be represented by the sides of a square $\alpha : I^2 \to A$. We have to prove that such a square is unique in ρ_2.

Let $\alpha' : I^2 \to A$ be another square whose edges a', b', c', d' are equivalent in $\pi_1(A, C)$ to a, b, c, d respectively. Choose maps $h, k, l : I^2 \to A$ giving homotopies rel end points $a \simeq a'$, $b \simeq b'$, $c \simeq c'$; these homotopies and α and α' can be represented as



Folding the diagram gives a map H from five 2-faces of I^3 to A.

Now, using the retraction from I^3, we can extend this to a map $I^3 \to A$, which gives a thin homotopy as required.

Note that this is where we use the fact that a thin homotopy is allowed to move the edges of the square within A. $\square$

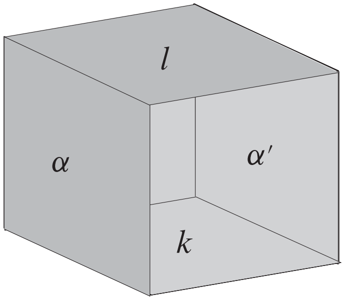


Figure 6.2. Box without a lateral face.

Since this important example has this structure, it is reasonable to call them double groupoids. This leads to:

Definition 6.4.4. A *double groupoid* is a double category such that all three structures are groupoids, together with a thin structure. We write DGpds for the category of double groupoids taking as morphisms the double functors that preserve the given thin structures.

We are interested in the restriction to this category of the functor γ defined in Section 6.2, and which we write now as

$$\gamma\colon \mathsf{DGpds} \to \mathsf{XMod}. \qquad \square$$

Notice that the thin elements $\ulcorner$, $\lrcorner$ in $\rho(X, A, C)$ are determined by specific maps on I^2, namely composition of a path $I \to A$ with the maps $\max, \min\colon I^2 \to I$. We will say more on this in the next section, and give further elaboration in all dimensions in Section 13.1.

An important consequence of the existence of a thin structure in a double groupoid is that the vertical and horizontal groupoid structures in dimension 2 are isomorphic. The isomorphism is given by 'rotation' maps $\sigma, \tau\colon G_2 \to G_2$ which correspond to a clockwise and an anticlockwise rotation through $\pi/2$.

Definition 6.4.5. Let G be a double groupoid. We define functions $\sigma, \tau\colon G_2 \to G_2$ as follows: if $u \in G_2$ then

$$\sigma(u) = \begin{bmatrix} \| & \ulcorner & = \\ \llcorner & u & \urcorner \\ = & \lrcorner & \| \end{bmatrix} \quad \text{and} \quad \tau(u) = \begin{bmatrix} = & \urcorner & \| \\ \ulcorner & u & \lrcorner \\ \| & \llcorner & = \end{bmatrix}.$$

Notice that the boundary say $\left(a \begin{smallmatrix} c \\ b \end{smallmatrix} d\right)$ of u is transformed by σ, τ to respectively:

$$\left(b \begin{matrix} a^{-1} \\ d^{-1} \end{matrix} c\right), \quad \left(c^{-1} \begin{matrix} d \\ a \end{matrix} b^{-1}\right). \qquad \square$$

The main properties of these operations are not essential for our main aims, but the proofs illustrate the idea of 'rewriting' in higher dimensional algebra, and such arguments occur frequently later in proving the main algebraic properties of the structures we use.[82]

Since thin elements are determined by their boundaries, the next result follows immediately – but these formulae need interpretation: when we write for example $\sigma(\ulcorner) = \urcorner$ the boundary is transformed as $\begin{pmatrix} & 1 & \\ 1 & & a \\ & a & \end{pmatrix} \mapsto \begin{pmatrix} & 1 & \\ a & & 1 \\ & a^{-1} & \end{pmatrix}$. Thus distinct occurrences of say $\llcorner$ do not necessarily represent the same element of G_2. Nonetheless this abbreviated notation is very useful in writing down and understanding the complicated compositions which we use in proving say Theorem 6.4.10.

Proposition 6.4.6. *The images of thin elements under σ and τ are as follows*

$$\sigma\colon \square \mapsto \square, \quad \overline{\underline{}} \mapsto |\,| \mapsto \overline{\underline{}}, \quad \ulcorner \mapsto \urcorner \mapsto \lrcorner \mapsto \llcorner \mapsto \ulcorner,$$

$$\tau\colon \square \mapsto \square, \quad \overline{\underline{}} \mapsto |\,| \mapsto \overline{\underline{}}, \quad \ulcorner \mapsto \llcorner \mapsto \lrcorner \mapsto \urcorner \mapsto \ulcorner. \qquad \square$$

The following result is also clear from the definitions, and facts such as $-_2 \ulcorner = \urcorner$.

Proposition 6.4.7. *The following relations hold:*

$$-_1\sigma = \tau -_1, \quad -_1\tau = \sigma -_1, \quad -_2\sigma = \tau -_2, \quad -_2\tau = \sigma -_2\,. \qquad \square$$

We will prove in Theorem 6.4.10 that σ is a homomorphism from the horizontal to the vertical composition, while τ is a homomorphism from the vertical to the horizontal composition; and in Theorem 6.4.11 that τ is an inverse to σ. It follows that in the case of a double groupoid the horizontal and the vertical groupoid structures in dimension 2 are isomorphic. Assuming these theorems for the moment, we have the following result:

Proposition 6.4.8.

$$\sigma^2 = \tau^2 = -_1 -_2, \quad \sigma^4 = \tau^4 = 1.$$

Proof. We have

$$\overline{\underline{}} = \sigma(-_2\alpha) -_2 \sigma(-_2\alpha) = \sigma(-_2\alpha) +_2 \tau(\alpha).$$

Applying σ and using the facts stated above we get

$$|\,| = \sigma^2(-_2\alpha) +_1 \alpha,$$

from which follows

$$\sigma^2(\alpha) = -_1 -_2\, \alpha.$$

The other relations follow similarly and easily. $\square$

Remark 6.4.9. It is from this easy to see that the operations $-_1, -_2, \sigma, \tau$ generate a group isomorphic to D_8 since they operate faithfully on the boundary of a generic element α. □

Now we prove the facts stated before the last proposition. Here again we refer you to Remark 6.4.2 on checking for yourself how the boundaries of u and v are transformed during the proofs.

Theorem 6.4.10. *For any $u, v, w \in G_2$,*

$$\sigma([u, v]) = \begin{bmatrix} \sigma u \\ \sigma v \end{bmatrix} \quad \textit{and} \quad \sigma\left(\begin{bmatrix} u \\ w \end{bmatrix}\right) = [\sigma w, \sigma u],$$

$$\tau([u, v]) = \begin{bmatrix} \tau v \\ \tau u \end{bmatrix} \quad \textit{and} \quad \tau\left(\begin{bmatrix} u \\ w \end{bmatrix}\right) = [\tau u, \tau w]$$

whenever the compositions are defined.

Proof. We prove only the first rule and leave the others to the reader.

By definition, the element $\sigma([u, v])$ is the composition of the array

$$\begin{bmatrix} \vert\,\vert & \ulcorner & = \\ \llcorner & u +_2 v & \urcorner \\ = & \lrcorner & \vert\,\vert \end{bmatrix}.$$

We get a refinement of this array by substituting each element for a box which has the initial element as its composition as follows:

$$\left[\begin{array}{c|cc|c} \vert\,\vert & \ulcorner & = & = \\ \vert\,\vert & \vert\,\vert & \ulcorner & = \\ \hline \llcorner & u & v & \urcorner \\ \hline = & \lrcorner & \vert\,\vert & \vert\,\vert \\ = & = & \lrcorner & \vert\,\vert \end{array}\right].$$

By Remark 6.1.6 this new array has the same composition as the initial one. We now subdivide the second column horizontally in two, getting a new refinement

$$\left[\begin{array}{c|cc|cc}
\cline{2-3}
|\,| & \ulcorner & = & = & = \\
\cline{2-3}
|\,| & |\,| & \square & \ulcorner & = \\
\cline{2-3}
\llcorner & u & = & v & \urcorner \\
\cline{2-3}
= & \lrcorner & \square & |\,| & |\,| \\
\cline{2-3}
= & = & = & \lrcorner & |\,| \\
\cline{2-3}
\end{array}\right]$$

which still has the same composition. Finally, we expand the three middle rows into six in such a way that we do not change the vertical composition of each column getting

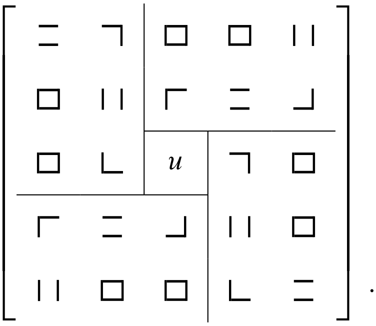
$$\left[\begin{array}{c|c|c|c|c}
\multicolumn{1}{c}{|\,|} & \multicolumn{1}{c}{\ulcorner} & \multicolumn{1}{c}{=} & \multicolumn{1}{c}{=} & \multicolumn{1}{c}{=} \\
\hline
|\,| & |\,| & \square & \square & \square \\
\llcorner & u & \urcorner & \square & \square \\
= & \lrcorner & |\,| & \square & \square \\
\square & \square & |\,| & \ulcorner & = \\
\square & \square & \llcorner & v & \urcorner \\
\square & \square & \square & |\,| & |\,| \\
\hline
\multicolumn{1}{c}{=} & \multicolumn{1}{c}{=} & \multicolumn{1}{c}{=} & \multicolumn{1}{c}{\lrcorner} & \multicolumn{1}{c}{|\,|}
\end{array}\right].$$

The composition of this array still is $\sigma([u, v])$ by Remark 6.1.6. To get the result, we now see that the composition of the block given by the first four rows is σu and the composition of the other four is σv. □

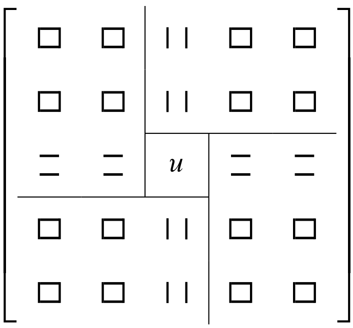
It is a nice exercise to extend this result to any rectangular array using associativity.

Theorem 6.4.11. *The rotations σ, τ satisfy $\tau\sigma = \sigma\tau = 1$.*

Proof. It is easily seen that $\tau\sigma(u)$ is the composition of the array



Using Remark 6.1.6 four times, we can change the four blocks one by one and substitute them for another four having the same boundary and composition, getting that $\tau\sigma(u)$ is also the composition of the array



whose composition reduces to u. We leave you to give a similar proof that $\sigma\tau = 1$ or to deduce that from the rule $-_1\tau = \sigma_1$. □

Remark 6.4.12. When these results are applied to $\rho(X, A, C)$, the fundamental double groupoid of (X, A, C), they imply the existence of specific thin homotopies. Indeed one of the aims of higher order groupoid theory is to give an algebraic framework for calculating with homotopies and higher homotopies. □

6.5 Connections in a double category: equivalence with thin structure

The extension of the notion of thin structure to higher dimensions is not straightforward since it would require the notion of commutative n-cube and this notion is not easy even for a 3-cube. We shall return to this at the end of this section.

We must look for an alternative which generalises more easily to higher dimensions. We take as basic the two maps $\Gamma^-, \Gamma^+ \colon D_1 \to D_2$, that correspond to the thin elements $\lrcorner$, $\ulcorner$, satisfying the properties we have seen in (6.4.3) and (6.4.4) after Definition 6.4.1. We make this concept clear and develop the equivalence between the two notions in this section.

Definition 6.5.1. A *connection pair* on a double category D is a pair of maps

$$\Gamma^-, \Gamma^+ \colon D_1 \to D_2$$

satisfying the four properties below.

• The first one is that the shells are what one expects, i.e., if $a \colon x \to y$ in D_1 then $\Gamma^-(a), \Gamma^+(a)$ shells are

$$\Gamma^-(a) = \begin{array}{ccc} & a & \\ a & \boxed{\ \lrcorner\ } & 1_y \\ & 1_y & \end{array} \qquad \Gamma^+(a) = \begin{array}{ccc} & 1_x & \\ 1_x & \boxed{\ \ulcorner\ } & a \\ & a & \end{array}$$

which can be more formally stated as

$$\partial_2^-\Gamma^-(a) = \partial_1^-\Gamma^-(a) = a \quad \text{and} \quad \partial_2^+\Gamma^-(a) = \partial_1^+\Gamma^-(a) = \varepsilon\partial^+ a, \tag{CON 1}$$
$$\partial_2^+\Gamma^+(a) = \partial_1^+\Gamma^+(a) = a \quad \text{and} \quad \partial_2^-\Gamma^+(a) = \partial_1^-\Gamma^+(a) = \varepsilon\partial^- a. \tag{CON$'$ 1}$$

• We also assume that the connections associate to a degenerate element a double degenerate one:

$$\Gamma^-\varepsilon(x) = 0_x, \tag{CON 2}$$
$$\Gamma^+\varepsilon(x) = 0_x. \tag{CON$'$ 2}$$

• The relation with composition is given by the ‘transport laws’ (see (6.4.4)):

$$\Gamma^-(ab) = \begin{bmatrix} \Gamma^- a & \mid\mid \\ = & \Gamma^- b \end{bmatrix} = \begin{array}{|c|c|}\hline \lrcorner & \mid\mid \\ \hline = & \lrcorner \\ \hline \end{array} \tag{CON 3}$$

$$\Gamma^+(ab) = \begin{bmatrix} \Gamma^+ a & = \\ \mid\mid & \Gamma^+ b \end{bmatrix} = \begin{array}{|c|c|}\hline \ulcorner & = \\ \hline \mid\mid & \ulcorner \\ \hline \end{array} \tag{CON$'$ 3}$$

Intuitively, a feature which 2-dimensional movements can have extra to 1-dimensional movements is the possibility of turning left or right. The transport laws state intuitively that 'turning left with one's right arm outstretched is the same as turning left', and similarly for turning right.

• A final condition deduced from the same idea is that the connections can cancel each other in a way corresponding to (6.4.3), i.e.

$$\Gamma^{+}(a) +_2 \Gamma^{-}(a) = \varepsilon_1(a), \tag{CON 4}$$

$$\Gamma^{+}(a) +_1 \Gamma^{-}(a) = \varepsilon_2(a). \tag{CON$'$ 4}$$

□

It is interesting to notice that for double categories where all structures are groupoids we need only a map Γ^{-} satisfying the conditions (CON 1)–(CON 3) since Γ^{+} can be defined using (CON 4).

Proposition 6.5.2. *For a double category in which all structures are groupoids, Γ^{-} and Γ^{+} may be obtained from each other by the formula*

$$\Gamma^{+}(a) = -_2 -_1 \Gamma^{-}(a^{-1}).$$

Proof. Let us define

$$\Gamma''(a) = -_2 -_1 \Gamma^{-}(a^{-1}).$$

Since $\Gamma^{-}(aa^{-1}) = \Gamma^{-}(1) = \square$, we obtain from the transport law (CON 3) that $\Gamma^{-}(a^{-1}) = -_1[\Gamma^{-}a,\ \ (\varepsilon_1 a^{-1})]$. Hence $\Gamma''(a) = [(\varepsilon_1 a),\ \ -_2\Gamma^{-}a]$.

This implies that $\Gamma''(a) +_2 \Gamma^{-}(a) = \varepsilon_1(a)$, and so by (CON 4) $\Gamma''(a) = \Gamma^{+}(a)$.

□

If we use an analogue of our previous notations $\lrcorner$, $\ulcorner$, for Γ^{-}, Γ^{+} respectively then of course we see that all these laws except the last are the ones we have given before for thin elements. So it is not very difficult to see that any thin structure has associated a unique connection, and that the given thin structure is determined by this connection.

Proposition 6.5.3. *If there is a thin structure Θ on D we have an associated connection defined by*

$$\Gamma^{-}a = \Theta\begin{pmatrix} & a & \\ a & & 1 \\ & 1 & \end{pmatrix} \quad \textit{and} \quad \Gamma^{+}a = \Theta\begin{pmatrix} & 1 & \\ 1 & & a \\ & a & \end{pmatrix}.$$

Moreover, the morphism Θ can be recovered from the connection, since

$$\Theta\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix} = (\varepsilon_2 a +_1 \Gamma^{+} b) +_2 (\Gamma^{-} c +_1 \varepsilon_2 d) = (\varepsilon_1 c +_2 \Gamma^{+} d) +_1 (\Gamma^{-} a +_2 \varepsilon_1 b). \tag{CON 5}$$

Proof. The results on the behaviour of Γ^{-} and Γ^{+} with respect to boundaries and degeneracies are immediate.

Before proving the relation with the compositions, it is worth mentioning that the values of Θ on degenerate elements are determined by the fact that Θ is a morphism of double categories, so, $\Theta\varepsilon_1(b) = \varepsilon_1(b)$ and $\Theta\varepsilon_2(b) = \varepsilon_2(b)$.

Applying Θ to the equation

$$\begin{pmatrix} & ab & \\ ab & & 1 \\ & 1 & \end{pmatrix} = \left(\begin{matrix} \begin{pmatrix} & a & \\ a & & 1 \\ & 1 & \end{pmatrix} & \begin{pmatrix} & b & \\ 1 & & 1 \\ & b & \end{pmatrix} \\[2ex] \begin{pmatrix} & 1 & \\ b & & b \\ & 1 & \end{pmatrix} & \begin{pmatrix} & b & \\ b & & 1 \\ & 1 & \end{pmatrix} \end{matrix} \right)$$

we get the transport law

$$\Gamma^-(ab) = \begin{bmatrix} \Gamma^- a & \varepsilon_1 b \\ \varepsilon_2 b & \Gamma^- b \end{bmatrix}.$$

and the law for Γ^+ is obtained along the same lines.

Moreover, it is easy to see that in $\square\, D$, the element

$$\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix}$$

may by decomposed as the product of any of the two arrays

$$\left(\begin{matrix} \begin{pmatrix} & 1 & \\ a & & a \\ & 1 & \end{pmatrix} & \begin{pmatrix} & c & \\ c & & 1 \\ & 1 & \end{pmatrix} \\[2ex] \begin{pmatrix} & 1 & \\ 1 & & b \\ & b & \end{pmatrix} & \begin{pmatrix} & 1 & \\ d & & d \\ & 1 & \end{pmatrix} \end{matrix} \right) \quad \text{or} \quad \left(\begin{matrix} \begin{pmatrix} & c & \\ 1 & & 1 \\ & c & \end{pmatrix} & \begin{pmatrix} & 1 & \\ 1 & & d \\ & d & \end{pmatrix} \\[2ex] \begin{pmatrix} & a & \\ a & & 1 \\ & 1 & \end{pmatrix} & \begin{pmatrix} & b & \\ 1 & & 1 \\ & b & \end{pmatrix} \end{matrix} \right)$$

where in the first one we have to compose first columns then rows and in the second one the other way about.

Applying Θ to these expressions, we get the formula (CON 5). $\square$

Remark 6.5.4. As we have seen in the proof of the preceding property, the thin elements are composites of degenerate elements and connections. Conversely, all degeneracies and connections lie in the image of Θ, so any composition of such elements is a thin element. Thus we have an easy characterisation of the thin elements. The higher dimensional versions of these results are given in Section 13.4.[83] $\square$

There is more work in obtaining the other implication, i.e. getting the thin structure from the connection maps.

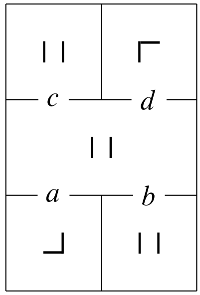
Proposition 6.5.5. *If there is a connection on D, we have an associated thin structure Θ defined by the formula* (CON 5) *in Proposition* 6.5.3*. Moreover, the connection can be recovered from Θ, since*

$$\Gamma^-(a) = \Theta \begin{pmatrix} & a & \\ a & & 1 \\ & 1 & \end{pmatrix} \quad \textit{and} \quad \Gamma^+(a) = \Theta \begin{pmatrix} & 1 & \\ 1 & & a \\ & a & \end{pmatrix}.$$

Proof. Let us first prove that either formulae gives the same function. This will make it easier to prove the morphism property.

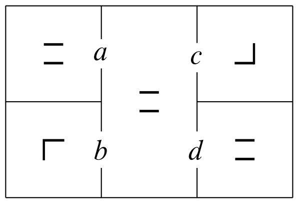
We write

$$\Theta_1\begin{pmatrix} a & \begin{matrix} c \\ b \end{matrix} & d \end{pmatrix} = (\varepsilon_1 c +_2 \Gamma^+ d) +_1 (\Gamma^- a +_2 \varepsilon_1 b) =$$

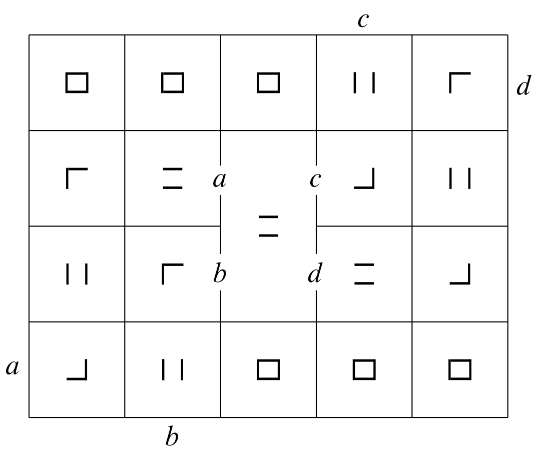


where the last diagram is obtained adding the degenerate middle row, and

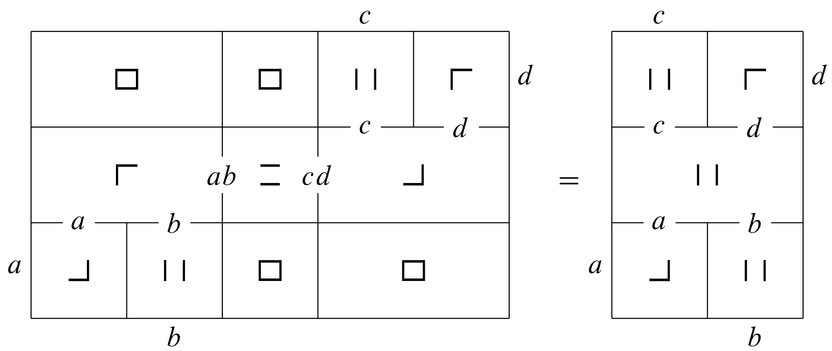
$$\Theta_2\begin{pmatrix} a & \begin{matrix} c \\ b \end{matrix} & d \end{pmatrix} = (\varepsilon_2 a +_1 \Gamma^+ b) +_2 (\Gamma^- c +_1 \varepsilon_2 b) =$$



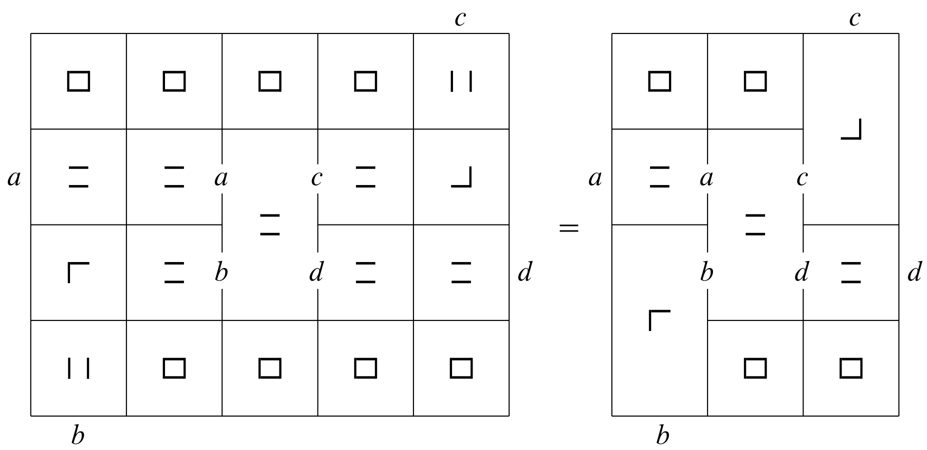
Then we want to prove $\Theta_1 = \Theta_2$. A usual way of proving that two compositions of arrays produce the same result is to construct a common subdivision. One that is appropriate for this case is the following:



From this diagram, we may compose the second and third row using the transport law and then rearrange things, getting Θ_1 as indicated:



Similarly, operating in the bottom left and the top right corner, we get



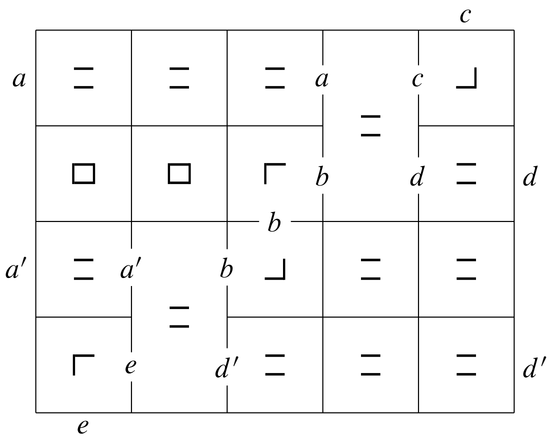
and this last diagram clearly is Θ_2. We write Θ for the common value.

We would like to prove that Θ is a morphism. From any of its representations, it is clear that Θ commutes with faces and degeneracies. The only point we have to prove is that it commutes with both compositions. In this direction, it is good to have two definitions of Θ. First, we use $\Theta = \Theta_2$ to prove that Θ preserves the vertical composition. The use of $\Theta = \Theta_1$ to prove that it preserves the horizontal composition is similar.

So we want to prove

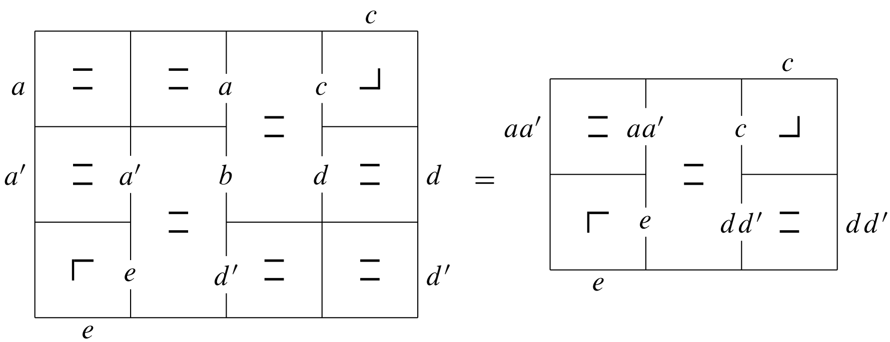
$$\Theta_2 \begin{pmatrix} a & \begin{smallmatrix} c \\ b \end{smallmatrix} & d \end{pmatrix} +_1 \Theta_2 \begin{pmatrix} a' & \begin{smallmatrix} b \\ e \end{smallmatrix} & d' \end{pmatrix} = \Theta_2 \begin{pmatrix} aa' & \begin{smallmatrix} c \\ e \end{smallmatrix} & dd' \end{pmatrix}.$$

As before we compute a common subdivision in two ways, and we choose as follows:



If we compose the first two rows, they produce $\Theta_2\begin{pmatrix} a & \begin{smallmatrix} c \\ b \end{smallmatrix} & d \end{pmatrix}$. Similarly, the two last rows give $\Theta_2\begin{pmatrix} a' & \begin{smallmatrix} b \\ e \end{smallmatrix} & d' \end{pmatrix}$.

On the other hand, making some easy adjusts on the two middle rows, we get



which clearly is $\Theta_2\begin{pmatrix} aa' & \begin{smallmatrix} c \\ e \end{smallmatrix} & dd' \end{pmatrix}$. □

6.6 Equivalence between crossed modules and double groupoids: folding

In this section, we prove the equivalence between the category DGpds of double groupoids of Definition 6.4.4 and that of crossed modules of groupoids XMod of Definition 6.2.1. Another aspect of this equivalence of categories is that it gives us a large source of examples of double groupoids.[84]

On the one hand, the crossed module associated to a double groupoid is given by the functor

$$\gamma\colon \mathsf{DGpds} \to \mathsf{XMod},$$

restriction of the one defined in Section 6.2.

On the other hand, there is a double groupoid associated to each crossed module as was already hinted at the end of Section 6.1. We shall develop this idea in this section. We recall that to generalise the category of commutative 2-shells in a category, we use 2-shells which commute up to some element in the image of the crossed module.

Let $\mathcal{M} = (\mu\colon M \to P)$ be a crossed module over a groupoid. We claim there is an associated double groupoid $G = \lambda\mathcal{M}$ whose sets are $G_0 = P_0$, $G_1 = P_1$ and

$$G_2 = \{(m; \begin{pmatrix} a & \begin{smallmatrix} c \\ b \end{smallmatrix} & d \end{pmatrix}) \mid m \in M,\ a, b, c, d \in P_1,\ \mu m = b^{-1}a^{-1}cd\}.$$

The elements of G_2 may also be represented by

$$\begin{array}{ccc} & c & \\ a & \boxed{\;m\;} & d \\ & b & \end{array}$$

so that m measures the lack of commutativity of the boundary, giving the composition of the sides of the boundary in clockwise direction starting from the bottom right corner, considered as base point of the square.

Exercise 6.6.1. This choice out of 8 possibilities is conventional, and will influence many later formulae. You are invited to consider the effect of other conventions on such formulae. □

The category structure in (G_1, G_0) is the same as that of (P_1, P_0), so it is a groupoid. The horizontal and vertical structures on (G_2, G_1) have source, target and identities defined as in $\square P$. The definition of the compositions in dimension 2 is the key to the work.

For the 'horizontal' composition we require the boundaries to be given as follows:

$$\begin{array}{ccc} & g & \\ h & \boxed{\;n\;} & a \\ & k & \end{array} +_2 \begin{array}{ccc} & c & \\ a & \boxed{\;m\;} & d \\ & b & \end{array} = \begin{array}{ccc} & gc & \\ h & \boxed{\;A\;} & d \\ & kb & \end{array} \qquad (6.6.1)$$

and for the ‘vertical composition’ we require

$$\begin{array}{ccc} & g & \\ f & \boxed{\quad u \quad} & e \\ & c & \end{array} \;+_1\; \begin{array}{ccc} & c & \\ a & \boxed{\quad m \quad} & d \\ & b & \end{array} \;=\; \begin{array}{ccc} & g & \\ fa & \boxed{\quad B \quad} & ed. \\ & b & \end{array} \tag{6.6.2}$$

The problem is to find reasonable values in M for A, B. With our convention the boundary of the square A is:

$$(kb)^{-1}h^{-1}gcd = b^{-1}k^{-1}h^{-1}gabb^{-1}a^{-1}cd = b^{-1}\mu(n)b\mu(m).$$

So a good choice is

$$A = n^b\, m.$$

This agrees with intuition since n has to be ‘moved to the right’ by the edge b to have the same base point as m. Similarly, and the calculation is left to you, a good choice is

$$B = m\, u^d,$$

since u has to be ‘moved down’ by the edge d. Notice that we use the rule CM1) for a crossed module.

It is not difficult to check that with these compositions all three categories are groupoids. We now verify the interchange law, using the following diagram:

$$\begin{array}{|c|c|} \hline \quad v \quad & \quad u \quad \\ \hline n & a \quad m \quad d \\ \hline \end{array}$$

with the lower right square having top edge c, left edge a, right edge d and bottom edge b.

Evaluating the rows first gives the first component of the composition, in an abbreviated notation since the edges are omitted, as

$$\begin{bmatrix} v^c u \\ \\ n^b m \end{bmatrix} = (n^b m)(v^c u)^d$$

while evaluating the columns first gives the first component of the composition, in a similar notation, as

$$\begin{bmatrix} nv^a & mu^d \end{bmatrix} = (nv^a)^b m u^d.$$

So to prove the interchange law we have to verify that

$$mv^{cd} = v^{ab} m.$$

This follows from CM2) since $\mu m = b^{-1}a^{-1}cd$ and then

$$m^{-1}v^{ab}m = (v^{ab})^{\mu m} = v^{cd}.$$

Remark 6.6.2. These 'childish calculations' were a key to the whole theory, and will be part of the higher dimensional theory in Chapter 13. □

To finish, we define a thin structure on G by the obvious morphism

$$\Theta\colon \square P \to G_2$$

given by $\Theta\left(a \begin{smallmatrix} c \\ b \end{smallmatrix} d\right) = (1; \left(a \begin{smallmatrix} c \\ b \end{smallmatrix} d\right))$.

Definition 6.6.3. The previous construction of the *double groupoid associated to a crossed module* gives a functor

$$\lambda\colon \mathsf{XMod} \to \mathsf{DGpds}.$$ □

It is immediate that $\gamma\lambda\mathcal{M}$ is naturally isomorphic to $\mathcal{M}$ in dimensions 0, 1 and also in dimension 2 as follows from

$$(\gamma\lambda\mathcal{M})_2 = \{(m\colon \left(1 \begin{smallmatrix} \mu m \\ 1 \end{smallmatrix} 1\right)) \mid m \in M\} \cong M.$$

It is rather more involved to get for any double groupoid G a natural isomorphism from G to $\lambda\gamma G$. In order to do this, we shall see first that a double groupoid is 'generated' by the thin elements and those that have only one nondegenerate face, which we assume to be the top face. To this end we 'fold' all faces to the chosen one.

Definition 6.6.4. Let G be a double groupoid. We define the *folding* map

$$\Phi\colon G_2 \to (\gamma G)_2 \subseteq G_2$$

by the formula $\Phi u = [-_2\varepsilon_1\partial_1^+ u,\ -_2\Gamma^-\partial_2^- u,\ u,\ \Gamma^-\partial_2^+ u]$. Notice that this can be defined only in the groupoid case because we are using $-_2$. □

In the usual description

$$\Phi u \;=\; \begin{array}{cccc} b^{-1} & a^{-1} & c & d \\ \hline \multicolumn{1}{|c|}{\mid\,\mid} & \multicolumn{1}{c|}{\llcorner} & \multicolumn{1}{c|}{a \quad u \quad d} & \multicolumn{1}{c|}{\lrcorner} \\ \hline b^{-1} & & b & \end{array}$$

Now let us see that the boundary of Φu is such that $\Phi u \in \gamma G_2$.

Proposition 6.6.5. *All faces of Φu are identities except the first one in the vertical direction, and*

$$\partial_1^- \Phi u = \partial_1^+ u^{-1} \partial_2^- u^{-1} \partial_1^- u \partial_2^+ u.$$

Thus $\Phi u \in \gamma G_2$ and $\operatorname{Im} \Phi \subseteq \gamma G_2$.

Proof. All are easy calculations which are left as exercises. □

Also from the definition, the following property is clear.

Proposition 6.6.6. *All $u \in \gamma G_2$ satisfy $\Phi u = u$. Thus $\gamma G_2 = \operatorname{Im} \Phi$ and $\Phi\Phi = \Phi$.*

Proof. This is immediate since in this case all the elements making up Φu except u itself are identities. □

Definition 6.6.7. We are now able to define a morphism of double groupoids

$$\Psi \colon G \to (\lambda\gamma)G$$

which is the identity in dimensions 0 and 1 and acts on dimension 2 by mapping any element $u \in G_2$ to the element given by the folded element Φu and the shell of u:

$$\begin{array}{ccc} & c & \\ a & \boxed{\;u\;} & d \\ & b & \end{array} \quad \mapsto \quad \left(\Phi(u) \colon \left(a \;\begin{smallmatrix} c \\ b \end{smallmatrix}\; d\right)\right). \qquad \square$$

We shall see that Ψ is an isomorphism of double groupoids.

It is clear that Ψ preserves faces and degeneracies.

The most delicate part of the proof is the behaviour of the folding map Φ with respect to compositions. We obtain not a homomorphism but a kind of 'derivation', involving conjugacies, or, equivalently, the action in the crossed module γG.

Proposition 6.6.8. *Let $u, v, w \in G_2$ be such that $u +_1 v$, $u +_2 w$ exist. Then*

$$\Phi(u +_1 v) = [\Phi v, -_2\varepsilon_1 g, \Phi u, \varepsilon_1 g] = \Phi v +_2 (\Phi u)^g,$$
$$\Phi(u +_2 w) = [-_2\varepsilon_1 b, \Phi u, \varepsilon_1 b, \Phi w] = (\Phi u)^b +_2 \Phi w,$$

where $g = \partial_2^+ v$, $b = \partial_1^+ u$.

Proof. The proof of the second rule is simple, involving composition and cancelation in direction 2, so we prove in detail only the first rule. As before, this is done by constructing a common subdivision and computing it in two ways. Namely if both u, v are represented by

$$\begin{array}{ccc} & c & \\ a & \boxed{\;u\;} & d \\ & b & \end{array} \quad \text{and} \quad \begin{array}{ccc} & b & \\ e & \boxed{\;v\;} & g \\ & f & \end{array}$$

then

$$u +_1 v \quad = \quad \begin{array}{ccc} & c & \\ ae & \boxed{u +_1 v} & dg \\ & f & \end{array}$$

So we have

$$\Phi(u +_1 v) \quad = \quad \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{f^{-1}} & \multicolumn{1}{c}{(ae)^{-1}} & \multicolumn{1}{c}{c} & \multicolumn{1}{c}{dg} \\ \hline \vert\,\vert & \llcorner & u +_1 v & \lrcorner \\ \hline \end{array}$$

Applying both transport laws to the second and fourth square, we get a refinement

$$\begin{array}{|c|c|c|c|c|c|} \multicolumn{1}{c}{f^{-1}} & \multicolumn{1}{c}{e^{-1}} & \multicolumn{1}{c}{a^{-1}} & \multicolumn{1}{c}{c} & \multicolumn{1}{c}{d} & \multicolumn{1}{c}{g} \\ \hline \vert\,\vert & \vert\,\vert & \llcorner & u & \lrcorner & \vert\,\vert \\ \hline \vert\,\vert & \llcorner & = & v & = & \lrcorner \\ \hline \end{array}$$

having the same composition by Remark 6.1.6. Next we get another array

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|} \multicolumn{1}{c}{f^{-1}} & \multicolumn{1}{c}{e^{-1}} & \multicolumn{1}{c}{b} & \multicolumn{1}{c}{g} & \multicolumn{1}{c}{g^{-1}} & \multicolumn{1}{c}{b^{-1}} & \multicolumn{1}{c}{a^{-1}} & \multicolumn{1}{c}{c} & \multicolumn{1}{c}{d} & \multicolumn{1}{c}{g} \\ \hline \vert\,\vert & \vert\,\vert & \vert\,\vert & \vert\,\vert & \vert\,\vert & \vert\,\vert & \llcorner & u & \lrcorner & \vert\,\vert \\ \hline \vert\,\vert & \llcorner & v & \lrcorner & \vert\,\vert & \vert\,\vert & \square & \vert\,\vert & \square & \vert\,\vert \\ \hline \end{array}$$

having the same composite because each row has same composite in both cases (apply Remark 6.1.6). Now we can compose vertically in this last diagram to get

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|} \multicolumn{1}{c}{f^{-1}} & \multicolumn{1}{c}{e^{-1}} & \multicolumn{1}{c}{b} & \multicolumn{1}{c}{g} & \multicolumn{1}{c}{g^{-1}} & \multicolumn{1}{c}{b^{-1}} & \multicolumn{1}{c}{a^{-1}} & \multicolumn{1}{c}{c} & \multicolumn{1}{c}{d} & \multicolumn{1}{c}{g} \\ \hline \vert\,\vert & \llcorner & v & \lrcorner & \vert\,\vert & \vert\,\vert & \llcorner & u & \lrcorner & \vert\,\vert \\ \hline \end{array}$$

and this is clearly $\Phi v +_2 (\Phi u)^g$ as stated. $\square$

Since the equations proved in the preceding property are part of the definition of the compositions in $(\lambda\gamma G)_2$, we have the following:

Corollary 6.6.9. *$\Psi\colon G_2 \to (\lambda\gamma G)_2$ is a homomorphism with respect to both compositions.*

To end our proof of the equivalence between the categories of crossed modules over groupoids and double groupoids, it just remains to prove that the map Ψ is bijective, and preserves the thin structures. Let us start by characterising the thin elements of G_2 using the folding map.

Proposition 6.6.10. *An element $u \in G_2$ is thin if and only if $\Phi u = 1$.*

Proof. As we pointed out in the Remark 6.5.4 an element $u \in G_2$ is thin if and only if it is a composition of identities and connections. By the preceding properties, it is clear that both identities and connections map to 1 under the folding map, so the same remains true for their compositions.

Conversely, if $u \in G_2$ satisfies $\Phi u = 1$, and u has boundary $\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix}$ so that $b^{-1}a^{-1}cd = 1$, then

$$1 = \begin{array}{cccc} b^{-1} & a^{-1} & c & d^{-1} \\ \hline \multicolumn{1}{|c|}{\mid\,\mid} & \multicolumn{1}{c|}{\llcorner} & \multicolumn{1}{c|}{u} & \multicolumn{1}{c|}{\lrcorner} \\ \hline b^{-1} & & b & \end{array}$$

and so solving for u we have

$$u = \begin{array}{cccccc} & a & b & 1 & d^{-1} & \\ \cline{2-5} a & \multicolumn{1}{|c|}{\lrcorner} & \multicolumn{1}{c|}{\mid\,\mid} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{\llcorner} & d \\ \cline{2-5} & & b & 1 & & \end{array}$$

which gives the correct boundary for u since $c = abd^{-1}$, and gives u as a composition of identities and connections. $\square$

Corollary 6.6.11. *The map Ψ preserves the thin structures.*

Thus we can conclude that an element $u \in G_2$ is uniquely determined by its boundary and its image under the folding map.

Proposition 6.6.12. *Given an element $\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix} \in \square\, G_2$ and $m \in \gamma G_2$, there is an element $u \in G_2$ with boundary $\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix}$ and $\Phi u = m$ if and only if $\partial_1^- m = b^{-1}a^{-1}cd$. Moreover, in this case u is unique.*

Proof. We can solve the equation for u getting

$$u = \begin{array}{c|c|c|c|c|c} & a & b & \partial_1^- m & d^{-1} & \\ \hline a & \lrcorner & |\;| & m & \llcorner & d \\ \hline & 1 & b & 1 & 1 & \end{array}$$

thus giving the construction of such an element u. Uniqueness follows since this is an equation in groupoids. □

Corollary 6.6.13. *The map $\Psi_2 \colon G_2 \to (\lambda\gamma G)_2$ is bijective and determines a natural equivalence of functors $1 \simeq \lambda\gamma$.*

Thus we have completed the proof that the functors γ and λ give an equivalence of categories.[85]

Corollary 6.6.14. *The functor γ preserves pushouts and, more generally, colimits.* □

This allows us to prove a 2-dimensional Seifert–van Kampen Theorem first for the fundamental double groupoid and then deduce a corresponding theorem for the fundamental crossed module.

Remark 6.6.15. This equivalence also gives another way of checking some equalities in double groupoids. To see that two elements are equal we just need to know that they have the same boundary and that they fold to the same element. Alternatively, we can just check the equations in a double groupoid of the form $\lambda(M \to P)$. This is another way of proving the properties of rotations. □

6.7 Homotopy Commutativity Lemma

As we saw in Chapter 1, the desire for the generalisation to higher dimensions of the concept of commutative square was one of the motivations behind the search for higher dimensional group theory.

Recall that when proving the classical Seifert–van Kampen Theorem 1.6.1, the main idea in the second part was to divide a homotopy into smaller squares and change each one to give a commutative square in π_1. Then we applied the morphisms and got composable commutative 2-shells in K; the fact that in a groupoid any composition of commutative 2-shells is commutative gave the result.

To generalise this to prove a 2-dimensional Seifert–van Kampen Theorem, we need at least the following:

– a concept of commutative 3-shell;

– to prove that the composition of 3-shells is commutative; and

– to relate commutative 3-shells with homotopy invariants.

Those are the objectives of this section.

Before getting down to business, let us point out that there is a further generalisation to commutative n-shells for all n which will be explained in Part III (Chapter 13) using the notion of *thin n-cube*. Nevertheless, in the 3-dimensional case the above needs can be met by some careful handling of connections.

The method of course generalises the construction of the double categories of 2-shells and commutative 2-shells seen in Example 6.1.8. In the 3-dimensional case we get what could be labeled a 'triple category' but we are not formalising this concept at this stage because that can be done in a more natural way in the more general setting of Chapter 13 and is not necessary now.

First we consider the definition of 3-shells: this does not use the thin structure. Let us start with the picture of a 3-cube where we have drawn the directions to make things a bit easier to follow:

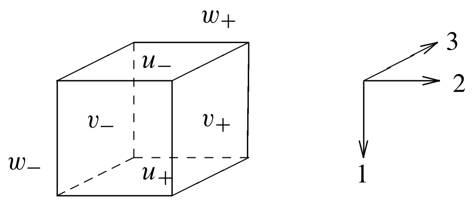


Figure 6.3. 3-shell.

Definition 6.7.1. Let D be a double category. A *cube* or (3-*shell*) s in D is a 6-tuple

$$s = (u_-, u_+, v_-, v_+, w_-, w_+)$$

of squares in D_2 which fit together as do the faces of a 3-cube. To understand what this means, we first define the *faces* of the above shell s to be $\partial_1^\sigma s = u_\sigma$, $\partial_2^\sigma s = v_\sigma$, and $\partial_3^\sigma s = w_\sigma$ for $\sigma = \pm$. Among these, the *even* faces are u_+, v_-, w_+ and the *odd* faces are u_-, v_+, w_-; thus the parity of a face ∂_i^σ is the parity of $i + l(\sigma)$ where $l(-) = 0$, $l(+) = 1$. Next we note that since u_-, u_+ are the faces of the shell in direction 1, then their edges as embedded in a possible cube are given in the picture by the directions 2, 3; for v_-, v_+ the edges are given by the directions 1, 3; for w_-, w_+ the edges are given by the directions 1, 2. Thus the conditions for the faces to 'fit together' are the following:

$$\begin{array}{llll}
\partial_1^- u_- = \partial_1^- v_-, & \partial_1^+ u_- = \partial_1^- v_+; & \partial_1^- u_+ = \partial_1^+ v_-, & \partial_1^+ u_+ = \partial_1^+ v_+; \\
\partial_2^- u_- = \partial_1^- w_-, & \partial_2^+ u_- = \partial_1^- w_+; & \partial_2^- u_+ = \partial_1^+ w_-, & \partial_2^+ u_+ = \partial_1^+ w_+; \\
\partial_2^- v_- = \partial_2^- w_-, & \partial_2^+ v_- = \partial_2^- w_+; & \partial_2^- v_+ = \partial_2^+ w_-, & \partial_2^+ v_+ = \partial_2^+ w_+.
\end{array}$$ □

We now make the set of these 3-shells with the above face maps into a triple category by defining three partial compositions of 3-shells as follows:

Definition 6.7.2. Let $s = (u_-, u_+, v_-, v_+, w_-, w_+)$ and $t = (x_-, x_+, y_-, y_+, z_-, z_+)$ be 3-shells in D. We define

$$s +_i t = \begin{cases} (u_-, x_+, v_- +_1 y_-, v_+ +_1 y_+, w_- +_1 z_-, w_+ +_1 z_+) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } i = 1 \text{ and } u_+ = x_-, \\ (u_- +_1 x_-, u_+ +_1 x_+, v_-, y_1, w_- +_2 z_0, w_+ +_2 z_+) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } i = 2 \text{ and } v_+ = y_-, \\ (u_- +_2 x_-, u_+ +_2 x_+, v_- +_2 y_-, v_+ +_2 y_+, w_-, z_+) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } i = 3 \text{ and } w_+ = z_-. \end{cases}$$

□

You should draw pictures and check that these compositions are well defined and yield a triple category, in the obvious sense. This construction will be extended to all dimensions in Chapter 13, Definition 13.5.4, using a notation more suitable for the general case.

Now we have to formulate the notion of *commutative* 3-*shell*. From the 2-dimensional case it might seem that the proper generalisation would be that the composition of the even faces of the shell equals the composition of the odd faces. We shall take a different route which works in the groupoid case, explaining briefly the categorical route in Remark 6.7.7.

Let us try to give some meaning to one face of a cube being the composition of the remaining five. We can start by thinking of the picture we get by folding flat those five faces of the cube:

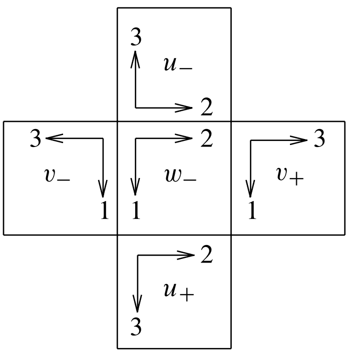


Figure 6.4. Five faces of a cube folded flat.

First, notice that already in this figure we need that the double category we are using has all structures groupoids since we are using inverses of u_- and v_- in different directions. Also, this Figure 6.4 does not give a composable array in any obvious sense. However we can use the connections in a double groupoid with thin structure to fill the corners of the diagram to give a composable array, and ask if:

$$w_+ \;=\; \begin{array}{|c|c|c|} \hline \ulcorner & -_1 u_- & \urcorner \\ \hline -_2 v_- & w_- & v_+ \\ \hline \llcorner & u_+ & \lrcorner \\ \hline \end{array} \tag{6.7.1}$$

If the face $w_+ = \partial_3^+ s$ is the composition of the previous array involving the other five faces, then we shall say that *the above* 3*-shell* s *in a double groupoid commutes*.

Remark 6.7.3. It is explained in Chapter 13 how to develop a corresponding theory in higher dimensions by taking the connections rather than thin structure as basic, since the properties of connections in all dimensions are easily expressed in terms of a finite number of axioms, each of which expresses simple geometric features of mappings of cubes. It is then a main feature of the algebra to develop the related notion of thin structure. The chief advantage of thin structures is that potentially complicated arguments involving multiple compositions of commuting shells of cubes are replaced by simple arguments on the composition of thin elements: see Definition 13.4.17 and the comments following it, as well as Section 13.7. □

Now we prove two results on commuting cubes which are key to the proof of our 2-dimensional Seifert–van Kampen Theorem 6.8.2. Our first result is about the composition of commutative 3-shells.[86]

Theorem 6.7.4. *In a double groupoid with connections, any composition of commutative* 3*-shells is commutative.*

Proof. It is enough to prove that any composition of two commutative 3-shells is commutative.

So, let us consider two commutative 3-shells

$$s = (u_-, u_+, v_-, v_+, w_-, w_+) \quad \text{and} \quad t = (x_-, x_+, y_-, y_+, z_-, z_+)$$

in a double groupoid G. This implies that w_+ and z_+ are given respectively by

$$
w_+ \;=\; \begin{array}{|c|c|c|}\hline \ulcorner & -_1 u_- & \urcorner \\ \hline -_2 v_- & w_- & v_+ \\ \hline \llcorner & u_+ & \lrcorner \\ \hline \end{array}
\qquad
z_+ \;=\; \begin{array}{|c|c|c|}\hline \ulcorner & -_1 x_- & \urcorner \\ \hline -_2 y_- & z_- & y_+ \\ \hline \llcorner & x_+ & \lrcorner \\ \hline \end{array}
$$

We now check that composing in any of the three possible directions gives a commutative 3-shell.

If $v_+ = y_-$, the face $\partial_3^+(s +_2 t) = w_+ +_2 z_+$ of $s +_2 t$ is given by

$$
w_+ +_2 z_+ \;=\; \begin{array}{|c|c|c|c|c|c|}\hline \ulcorner & -_1 u_- & \urcorner & \ulcorner & -_1 x_- & \urcorner \\ \hline -_2 v_- & w_- & v_+ & -_2 v_+ & z_- & y_+ \\ \hline \llcorner & u_+ & \lrcorner & \llcorner & x_+ & \lrcorner \\ \hline \end{array}
$$

Adding first the central two columns of this array and then the central three columns of the resulting array, we get

$$
\begin{array}{|c|c|c|c|c|}\hline \ulcorner & -_1 u_- & \equiv & -_1 x_- & \urcorner \\ \hline -_2 v_- & w_- & \equiv & z_- & y_+ \\ \hline \llcorner & u_+ & \equiv & x_+ & \lrcorner \\ \hline \end{array}
\;=\;
\begin{array}{|c|c|c|}\hline \ulcorner & -_1(u_- +_2 x_-) & \urcorner \\ \hline -_2 v_- & w_- +_2 z_- & y_+ \\ \hline \llcorner & u_+ +_2 x_+ & \lrcorner \\ \hline \end{array}
$$

Thus $s +_2 t$ is a commutative 3-shell.

Working vertically in the same way you can prove that $s +_1 t$ is commutative when it is defined and both s and t are commutative.

The case $s +_3 t$ is a bit different. In this case $w_+ = z_-$, thus we have

$$
w_+ \;=\; \begin{array}{|c|c|c|}\hline \ulcorner & -_1 u_- & \urcorner \\ \hline -_2 v_- & w_- & v_+ \\ \hline \llcorner & u_+ & \lrcorner \\ \hline \end{array}
\qquad
z_+ \;=\; \begin{array}{|c|c|c|}\hline \ulcorner & -_1 x_- & \urcorner \\ \hline -_2 y_- & w_+ & y_+ \\ \hline \llcorner & x_+ & \lrcorner \\ \hline \end{array}
$$

Substituting the first array for w_+ in the second array and subdividing the other blocks to produce a composable array, we get that

$$
z_+ \;=\; \begin{array}{|c|c|c|c|c|}\hline \ulcorner & = & -_1 x_- & = & \urcorner \\ \hline \| & \ulcorner & -_1 u_- & \urcorner & \| \\ \hline -_2 y_- & -_2 v_- & w_- & v_+ & y_+ \\ \hline \| & \llcorner & u_+ & \lrcorner & \| \\ \hline \llcorner & = & x_+ & = & \lrcorner \\ \hline \end{array}
$$

Now, we can compose by blocks and, using the transport law, we get

$$
z_+ \;=\; \begin{array}{|c|c|c|}\hline \ulcorner & -_1(u_- +_1 x_-) & \urcorner \\ \hline -_2(v_- +_2 y_-) & w_- & v_+ +_2 y_+ \\ \hline \llcorner & u_+ +_1 x_+ & \lrcorner \\ \hline \end{array}
$$

Thus $s +_3 t$ is also a commutative 3-shell. $\square$

Our second result is how we show independence of choices in the proof of our 2-dimensional Seifert–van Kampen Theorem 6.8.2, in particular in Lemma 6.8.4: it shows that two nondegenerate faces of a 'degenerate' commutative 3-shell are equal.

Theorem 6.7.5. *Let s be a commutative 3-shell in a double groupoid G. Suppose that all the faces of s not involving direction 3 are degenerate. Then $\partial_3^- s = \partial_3^+ s$.*

Proof. In this case the array containing the five faces is

$$\begin{array}{|c|c|c|}\hline \square & \mid\;\mid & \square \\ \hline = & \partial_3^+ s & = \\ \hline \square & \mid\;\mid & \square \\ \hline \end{array} \tag{6.7.2}$$

whose composition is clearly $\partial_3^+ s$. Thus the commutativity of the 3-shell implies that $\partial_3^- s = \partial_3^+ s$. □

We now apply these ideas to the case of the fundamental double groupoid of a triple (X, A, C). In particular, we will see that the 3-cubes $h: I^3 \to X$ which map the edges of I^3 into A and its vertices to C produce a commutative 3-shell in $\rho(X, A, C)$. We call this result a 'Homotopy Commutativity Lemma'. Later, in Section 9.9, we will prove, using the language of crossed complexes, a more traditional type of formula giving the boundary of a cube or simplex in terms of a 'sum' of the faces, and we call this formula a 'Homotopy Addition Lemma'.

Proposition 6.7.6 (Homotopy Commutativity Lemma). *Let $h: I^3 \to X$ be a cube in X with edges in A and vertices in C, where $C \subseteq A \subseteq X$. Then the boundary of h determines a 3-shell in $\rho(X, A, C)$, and this 3-shell is commutative.*

Proof. The fact that the boundary of h determines a 3-shell is clear. The idea of the proof of commutativity is intuitively the reverse of the folding flat process we discussed for diagram (6.7.1). To this end we define two maps $\varphi_0, \varphi_1: I^2 \to I^3$ which agree on the boundary of I^2 and such that in terms of the following diagrams ϕ_1 maps onto the face ∂_3^+ of I^3 according to the second subdivision and ϕ_0 maps onto the other five faces of I^3, with parts labelled by connections mapped to edges. Since I^3 is convex, and ϕ_0, ϕ_1 agree on the boundary of I^2, they are homotopic rel the boundary of I^2 and hence $h\phi_0$, $h\phi_1$ are homotopic rel boundary, which gives the result.

For the details we introduce some notation that represents the changes of coordinates suggested by Figure 6.4 (page 185). So the faces of h are given by restriction to the corresponding faces of the cube, i.e.

$$\partial_i^\alpha h = h \circ \delta_i^\alpha,$$

$$\begin{array}{|c|c|c|}\hline \ulcorner & -_1 u_- & \urcorner \\ \hline -_2 v_- & w_- & v_+ \\ \hline \llcorner & u_+ & \lrcorner \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline \square & \mid\mid & \square \\ \hline = & w_+ & = \\ \hline \square & \mid\mid & \square \\ \hline \end{array}$$

Figure 6.5. Two arrays with the same boundary.

where $\delta_i^\alpha(r_1, r_2) = (t_1, t_2, t_3)$, the t_j being defined by $t_j = r_j$ for $j < i$, $t_j = r_{j-1}$ for $j > i$, while $t_i = 0, 1$ whenever $\alpha = -, +$ respectively. For a discussion of these ideas in all dimensions, see Section 11.1.

Also in some of the cases we are going to need some switching of coordinates, so we set

$$\tilde{\delta}_1^-(r_1, r_2) = (0, r_2, r_1) \quad \text{and} \quad \tilde{\delta}_1^+(r_1, r_2) = (1, r_2, r_1).$$

What the proposition says is that if we define the elements $u_\alpha, v_\alpha, w_\alpha$ of ρ_2 represented by the faces of h to be respectively the classes of $h \circ \tilde{\delta}_1^\alpha, h \circ \delta_2^\alpha, h \circ \delta_3^\alpha$ $(\alpha = 0, 1)$, then

$$w_+ = \begin{bmatrix} \ulcorner & -_1 u_- & \urcorner \\ -_2 v_- & w_- & v_+ \\ \llcorner & u_+ & \lrcorner \end{bmatrix}$$

in ρ_2 where the corner elements are thin elements as above.

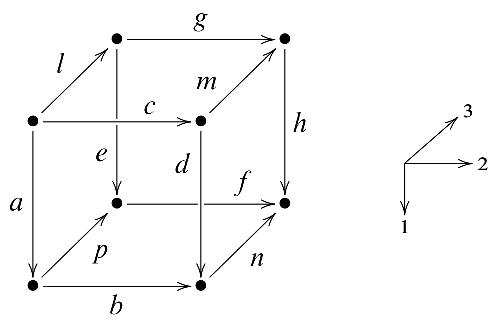
Consider the maps defined by

$$\varphi_0 = \begin{bmatrix} -_2 -_1 \Gamma & -_1(\tilde{\delta}_1^-) & -_1 \Gamma \\ -_2 \delta_2^- & \delta_3^- & \delta_2^+ \\ -_2 \Gamma & \tilde{\delta}_1^+ & \Gamma \end{bmatrix}, \quad \varphi_1 = \begin{bmatrix} -_2 -_1 \Gamma & 1 & -_1 \Gamma \\ 0 & \delta_3^+ & 0 \\ -_2 \Gamma & 1 & \Gamma \end{bmatrix}$$

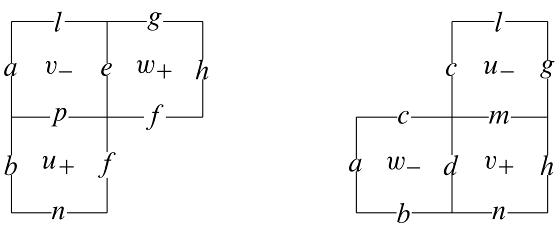
where Γ is the map induced by $\gamma \colon I^2 \to I$ given by $\gamma(r_1, r_2) = \max(r_1, r_2)$. Notice that φ_0, φ_1 agree on ∂I^2. Hence $\langle\langle h\varphi_0 \rangle\rangle = \langle\langle h\varphi_1 \rangle\rangle$ in ρ_2. So the 3-shell in $\rho(X, A, C)$ determined by h is commutative. □

Remark 6.7.7. In the case where D is a double category with thin structure, we cannot get a formula of the above type, because of the lack of inverses. What we can expect as commuting boundary is a formula saying that some composition of the even faces is the same as a composition of the odd faces.

Here is a diagram of a 3-cube with labelled and directed edges:



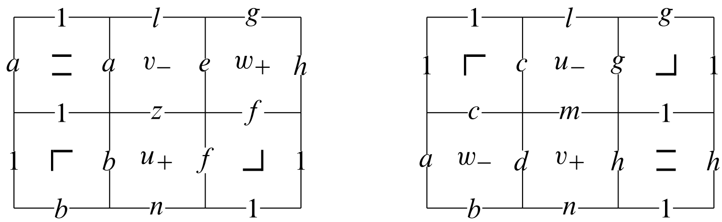
The 6 faces divide into even and odd ones which separately fit together as shown in Diagram (∗):



(∗)

even faces odd faces

Unfortunately, the possible ‘compositions of the even and of the odd faces’ as above do not make sense, and the edges along the two ‘boundaries’ do not agree. So we add to these diagrams thin squares to complete the diagram (∗) as shown:



(∗∗)

We say that the original cube is *commutative*, or, more precisely, has commutative shell, if the completed composite elements in diagram (∗∗) are equal.[87]

Since the subdivision of I^3 corresponding to folding flat a 4-cube analogously to diagram (6.7.1) would have 27 sub 3-cubes, it is clear that new ideas are needed to carry out similar methods to the above in higher dimensions. □

6.8 Proof of the 2-dimensional Seifert–van Kampen Theorem

In this last section of Part I we shall prove a 2-dimensional Seifert–van Kampen Theorem 6.8.2 which includes as a particular case Theorem 2.3.1 some of whose algebraic consequences have been studied in Chapters 4 and 5.[88]

We note again that all the results contained in Chapters 2–5 are about crossed modules over groups, while in this chapter we generalise to crossed modules over groupoids to prove the 2-dimensional Seifert–van Kampen Theorem. The fact that pushouts, and coequalisers, give the same results in these two contexts follows from the fact that these two types of colimit are defined by connected diagrams, and then applying Theorem B.1.7 of Appendix B.

Theorem 6.8.2 is true for triples of spaces (X, A, C) satisfying some connectivity conditions which can be expressed as algebraic conditions on the π_0 and π_1 functors.

Definition 6.8.1. We say that the triple (X, A, C) is *connected* if the following conditions hold:

(ϕ_0) The maps $\pi_0(C) \to \pi_0(A)$ and $\pi_0(C) \to \pi_0(X)$ are surjective.

(ϕ_1) The morphism of groupoids $\pi_1(A, C) \to \pi_1(X, C)$ is piecewise surjective.

Notice that condition (ϕ_0) is equivalent to saying that C intersects all path components of X and all those of A. Also condition (ϕ_1) just says that the function $\pi_1(A)(x, y) \to \pi_1(X)(x, y)$ induced by inclusion is surjective for all $x, y \in C$. It may be shown that given (ϕ_0), condition (ϕ_1) may be replaced by

(ϕ_1') For each $x \in C$, the homotopy fibre over x of the inclusion $A \to X$ is path connected.

That both conditions can be stated in terms of connectivity, explains the origin of the term 'connected'. □

Let us introduce some notation which will be helpful in both the statement and the proof of the 2-dimensional Seifert–van Kampen Theorem 6.8.2.

Suppose we are given a cover $\mathcal{U} = \{U^\lambda\}_{\lambda\in\Lambda}$ of X such that the interiors of the sets of $\mathcal{U}$ cover X. For each $\nu = (\lambda_1, \dots, \lambda_n) \in \Lambda^n$ we write

$$U^\nu = U^{\lambda_1} \cap \dots \cap U^{\lambda_n}.$$

An important property of this situation is that a continuous function f on X is entirely determined by a family of continuous functions $f^\lambda \colon U^\lambda \to X$ which agree on all pairwise intersections $U^{\lambda_1} \cap U^{\lambda_2}$. In formal terms, this states that the following diagram

$$\bigsqcup_{\nu\in\Lambda^2} U^\nu \overset{a}{\underset{b}{\rightrightarrows}} \bigsqcup_{\lambda\in\Lambda} U^\lambda \xrightarrow{c} X$$

is a coequaliser in the category of topological spaces. The functions a, b are determined by the inclusions $U^\nu = U^\lambda \cap U^\mu \to U^\lambda$, and $U^\nu \to U^\mu$ for each $\nu = (\lambda, \mu) \in \Lambda^2$, and c is determined by the inclusions $U^\lambda \to X$ for each $\lambda \in \Lambda$. (Note that in the disjoint union U^ν is indexed by ν so a, b are well determined. Also in the first disjoint union we get some intersections repeated, but this does not matter.)

It is not difficult to extend this last coequaliser diagram to the case of a triple (X, A, C): thus we define $A^\nu = A \cap U^\nu$ and $C^\nu = C \cap U^\nu$ and get an analogous coequaliser diagram in the category of triples of spaces:

$$\bigsqcup_{\nu\in\Lambda^2}(U^\nu, A^\nu, C^\nu) \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_{\lambda\in\Lambda}(U^\lambda, A^\lambda, C^\lambda) \xrightarrow{\ c\ } (X, A, C).$$

Now we move from this to the homotopical situation, by applying ρ to this coequaliser diagram of triples. So the homotopy double groupoids in the following *ρ-sequence of the cover* are well defined:

$$\bigsqcup_{\nu\in\Lambda^2}\rho(U^\nu, A^\nu, C^\nu) \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_{\lambda\in\Lambda}\rho(U^\lambda, A^\lambda, C^\lambda) \xrightarrow{\ c\ } \rho(X, A, C). \quad (6.8.1)$$

Here $\bigsqcup$ denotes disjoint union, which is the coproduct in the category of double groupoids. It is an advantage of the approach using a set of base points that the coproduct in this category is simple to describe. The morphisms a, b are determined by the inclusions

$$U^\nu \to U^\lambda, \quad U^\nu \to U^\mu$$

where $U^\nu = U^\lambda \cap U^\mu$, for each $\nu = (\lambda, \mu) \in \Lambda^2$, and c is determined by the inclusions $U^\lambda \to X$ for each $\lambda \in \Lambda$. We sometimes use the notation U^ν when ν is a finite subset of Λ, when of course it means the intersection of the U^λ for λ in ν.

Theorem 6.8.2 (Seifert–van Kampen Theorem for the fundamental double groupoid). *Assume that for each $n \geqslant 2$ and $\nu = (\lambda_1, \dots, \lambda_n) \in \Lambda^n$ the triple (U^ν, A^ν, C^ν) is connected. Then*

(Con) *the triple (X, A, C) is connected, and*

(Iso) *in the ρ-sequence (6.8.1), c is the coequaliser of a, b in the category of double groupoids.*

Proof. The proof follows the pattern of the 1-dimensional case (Theorem 1.6.1) and is in three stages.

We shall be aiming for the coequaliser result (Iso) because the connectivity part (Con) is obtained along the way. So we start with a double groupoid G and a morphism of double groupoids

$$f' \colon \bigsqcup_{\lambda\in\Lambda}\rho(U^\lambda, A^\lambda, C^\lambda) \to G$$

such that $f'a = f'b$. We have to show that there is a unique morphism of double groupoids

$$f : \rho(X, A, C) \to G$$

such that $fc = f'$.

Recall that by the structure of coproduct in the category of double groupoids, the map f' is just the disjoint union of maps $f^\lambda\colon \rho(U^\lambda, A^\lambda, C^\lambda) \to G$ and the condition $f'a = f'b$ translates to f^λ and f^μ being the same when restricted to $\rho(U^\nu, A^\nu, C^\nu)$ for $\nu = (\lambda, \mu)$.

To define f on $\rho(X, A, C)$ we shall describe how to construct an $F(\alpha) \in G_2$ for all $\alpha \in R_2(X, A, C)$. Then we define $f(\langle\!\langle \alpha \rangle\!\rangle) = F(\alpha)$ and prove independence of all choices.

Stage 1. Define $F(\alpha) \in G_2$ when α has a decomposition $\alpha = [\alpha_{ij}]$ such that each α_{ij} lies in some $R_2(U^\lambda, A^\lambda, C^\lambda)$.

The easiest case is when the image of α lies in some U^λ of $\mathcal{U}$. Then α determines uniquely an element $\alpha^\lambda \in R_2(U^\lambda, A^\lambda, C^\lambda)$. The only way to have $fc = f'$ is by defining

$$F(\alpha) = f^\lambda(\langle\!\langle \alpha^\lambda \rangle\!\rangle).$$

The definition does not depend on the choice of U^λ, because of the condition $f'a = f'b$.

Next, suppose that the element $\alpha \in R_2(X, A, C)$ may be expressed as the composition of an array

$$\alpha = [\alpha_{ij}]$$

such that each α_{ij} belongs to $R_2(X, A, C)$, and also the image of α_{ij} lies in some U^λ of $\mathcal{U}$ which we shall denote by U^{ij}.

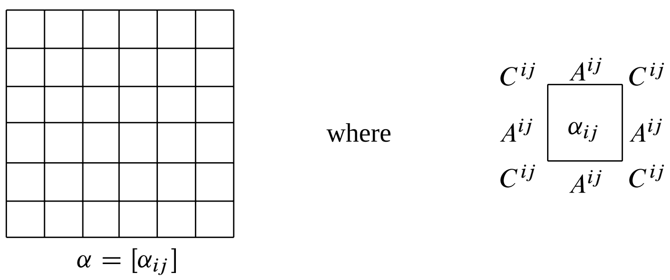


Figure 6.6. Case $\alpha = [\alpha_{ij}]$ with $\alpha_{ij} \in R_2(U^{ij}, A^{ij}, C^{ij})$.

We can define $F(\alpha_{ij})$ for each ij as before. Since the composite $[\alpha_{ij}]$ is defined, it is easy to check using $f'a = f'b$, that the elements $F(\alpha_{ij})$ compose in G_2. We define $F(\alpha)$ to be the composite of these elements of G_2, i.e.

$$F(\alpha) = F([\alpha_{ij}]) = [F(\alpha_{ij})],$$

although *a priori* this definition could depend on the subdivision chosen.

Stage 2. Define $F(\alpha) \in G_2$ by changing α by a thin homotopy to a map of the type used in Stage 1.

This is done analogously to the 1-dimensional case (Theorem 1.6.1). So, first we apply the Lebesgue covering lemma to get a subdivision $\alpha = [\alpha_{ij}]$ such that for each i, j, α_{ij} lies in some element U^{ij} of the covering. In general, we will not have $\alpha_{ij} \in R_2(U^{ij}, A^{ij}, C^{ij})$, so we have to deform α to another β satisfying this condition. The homotopy for this is given by the next lemma. In this we use the cell-structure on I^2 determined by a subdivision of α as in Remark 6.3.2, and also refer to the 'domain' of α_{ij} as defined there.

Lemma 6.8.3. *Let $\alpha \in R_2(X, A, C)$ and let $\alpha = [\alpha_{ij}]$ be a subdivision of α such that each α_{ij} lies in some U^{ij} of $\mathcal{U}$. Then there is a thin homotopy $h\colon \alpha \equiv \alpha'$, with $\alpha' \in R_2(X, A, C)$, such that, in the subdivision $h = [h_{ij}]$ determined by that of α, each homotopy $h_{ij} : \alpha_{ij} \simeq \alpha'_{ij}$ satisfies:*

(i) *h_{ij} lies in U^{ij};*

(ii) *α'_{ij} belongs to $R_2(X, A, C)$, and so it is an element of $R_2(U^{ij}, A^{ij}, C^{ij})$;*

(iii) *if a vertex v of the domain of α_{ij} is mapped into C, then h is constant on v;*

(iv) *if a cell e of the domain of α_{ij} is mapped by α into A (resp. C), then $e \times I$ is mapped by h into A (resp. C), and hence $\alpha'(e)$ is contained in A (resp. C).*

Proof. Let K be the cell-structure on I^2 determined by the subdivision $\alpha = [\alpha_{ij}]$, as in Remark 6.3.2. We define h inductively on $K^n \times I \cup K \times \{0\} \subseteq K \times I$ using the connectivity conditions of the statement, where K^n is the n-skeleton of K for $n = 0, 1, 2$.

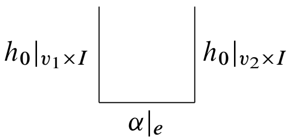
Step 1. Extend $\alpha|_{K^0\times\{0\}}$ to $h_0\colon K^0 \times I \to C$.

Since the triples (U^ν, A^ν, C^ν) are connected for all finite sets $\nu \subseteq \Lambda$, the map $\pi_0(C^\nu) \to \pi_0(U^\nu)$ is surjective. For each vertex $v \in K$ we can choose a path lying in the intersection of all the U^λ corresponding to all the 2-cells of K containing v (one to four according to the situation of v) and going from $\alpha(v)$ to a point of C.

In particular, when $\alpha(v) \in C$ we choose the constant path and if $\alpha(v) \in A$, using that $\pi_0(C^\nu) \to \pi_0(A^\nu)$ is also surjective, we choose the path lying in A. These paths give a map $h_0\colon K^0 \times I \to C$.

Step 2. Extend $\alpha|_{K^1\times\{0\}} \cup h_0$ to $h_1\colon K^1 \times I \to A$.

For each 1-cell $e \in K$ with vertices v_1 and v_2, we have the following diagram



where on the three sides of $e \times I$ the definition of h_1 is given as indicated. We proceed to extend to $e \times I$ with some care.

If $\alpha(e) \subseteq A$ we consider two cases. When v_1, v_2 are mapped into C, we extend to $e \times I$ using α at each level $e \times \{t\}$. If $\alpha(e) \subseteq A$, and v_1, v_2 are not both mapped into C, since all edges go to A, then we can use a retraction to extend the homotopy.

Otherwise, the product of these three paths defines an element of $\pi_1(U^\nu, C^\nu)$ where U^ν is the intersection of the U^λ corresponding to all the 2-cells containing e (1 or 2 according to the situation of e). Using the condition on the surjectivity of the π_1, we have a homotopy rel $\{0,1\}$ to a path in (A^ν, C^ν). This homotopy gives $h_1|_{e\times\{1\}}$.

Step 3. Extend $\alpha|_{K\times\{0\}} \cup h_1$ to $h\colon K \times I \to X$.

This is done using for each 2-cell e the retraction of $e \times I$ to $\partial e \times I \cup e \times 0$ given

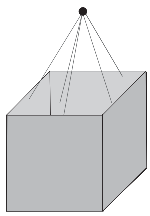


Figure 6.7. Projecting from above in a 3-cube.

by projecting from a point above the centre of the top face. □

The three steps in the construction of h in this lemma are indicated in Figure 6.8 where the third one looks from the back like a hive with square cells.

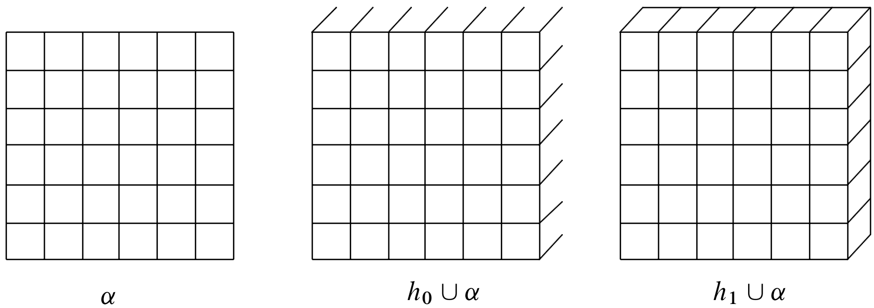


Figure 6.8. Steps in constructing h in Lemma 6.8.3.

Notice that the connectivity result (Con) follows immediately from this lemma, particularly (iv), applied to doubly degenerate or to degenerate squares representing elements of an appropriate π_0 or π_1.

We can now define F for an arbitrary element $\alpha \in R_2(X, A, C)$ as follows. First we choose a subdivision $[\alpha_{ij}]$ of α such that for each i, j, α_{ij} lies in some U^{ij}. Then we apply Lemma 6.8.3 to get an element $\alpha' = [\alpha'_{ij}]$ and a thin homotopy $h\colon \alpha \equiv \alpha'$ decomposing as $h = [h_{ij}]$, the image of each h_{ij} lying in some U^{ij}.

We define

$$F(\alpha) = F(\alpha') = [F({\alpha'}_{ij})],$$

i.e. the composition of the array in G got by applying the appropriate f^{λ} to the decomposition resulting on the back face of the last diagram in Figure 6.8. Since this in principle depends on the subdivision and the homotopy h we will sometimes write this element as $F(\alpha, (h_{ij}))$.

Stage 3. Key lemmas.

The tools for our independence of choices are going to be proved at this stage: they are two lemmas considering a homotopy H of elements $\alpha, \beta \in R_2(X, A, C)$, such that H has a given subdivision $H = [H_{ijk}]$ represented in Figure 6.9. These lemmas should be compared with corresponding steps in the proof of the 1-dimensional Seifert–van Kampen Theorem.

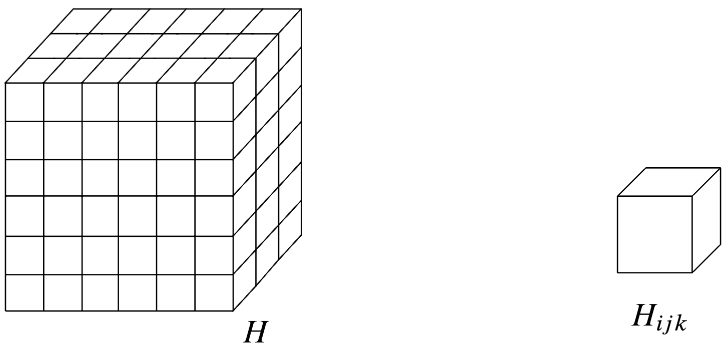


Figure 6.9. Decomposition of a homotopy $H = [H_{ijk}]$.

The first lemma is a rather short application of previous results on commutative cubes and states that $F(\alpha) = F(\beta)$ gives particular conditions on α, β and on a thin homotopy $H : \alpha \equiv \beta$.

Lemma 6.8.4. *Let $H : I^3 \to X$ be a thin homotopy of maps*

$$\alpha, \beta : (I^2, \partial I^2, \partial^2 I^2) \to (X, A, C).$$

Suppose given a subdivision $H = [H_{ijk}]$ of H such that each H_{ijk} maps its domain D^{ijk} of I^3 into a set U^{ijk} of the cover and maps the vertices and edges of D^{ijk} into C and A respectively, i.e. all its faces lie in $R_2(U^{ijk}, A^{ijk}, C^{ijk})$. Then for the induced subdivisions $\alpha = [\alpha_{ij}]$, $\beta = [\beta_{ij}]$ we have in G that

$$F(\alpha) = F(\beta).$$

Proof. The assumptions imply that each H_{ijk} satisfy the conditions of the Homotopy Commutativity Lemma (6.7.6) and thus defines a commutative 3-shell in the double groupoid $\rho(U^{ijk}, A^{ijk}, C^{ijk})$). This 3-shell is mapped by f^{ijk} to give a commutative

3-shell s_{ijk} in G. The condition $f'a = f'b$ implies that these 3-shells s_{ijk} are composable in G, and so, by Theorem 6.7.4, their composition is a commutative 3-shell $s = [s_{ijk}]$ in G. The assumption that H is a thin homotopy allows us to apply Theorem 6.7.5, and to deduce $F(\alpha) = F(\beta)$, as required. □

Now we have to prove that we can always obtain from a general thin homotopy between two maps a thin homotopy between associated maps that satisfies the conditions of the previous lemma. This is where our connectivity assumptions are used again.

Lemma 6.8.5. *Let $H: I^3 \to X$ be a thin homotopy of maps*

$$\alpha, \beta \colon (I^2, \partial I^2, \partial^2 I^2) \to (X, A, C).$$

Suppose given a subdivision $H = [H_{ijk}]$ of H such that each H_{ijk} maps its domain D^{ijk} of I^3 into a set U^{ijk} of the cover. Then there is a homotopy Φ of H to a homotopy H' such that in the cell structure K determined by the subdivision of H,

(i) *H' maps the 0-cells of K into C and the 1-cells into A;*

(ii) *if a 0-cell v of K is mapped by H into C, then Φ is constant on v, and if v is mapped into A by H, then so also is $v \times I$ by Φ;*

(iii) *if a 1-cell e of K is mapped by H into C, then Φ is constant on e, and if e is mapped into A by H, then so also is $e \times I$ by Φ.*

Proof. As in Remark 6.3.2, but now in dimension 3, there is a cell structure K on I^3 appropriate to the subdivision of H. We define a homotopy $\Phi\colon K \times I \to X$ of H by induction on $K^n \times I \cup k \times \{0\} \subseteq K$. The first two steps are as in Lemma 6.8.3. This takes us up to $K^1 \times I \cup K \times \{0\}$. Finally, we extend Φ over the 2- and 3-skeleta of K by using retractions, i.e. by a careful use of the Homotopy Extension Property. □

Remark 6.8.6. The map H' constructed in the lemma gives a thin homotopy from $\alpha' = H'_0$ to $\beta' = H'_1$. Also there is a decomposition of $\alpha' = [\alpha'_{ij}]$ and $\beta' = [\beta'_{ij}]$ which has each element lying in some $R_2(U^\lambda, A^\lambda, C^\lambda)$. Moreover, the homotopy Φ induces homotopies $h\colon \alpha \equiv \alpha'$ and $h'\colon \beta \equiv \beta'$ of the type described in Lemma 6.8.3 and later used to define $F(\langle\!\langle \alpha \rangle\!\rangle)$.

In particular, if all the maps in the induced subdivisions $\alpha = [\alpha_{ij}]$ and $\beta = [\beta_{ij}]$ lie in some $R_2(U^\lambda, A^\lambda, C^\lambda)$, the map H' constructed in the lemma is a thin homotopy $H'\colon \alpha \equiv \beta$. □

Stage 4. Independence of choices inside the same thin homotopy class.

Now we can prove that f is well defined, proving independence of two choices.

1. Independence of the subdivision and the homotopy h of Lemma 6.8.3.

Let us consider two subdivisions of the same map $\alpha \in R_2(X, A, C)$. As there is a common refinement we can assume that one is a refinement of the other. We shall write them $\alpha = [\alpha_{ij}]$ and $\alpha = [\alpha^{ij}_{kl}]$ where for a fixed ij we have $\alpha_{ij} = [\alpha^{ij}_{kl}]$.

Using Lemma 6.8.3, we get thin homotopies $h: \alpha \equiv \alpha'$, with $\alpha' \in R_2(X, A, C)$, such that, in the subdivision $h = [h_{ij}]$ determined by that of α, each homotopy $h_{ij}: \alpha_{ij} \simeq \alpha'_{ij}$ and $h': \alpha \equiv \alpha''$, with $\alpha'' \in R_2(X, A, C)$, such that, in the subdivision $h' = [h'^{ij}_{kl}]$ determined by that of α, each homotopy $h'^{ij}_{kl}: \alpha^{ij}_{kl} \simeq \alpha''^{ij}_{kl}$. We want to prove that

$$[F(\alpha'_{ij})] = [F(\alpha''^{ij}_{kl})].$$

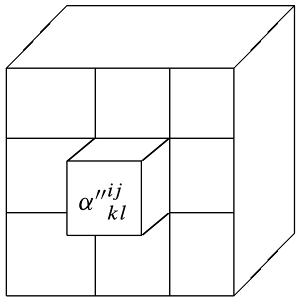


Figure 6.10. Independence of subdivision.

The situation for a fixed ij is described in Figure 6.10 where the smaller cube at the front represents h'^{ij}_{kl} and the larger cube at the back is h_{ij}.

If we denote by h'_{ij} the composition of the array $h'_{ij} = [h'^{ij}_{kl}]$ and by α'_{ij} the composition of the array $\alpha'_{ij} = [\alpha'^{ij}_{kl}]$, we have $h'_{ij}: \alpha_{ij} \simeq \alpha''_{ij}$.

Now $\bar{h}h'$ gives a thin homotopy satisfying the conditions of Lemma 6.8.5 if we denote by $\bar{h}$ the homotopy given by $\bar{h}(x, y, t) = h(x, y, 1-t)$. First, we change this homotopy using Lemma 6.8.5 and we then apply Lemma 6.8.4, to get

$$[F(\alpha'_{ij})]) = [F(\alpha''_{ij})].$$

On the other hand since the second is a refinement of the first, we have

$$[F(\alpha''_{ij})]) = [F(\alpha''^{ij}_{kl})].$$

As a consequence to define the element $F(\alpha)$ we can choose whatever subdivision and homotopy we want insofar as the conditions of Lemma 6.8.3 are met.

2. Independence of the choice inside the same thin homotopy class.

Let $H: \alpha \equiv \beta$ be a thin homotopy of elements of $R_2(X, A, C)$. We choose a subdivision $H = [H_{ijk}]$ of H so that each H_{ijk} maps into a set of $\mathcal{U}$. On both extremes there are induced subdivisions $\alpha = [\alpha_{ij}], \beta = [\beta_{ij}]$. We apply Lemma 6.8.3 to H, getting $H': \alpha' \equiv \beta'$.

As indicated in the Remark 6.8.6, these α', β' satisfy the conditions to be used when defining $F(\alpha)$ and $F(\beta)$. Also H' satisfies the conditions of Lemma 6.8.4. Thus

$$F(\alpha) = [F(\alpha'_{ij})] = [F(\beta'_{ij})] = F(\beta).$$

Stage 5. End of proof

Now we have proved that there is a well-defined map $f : \rho(X, A, C)_2 \to G_2$, given by $f(\langle\!\langle \alpha \rangle\!\rangle) = F(\alpha, (h_{ij}))$, which satisfies $fc = f'$ at least on the 2-dimensional elements of ρ.

The remainder of the proof of (Iso), that is the verification that f is a morphism, and is the only such morphism, is straightforward. It is easy to check that f preserves addition and composition of squares, and it follows from (iii) of Lemma 6.8.3 that f preserves thin elements.

It is now easy to extend f to a morphism $f : \rho(X, A, C) \to G$ of double groupoids, since the 1- and 0-dimensional parts of a double groupoid determine degenerate 2-dimensional parts. Clearly this f satisfies $fc = f'$ and is the only such morphism.[89]

This completes the proof of Theorem 6.8.2. □

Remark 6.8.7. Of especial interest (but not essentially easier to prove) is the case of the theorem in which the cover $\mathcal{U}$ has only two elements; in this case Theorem 6.8.2 gives a push-out of double groupoids.[90] In the applications in previous chapters we have considered only path-connected spaces and assumed that $C = \{x\}$ is a singleton. Taking x as base point, the double groupoids can then be interpreted as crossed modules of groups to give the 2-dimensional analogue of the Seifert–van Kampen Theorem given as Theorem 2.3.1 earlier. We do not know how to prove that theorem without using higher order groupoids in some form. A higher dimensional form of this proof and theorem is given in Part III, namely Theorem 14.3.1. □

Proof of Theorem 2.3.1 In the case where (X, A) is a based pair with base point x, $\rho(X, A, x)$ is abbreviated to $\rho(X, A)$. That we obtain a pushout of crossed modules under the hypothesis of Theorem 2.3.1 is simply the previous remark, together with Proposition 6.3.8, which gives the equivalence between double groupoids and crossed modules.

The corresponding result of Theorem 2.3.3 follows from Theorem 2.3.1 by standard techniques using mapping cylinders. For analogues of these techniques for the fundamental groupoid, see Chapter 8 of [Bro06]. □

Remark 6.8.8. Theorem 6.8.2 contains 1-dimensional information which includes most known results expressing the fundamental group of a space in terms of an open cover, but it does not assume that the spaces of the cover or their intersections are path-connected. That is, it contains the classical Seifert–van Kampen Theorem on $\pi_1(X, A)$ given in Chapter 1. □

Thus we have completed the aims of Part I, to give a reasonably full and we hope comprehensible account of what we understand as 2-dimensional nonabelian algebraic topology, which is essentially the theory and application to algebraic topology of crossed modules, double groupoids and related structures.

Now in Parts II and III we move on to the higher dimensional theory. The situation is more complicated because there are two generalisations of crossed modules and double groupoids with applications to algebraic topology, basically in terms of crossed complexes, or in terms of crossed n-cubes of groups, for an introduction to which see [Bro92], [Por11]. The theory of crossed complexes is limited in its applications, because it starts as being a purely 'linear' theory. However, even this theory has advantages, namely:

- the range of applications;
- the relation to well-known theories, such as chain complexes with a group of operators;
- the use of groupoids;
- its intuitive basis as a development of the methods of Part I.

So this is the account we give, in the space we have here. The theory of crossed n-cubes of groups requires another account! A hint on crossed squares is given in Section B.4 of Appendix B.

Notes

68 p. 142 These intuitions led Brown to announce in the Introduction to [Bro67] a higher dimensional version of the Seifert–van Kampen Theorem. All these intuitions were encapsulated in the paper [BH78a].

69 p. 144 A general discussion of partial algebraic structures was given by Higgins in [Hig63], and this idea was taken up by Birkoff–Lipson in [BL70]. The importance of this idea is that we can think of 'higher dimensional algebra' as the study of partial algebraic structures with algebraic operators whose domains of definition are given by geometric conditions.

70 p. 145 Higgins in [Hig71] adopts the more consistent 'algebraist's' convention and notation of writing all functions on the right of their arguments.

71 p. 146 The term double category is also used more generally for a 'category internal to a category', or a set with two distinct category structures each of which is a morphism for the other, a condition which is equivalent to the interchange law given below. This idea was given by Ehresmann in [Ehr65], and is pursued to higher dimensions in [BH81b]. However we have need in this work only of the more special notion. A web search on 'double category' shows a lot of work on the more general concept.

72 p. 149 The term ‘rewriting’ is used for the manipulation of formulae according to certain rules. There are substantial expositions of this theory. Usually they deal only with 1-dimensional formulae. The idea of higher dimensional rewriting occurs in knot theory, with Reidemeister moves, and in higher dimensional category theory. The article [GM09] gives some current ideas, references and results in this area. The papers [AABS02], [Hig05] use some 3-dimensional rewriting!

73 p. 150 A construction of double categories of a more general kind is given in [Bro09a].

74 p. 151 We emphasise again that these are not the most general kinds of double groupoids or double categories, see for example [BS76a], [AN09], [BJ04], but they are the only ones we consider in this book.

75 p. 152 This normality concept is studied in [Bro06], Sections 8.3 and 11.3.

76 p. 152 A key step in developing the current theory was the recognition in the early 1970s of the relation between double groupoids and crossed modules, which was published as [BS76a], [BS76b]. Further work on double groupoids is in for example [BM92], [AN09], [Bro09a]. There is now considerable work on the closely related, even sometimes equivalent, concept of ‘2-group’, see for example [BL04], but these should not be confused with the notion of 2-group in group theory. There is a weaker structure of ‘categorical group’ for which we give [CGV06] as an entry to the literature.

77 p. 154 We also mention that there are significant algebraic examples of crossed modules over groupoids. Thus the article [Lab99] gives many examples of the notion of *crossed group*: this is precisely a crossed module $\mu\colon M \to P$ in which P is an action groupoid of the form $G \ltimes X$ where G is a group acting on the set X.

78 p. 157 It is possible to use ‘hyperrectangles’ of varying length at this stage, see for example [Bro09b], and so obtain strict identities and strict associativity. We leave you to investigate this further.

79 p. 158 Granted the success of the fundamental groupoid and the known definition of double groupoid, it seemed natural to Brown in 1965 to attempt to define a fundamental or homotopy double groupoid of a space by considering maps $I^2 \to X$ of a square. He was surely not the only topologist to have considered this idea. Experiments over 9 years proved abortive, though the ideas would not go away, and the success of the algebraic theory relating double groupoids and crossed modules and published finally in [BS76a] was suggestive, and gave the key notions of commutative cube in a double groupoid with connections.

A clue considered seriously in June, 1974, was that Whitehead’s theorem on free crossed modules was an example of the existence of a universal property in 2-

dimensional homotopy theory. Further, if a 2-dimensional Seifert–van Kampen Theorem was to be any good it should surely recover Whitehead's theorem. But, Whitehead's theorem was about *relative* homotopy groups! So one ought to look for a homotopy double groupoid in a relative situation. Then the ideas fell into place! In a few days, Brown and Higgins realised that a successful theory could be obtained by considering a triple (X, A, C), i.e. a space X and two subspaces $C \subseteq A \subseteq X$.

In the published papers up to 1981 the condition of homotopy rel vertices was not used but instead an additional condition that each loop in (A, C) should be contractible in A rel vertices. This condition turned out to be inconvenient for application to function spaces.

80 p. 160 This idea may be generalised from a pair (X, A) to a triad $(X; A, B)$, where A, B are subspaces of X with C contained in $A \cap B$, but under the extra condition that the induced morphisms $\pi_2(A, c) \to \pi_2(X, c), \pi_2(B, c) \to \pi_2(X, c)$ have the same image for all $c \in C$ in [Bro09a]. We leave the proof as an exercise. See also the comments related to p. 586 on p. 596.

81 p. 164 The name 'transport laws' was given because they were borrowed from a transport law for path connections in differential geometry, as explained in [BS76a].

82 p. 167 These proofs of the properties of rotations are due to Higgins and appeared in [Bro82]. The results on rotations in the single base point case appeared in [BS76a], using the equivalence of categories. In this book we use only 2-dimensional rewriting, but 3-dimensional arguments occur in [AABS02], to prove a braid relation, and in [Hig05]. There would seem to be a problem in managing even higher dimensional rewriting! Perhaps this could be done with computers and appropriate data tools, but it is clear that there is no algorithmic approach.

83 p. 173 These results have been generalised to higher categorical versions by Higgins in [Hig05]. See also [Ste06]. Proposition 6.5.5 was a part of the thesis of G. H. Mosa, [Mos87], and published separately as [BM99]. It is also related to results of [BS76a], and of [BH78a].

84 p. 177 Indeed one motivation for establishing the equivalence in the work of [BS76b], [BS76a] was simply to find new examples of double groupoids, and so to see how interesting these structures might be. The construction of double groupoids from crossed modules thus gave a large, new source of double groupoids. More general kinds of double groupoids are discussed in for example [BM92], [AN09].

85 p. 183 The style of proof of this equivalence of categories follows that of [BM99]. The result on the equivalence of 2-categories and double categories with connection

is due to Chris Spencer in [Spe77], and these ideas were developed in [SW83]. The generalisation of this categorical result to all dimensions is in [AABS02].

86 p. 186 For another exposition of this material, see [BKP05] for double groupoids, and [Hig05] for the general and ω-categorical case.

87 p. 191 This definition is taken from [BKP05], and the ideas are developed in [Hig05]. The latter paper writes more generally as follows: "If $x \in \mathsf{C}_n$ is an n-cube in C one may ask which of its $(n-1)$-faces have common $(n-2)$-faces and can be composed in C_{n-1}. The answer is that the following pairs of faces (and in general only these pairs) can be composed:

$$(\partial_i^- x, \partial_{i+1}^+ x), \quad (\partial_i^+ x, \partial_{i+1}^- x), \quad i = 1, 2, \ldots, n-1.$$

Thus the faces of x (by which we mean its $(n-1)$-faces) divide naturally into two sequences

$$(\partial_1^- x, \partial_2^+ x, \partial_3^- x, \ldots, \partial_n^{\pm} x) \quad \text{and} \quad (\partial_1^+ x, \partial_2^- x, \partial_3^+ x, \ldots, \partial_n^{\mp} x)$$

in which neighbouring pairs can be composed. We call these respectively the *negative* and the *positive* faces of x." We may also call them 'odd' and 'even' faces.

88 p. 192 This theorem was stated and proved in [BH78a], but is not referred to in any text on topology except [Bro06].

89 p. 200 Actually, it is aesthetically desirable to write out a proof of a universal property by first verifying uniqueness, and then using the resulting description to prove existence. We leave it as an exercise to the reader to reorder the proof in this way.

90 p. 200 An examination of the proof of Theorem 6.8.2 shows that conditions (ϕ_0) and (ϕ_1) are required only for 8-fold intersections of elements of $\mathcal{U}$. However, it has been shown by Razak Salleh [RS76] that in fact one need assume (ϕ_0) only for 4-fold intersections and (ϕ_1) only for 3-fold intersections. Further, these conditions are best possible. The reader may like to try to recover these results using the tool of Lebesgue covering dimension as in the paper [BRS84].

Part II

Crossed complexes

Introduction to Part II

The utility of crossed modules for certain nonabelian homotopical calculations in dimension 2 has been shown in Part I, mainly as applications of a 2-dimensional Seifert–van Kampen Theorem. In Part II, we obtain homotopical calculations using *crossed complexes*, which are a kind of combination of crossed modules of groupoids with chain complexes, but keeping the operations in all dimensions. Again, a Higher Homotopy Seifert–van Kampen Theorem (HHSvKT) plays a key role, but we have to cover also a range of new techniques.

Chapter 7 sets out the basic structures we need to consider, including their important relation with chain complexes with a groupoid of operators. Also included is a brief account of homotopies, an area which is developed fully in Chapter 9.

Chapter 8 is devoted to the statement and immediate applications of the Higher Homotopy Seifert–van Kampen Theorem for crossed complexes.

Chapter 9 introduces a crucial monoidal closed structure on the category of crossed complexes. This structure gives notions of homotopy and higher homotopy for crossed complexes.

Chapter 10 develops the notion of *free crossed resolution* of a group or groupoid, including an outline of a method of computation for finitely presented finite groups, by the method of 'constructing a home for a contracting homotopy'. This uses the notion of covering morphism of crossed complexes. Also included is an account of acyclic models for crossed complexes; this is a basic technique for chain complexes in algebraic topology, and the version for crossed complexes has a few twists to make it work.

Chapter 11 deals with the cubical classifying space of a crossed complex. We give this cubical version because it has convenient properties, and also because cubical methods fit better with other techniques of this book, used extensively in Part III. Thus basic results on collapsing in cubical sets are useful for establishing properties of the category of cubical sets, and are also used essentially in Part III for the proof of the Higher Homotopy Seifert–van Kampen Theorem.

In Chapter 12 we begin with the theory of fibrations of crossed complexes and their long exact sequences. These are used with the methods of the classifying space to discuss the homotopy classification of maps of spaces. A notable feature of our methods is that we are able to make some explicit computations, for example of the k-invariants of certain crossed modules. As another sample calculation, we compute in Example 12.3.13 certain homotopy classes of maps from $\mathbb{R}P^2 \times \mathbb{R}P^2$ to the space $\mathbb{R}P^3$ with homotopy groups in dimensions higher than 3 killed. We also relate the crossed complex methods with those of Čech cohomology and with the cohomology of groupoids.

In this way we fulfill the promise of the Introduction to the book, except for the proofs of those results which require the techniques of Part III.

Chapter 7
The basics of crossed complexes

Introduction

This first chapter of Part II gives the background on crossed complexes which is required for the statement and applications given in the next chapter of the Higher Homotopy Seifert–van Kampen Theorem (HHSvKT) for the functor

$$\Pi : \text{(filtered spaces)} \to \text{(crossed complexes)}.$$

This is a substantial chapter, so you are encouraged to read the definition of crossed complex and of the functor Π, and then skip to the next chapters, returning to this one for further information as required.

The first section of this chapter contains:

- a quick introduction of the category FTop of filtered topological spaces paying special attention to the standard example, the skeletal filtration of a CW-complex, and using the tensor product of filtered spaces as a source of examples;
- an introduction to the category Crs of crossed complexes;
- the definition using relative homotopy groups of the fundamental crossed complex functor $\Pi : \mathsf{FTop} \to \mathsf{Crs}$.

We leave to the next chapter the statement and applications of the Higher Homotopy Seifert–van Kampen Theorem (HHSvKT), which says in general terms that the functor Π preserves some colimits of filtered spaces, but phrased in terms of coequalisers. This theorem allows calculation with the homotopically defined functor Π and is central to the main theme of this book. Analogously to Chapter 6 of Part I, the proof of the HHSvKT requires cubical techniques of what are called ω-groupoids; this is complicated to set up and so is delayed to Part III.

We define these categories FTop, Crs and this functor Π in the first section of this chapter, and proceed to explain in following sections how to compute the colimits which arise naturally in the applications. The HHSvKT and immediate applications are given in the next chapter. It is therefore quite reasonable for the reader to make sure of the basic definitions, and then skip to the next chapter, returning to this chapter as necessary.

Also in the first section, some subsections detail algebraic facts necessary to apply the HHSvKT. In Section 7.1.iii, in order later to analyse colimits of crossed complexes, we define some categories related to Crs, for example Crs_n the category of *n-truncated crossed complexes*: the category XMod of crossed modules over groupoids has been defined in Chapter 6, and the category Mod of *modules over groupoids* is introduced in

Section 7.2.ii before the definition of crossed complex. The objects of Mod are pairs (M, G) where G is a groupoid acting on the family M of abelian groups. This category is not quite so standard even for the case of groups G, since we are varying G.[91] The generalisation to groupoids is necessary for many of the applications of the fundamental crossed complex since it allows the use of several base points and takes into account the full action of the fundamental groupoid.

The n-truncated crossed complexes have the structure of the first n dimensions of a crossed complex. In particular, Crs_2, the 2-truncated crossed complexes, are simply the crossed modules over groupoids as seen in Chapter 6. This category thus includes the crossed modules over groups studied in Chapters 2–5.

The second Section 7.2 is devoted to the study of colimits in the category Crs: the results are necessary to do any computations with the HHSvKT.

It is an easy consequence of the appropriate functors having right adjoints (and hence preserving colimits) that the colimits in Crs can be computed by taking colimits in three categories. First a colimit in groupoids, then a colimit of crossed modules and, last, colimits of modules in all dimensions $n \geqslant 3$.

Moreover, the computation of colimits in Mod and in XMod can be done in two steps. First a change of base groupoid, via the *induced module construction*, and then, a colimit in the category of modules over a fixed groupoid.

We proceed to explain a bit further how to compute induced modules and how to define free modules as a special kind of induced module (indicating the same results for crossed modules) and end the section with some examples of colimits.

Section 7.3 gives the notion of *free crossed complex*. This is a basic concept for many homotopy classification results, since a morphism from a free crossed complex, and also a homotopy, can be constructed in terms of values on a free basis. A consequence of our results is that the skeletal filtration of a CW-complex X is a connected filtration, and that its fundamental crossed complex is a free crossed complex on a basis determined by the characteristic maps of the cells of X.

For these results it is essential to use groupoids rather than groups, and so we set up enough of the general theory of fibred categories to handle the notions of pullback and induced constructions which arise in a variety of situations.

In a final Section 7.4 we relate the notion of crossed complex to the more widely familiar notion of chain complex with operators. The usual notion is that of a group of operators, but in order to model the geometry, and to have better properties, it is again *essential* to generalise this to a *groupoid of operators*.

7.1 Our basic categories and functors

7.1.i The category of filtered topological spaces

By a *space* is meant a compactly generated topological space: these spaces are called k-*spaces* in [Bro06].[92] We write Top for the category of compactly generated topological spaces and continuous maps.

Definition 7.1.1. A *filtered space* X_* consists of a space X and an increasing sequence of subspaces of X:

$$X_* := X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X$$

which we call a *filtration* of X. It is also common to assume that X is the union of the X_n for all n, as happens in our main examples; however we rarely use this condition.

A *filtration preserving map*

$$f : X_* \to Y_*$$

is a continuous map $f : X \to Y$ such that $f(X_n) \subseteq Y_n$ for all $n \geqslant 0$.

These objects and morphisms form the *category* FTop *of filtered spaces and filtered maps*. □

Definition 7.1.2. A standard way of constructing a new filtered space from given ones X_*, Y_* is the *tensor product* with total space $X \times Y$ and filtration given by

$$(X_* \otimes Y_*)_n = \bigcup_{p+q=n} X_p \times Y_q. \qquad \square$$

Remark 7.1.3. The category FTop is, like the category Top, both complete and complete, that is it admits all limits and colimits. Colimits are calculated filtration wise: that is, if $T : \mathsf{C} \to$ FTop is a small diagram in FTop, then $T_n : \mathsf{C} \to$ Top is well defined, and $L = \operatorname{colim} T$ is the filtered space with $L_n = \operatorname{colim} T_n$ in Top, provided L_n is a subspace of L_{n+1}. This will happen in the cases we use. □

Example 7.1.4. Here are some standard filtered spaces.

1) We denote the standard n-*simplex* by Δ^n. We take this to be the subset of $\mathbb{R}^{n+1}$ of points $(x_0, x_1, \ldots, x_n)$ for which $x_i \geqslant 0$ and $x_0 + \cdots + x_n = 1$. We set $\Delta^n_r = \Delta^n$ for $r \geqslant n$, and for $0 \leqslant r < n$ we let it be the set of points $(x_0, \ldots, x_n)$ for which at least $n - r$ of the x_i are 0. This defines the filtered space Δ^n_*.
2) The filtered space I_* has $I_0 = \{0, 1\}$ and $I_1 = I = [0, 1]$, and we write I^n for the n-fold product of I with itself and I^n_* for the corresponding tensor product filtered space, which we call the skeletal filtration of the standard n-*cube*.
3) To define the filtered n-ball, we fix some notation. The standard n-*ball and* $(n-1)$-*sphere* are the usual subsets of the Euclidean space of dimension n:

$$E^n = \{x \in \mathbb{R}^n \mid \|x\| \leqslant 1\}, \quad S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$$

where $\|x\|$ is the standard Euclidean norm.

We write E^n_* for the filtered space of the filtration of the n-ball given by the base point up to dimension $n-2$, S^{n-1} in dimension $n-1$ and E^n in dimensions $\geqslant n$. Thus

$$E^1_* = I_* = \Delta^1_*,$$

and for $n \geqslant 2$, E^n_* is the filtration

$$\underset{0}{\{1\}} = \cdots = \underset{n-2}{\{1\}} \subseteq \underset{n-2}{S^{n-1}} \subseteq \underset{n}{E^n}. \qquad \square$$

Example 7.1.5 (CW-filtrations). Further standard examples of filtered spaces which include all the previous ones are the skeletal filtrations of *CW-complexes*. These are spaces built up in inductivefashion by attaching cells. We recall their construction, which also gives a preparation for Section 7.3.iii where we work analogously with crossed complexes.

We begin by explaining the process of attaching cells to a space. For background to this idea, we refer to Section 4.7 in [Bro06], and to the discussion for 2-cells on p. 42. Let A be a space, Λ a set of indexes, $\{f_\lambda\}_{\lambda\in\Lambda}$ a family of continuous maps $f_\lambda\colon S^{m_\lambda-1} \to A$. We form the adjunction space

$$X = A \cup_{\{f_\lambda\}} \{e^{m_\lambda}\}_{\lambda\in\Lambda},$$

given by the pushout diagram,

$$\begin{tikzcd} \bigsqcup_{\lambda\in\Lambda} S^{m_\lambda-1} \arrow[r, "(f_\lambda)"] \arrow[d] & A \arrow[d, "j"] \\ \bigsqcup_{\lambda\in\Lambda} E^{m_\lambda} \arrow[r, "(h_\lambda)"] & X. \end{tikzcd}$$

Then we say that the space X is *obtained from A by attaching cells*. By standard properties of adjunction spaces (see [Bro06], Chapter 4), the map j is a closed injection, and so we usually assume it is an inclusion. As examples, we have

$$E^1 = e^0_\pm \cup e^1 \quad \text{and} \quad E^n = e^0 \cup e^{n-1} \cup e^n \quad \text{for } n \geqslant 2.$$

The maps $h_\lambda\colon E^{m_\lambda} \to X$ are called the *characteristic maps* of the cells. It is a standard fact that they are homeomorphisms on the interior of each E^{m_λ}. The images $e^{m_\lambda} = h_\lambda(E^{m_\lambda})$ in X are called the closed *cells of X relative to A*. We say that X is obtained from A by attaching the cells $\{e^{m_\lambda}\}_{\lambda\in\Lambda}$. It is important to notice that a map $f\colon X \to Y$ is continuous if and only $f|_A$ is continuous and each composite fh_λ is continuous.

We construct a *relative CW-complex* (X^*, A) by attaching cells in the following inductive process. We start with a space A and form a sequence of spaces X^n by setting X^0 to be the disjoint union of A and a discrete space Λ_0. Then, inductively, we form

X^n by 'attaching' to X^{n-1} a family of n-cells indexed by a set Λ_n. That is for each $n \geqslant 0$ we choose a family of maps $f_\lambda \colon S^{n-1} \to X^{n-1}$, $\lambda \in \Lambda_n$, and define

$$X^n = X^{n-1} \cup_{\{f_\lambda\}} \{e^n_\lambda\}_{\lambda \in \Lambda_n} \quad \text{and} \quad X = \operatorname{colim} X^n.$$

The canonical map $j : A \to X$ is also called a *relative CW-complex*. Clearly, the X^n (called the *relative n-skeleton*) form a filtration of X which we write X_*. If $A = \emptyset$, we say that X is a *CW-complex*.

The cells, characteristic maps, etc., are regarded as part of the structure of a relative CW-complex. The advantage of this structure is that it allows proofs by induction on n. For example, a map $f : X \to Y$ is continuous if and only each restriction $f_n : X^n \to Y$ is continuous and this holds if and only if $f|A$ is continuous and each composite $f h_\lambda$ is continuous, for all $\lambda \in \Lambda_n$ and all $n \geqslant 0$. Thus we may construct a map $f : X \to Y$ by induction on skeleta starting with X_0, which is just the disjoint union of A and Λ_0.

We can conveniently write

$$X = A \cup \{e^n_\lambda\}_{\lambda \in \Lambda_n, n \geqslant 0},$$

and may abbreviate this in some cases, for example to $X = A \cup e^n \cup e^m$.

All filtered spaces given in Example 7.1.4 are CW-complexes. More detail of the above, including the characteristic maps, is given, for example, in [Bro06] for the finite case, and in many books for the general case.[93] □

Remark 7.1.6. We recall from the section on filtered spaces (see p. xxv) of the Introduction to this book that there are many examples of filtered spaces other than the skeletal filtration of a CW-complex, and some of these will be used crucially later. The emphasis on homotopical invariants of filtered spaces, and on colimits of filtered spaces, is fundamental for the methods of this book. □

7.1.ii Modules over groupoids

We have introduced the category XMod of crossed modules over groupoids in Section 6.2. Here we give the simpler category Mod of modules over groupoids. They are a useful generalisation of the well-known modules over groups,[94] and also form part of the basic structure of crossed complexes. Homotopy groups $\pi_n(X; X_0), n \geqslant 2$, of a space X with a set X_0 of base points form a module over the fundamental groupoid $\pi_1(X, X_0)$, as do the relative homotopy groups $\pi_n(Y, X, X_0), n \geqslant 3$, of a pair (Y, X).

Definition 7.1.7. A *module over a groupoid* is a pair (M, G), where G is a groupoid with set of objects G_0, M is a totally disconnected abelian groupoid with the same set of objects as G, and with a given action of G on M. Thus M comes with a *target function* $t \colon M \to G_0$, and each $M(x) = t^{-1}(x), x \in G_0$, has the structure of abelian group. The G-action is given by a family of maps

$$M(x) \times G(x, y) \to M(y)$$

for all $x, y \in G_0$. These maps are denoted by $(m, p) \mapsto m^p$ and satisfy the usual properties, i.e. $m^1 = m$, $(m^p)^{p'} = m^{(pp')}$ and $(m + m')^p = m^p + m'^p$, whenever these are defined. In particular, any $p \in G(x, y)$ induces an isomorphism $m \mapsto m^p$ from $M(x)$ to $M(y)$. If (M, G) is a module, then $(M, G)_0$ is defined to be G_0.

A *morphism of modules* is a pair $(f, \theta)\colon (M, G) \to (N, H)$, where $\theta\colon G \to H$ and $f\colon M \to N$ are morphisms of groupoids and preserve the action. That is, f is given by a family of group morphisms

$$f(x)\colon M(x) \to N(f(x))$$

for all $x \in G_0$ satisfying $f(y)(m^p) = (f(x)(m))^{\theta(p)}$, for all $p \in G(x, y), m \in M(x)$.

This defines the category Mod having modules over groupoids as objects and the morphisms of modules as morphisms.

In the case when G_0 is a single point we recover the category of modules over groups.

We can also fix the groupoid G and restrict the morphisms between modules to those inducing the identity on G, getting then the category Mod_G of *modules over* G. □

In Appendix A, Section A.8, we discuss the notion of abelianisation M^{ab} of a groupoid M: in the case M is just a family of groups, as in the case when $\mu\colon M \to P$ is a crossed module over the groupoid P, then M^{ab} is just the family of abelianisations $M(x)^{\mathrm{ab}}, x \in P_0$ of the groups $M(x)$. This family inherits an action of $\operatorname{Cok} \mu$.

Proposition 7.1.8. *The natural inclusion functor* $i\colon \mathsf{Mod} \to \mathsf{XMod}$ *which to a module* (M, G) *assigns the crossed module* $0\colon M \to G$ *with trivial boundary has a left adjoint which assigns to the crossed module* $\mathcal{M} = (\mu\colon M \to P)$ *the module* $\mathcal{M}^{\mathrm{mod}} = (M^{\mathrm{ab}}, \operatorname{Cok} \mu)$*, called* ***modulisation of the crossed module****.*

The proof is left as an exercise.

7.1.iii The category of crossed complexes

The structure of *crossed complex* is suggested by the canonical example, the *fundamental crossed complex* ΠX_* of the filtered space X_*, which we explain in Section 7.1.v. Here we give the purely algebraic definition.[95]

Definition 7.1.9. A *crossed complex* C over C_1 is written as a sequence

$$\cdots \longrightarrow C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots\cdots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1$$

and it is given by the following three sets of data:

1. For $n \geqslant 2$, C_n is a totally disconnected groupoid (abelian if $n \geqslant 3$) with the same set of objects as C_1, namely C_0. This is equivalent to saying that C_n is a family of groups $\{C_n(x)\}_{x \in C_0}$ and for $n \geqslant 3$, the groups $C_n(x)$ are abelian.

We shall generally use additive notation for all groups $C_n(x), n \geqslant 3$, and multiplicative notation for $n = 1, 2$, and we shall use the symbol 0 or 1 for their respective identity elements. However, in dealing with the tensor product in the next chapter, it is often convenient to use additive notation in all dimensions $\geqslant 1$.

2. For $n \geqslant 2$, an action of the groupoid C_1 on the right on each C_n,

$$C_n \times C_1 \to C_n$$

denoted $(c, c_1) \mapsto c^{c_1}$, such that if $c \in C_n(x)$ and $c_1 \in C_1(x, y)$ then $c^{c_1} \in C_n(y)$. For $n \geqslant 3$, this property is equivalent to say that C_n is a C_1-module (see Definition 7.1.7).

We shall always consider C_1 as acting on its vertex groups $C_1(x)$ by conjugation, i.e. $c^{c_1} = c_1^{-1} c c_1 \in C_1(y)$ for all $c \in C_1(x)$ and $c_1 \in C_1(x, y)$.

As an example of our use of notation, two of the conditions for an action are written $c^{c_1 c_1'} = (c^{c_1})^{c_1'}$ and $c^1 = c$ in all dimensions, but the third condition is expressed as $(cc')^{c_1} = c^{c_1} c'^{c_1}$ for $n = 1, 2$, and $(c + c')^{c_1} = c^{c_1} + c'^{c_1}$ for $n \geqslant 3$.

A consequence of the existence of this action is that $C_n(x) \cong C_n(y)$ if there is a morphism in $C_1(x, y)$, i.e. when x and y lie in the same component of the groupoid C_1.

3. For $n \geqslant 2$, $\delta_n : C_n \to C_{n-1}$ is a morphism of groupoids over C_0 and preserves the action of C_1.

These three sets of data have to satisfy two conditions:

CX1) $\delta_{n-1}\delta_n = 0 : C_n \to C_{n-2}$ for $n \geqslant 3$ (thus C has analogies with chain complexes);

CX2) $\operatorname{Im} \delta_2$ acts by conjugation on C_2, and trivially on C_n for $n \geqslant 3$.

Notice that CX2) actually has two parts. The first part, together with the condition that δ_2 preserves the action of C_1, says that C_2 is a crossed module over the groupoid C_1, since for $c, c' \in C_2$, $c^{\delta_2 c'} = c'^{-1} c c'$. The second part implies that for $n \geqslant 3$, C_1 acts on C_n through

$$\pi_1(C) = \operatorname{Cok} \delta_2 = \frac{C_1}{\operatorname{Im} \delta_2},$$

which we call the *fundamental groupoid* of C.

These two axioms give a good reason for the name ‘crossed complex’: it has a ‘root’ which is a crossed module (over C_1) and a ‘trunk’ that is a (kind of) chain complex (over $\pi_1(C)$). The interplay of these two actions is important in what follows.

We write $s, t : C_1 \to C_0$ for the source and target maps of the groupoid C_1 and $t : C_n \to C_0$ denotes for $n \geqslant 2$ the target, or base point map, of the totally disconnected groupoid C_n.

A *morphism of crossed complexes* $f : C \to D$ is a family of morphisms of groupoids $f_n : C_n \to D_n$ $(n \geqslant 1)$ all inducing the same map of vertices $f_0 : C_0 \to D_0$, and compatible with the boundary maps and the actions of C_1 and D_1. This means that $\delta_n f_n(c) = f_{n-1}\delta_n(c)$ and $f_n(c^{c_1}) = f_n(c)^{f_1(c_1)}$ for all $c \in C_n$ and $c_1 \in C_1$. We represent a morphism of crossed complexes by the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_n \arrow[r, "\delta_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \arrow[r, "\delta_3"] & C_2 \arrow[r, "\delta_2"] \arrow[d, "f_2"] & C_1 \arrow[d, "f_1"] \\
\cdots \arrow[r] & D_n \arrow[r, "\delta_n"'] & D_{n-1} \arrow[r, "\delta_{n-1}"'] & \cdots \arrow[r, "\delta_3"'] & D_2 \arrow[r, "\delta_2"'] & D_1 .
\end{tikzcd}$$

We denote by Crs the resulting *category of crossed complexes*.

A morphism $f : C \to D$ of crossed complexes induces a morphism $f_* : \pi_1(C) \to \pi_1(D)$ of fundamental groupoids, giving a functor

$$\pi_1 : \mathsf{Crs} \to \mathsf{Gpds}. \qquad \square$$

In the case when C_0 is a single point we call C a *reduced crossed complex*, or a *crossed complex over a group*. These crossed complexes give a full subcategory of Crs, which we write $\mathsf{Crs}_{\mathrm{red}}$.

We can also fix the groupoid C_1 to be a groupoid G and restrict the morphisms to those inducing the identity on G, getting then the category Crs_G of *crossed complexes over* G. □

Remark 7.1.10. Although there are many similarities between crossed complexes and chain complexes there are key differences. In dimension 0 a crossed complex C is just a set C_0, so even if this is a singleton, it is not seen as an abelian group. Thus a chain complex of abelian groups is not a special case of a crossed complex. Relations between crossed complexes and chain complexes with a groupoid of operators are studied later in Sections 7.4, 8.4 and 9.5. □

Definition 7.1.11. There are some simple examples of crossed complexes which will recur often. If G is a groupoid, then there is a crossed complex which we write $\mathbb{K}(G,1)$ which is G in dimensions 0 and 1, and is trivial in dimensions $n \geqslant 2$. We use this notation to keep an analogy with Eilenberg–Mac Lane space in algebraic topology. In fact the crossed complex $\mathbb{K}(G,1)$ is identical to the crossed complex $\mathrm{sk}^1(G)$ of Section 7.1.vi. The crossed complex $\mathbb{K}(G,1)$ is often written simply as G. Next, for $n \geqslant 2$ the functor

$$\mathbb{K}_n : \mathsf{Mod} \to \mathsf{Crs}$$

is defined on objects by

$$\mathbb{K}_n(M,G) := \quad \cdots \longrightarrow 0 \longrightarrow \underset{n}{M} \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \underset{1}{G}$$

and gives an embedding of Mod as a full subcategory of Crs for any dimension $n \geqslant 2$. This functor is also relevant to later results on the homotopy classification of maps, and the notion of local coefficients, in Chapter 12. If $n \geqslant 1$, $G = 1$ is the trivial group and M is an abelian group we write $\mathbb{K}(M,n)$ for $\mathbb{K}_n(M,1)$ and we also write $\mathbb{S}(n)$ for $\mathbb{K}(\mathbb{Z},n)$, while $\mathbb{S}(0)$ denotes the crossed complex with object set $\{0,1\}$ and everything else trivial. □

Definition 7.1.12. For each $n \geqslant 3$, we define the functor

$$\mathbb{F}_n \colon \mathsf{Mod} \to \mathsf{Crs}$$

to have value on a module (M, G) the crossed complex

$$\cdots \longrightarrow 0 \longrightarrow \underset{n}{M} \xrightarrow{1_M} \underset{n-1}{M} \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \underset{1}{G} \ ,$$

where the two Ms are in dimensions n and $n-1$, the map between them is the identity, and all other boundary maps are 0. The value of $\mathbb{F}_n$ on morphisms is defined similarly. In particular, with $G = 1$, the trivial group, and $M = \mathbb{Z}$, we write $\mathbb{F}(n)$ for $\mathbb{F}_n(\mathbb{Z}, 1)$, and extend this to the cases $n = 2, 1$ by setting

$$\mathbb{F}(2)_i = \mathbb{Z}, \quad i = 1, 2$$

with boundary the identity, and $\mathbb{F}(1)$ is another name for the crossed complex $\mathcal{I}$, these crossed complexes being understood to be trivial except in the stated dimensions. Note that $\mathbb{F}(n)$ has one object for $n \geqslant 2$ and two objects for $n = 1$. We also set $\mathbb{F}(0)$ to be the trivial crossed complex on one object. □

Definition 7.1.13. For $n \geqslant 2$, we define the *restriction to dimension n functor*

$$\mathrm{res}_n \colon \mathsf{Crs} \to \mathsf{Mod}$$

to be given on objects by

$$\mathrm{res}_n(C) = \begin{cases} (C_n, \pi_1 C) & \text{if } n \geqslant 3, \\ (C_2^{\mathrm{ab}}, \pi_1 C) & \text{if } n = 2. \end{cases}$$

and with the obvious extension to morphisms. □

Proposition 7.1.14. *For $n \geqslant 2$, the functor* res_n *is left adjoint to* $\mathbb{F}_{n+1}$.

Proof. Suppose $n \geqslant 3$. We need to study $\mathsf{Crs}(C, \mathbb{F}_{n+1}(M, G))$, i.e. morphisms of crossed complexes

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+2} \arrow[r, "\delta_{n+2}"] \arrow[d, "0"] & C_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\delta_n"] \arrow[d, "f_n"] & \cdots \arrow[r, "\delta_3"] & C_2 \arrow[r, "\delta_2"] \arrow[d, "0"] & C_1 \arrow[d, "f_1"] \\
\cdots \arrow[r] & 0 \arrow[r, "0"'] & M \arrow[r, "1_M"'] & M \arrow[r, "0"'] & \cdots \arrow[r, "0"'] & 0 \arrow[r, "0"'] & G .
\end{tikzcd}$$

Since $f_1 \delta_2 = 0$ this diagram produces a morphism of modules

$$(f_n, \theta) \colon (C_n, \pi_1(C)) \to (M, G)$$

where $\theta\colon \pi_1(C) \to G$ is induced by f_1.

On the other hand, given a morphism of modules

$$(f', \theta)\colon (C_n, \pi_1(C)) \to (M, G)$$

we get a morphism of crossed modules on putting $f_1 = \theta\phi$ (ϕ being the projection $\phi\colon C_1 \to \pi_1 C$), $f_n = f'$ and $f_{n+1} = f_n \delta_{n+1}$.

These correspondences give the adjointness for this case. We leave the case $n = 2$ to the reader. □

Corollary 7.1.15. *For $n \geqslant 2$, the functor* $\mathrm{res}_n\colon \mathsf{Crs} \to \mathsf{Mod}$ *preserves colimits.*

Exercise 7.1.16. Give the proof of the case $n = 2$ of the last proposition. □

Exercise 7.1.17. There is another restriction functor for $n \geqslant 3$

$$\mathrm{res}'_n\colon \mathsf{Crs} \to \mathsf{Mod}$$

given by

$$\mathrm{res}'_n C = \left(\frac{C_n}{\delta_{n+1} C_{n+1}}, C_1\right).$$

Show that res'_n is right adjoint to $\mathbb{K}_n$. □

7.1.iv Homotopy and homology groups of crossed complexes

Let us recall some definitions and define some new functors giving direct algebraic and set theoretic invariants of crossed complexes. The first one expresses the connectivity of the basic groupoid C_1.

Definition 7.1.18. The *set of components* of the crossed complex C, written $\pi_0(C)$, is just the set of components of the groupoid C_1. This definition gives a functor

$$\pi_0\colon \mathsf{Crs} \to \mathsf{Set}.$$ □

Exercise 7.1.19. Prove that $\pi_0\colon \mathsf{Crs} \to \mathsf{Set}$ has left and right adjoints. □

Example 7.1.20. It is easy to see that for the skeletal filtration of a CW-complex, $\pi_0 \Pi(X_*)$ is bijective with $\pi_0(X)$. □

We have earlier defined the fundamental groupoid $\pi_1(C)$, the cokernel of the crossed module part of the crossed complex.

Example 7.1.21. By the homotopy long exact sequence of a pair, it is clear that if X_* is a filtered space such that the morphism induced by inclusion $\pi_1(X_1, x) \to \pi_1(X_2, x)$ is surjective for all $x \in X_0$, then $\pi_1 \Pi(X_*) \cong \pi_1(X_2, X_0)$. The Seifert–van Kampen Theorem 1.6.1 for the fundamental groupoid implies, in particular, that for the skeletal filtration of a CW-complex, we have $\pi_1 \Pi(X_*) \cong \pi_1(X, X_0)$. □

Now we consider the homology of the abelian part of a crossed complex, getting $\pi_1(C)$-modules associated to a crossed complex C.

Definition 7.1.22. For any crossed complex C and for $n \geqslant 2$ there is a totally disconnected groupoid $H_n(C)$ given by the family of abelian groups

$$H_n(C,x) = \frac{\operatorname{Ker} \delta_n(x)}{\operatorname{Im} \delta_{n+1}(x)}$$

for all $x \in C_0$. This is called the family of *n-homology groups* of the crossed complex C.

A morphism $f : C \to D$ of crossed complexes induces morphisms

$$f_* \colon H_n(C) \to H_n(D)$$

for all $n \geqslant 2$. □

Exercise 7.1.23. Prove that for a crossed complex C and $n \geqslant 2$, the homology groups are a family of abelian groups, and that there is an induced action of $\pi_1(C)$ on the family $H_n(C)$ of homology groups making $H_n(C)$ a $\pi_1(C)$-module. Thus each such homology group gives a functor

$$H_n : \mathsf{Crs} \to \mathsf{Mod}.$$ □

Definition 7.1.24. A morphism $f : C \to D$ is a *weak equivalence* if it induces a bijection $\pi_0(C) \to \pi_0(D)$ and isomorphisms $\pi_1(C,x) \to \pi_1(D,fx)$, $H_n(C,x) \to H_n(D,fx)$ for all $x \in C_0$ and $n \geqslant 2$. □

Example 7.1.25. We shall see in the next chapter in Section 8.4 that if X_* is the skeletal filtration of a CW-complex, then $H_n(\Pi X_*, x)$ is isomorphic to $H_n(\widetilde{X}_x)$, the n-th homology group of the universal cover of X based at x. □

Remark 7.1.26. In Section 7.1.vii we shall introduce the notion of homotopy of morphisms of crossed complexes. It is then an easy exercise to define homotopy equivalences of crossed complexes and check that they are weak equivalences. The converse is true for the free crossed complexes which we define in Section 7.3.iii, but this is a nontrivial result. □

7.1.v The fundamental crossed complex functor

For any filtered space X_* and any $x \in X_0$, there is a sequence of groups and homomorphisms (abelian for $n \geqslant 3$):

$$\cdots \xrightarrow{\delta_{n+1}} \pi_n(X_n, X_{n-1}, x) \xrightarrow{\delta_n} \pi_{n-1}(X_{n-1}, X_{n-2}, x)$$
$$\longrightarrow \cdots \xrightarrow{\delta_3} \pi_2(X_2, X_1, x) \xrightarrow{\delta_2} \pi_1(X_1, x).$$

In this sequence, the $\pi_n(X_n, X_{n-1}, x), n \geqslant 2$, are the relative homotopy groups; the composition in these groups and the action of the groupoid $\pi_1(X_1, X_0)$ on these relative groups for varying $x \in X_0$ were studied in Section 2.1; $\delta_2\colon \pi_2(X_2, X_1, x) \to \pi_1(X_1, x)$ is the standard boundary map considered in Equation (2.1.3), and for $n \geqslant 3$ the boundary maps δ_n are defined as on p. 36 as the composition

$$\pi_n(X_n, X_{n-1}, x) \xrightarrow{\partial_n} \pi_{n-1}(X_{n-1}, x) \xrightarrow{i_{n-1}} \pi_{n-1}(X_{n-1}, X_{n-2}, x).$$

It is convenient to combine these structures over all base points $x \in X_0$ and so to use crossed complexes over groupoids. So we get groupoids $\pi_n(X_n, X_{n-1}, X_0)$ for $n \geqslant 2$, and the groupoid $\pi_1(X_1, X_0)$, all having the same set X_0 of objects.

Definition 7.1.27. The fundamental crossed complex ΠX_* of the filtered space X_*

$$\cdots \xrightarrow{\delta_{n+1}} (\Pi X_*)_n \xrightarrow{\delta_n} (\Pi X_*)_{n-1} \to \cdots \to (\Pi X_*)_2 \to (\Pi X_*)_1$$

is defined by

$$(\Pi X_*)_n = \pi_n(X_n, X_{n-1}, X_0), n \geqslant 2 \quad \text{and} \quad (\Pi X_*)_1 = \pi_1(X_1, X_0).$$

This structure of groupoids, boundary morphisms, and actions define the *fundamental crossed complex of the filtered space* X_*.[96] □

That ΠX_* has the properties of a crossed complex can be proved directly, in a manner similar to proofs in Section 2.1. Instead, we shall deduce these properties from the full construction and properties of the *homotopy ω-groupoid* $\rho(X_*)$, since the relation between these constructions, given in Part III, is a kind of engine which drives this book. It turns out that ΠX_* can be considered as a substructure $\gamma\rho X_*$ of ρX_*, and in this way we obtain in Chapter 14 a verification that ΠX_* is a crossed complex. In fact the definition of the required composition structure on $\rho(X_*)$ is quite simple, but the proof that this composition is well defined is not simple, as is seen in Section 6.3 in dimension 2. The relations between these two functors ρ and Π form a basis for this whole book, even though this may be disguised in Part II, in which our main object is the study and use of Π.

The homotopical definition of this crossed complex implies immediately that it gives a functor

$$\Pi\colon \mathsf{FTop} \to \mathsf{Crs}.$$

Note also that in each of the categories FTop, Crs, disjoint unions are the coproducts: this is partly because the empty groupoid exists which is but one of the advantages of a groupoid approach. The homotopical definition of the functor Π implies easily that it preserves disjoint unions.

An obvious property of Π is that it is preserved by isomorphisms of filtered spaces. This is analogous to the fact that singular homology is pretty obviously preserved by homeomorphisms of spaces; separate methods have to be developed to calculate with singular homology, and quite different methods are used to calculate with Π.

Proposition 7.1.28. *An isomorphism $f : X_* \to Y_*$ in the category* FTop *induces an isomorphism of crossed complexes $\Pi f : \Pi X_* \to \Pi Y_*$.*

A more subtle property is the following:

Proposition 7.1.29. *Let $f : X_* \to Y_*$ be a map of filtered space such that $f_0 : X_0 \to Y_0$ is a bijection, and for $n \geqslant 1$, $f_n : X_n \to Y_n$ is a homotopy equivalence. Then $\Pi f : \Pi X_* \to \Pi Y_*$ is an isomorphism.*

Proof. This follows from basic properties of relative homotopy groups.[97] □

Thus one advantage of the functor Π is that its topological and indeed homotopical invariance in the above sense is clear. The fact that we can calculate to some extent with Π comes from the Higher Homotopy Seifert–van Kampen Theorem in the next chapter.

We will show in Chapter 9 how the functor Π behaves with respect to homotopies of filtered maps; such a homotopy is a map $I_* \otimes X_* \to Y_*$ in FTop. This discussion requires the development of more algebraic machinery, and in particular the tensor product of crossed complexes.

We emphasise that the use of crossed complexes over groupoids is central to this theory, both for the development of the algebra and for the modelling of the topology.[98]

7.1.vi Substructures

We will use finite-dimensional versions of crossed complexes.

Definition 7.1.30. An *n-truncated crossed complex* over a groupoid C_1 is a finite sequence

$$C_n \xrightarrow{\delta_n} C_{n-1} \longrightarrow \cdots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1$$

satisfying all the axioms for a crossed complex in so far as they make sense. In a similar way, we define *morphisms between n-truncated crossed complexes*. They define the category Crs_n of n-truncated crossed complexes. □

Notice that a 1-truncated crossed complex is just a groupoid, and a 2-truncated crossed complex is a crossed module over a groupoid, as defined in Chapter 6. Thus we can write $\mathsf{Crs}_1 = \mathsf{Gpds}$ and $\mathsf{Crs}_2 = \mathsf{XMod}$.

Definition 7.1.31. We define the *n-truncation functor*

$$\mathrm{tr}_n : \mathsf{Crs} \to \mathsf{Crs}_n$$

which applied to a crossed complex C gives its part in dimensions $\leqslant n$. □

There is also a functor in the other direction:

Definition 7.1.32. The *n-skeleton functor*

$$\mathrm{sk}^n \colon \mathsf{Crs}_n \to \mathsf{Crs}$$

maps an n-truncated crossed complex

$$C\colon = \quad C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots \longrightarrow C_2 \xrightarrow{\delta_2} C_1$$

to

$$\mathrm{sk}^n(C)\colon \ \cdots \longrightarrow 0 \longrightarrow C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots \longrightarrow C_2 \xrightarrow{\delta_2} C_1,$$

which agrees with C up to dimension n and is trivial thereafter.

It is also convenient to write $\mathrm{Sk}^n = \mathrm{sk}^n\,\mathrm{tr}_n$, so that an n-truncated crossed complex is also thought of as a crossed complex C with $C_k = 0$ for $k > n$. This n-skeleton functor allows us to consider Crs_n as a full subcategory of Crs. Conversely, for a crossed complex C, $\mathrm{Sk}^n\,C$ is also thought of as a 'truncation' of C. □

Proposition 7.1.33. *The n-skeleton functor* sk^n *is left adjoint to the n-truncation functor* tr_n.

Proof. For any crossed complex D and n-truncated crossed complex C there is an obvious bijection

$$\mathsf{Crs}(\mathrm{sk}^n(C), D) \to \mathsf{Crs}_n(C, \mathrm{tr}_n(D))$$

because a morphism of crossed complexes

$$f\colon\ \mathrm{sk}^n(C) \to D$$

is given just by the first n maps f_i, since all the others are the 0 maps as in the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & C_n \arrow[r, "\delta_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots\cdots \arrow[r, "\delta_3"] & C_2 \arrow[r, "\delta_2"] \arrow[d, "f_2"] & C_1 \arrow[d, "f_1"] \\
\cdots \arrow[r] & D_{n+1} \arrow[r] & D_n \arrow[r, "\delta'_n"'] & D_{n-1} \arrow[r, "\delta'_{n-1}"'] & \cdots\cdots \arrow[r, "\delta'_3"'] & D_2 \arrow[r, "\delta'_2"'] & D_1.
\end{tikzcd}$$

□

The n-truncation functor has also a right adjoint which is a modification of the n-skeleton functor.

Definition 7.1.34. We define the *n-coskeleton functor*

$$\mathrm{cosk}^n \colon \mathsf{Crs}_n \to \mathsf{Crs},$$

on an n-truncated crossed complex

$$C: \; C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots\cdots \longrightarrow C_2 \xrightarrow{\delta_2} C_1$$

by

$$\operatorname{cosk}^n(C): = \; \cdots \longrightarrow 0 \longrightarrow \operatorname{Ker}\delta_n \longrightarrow C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots \longrightarrow C_1$$

for $n \geqslant 2$ and by

$$\operatorname{cosk}^1(C): = \; \cdots \longrightarrow 0 \longrightarrow \operatorname{Inn}(C_1) \longrightarrow C_1,$$

where $\operatorname{Inn}(C_1)$ is the totally disconnected groupoid formed by the object groups of C_1. □

We also write $\operatorname{Cosk}^n = \operatorname{cosk}^n \operatorname{tr}_n$ as a functor $\mathsf{Crs} \to \mathsf{Crs}$. Notice that the only difference of the coskeleton from the skeleton functor is in the existence of elements of dimension $n+1$. The importance of this is realised when proving adjointness.

Proposition 7.1.35. *The functor* cosk^n *is right adjoint to the n-truncation functor* tr_n.

Proof. Let $n \geqslant 2$. For any crossed complex C and n-truncated crossed complex D there is an obvious bijection

$$\mathsf{Crs}(C, \operatorname{cosk}^n(D)) \to \mathsf{Crs}_n(\operatorname{tr}_n(C), D)$$

because a morphism f from C to $\operatorname{cosk}^n(D)$ is just given by the first n maps since the $(n+1)$-st has to be the restriction of $f_n\delta_{n+1}$ to its image and all others have to be the 0 maps as in the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+2} \arrow[r] \arrow[d, "0"'] & C_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "f_n\delta_{n+1}"'] & C_n \arrow[r, "\delta_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \arrow[r, "\delta_3"] & C_2 \arrow[r, "\delta_2"] \arrow[d, "f_2"] & C_1 \arrow[d, "f_1"] \\
\cdots \arrow[r] & 0 \arrow[r] & \operatorname{Ker}\delta'_n \arrow[r] & D_n \arrow[r, "\delta'_n"'] & D_{n-1} \arrow[r, "\delta'_{n-1}"'] & \cdots \arrow[r, "\delta'_3"'] & D_2 \arrow[r, "\delta'_2"'] & D_1.
\end{tikzcd}$$

Notice that in order to be able to define f_{n+1} as above we need the $(n+1)$-st dimensional part of $\operatorname{cosk}^n(D)$ to be $\operatorname{Ker}\delta'_n$, because $\delta'_n f_n \delta_{n+1} = f_{n-1}\delta_n\delta_{n+1} = 0$.

We leave the case $n = 1$ to the reader. □

We will use later also a left adjoint to sk^n.[99]

Proposition 7.1.36. *For $n \geqslant 1$, the cotruncation functor* $\operatorname{cotr}_n : \mathsf{Crs} \to \mathsf{Crs}_n$ *which assigns to a crossed complex C the n-truncated crossed complex over C_0,*

$$\operatorname{cotr}_n C: = \frac{C_n}{\operatorname{Im}\delta_{n+1}} \to C_{n-1} \to \cdots \to C_2 \to C_1,$$

is left adjoint to sk^n.

Proof. We leave details to the reader, including the proof that $\operatorname{cotr}_n C$ inherits the structure of crossed complex. □

In summary, we have functors

$$\operatorname{tr}_n, \operatorname{cotr}_n : \mathsf{Crs} \to \mathsf{Crs}_n, \quad \operatorname{sk}^n, \operatorname{cosk}^n : \mathsf{Crs}_n \to \mathsf{Crs}$$

such that tr_n has left adjoint sk^n, and right adjoint cosk^n, while sk^n has right adjoint tr_n and left adjoint cotr_n.

Corollary 7.1.37. *The functors* sk^n, tr_n *preserve limits and colimits;* cosk^n *preserves limits;* cotr_n *preserves colimits. In particular, the fundamental groupoid functor* $\pi_1 : \mathsf{Crs} \to \mathsf{Gpds}$, *which coincides with* cotr_1, *preserves colimits.*

7.1.vii Homotopies of morphisms of crossed complexes

The reasons for placing the notion of homotopy here are: (i) it is a basic notion for the theory, (ii) the notion of homotopy will be familiar to many readers in other contexts such as chain complexes, and so we wanted it early on; (iii) we wanted to show at this stage the homotopy relationship of a connected crossed complex to the part of it over a single point.

Because of the amount of information encoded in crossed complexes, particularly on base points, the definition of homotopy is more complicated than that in the chain complex case, but this complication is necessary to reflect the geometry of the cylinder $E^1 \times E^m$.[100] However the justification of the conventions in this definition of homotopy will have to wait till Chapter 9 where the definition is put in the context of a tensor product structure on Crs and a homotopy can then be seen as a morphism $\mathcal{I} \otimes C \to D$ from a 'cylinder object' $\mathcal{I} \otimes C$ to D, where $\mathcal{I}$ is the groupoid defined on p. 26, but regarded as a crossed complex trivial above dimension 1.

The only result of this section is Proposition 7.1.46 which shows usefully how the homotopy type of a connected crossed complex is related to that of its reduced part over a vertex.

Recall that we write $s, t : C_1 \to C_0$ for the source and target maps of the groupoid C_1, and $t : C_n \to C_0$ denotes for $n \geqslant 2$ the target, or base point map, of the totally disconnected groupoid C_n.

In chain complexes a homotopy $C \to D$ can be defined simply as a family of morphisms $C_n \to D_{n+1}$, and this can relate a pair of morphisms f^-, f^+. For crossed complexes, the homotopy has to be associated with a morphism f^+. In the following we use the term 'homotopy to f^+', but in Chapter 9, we will need to call this a *left 1-homotopy to* f^+, since we then discuss left and right m-homotopies for all $m \geqslant 1$.

Definition 7.1.38. Let $f^-, f^+ : C \to D$ be morphisms between two crossed complexes. A *homotopy* from f^- to f^+,

$$H : f^- \simeq f^+,$$

is given by a pair (H, f^+) where H is a sequence of maps

$$H_n : C_n \to D_{n+1}, \quad n \geqslant 0$$

which satisfy the following properties:

1. For $c \in C_n, n \geqslant 0$,

$$H_n(c) \in \begin{cases} D_1(f^-(c), f^+(c)) & \text{if } n = 0, \\ D_{n+1}(tf^+(c)) & \text{if } n > 0. \end{cases} \tag{i}$$

2. If $c, c' \in C_n$ and cc' or $c + c'$ is defined according as $n = 1$ or $n \geqslant 2$ respectively then:

$$H_1(cc') = H_1(c)^{f^+c'} H_1(c') \quad \text{if } n = 1, \tag{iia}$$
$$H_n(c + c') = H_n(c) + H_n(c') \quad \text{if } n \geqslant 2. \tag{iib}$$

Thus H_1 is a derivation over f^+ (see Remark 7.1.42 for more details) and for $n \geqslant 2$ H_n is linear.

3. For $n \geqslant 2$, H_n preserves the action over f_1^+, i.e.

$$H_n(c^{c_1}) = (H_n c)^{f_1^+ c_1} \quad \text{if } c \in C_n, n \geqslant 2,\ c_1 \in C_1, \text{ and } c^{c_1} \text{ is defined.} \tag{iii}$$

4. The pair (H, f^+) determines the initial morphism f^-; if $c \in C_n, n \geqslant 0$ then:

$$f^-(c) = \begin{cases} sH_0(c) & \text{if } n = 0, \\ (H_0sc)(f^+c)(\delta_2 H_1 c)(H_0 tc)^{-1} & \text{if } n = 1, \\ \{f^+c + H_{n-1}\delta_n c + \delta_{n+1} H_n c\}^{(H_0 tc)^{-1}} & \text{if } n \geqslant 2. \end{cases} \tag{iv}$$

□

It is useful to visualise (iv) for the case $c : x \to y$ in C_1 in terms of the diagram:[101]

$$\begin{tikzcd} f^- x \arrow[r, "H_0 x"] \arrow[d, "f^- c"'] & f^+ x \arrow[d, "f^+ c"] \\ f^- y \arrow[r, "H_0 y"'] & f^+ y \arrow[loop, out=330, in=270, distance=2em, "\delta_2 H_1 c"] \end{tikzcd}$$

Exercise 7.1.39. Prove that if (H, f^+) is a homotopy and we define f^- using (iv), then f^- is a morphism of crossed modules. □

Remark 7.1.40. An important observation which we will use later is that if f^-, f^+ are given and $c \in C_n$ then $\delta_{n+1} H_n(c)$ is determined by $H_0 tc$ and $H_{n-1}\delta_n(c)$. This is a key to later inductive constructions of homotopies. □

Exercise 7.1.41. Prove directly from this definition that homotopy of morphisms is an equivalence relation. □

Remark 7.1.42. Let us expand a bit on the fact that H_1 is an f^+-*derivation*. Note that C_1 operates on D_2 via f^+ and so we can form the semidirect product groupoid $C_1 \ltimes D_2$ with projection pr_1 to C_1. This groupoid has objects C_0 and arrows pairs $(c, d) \in C_1 \times D_2$, such that $f_0^+\delta_1(c) = t(d)$, with composition $(c, d)(c', d') = (cc', d^{f^+c'}d')$. This can be seen in the picture:

$$\begin{array}{ccccc} & & \bullet d' & & \\ & \bullet d & \bullet d^{f^+c'} & = & \bullet d^{f^+c'}d' \\ \bullet \xrightarrow{\ c\ } & \bullet \xrightarrow{\ c'\ } & \bullet & & \bullet \xrightarrow{\ cc'\ } \bullet \end{array}$$

It is then easily seen that an f^+-derivation H_1 is determined completely by a morphism $H_1' \colon C_1 \to C_1 \ltimes D_2$ such that $\mathrm{pr}_1 H_1' = 1_{C_1}$. A corollary is that if C_1 is a free groupoid, then an f^+-derivation is completely determined by its values on a set of free generators of C_1. □

Remark 7.1.43. Notice that the definition of the derivation has been on the left. Sometimes 'right derivations' are useful governed by the rule

$$H_1(cc') = H_1(c')H_1(c)^{f_1^+(c')}. \qquad \square$$

Example 7.1.44 (Contracting homotopies). From the above we can deduce formulae for a contraction. Suppose then in the above formulae we take $C = D$, $f^- = 1_C$, $f = 0$ where 0 denotes the constant morphism on C mapping everything to a base point 0 and the corresponding identities. Then the homotopy $H : 1 \simeq 0$ must satisfy:

$$\begin{aligned}
H_0 c &\in C_1(c, 0) && \text{if } c \in C_0, && \text{(ri)}\\
H_1(cc') &= H_1(c)H_1(c') && \text{if } c, c' \in C_1, \text{ and } cc' \text{ is defined}, && \text{(rii)}\\
H_n(c_1^c) &= H_n(c_1) && \text{if } c_1 \in C_n, c \in C_1, \text{ and } c_1^c \text{ is defined}, && \text{(riii)}\\
\delta_2 H_1 c &= (H_0 sc)^{-1} c\,(H_0 tc) && \text{if } c \in C_1, && \text{(riva)}\\
\delta_{n+1} H_n c &= -H_{n-1}\delta_n c + c^{H_0 tc} && && \\
&= c^{H_0 tc} - H_{n-1}\delta_n c && \text{if } c \in C_n, n \geqslant 2. && \text{(rivb)}
\end{aligned}$$

The reason for the last equality is that the groups $C_n(x)$ are abelian for $n \geqslant 3$ while for $n = 2$ the image of δ_3 is central in C_2. All these conditions (ri)–(riv) are necessary and sufficient for $(H, 0)$ to be a contracting homotopy. □

Exercise 7.1.45. Define the notion of homotopy equivalence $f : C \to D$ of crossed complexes. Recall that a morphism $f : C \to D$ of crossed complexes induces morphisms of the fundamental groupoids and homology groups. Prove that a homotopy equivalence of crossed complexes induces an equivalence of fundamental groupoids. What can you say about the induced morphism of homology groups? □

In analogy with the relation between groups and connected groupoids discussed in Section 1.7, we now give the relation between a connected crossed complex and its reduced version. However, as for groupoids, this is not an argument for considering only the reduced case, and the value of the general case will appear as we proceed (HHSvKT, monoidal closed structure, covering crossed complexes, . . .).

A connected groupoid G is known to be isomorphic to $G(x_0) * T$ where $x_0 \in \mathrm{Ob}\, G$ and T is a wide tree subgroupoid of G, [Bro06], 8.1.5. See also Equation (1.7.1). Further, T determines a strong deformation retraction $G \to G(x_0)$. We now show the same applies to crossed complexes. We extend the term 'wide tree subgroupoid' to 'wide tree subcrossed complex' of C, namely a wide subcrossed complex T of C such that T_1 is a wide tree sub groupoid of C_1, and $T_n(x)$ is trivial for all $x \in T_0$ and $n \geqslant 2$. The final part of the following proposition generalises [Bro06], 6.7.3, and is related to Proposition 1.7.1.

Proposition 7.1.46. *Let C be a connected crossed complex, let $x_0 \in C_0$ and let T be a wide tree subcrossed complex of C. Let $C(x_0)$ be the subcrossed complex of C at the base point x_0. Then the natural morphism*

$$\phi\colon C(x_0) * T \to C$$

determined by the inclusions is an isomorphism, and T determines a strong deformation retraction

$$r\colon C \to C(x_0).$$

Further, if $f\colon C \to D$ is a morphism of crossed complexes which is the identity on $C_0 \to D_0$ then we can find a retraction $s\colon D \to D(x_0)$ giving rise to a pushout square

$$\begin{tikzcd} C \arrow[r, "r"] \arrow[d, "f"'] & C(x_0) \arrow[d, "f'"] \\ D \arrow[r, "s"'] & D(x_0) \end{tikzcd} \tag{7.1.1}$$

in which f' is the restriction of f.

Proof. Let $i\colon C(x_0) \to C$, $j\colon T \to C$ be the inclusions. We verify the universal property of the free product. Let $\alpha\colon C(x_0) \to E$, $\beta\colon T \to E$ be morphisms of crossed complexes agreeing on x_0. Suppose $g\colon C \to E$ satisfies $gi = a$, $gj = b$. Then g is determined on C_0. Let $c \in C_1(x, y)$. Then

$$c = (\tau x)((\tau x)^{-1}c(\tau y))(\tau y)^{-1} \tag{$*$}$$

and so

$$\begin{aligned} gc &= g(\tau x)g((\tau x)^{-1}c(\tau y))g(\tau y)^{-1} \\ &= \beta(\tau x)\alpha((\tau x)^{-1}c(\tau y))\beta(\tau y)^{-1}. \end{aligned}$$

If $c \in C_n(x), n \geqslant 2$, then

$$c = (c^{\tau x})^{(\tau x)^{-1}} \qquad (**)$$

and so

$$g(c) = \alpha(c^{\tau x})^{\beta(\tau x)^{-1}}.$$

This proves uniqueness of g, and conversely one checks that this formula defines a morphism g as required.

In effect, equations $(*)$ and $(**)$ give for the elements of C normal forms in terms of elements of $C(x_0)$ and of T.

This isomorphism and the constant map $T \to \{x_0\}$ determine the strong deformation retraction $r\colon C \to C(x_0)$.

The retraction s is defined by the elements $f\tau(x), x \in C_0$, and then the diagram (7.1.1) is a pushout since it is a retract of the pushout square

$$\begin{tikzcd} C \arrow[r, "1"] \arrow[d, "f"'] & C \arrow[d, "f"] \\ D \arrow[r, "1"'] & D. \end{tikzcd}$$

□

7.2 Colimits of crossed complexes

The HHSvKT Theorem 8.1.5 in the next chapter states that the functor $\Pi\colon \mathsf{FTop} \to \mathsf{Crs}$ preserves certain colimits. The proof, which we give in Part III, does not require knowledge of the existence of colimits in the category Crs. It is true that these colimits exist: this follows from general facts on algebraic theories which do not need to go into here.[102] However, in order to apply the HHSvKT we need to know, not that colimits exist in general, but how to compute colimits of crossed complexes in more familiar terms and in specific situations.

It is an important feature of the HHSvKT that it deals with algebraic structures with structure in a range of dimension. This enables the theorem to give information on how low dimensional identifications influence higher dimensional homotopy information. For groupoids, using this information requires study of the functor $\mathrm{Ob}\colon \mathsf{Gpds} \to \mathsf{Set}$, and this is done in Appendix B using notions of fibration and cofibration of categories.

In this section, we carry out a similar study on crossed complexes, using the idea that crossed complexes have algebraic structure in a range of dimensions. The particular functors of truncation, skeleton, coskeleton, cotruncation, have importance not only for particular calculations of colimits of crossed complexes, but also for the theoretical studies of Part III on the equivalence of crossed complexes and ω-groupoids, whose utility is at the heart of this book, even if at this stage in a way which may be mysterious.

We can now easily show that the determination of colimits in Crs can be reduced to the determination of colimits in:

(i) the category Gpds of groupoids;
(ii) the category XMod of crossed modules over groupoids, and
(iii) the category Mod of modules over groupoids.

That is explained in Section 7.2.i.

In order to describe colimits in groupoids, crossed modules, and modules, it is convenient to use the language of 'fibred and cofibred categories', also called 'fibrations and cofibrations'. These notions are developed in Appendix B, and we assume that language. The construction of colimits of connected diagrams in either of Mod and XMod may be done in two steps. First, we change the base groupoids of the modules of a diagram so that they become the same for all modules or crossed modules and then we take the colimit in Mod_G or XMod_G. This is proved in Sections 7.2.ii and 7.2.iii. We leave to the reader to analyse the 'universal groupoids' $U_\sigma G$ obtained from a groupoid G and a set function $\sigma\colon \operatorname{Ob} G \to Y$ in the light of the theory of cofibrations of categories.

7.2.i Computation of colimits of crossed complexes dimensionwise

We recapitulate how colimits of crossed complexes may be computed piecewise, i.e. in terms of the truncation functors defined in Section 7.1.vi.

Proposition 7.2.1. *Let $C = \operatorname{colim} C^\lambda$ be a colimit in the category Crs of crossed complexes. Then*

(i) *for $n = 1$, the groupoid $\operatorname{tr}_1 C = C_1$ is the colimit in Gpds of the groupoids $\operatorname{tr}_1 C^\lambda = C_1^\lambda$, i.e.*
$$C_1 = \operatorname{colim}_{\mathsf{Gpds}} C_1^\lambda;$$

(ii) *also for $n = 1$, the groupoid $\pi_1 C$ is the colimit in Gpds of the groupoids $\pi_1 C^\lambda$, i.e.*
$$\pi_1 C = \operatorname{colim}_{\mathsf{Gpds}} \pi_1 C^\lambda;$$

(iii) *for $n = 2$, the crossed complex $\operatorname{tr}_2 C$ is the colimit in XMod of the crossed modules $\operatorname{tr}_2 C^\lambda$, i.e.*
$$(C_2 \to C_1) = \operatorname{colim}_{\mathsf{XMod}} (C_2^\lambda \to C_1^\lambda);$$

(iv) *for each $n \geqslant 3$, the groupoid C_n as a module over $\pi_1(C)$ is the colimit in the category Mod of the groupoids C_n^λ as modules over $\pi_1(C_n^\lambda)$, i.e.*
$$(C_n, \pi_1(C)) = \operatorname{colim}_{\mathsf{Mod}} (C_n^\lambda, \pi_1(C^\lambda)).$$

Proof. All these facts hold because the functors appropriate to each case have left adjoints and, consequently, they preserve colimits:

(i) follows since the truncation functor $\mathrm{tr}_1\colon \mathsf{Crs} \to \mathsf{Gpds}$ preserves colimits because the coskeleton functor $\mathrm{cosk}_1\colon \mathsf{Gpds} \to \mathsf{Crs}$ is its left adjoint (see Proposition 7.1.35).

(ii) follows from (i) because π_1 preserves colimits since it has a left adjoint (see Corollary 7.1.37).

(iii) follows since the truncation functor $\mathrm{tr}_2\colon \mathsf{Crs} \to \mathsf{XMod}$ preserves colimits because the coskeleton functor $\mathrm{cosk}_2\colon \mathsf{XMod} \to \mathsf{Crs}$ is its left adjoint (see Proposition 7.1.35).

(iv) follows since the restriction functor $\mathrm{res}_n\colon \mathsf{Crs} \to \mathsf{Mod}$ preserves colimits because the functor $\mathbb{F}_{n+1}\colon \mathsf{Mod} \to \mathsf{Crs}$ is its left adjoint (see Proposition 7.1.14). $\square$

The previous description gives not only the groupoid, crossed module and modules, but also the boundary maps $\delta\colon C_n \to C_{n-1}$; these can be recovered as induced by the maps $\delta^\lambda\colon C_n^\lambda \to C_{n-1}^\lambda$, for all λ.

7.2.ii Groupoid modules bifibred over groupoids

We have a forgetful functor $\Phi_{\mathsf{M}}\colon \mathsf{Mod} \to \mathsf{Gpds}$ in which $(M, G) \mapsto G$.

Proposition 7.2.2. *The forgetful functor* $\Phi_{\mathsf{M}}\colon \mathsf{Mod} \to \mathsf{Gpds}$ *has a left adjoint and is fibred and cofibred.*

Proof. The left adjoint of Φ_{M} assigns to a groupoid G the module written $0 \to G$ which has only the trivial group over each $x \in G_0$.

Next, we give the pullback construction to prove that Φ_{M} is fibred. This is entirely analogous to the group case, but taking account of the geometry of the groupoid.

Let $\theta\colon G \to H$ be a morphism of groupoids,

First let N be an H-module. We construct $(M, G) = (\theta^* N, H)$ as follows. For $x \in G_0$ we set $M(x) = \{x\} \times N(\theta x)$ with addition given by that in $N(\theta x)$. The operation is given by $(x, n)^g = (y, n^{\theta g})$ for $g \in G(x, y)$.

Second, let M be a G-module. We construct $(N, H) = (\theta_* M, G)$ as follows.

For $z \in H_0$ we let $N(z)$ be the abelian group generated by pairs (m, h) with $m \in M, h \in H$, and $t(h) = z, s(h) = \theta(t(m))$, so that $N(z) = 0$ if no such pairs exist. The operation of H on N is given by $(m, h)^{h'} = (m, hh')$, addition is $(m, h) + (m', h) = (m + m', h)$ and the relations imposed are $(m^g, h) = (m, \theta(g)h)$ when these make sense. The cocartesian morphism over θ is given by $\psi\colon m \mapsto (m, 1_{\theta t(m)})$. $\square$

Corollary 7.2.3. *Let* $(M, G) = \operatorname{colim}(M^\lambda, G^\lambda)$ *be a colimit in the category* Mod *of modules. Make the following two stage construction:*

(i) *First calculate* $G = \operatorname{colim} G^\lambda$ *with canonical morphisms* $i_\lambda\colon G^\lambda \to G$;

(ii) *for each* λ *form the induced module* $((\pi_1 i_\lambda)_* M^\lambda, G)$, *where* $\pi_1(i_\lambda)\colon \pi_1 G^\lambda \to \pi_1 G$ *is the morphism induced by* i_λ.

Then the colimit in Mod_G *of* $(\pi_1(i_\lambda)_* M^\lambda, G)$ *is isomorphic to* (M, G).

Proof. This is just a restatement of Theorem B.3.4 □

Remark 7.2.4. The relation between a module over a groupoid and the restriction to the vertex groups is discussed in Theorem 7.1.46 in the general context of crossed complexes. However it is useful to give the general situation of many base points to describe the relative homotopy group $\pi_n(X, A, a_0)$ when X is obtained from A by adding n-cells at various base points. The natural invariant to consider is then $\pi_n(X, A, A_0)$ where A_0 is an appropriate set of base points.

7.2.iii Crossed modules bifibred over groupoids

The category XMod of crossed modules over groupoids has already been defined in Section 6.2. We have a forgetful functor $\Phi_1\colon \mathsf{XMod} \to \mathsf{Gpds}$ in which sends a crossed module $(\mu\colon M \to P)$ to the base groupoid P.

Proposition 7.2.5. *The forgetful functor* $\Phi_1\colon \mathsf{XMod} \to \mathsf{Gpds}$ *is fibred and has a left adjoint.*

Proof. The left adjoint of Φ_1 assigns to a groupoid P the crossed module $0 \to P$ which has only the trivial group over each $x \in P_0$.

Next, we give the pullback construction to prove that Φ_1 is fibred. So let $\theta\colon P \to Q$ be a morphism of groupoids, and let $\nu\colon N \to Q$ be a crossed module. We define $M = \theta^*(N)$ as follows.

For $x \in P_0$ we set $M(x)$ to be the subgroup of $P(x)\times N(\theta x)$ of elements (p,n) such that $\theta p = \nu n$. If $p_1 \in P(x,x'), n \in N(\theta x)$ we set $(p,n)^{p_1} = (p_1^{-1} p p_1, n^{f(p_1)})$, and let $\mu\colon (p,n) \mapsto p$. We leave the reader to verify that this gives a crossed module, and that the morphism $(p,n) \mapsto n$ is cartesian. □

The following result in the case of crossed modules of groups appeared in Chapter 5, described in terms of the crossed module $\partial\colon \theta_*(M) \to Q$ *induced* from the crossed module $\mu\colon M \to P$ by a morphism $\theta\colon P \to Q$.

Proposition 7.2.6. *The forgetful functor* $\Phi_1\colon \mathsf{XMod} \to \mathsf{Gpds}$ *is cofibred.*

Proof. We prove this by a direct construction, generalising that given earlier.

Let $\mu\colon M \to P$ be a crossed module, and let $\theta\colon P \to Q$ be a morphism of groupoids. The construction of $N = \theta_*(M)$ and of $\partial\colon N \to Q$ requires just care to the geometry of the partial action in addition to the construction for the group case.

Let $y \in Q_0$. If there is no $q \in Q$ from a point of $\theta(P_0)$ to y, then we set $N(y)$ to be the trivial group.

Otherwise, we define $F(y)$ to be the free group on the set of pairs (m,q) such that $m \in M(x)$ for some $x \in P_0$ and $q \in Q(\theta x, y)$. If $q' \in Q(y,y')$ we set $(m,q)^{q'} = (m,qq')$. We define $\partial' \colon F(y) \to Q(y)$ to be $(m,q) \mapsto q^{-1}(\theta m)q$. This gives a precrossed module over $\partial \colon F \to Q$, with function $i \colon M \to F$ given by $m \mapsto (m,1)$ where if $m \in M(x)$ then 1 here is the identity in $Q(\theta x)$.

We now wish to change the function $i \colon M \to F$ to make it an operator morphism. For this, factor F out by the relations

$$\begin{aligned}(m,q)(m',q) &= (mm',q),\\ (m^p,q) &= (m,(fp)q),\end{aligned}$$

whenever these are defined, to give a projection $F \to F'$ and $i' \colon M \to F'$. As in the group case, we have to check that $\partial' \colon F \to Q$ induces $\partial'' \colon F' \to H$ making this a precrossed module. To make this a crossed module involves factoring out Peiffer commutators, whose theory is analogous to that for the group case given in Section 3.4. This gives a crossed module morphism $(\phi, f) \colon (M,P) \to (N,Q)$ which is cocartesian. □

Corollary 7.2.7. *Let* $(\mu \colon M \to G) = \operatorname{colim}(\mu_\lambda \colon M^\lambda \to G^\lambda)$ *be a colimit in the category* XMod *of crossed modules. Make the following two stage construction:*

(i) *First calculate* $G = \operatorname{colim} G^\lambda$,

(ii) *for each* λ *construct the crossed module* $(i_\lambda)_* M^\lambda \to G$ *induced by* $i_\lambda \colon G \to G^\lambda$.

Then the colimit in XMod_G *of the* $(i_\lambda)_* M^\lambda \to G)$ *is isomorphic to* $\mu \colon M \to G$.

Proof. This is just a restatement of Theorem B.3.4. □

7.3 Free constructions

In Part I we have used free groups, and studied free crossed modules over groups; free modules over a group are common knowledge. Now we generalise all this to the groupoid case, in order to arrive at the notion of free crossed complexes. These are important in their own right in algebra, and also in topology because they gives a useful algebraic model CW-complexes.

In particular, free constructions given here model the process of attaching cells to a space.

Attaching 1-cells to a discrete space gives graphs, with the well-known free groupoids as algebraic models. In higher dimensions, for a space A we may form $X = A \cup_{f_i} e_i^2$ where the cells e_i^2 are attached by maps $f_i \colon S^1 \to A$. We may take the base point of S^1 to be say 1, and set $a_i = f_i(1)$, $A_0 = \{a_i\}$. We then want to express $\pi_2(X,A,A_0)$ as a free crossed module over the fundamental groupoid $\pi_1(A,A_0)$. We also want to see, if A has itself a base point say a_0, how to calculate the crossed module of groups $\pi_2(X,A,a_0) \to \pi_1(A,a_0)$.

So we must extend the notions of free groupoid to the higher dimensions of free crossed modules and free crossed complexes. Because of the given geometric structure of crossed complexes, this extension is quite simple.

We emphasise here the adjoint functor approach to free constructions since this shows how we bring to these constructions the underlying geometry of these algebraic structures.

We assume the notion of free groupoid as discussed in Appendix B, Section B.6.

7.3.i Free modules over groupoids

A module (M, G) comes equipped with a function $\omega = s = t \colon M \to G_0$. We define a category $\mathsf{Set}_0/\mathsf{Gpds}$ whose objects consist of functions $\omega \colon R \to G_0$ where G is a groupoid and R is a set, and whose morphisms are pairs (f, θ) such that $f : R \to R'$ is a function and $\theta \colon G \to G'$ is a morphism of groupoids giving a commutative diagram

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "\omega"'] & R' \arrow[d, "\omega'"] \\ G_0 \arrow[r, "\theta_0"'] & G'_0. \end{tikzcd}$$

Proposition 7.3.1. *There is a forgetful functor $U : \mathsf{Mod} \to \mathsf{Set}_0/\mathsf{Gpds}$ which forgets the structure of abelian groups on the family M in a module (M, G). This functor has a left adjoint which gives the free module on a function $\omega \colon R \to G_0$ for a set R and groupoid G.*

Proof. The left adjoint F is constructed on objects as follows. Suppose given the groupoid G and function $\omega \colon R \to G_0$. For $x \in G_0$ let $F(\omega)(x)$ be the free abelian group on pairs (r, g) for $r \in R$ and $g \colon \omega(r) \to x$ if such exist, and otherwise is the zero abelian group. The operation of G is given by $(r, g)^h = (r, gh)$ provided gh is defined.

It is easily checked that F extends to a functor which is the left adjoint as required. □

Remark 7.3.2. The counit of this adjunction gives for $\omega \colon R \to G_0$ in $\mathsf{Set}_0/\mathsf{Gpds}$ a function $i : R \to F(\omega)$ which gives the *free basis* for $F(\omega)$ and has the usual universal property of a free basis. Suppose we are given an isomorphism $(f, 1) \colon (F(\omega), G) \cong (M, G)$ and $i' = fi \colon R \to M$. By abuse of language we say then that M is a free G-module with free basis i'. (Recall that the counit of the adjunction with its universal property determines the adjunction.) □

Remark 7.3.3. 1. As is standard for universal constructions, any two free G-modules over the same map ω are isomorphic.

2. If $i: R \to M$ is a free basis for M as G-module and $\theta: G \to H$ is a morphism of groupoids then $\theta_* M$ is a free H-module with basis $\bar{\theta} i: R \to \theta_* M$, where $\bar{\theta}: M \to \theta_* M$. □

Since for any map $t: R \to G_0$ we have essentially a unique free P-module, we can construct one using the induced module construction. First, we get a kind of 'universal free module' over R.

Proposition 7.3.4. *Let R be an indexing set, and consider the module $\mathbb{Z}R = (\mathbb{Z} \times R, R)$ over the discrete groupoid R. The map $R \to \mathbb{Z} \times R$ mapping r to $(1, r)$ is a free basis of $\mathbb{Z}R$ as R-module.*

The proof is easy. Now we apply the induced module construction.

Corollary 7.3.5. *Let R be a set, regarded also as a discrete groupoid. Let P be a groupoid and $\omega: R \to P_0$ a function, defining also a groupoid morphism $\theta: R \to P$. Consider the module $\mathbb{Z}R = (\mathbb{Z} \times R, R)$ over the discrete groupoid R. Then the induced P-module*

$$(FM(\omega), P) = (\theta_* \mathbb{Z}R, P)$$

is free.

Proof. This is direct since the induced module construction preserves freeness. □

Example 7.3.6. Suppose R is the singleton $\{r\}$, $i(r) \in M(x)$ and M is the free P-module on i. Then $M(x)$ will be isomorphic to the free abelian group on elements r^q for all $q \in P(x)$, while if $y \in P_0$ then $M(y)$ will be the free abelian group on elements r^q for all $q \in P(x, y)$. In words, the contribution from r gets spread around the other objects of P by the action of P. You should now be easily able to see what happens if R has more than one element. □

We now move on to free crossed modules over groupoids. They generalise to groupoids the construction of free crossed modules over groups of Section 3.4.

7.3.ii Free crossed modules over groupoids

The geometry of crossed modules over groupoids involves a category we write $\mathsf{Set/Gpds}$ whose objects consist of a groupoid G and a function $\omega: R \to G$ such that $s\omega = t\omega: R \to G_0$, and whose morphisms are pairs (f, θ) such that $f: R \to R'$ is a function and $\theta: G \to G'$ is a morphism of groupoids giving a commutative diagram

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "\omega"'] & R' \arrow[d, "\omega'"] \\ G \arrow[r, "\theta"'] & G'. \end{tikzcd}$$

Proposition 7.3.7. *There is a forgetful functor $U : \mathsf{XMod} \to \mathsf{Set}/\mathsf{Gpds}$ which forgets the algebraic structures on the family M in a crossed module $\mu\colon M \to G$. This functor has a left adjoint FX which gives the free crossed module on a function $\omega\colon R \to G$ such that $s\omega = t\omega$ for a set R and groupoid G.*

Proof. The left adjoint FX is constructed on objects as follows. Suppose given the groupoid G and function $\omega\colon R \to G$ such that $s\omega = t\omega$. For $x \in G_0$ let $PFX(\omega)(x)$ be the free group on pairs (r, g) for $r \in R$ and $g\colon \omega(r) \to x$ in G if such exist, and otherwise is the trivial group. An operation of G on $PFX(\omega)$ is given on the generators by $(r, g)^h = (r, gh)$ provided gh is defined. An operator morphism $\theta\colon PFX(\omega) \to G$ is defined as usual by

$$\theta(r, g) = g^{-1}\omega(r)g.$$

This gives a precrossed module and the free crossed G-module on ω is defined to be

$$FX(\omega) = (PFX(\omega))^{\mathrm{cr}},$$

the associated crossed module obtained by factoring by the Peiffer elements.

It is easily checked that FX extends to a functor which is the left adjoint as required. □

Remark 7.3.8. 1. As usual in the universal constructions, any two free G-crossed modules over the same map ω are isomorphic.

2. If $i : R \to M$ is a free basis for $\mathcal{M} = (\mu\colon M \to G)$ as crossed G-module and $\theta\colon G \to H$ is a morphism of groupoids then $\theta_*\mathcal{M}$ is a free H-crossed module with basis the composition

$$R \xrightarrow{i} M \xrightarrow{\bar\theta} \theta_* M$$

where $(\bar\theta, \theta)\colon (M \to P) \to (\theta_* M \to H)$. □

Exercise 7.3.9. 1) Prove that an induced crossed module of a free crossed module is also free.

2) Generalise the construction in Chapter 4 to the case of crossed modules over groupoids, and prove that a coproduct of free crossed P-modules, where P is a groupoid, is also free. □

There is for Proposition 5.2.3 a groupoid version which is also a crossed module version of Proposition 7.3.4. First we point out:

Proposition 7.3.10. *Let R be a set, considered as discrete groupoid, let $R \times \mathbb{F}(2)$ be the disjoint union of copies of the crossed module $\mathbb{F}(2)$, considered as a crossed module of groupoids. This is a free crossed module with free basis the map $R \to R \times \mathbb{F}(2)_2$, $r \mapsto (r, 1)$.*

Now, we apply the induced crossed module construction.

Corollary 7.3.11. *Let R be an indexing set, G a groupoid and $\omega\colon R \to G$ a function such that $s\omega = t\omega$. Let $\theta\colon R \times \mathbb{F}(2)_1 \to G$ be the groupoid morphism determined by $(r, 1) \mapsto \omega(r)$. Then the induced crossed module*

$$\theta_*(R \times \mathbb{F}(2)_2)$$

is isomorphic to the free crossed P-module on ω.

Proof. This is clear since the induced module construction preserves freeness. □

Example 7.3.12. If the image of the function $\omega\colon R \to G$ consists only of identities, then the free crossed module on ω is just a free module. □

7.3.iii Free crossed complexes

We will first define the term *crossed complex of free type*, but then later abbreviate this with some abuse of language to *free crossed complex*.

Crossed complexes of free type model algebraically the topological notion of inductively attaching cells, as in relative CW-complexes.

Definition 7.3.13. A crossed complex C is *of free type* if:

– the groupoid C_1 is a free groupoid;

– the crossed module $\partial_2\colon C_2 \to C_1$ is a free crossed C_1-module; and

– for all $n \geqslant 3$, C_n is a free module over the groupoid $\pi_1 C$. □

Note that this includes the discrete crossed complex on a set S, which is just the disjoint union over S of trivial crossed complexes, and is also written S, as being of free type.

It is also convenient to see a crossed complex C of free type as built inductively from the discrete crossed complex on C_0 analogously to the construction of CW-complexes. The building blocks for C are the crossed complexes $\mathbb{F}(n)$ and $\mathbb{S}(n)$ of Definition 7.1.12, which are analogous to the cells and spheres of topology. We recall the definitions:

The crossed complex versions of 'n-cells', written $\mathbb{F}(n)$, are:

$\mathbb{F}(0)$ the trivial crossed complex with base point 0;

$\mathbb{F}(1) = \mathcal{I}$,

$\mathbb{F}(2)_i = \mathbb{Z}$ for $i = 1, 2$ with boundary the identity, and is otherwise trivial,

while for $n \geqslant 3$

$$\mathbb{F}(n)\colon = \quad \cdots \longrightarrow 0 \longrightarrow \underset{n}{\mathbb{Z}} \xrightarrow{1_{\mathbb{Z}}} \underset{n-1}{\mathbb{Z}} \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \underset{1}{1}\ .$$

The crossed complex versions of the 'n-spheres', written $\mathbb{S}(n)$, are:

$\mathbb{S}(0)$ is the discrete crossed complex on $\{0, 1\}$;

$\mathbb{S}(1) = \mathbb{K}(\mathbb{Z}, 1)$,

while for $n \geqslant 2$

$$\mathbb{S}(n) : = \quad \cdots \longrightarrow 0 \longrightarrow \underset{n}{\mathbb{Z}} \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \underset{1}{1}\ .$$

The crossed complexes $\mathbb{F}(n)$ are *freely generated by one generator c_n in dimension* n, and so satisfy the following property:

Proposition 7.3.14. *For any crossed complex C and any element $c \in C_n$ there is a unique morphism of crossed modules*

$$\hat{c} : \mathbb{F}(n) \to C$$

such that $\hat{c}(c_n) = c$. That is, there is a natural bijection of sets $C_n \cong \mathsf{Crs}(\mathbb{F}(n), C)$.

Remark 7.3.15. It is a straightforward consequence of convexity of the interval E^1 that $\Pi(E^1_*) \cong \mathbb{F}(1)$. That $\Pi(\mathbf{S}^1_*) \cong \mathbb{S}(1)$ follows from the fact that the fundamental group of S^1 is isomorphic to $\mathbb{Z}$, as has been proved in Section 1.7. It will be proved in Corollary 8.3.11 in the next chapter that $\Pi(S^n_*) \cong \mathbb{S}(n)$ for $n \geqslant 1$ and it follows from this that $\Pi(E^n_*) \cong \mathbb{F}(n)$. □

The crossed complex $\mathbb{F}(n)$ allows us to define the notion of 'adding to a crossed complex free generators in dimension n'.

Definition 7.3.16. Let A be a crossed complex. We say that a morphism of crossed complexes $j : A \to C$ is of *pure relative free type of dimension* $n \geqslant 0$ if there is a set of indexes Λ and a family of morphisms $f^\lambda : \mathbb{S}(n-1) \to A$ for $\lambda \in \Lambda$, such that the following square is a pushout in Crs:

$$\begin{tikzcd}
\bigsqcup_{\lambda\in\Lambda} \mathbb{S}(n-1) \arrow[r, "(f^\lambda)"] \arrow[d] & A \arrow[d, "j"] \\
\bigsqcup_{\lambda\in\Lambda} \mathbb{F}(n) \arrow[r] & C\,.
\end{tikzcd}$$

We write

$$C = A \cup \{x^n_\lambda\}_{\lambda\in\Lambda},$$

and may abbreviate this in some cases, for example to $C = A \cup x^n$. □

Remark 7.3.17. Despite the transparent formal property of adding generators, in dimensions > 0 the detailed effect on the algebraic structures of adding generators can be quite subtle, as is shown by the origins of the work on free crossed modules, and a considerable literature on combinatorial group theory.[103] □

We consider now the situation when the free generators are added in increasing order of dimension. In the limit we get a special kind of 'subcomplexes' $j : A \to C$ which we shall call a *crossed complex morphism of relative free type*.

Definition 7.3.18. Let A be any crossed complex. We define a sequence of complexes C^n and morphisms $j_n : C^{n-1} \to C^n$ starting with $C^0 = A$, and choosing a family of morphisms $f_n^\lambda : \mathbb{S}(n-1) \to C^{n-1}$ for $\lambda \in \Lambda_n$ such that C_n is got by forming the pushout

$$\begin{tikzcd} \bigsqcup_{\lambda \in \Lambda_n} \mathbb{S}(n-1) \arrow[r, "(f_n^\lambda)"] \arrow[d] & C^{n-1} \arrow[d, "j_n"] \\ \bigsqcup_{\lambda \in \Lambda_n} \mathbb{F}(n) \arrow[r] & C^n. \end{tikzcd}$$

Let $C = \operatorname{colim} C^n$, and let $j : A \to C$ be the canonical morphism. We call $j : A \to C$ a *crossed complex morphism of relative free type*, and also say j is *relatively free* . The images x^n of the elements c_n in C are called *basis elements of C relative to A*. We can conveniently write

$$C = A \cup \{x^n\}_{\lambda \in \Lambda_n, n \geqslant 0},$$

and may abbreviate this in some cases, for example to $C = A \cup x^n \cup x^m$, analogously to standard notation for CW-complexes. □

Example 7.3.19. It will be a corollary of the HHSvKT in the next chapter that for the skeletal filtration X_* of a CW-complex X, the crossed complex ΠX_* is free; and that if Y_* is a subcomplex of X_* then the induced morphism $\Pi Y_* \to \Pi X_*$ is relatively free. □

Of course, the advantage of a having a free basis X_* for a crossed complex C is that a morphism $f : C \to D$ is defined completely by the values of f on X_* provided the following conditions are satisfied:

(i) the values have the appropriate source and target, i.e. $sf_1 x = f_0 sx$ and $tf_1 x = f_0 tx$, for all $x \in X_1$, and $tf_n(x) = f_0(tx)$ for all $x \in X_n$, $n \geqslant 2$.

(ii) the values produce a morphism of crossed complexes, i.e. $\delta_n f_n(x) = f_{n-1}\delta_n(x)$, $x \in X_n$, $n \geqslant 2$.

Notice that in (ii), f_{n-1} has to be constructed on all of C_{n-1} from its values on the basis for C_{n-1}, before this condition can be verified.

If further D is free, then to specify $f_n(x)$ we simply have to give the expression of $f_n(x)$ in terms of the basis in dimension n for D_n.

Later we will see that homotopies can be specified similarly (see Corollary 9.6.6).

We end this section by stating some results on the preservation of relatively free morphisms of crossed complexes under composition, pushouts, and sequential colimits. Their proofs go by checking for the case when all generators have the same dimension and then for the general case by a colimit argument. These results are used in Section 9.6.

Proposition 7.3.20. *If $f : A \to B$ and $g : B \to C$ are morphisms of relative free type, so also is their composite $gf : A \to C$.*

Proposition 7.3.21. *If in a pushout square*

$$\begin{tikzcd} C' \arrow[r] \arrow[d] & D' \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

the morphism $C' \to C$ is of relative free type, so is the morphism $D' \to D$.

Proposition 7.3.22. *If in a commutative diagram*

$$\begin{tikzcd} C^0 \arrow[r] \arrow[d] & C^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & C^n \arrow[r] \arrow[d] & \cdots \\ D^0 \arrow[r] & D^1 \arrow[r] & \cdots \arrow[r] & D^n \arrow[r] & \cdots \end{tikzcd}$$

each vertical morphism is of relative free type, so is the induced morphism

$$\operatorname{colim}_n C^n \to \operatorname{colim}_n D^n.$$

In particular:

Corollary 7.3.23. *If in a sequence of morphisms of crossed complexes*

$$C^0 \to C^1 \to \cdots \to C^n \to \cdots$$

each morphism is of relative free type, then so are the composites $C^0 \to C^n$ and the induced morphism $C^0 \to \operatorname{colim}_n C^n$.

7.4 Crossed complexes and chain complexes

As we have seen in Section 7.1.iii, a crossed complex has some features of a kind of nonabelian chain complex with operators, the nonabelian features being confined to dimensions $\leqslant 2$. In this section, we begin to make the relation between the two kinds of complexes more precise. The adjoint constructions will be used later in Section 9.5 to help understand tensor products of crossed complexes.[104]

Definition 7.4.1. Let G be a groupoid. A *chain complex* $A = (A_n, \partial_n)_{n \geqslant 0}$ over G is a sequence

$$\cdots \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0$$

of G-modules and G-morphisms satisfying $\partial\partial = 0$. A *morphism of chain complexes* $(f, \theta) \colon (A, G) \to (B, H)$ is a family of morphisms $(f_n, \theta) \colon (A_n, G) \to (B_n, H)$ (over some $\theta \colon G \to H$) satisfying $\partial_n f_n = f_{n-1}\partial_n$. These form a category Chn and, for a fixed groupoid G, we have a subcategory Chn_G of chain complexes over G. The category Chn contains the full subcategory $\mathsf{Chn}_{\mathrm{red}}$ of chain complexes over groups. □

Our aim now is to construct a functor

$$\nabla \colon \mathsf{Crs} \to \mathsf{Chn}$$

which gives a kind of 'semiabelianisation' of the crossed module part of a crossed complex C, keeping information on the fundamental groupoid of C. It will be important later that this functor ∇ has a right adjoint (called Θ). We will use this adjoint pair in later chapters to investigate the tensor product of crossed complexes, and the homotopy classification of maps from a free crossed complex.

The definition of ∇ is easy in dimensions $\geqslant 3$, when we set $(\nabla C)_n = C_n$, with boundary ∂ just δ: we shall leave everything as it is where allowed. We are left with changing

$$C_3 \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1$$

where δ_2 is a crossed module with cokernel $\phi \colon C_1 \to G$ to get

$$C_3 \xrightarrow{\partial_3} A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0$$

where A_2, A_1 and A_0 are G-modules. We can use ϕ to associate to C_2 the G-module $A_2 = C_2^{\mathrm{ab}}$. It is more difficult to get the correct candidates for A_1 and A_0, but again they crucially involve ϕ.

In the first subsection, we study the candidates for A_0, the 'adjoint module' and the 'augmentation module' and prove that they give functors which have right adjoints. In Section 7.4.ii we study the candidate for A_1, the 'derived module'. A big advantage of working with the category Mod (which includes modules over all groupoids) is that we can exploit the formal properties of the functorial constructions used.

In the penultimate subsection we give the right adjoint Θ of $\nabla \colon \mathsf{Crs} \to \mathsf{Chn}$, and the last subsection illustrates with a specific calculation the fact that ∇ preserves colimits.

7.4.i Adjoint module and augmentation module

Basic constructions used to linearise the theory of groups in homological algebra are, for a group G, the group ring $\mathbb{Z}G$ and the augmentation ideal IG. We extend these constructions to the case of groupoids: however for a groupoid G we obtain not a 'groupoid ring' but what we call the 'adjoint module' $\vec{\mathbb{Z}}G$, and from this we get the 'augmentation module' $\vec{I}G$. We use the distinctive notation for the groupoid case, even though if G is a group then the constructions of $\mathbb{Z}G$ and $\vec{\mathbb{Z}}G$, IG and $\vec{I}G$, coincide: one reason is that they denote different structures, and another is that there is a second generalisation to a groupoid G of the usual group ring of a group, in which we obtain a 'ring with several objects' $\mathbb{Z}G$ which is an additive category with objects the same as those of G and in which $\mathbb{Z}G(p,q)$ is the free abelian group on $G(p,q)$.

Definition 7.4.2. Let G be a groupoid. For $q \in G_0$, we define $\vec{\mathbb{Z}}G(q)$ to be the free abelian group on the elements of G with target q. Thus an element has uniquely the form of a finite sum $\Sigma_i n_i g_i$ with $n_i \in \mathbb{Z}$ and $g_i \in G$ with $t(g_i) = q$.

Clearly, $\vec{\mathbb{Z}}G$ becomes a (right) G-module under the action

$$(a, g) \mapsto ag$$

of G on basis elements. Thus

$$\vec{\mathbb{Z}}G = \{\vec{\mathbb{Z}}G(q)\}_{q \in G_0}$$

is a G-module, which we call the *adjoint module* of G , since it involves the adjoint action of G on itself. This construction defines the functor

$$\vec{\mathbb{Z}}(-) \colon \mathsf{Gpds} \to \mathsf{Mod}. \qquad \Box$$

Notice that $\vec{\mathbb{Z}}G$ is 'G-free on G_0', i.e. it is freely generated as G-module by G_0 (embedded in $\vec{\mathbb{Z}}G$ as the set of identities of G).

There is also a generalisation to groupoids of constructions well known in the case of groups: the augmentation map and the augmentation ideal.

For a fixed groupoid G, is also useful to have a trivial G-module, corresponding to the integers $\mathbb{Z}$.

Definition 7.4.3. Let $\vec{\mathbb{Z}}$ be the (right) G-module consisting of the constant family $\vec{\mathbb{Z}}(p) = \mathbb{Z}$ for $p \in G_0$, with trivial action of G (which, as usual, means that each $g \in G(p,q)$ acts as the identity $\mathbb{Z}(p) \to \mathbb{Z}(q)$). We shall regard the $\mathbb{Z}(p)$ as distinct, so that $\vec{\mathbb{Z}}$ depends also on the object set G_0.

The *augmentation map*

$$\varepsilon \colon \vec{\mathbb{Z}}G \to \vec{\mathbb{Z}},$$

given by $\Sigma n_i g_i \mapsto \Sigma n_i$ is a morphism of G-modules and its kernel $\vec{I}G$ is called the (right) *augmentation module* of G.

Any morphism of groupoids $\theta \colon H \to G$ induces a module morphism $\vec{\mathbb{Z}}H \to \vec{\mathbb{Z}}G$ over θ which maps $\vec{I}H$ to $\vec{I}G$.

Since the augmentation map is natural, the augmentation module defines also a functor

$$\vec{I} \colon \mathsf{Gpds} \to \mathsf{Mod}. \qquad \Box$$

Exercise 7.4.4. Prove that for $q \in \operatorname{Ob} G$, the abelian group $\vec{I}G(q)$ has a $\mathbb{Z}$-basis consisting of all $g - 1_q$, for g a non-identity element of G with target q. $\Box$

We will prove that both $\vec{\mathbb{Z}}$ and $\vec{I}$ preserve colimits by giving right adjoints for them. That for $\vec{I}$ has a direct construction: the semidirect product.

Definition 7.4.5. Given a module (M, G), the *semidirect product* $G \ltimes M$ of G and M is the groupoid with the same set of objects as G, and

$$(G \ltimes M)(p, q) = G(p, q) \ltimes M(q),$$

i.e. as a set is $G(p, q) \times M(q)$ and the composition is given by

$$(x, m)(y, n) = (xy, m^y + n),$$

for $x \in G(p, q), y \in G(q, r)$, and $m \in M(q), n \in M(r)$. This semidirect product construction gives a functor

$$\ltimes \colon \mathsf{Mod} \to \mathsf{Gpds}. \qquad \Box$$

For the study of this, it is convenient to have a generalised notion of derivation, which will be used a lot later in connection with homotopies of morphisms of crossed complexes.

Definition 7.4.6. Let $\theta \colon H \to G$ be a morphism of groupoids, and let M be a G-module. A function $f \colon H \to M$ is called a θ-*derivation* if it maps $H(p, q)$ to $M(\theta q)$ and satisfies

$$f(xy) = (fx)^{\theta y} + fy$$

whenever xy is defined in H. In particular, if $H = G$, then a 1_G-derivation is called simply a *derivation*. $\Box$

Exercise 7.4.7. Let G be a groupoid. Prove that the mapping $\psi \colon G \to \vec{I}G$ sending $g \mapsto g - 1_{tg}$ is a derivation, and has the universal property: if $f \colon G \to N$ is a derivation to a G-module N, then there is a unique G-morphism $f' \colon \vec{I}G \to N$ such that $f'\psi = f$. $\Box$

Proposition 7.4.8. *The functor* $\ltimes \colon \mathsf{Mod} \to \mathsf{Gpds}$ *is a right adjoint of* $\vec{I} \colon \mathsf{Gpds} \to \mathsf{Mod}$. *Hence* $\vec{I}$ *preserves colimits.*

Proof. Let us begin by studying $\mathsf{Gpds}(H, G \ltimes M)$ for a groupoid H and module (M, G). A morphism

$$H \to G \ltimes M$$

is of the form $x \mapsto (\theta x, fx)$ where

$$\theta \colon H \to G$$

is a morphism of groupoids and

$$f : H \to M$$

is a θ-derivation. (In particular, all sections

$$G \to G \ltimes M$$

are of the form $x \mapsto (x, fx)$ where $f : G \to M$ is a derivation.)

By Exercise 7.4.7, the map $d : H \to \vec{I} H$, given by $d(x) = x - 1_q$ for $x \in H(p, q)$, is a universal derivation.

On the other hand, if $\theta : H \to G$ is a morphism of groupoids and M is a G-module, then any θ-derivation $f : H \to M$ is uniquely of the form $f = \hat{f}\psi$ where $\hat{f} : \vec{I} H \to M$ is a morphism of modules over θ. Thus we have a natural bijection

$$\mathsf{Mod}((\vec{I} H, H), (M, G)) \cong \mathsf{Gpds}(H, G \ltimes M). \qquad \square$$

The right adjoint to $\vec{\mathbb{Z}}$ comes from the pullback of a groupoid along a map defined in Example B.1.3 in conjunction with the adjoint module of a groupoid.

Definition 7.4.9. Given a module (M, G), we consider M as a set UM with the target map $t : UM \to \mathrm{Ob}\, G$. We may therefore form the pullback groupoid $P(M, G) = t^* G$. This construction gives a functor

$$P : \mathsf{Mod} \to \mathsf{Gpds}. \qquad \square$$

The groupoid $P(M, G)$, with its canonical morphism to G, $(m, g, n) \mapsto g$, is universal for morphisms $\theta : H \to G$ of groupoids such that $\mathrm{Ob}\, \theta$ factors through maps $\beta : M \to \mathrm{Ob}\, G$.

Proposition 7.4.10. *The functor* $P : \mathsf{Mod} \to \mathsf{Gpds}$ *is a right adjoint of* $\vec{\mathbb{Z}} : \mathsf{Gpds} \to \mathsf{Mod}$. *Hence* $\vec{\mathbb{Z}}$ *preserves colimits.*

Proof. By the definition of $P(M, G)$, the groupoid morphisms $H \to P(M, G)$ are naturally bijective with pairs (α, θ) where $\alpha : \mathrm{Ob}\, H \to UM$ is a map, $\theta : H \to G$ is a morphism and $\mathrm{Ob}\, \theta$ is of the form $\beta \circ \alpha$.

However, since $\vec{\mathbb{Z}} H$ is freely generated as H-module by H_0 (embedded in $\vec{\mathbb{Z}} H$ as the set of identities of H), such pairs (α, θ) are naturally bijective with morphisms of modules $(\gamma, \theta) : (\vec{\mathbb{Z}} H, H) \to (M, G)$. $\square$

These constructions are related as follows:

Proposition 7.4.11. *The inclusion,* $\vec{I} G \to \vec{\mathbb{Z}} G$, *regarded as a natural transformation, is conjugate under the above adjunction to the natural transformation* $\kappa = \kappa_{(M,G)}$ *where*

$$\kappa_{(M,G)} : P(M, G) \to G \ltimes M$$

is given by $\kappa(m, g, n) = (g, m^g - n)$. *For each module* (M, G), *this* $\kappa_{(M,G)}$ *is a covering morphism of groupoids.*

Proof. Any commutative triangle

$$
\begin{tikzcd}
(\vec{I}H, H) \arrow[rr, "i"] \arrow[dr, "(\alpha,\theta)"'] & & (\vec{\mathbb{Z}}H, H) \arrow[dl, "(\gamma,\theta)"] \\
& (M, G) &
\end{tikzcd}
$$

in Mod corresponds to a commutative triangle

$$
\begin{tikzcd}
G \ltimes M & & P(M, G) \arrow[ll, "\kappa"'] \\
& H \arrow[ul, "\xi"] \arrow[ur, "\eta"'] &
\end{tikzcd}
$$

in Gpds, where κ is natural and, if $h \in H(p, q)$, then

$$\xi h = (\theta h, \alpha(h - 1_q)) \quad \text{and} \quad \eta h = (\gamma 1_p, \theta h, \gamma 1_q).$$

Given $(m, g, n) \in P(M, G)$, we may take $G = H$, $\theta = \mathrm{id}$, and choose γ so that $\gamma 1_p = m$, $\gamma 1_q = n$. Then

$$
\begin{aligned}
\kappa(m, g, n) &= \xi g \\
&= (\theta g, \alpha(g - 1_q)) \\
&= (g, \gamma(g - 1_q)) \\
&= (g, \gamma(1_p h) - \gamma 1_q) \\
&= (g, m^g - n).
\end{aligned}
$$

Finally, let $(g, x) \in G \ltimes M$, with $g \in G(p, q)$ and $x \in M(q)$, and let $m \in M(p)$ be an object of $P(M, G)$ lying over the source p of (g, x). Then there is a unique $n \in M(q)$ such that $m^g - n = x$. Hence there is a unique arrow (m, g, n) over (g, x) with source n. □

Note that if one restricts attention to groups, and modules over groups, the restricted functor $\vec{\mathbb{Z}}(-)$ does not have a right adjoint since, for example, it converts the initial object 1 in the category of groups to the module $(\vec{\mathbb{Z}}, 1)$ which is not initial in the category of modules over groups. However, the functor $\vec{I}$, does, when restricted to groups, have a right adjoint given by the split extension as above.

7.4.ii The derived module

Another basic construction used to linearise the theory of groups in homological algebra is the derived module D_θ of a group morphism $\theta\colon H \to G$, usually appearing in the form $D_\theta = IH \otimes_H \mathbb{Z}G$. We extend this construction to the case of groupoids.

Definition 7.4.12. Let $\theta\colon H \to G$ be a morphism of groupoids. Its *derived module* is a G-module D_θ with a universal θ-derivation $h_\theta\colon H \to D_\theta$: that is, for any θ-derivation $f\colon H \to M$ to a G-module M, there is a unique G-morphism $f'\colon D_\theta \to M$ such that $f'h_\theta = f$. □

Exercise 7.4.13. Give a direct construction of the derived module as follows: for $q \in \operatorname{Ob} G$, let $F(q)$ be the free G-module on the family of sets of elements x of H such that $\theta(x)$ has target q. Then $F(q)$ has an additive basis of pairs (x, g) such that $\theta(x)g$ is defined in G, and the action of G is given by

$$(x, g)^{g'} = (x, gg')$$

when gg' is defined in G. There is a natural map

$$i\colon H \to F,$$

given by $i(x) = (x, 1_q)$, where $\theta(x)$ has target q. Now we impose on F the relations

$$i(xy) = i(x)^{\theta(y)} + i(y)$$

whenever xy is defined in H. This gives a quotient G-module D_θ, a quotient morphism $s\colon F \to D_\theta$ and a θ-derivation $h_\theta = si\colon H \to D_\theta$. □

Proposition 7.4.14. *Let $\theta\colon H \to G$ be a morphism of groupoids. If H is a free groupoid on X, then D_θ is a free G-module on $h_\theta(X)$.*

Proof. Let $Y = h_\theta(X)$. Let $f\colon Y \to M$ be graph morphism to a G-module M. Let $h'\colon X \to M$ be determined by h_θ and f. Since H is free on X, this graph morphism extends uniquely to a θ-derivation $f'\colon H \to M$. (We see this since a θ-derivation $H \to M$ is equivalent to a groupoid section of the projection $H \ltimes M \to H$.) This θ-derivation determines uniquely a G-morphism $f''\colon D_\theta \to M$ extending f as required. □

The universal property of the derived module construction shows that it yields a functor from the morphism category of Gpds

$$D\colon \mathsf{Gpds}^2 \to \mathsf{Mod}$$

given on objects by $D(H \xrightarrow{\theta} G) = (D_\theta, G)$.

Remark 7.4.15. For those familiar with the notion of Kan extensions, we can regard the category of G-modules as the functor category $(\mathsf{Ab})^G$, and any functor $M\colon H \to \mathsf{Ab}$ has a left Kan extension $\theta_* M\colon G \to \mathsf{Ab}$ along $\theta\colon H \to G$. Then the derived module D_θ is canonically isomorphic to $\theta_*(\vec{I}H)$, the G-module *induced* from $\vec{I}H$ by $\theta\colon H \to G$. In the case of a group morphism θ, this induced module is just $IH \otimes_H \mathbb{Z}G$, where $\mathbb{Z}G$ is viewed as a left H-module via θ and left multiplication. □

Now we obtain a right adjoint to D.

Proposition 7.4.16. *The functor D has a right adjoint* $\mathsf{Mod} \to \mathsf{Gpds}^2$ *given by*

$$(M, G) \mapsto (G \ltimes M \xrightarrow{\mathrm{pr}_1} G).$$

Proof. This is an immediate consequence of the adjointness of $\vec{I}$ and $\ltimes$ seen in Proposition 7.4.8 and the formula $D_\theta = \theta_*(\vec{I}H)$. □

Exercise 7.4.17. Verify that:

(i) The augmentation module $\vec{I}G$ is the derived module of the identity morphism $G \to G$.

(ii) If G is a totally disconnected groupoid on the set X, and $\theta\colon G \to X$ is the unique morphism over X to the discrete groupoid on X, then the derived module of θ is the abelianisation G^{ab} of G, see Section A.8 of Appendix A.

(iii) Discuss the derived module of a composition of morphisms $G \to H \to K$. □

7.4.iii The derived chain complex of a crossed complex

Now we can construct our functor[105]

$$\nabla\colon \mathsf{Crs} \to \mathsf{Chn}.$$

Theorem 7.4.18. *Let C be a crossed complex, and let $\phi\colon C_1 \to G$ be a cokernel of δ_2 of C. Then there are G-morphisms*

$$C_2^{\mathrm{ab}} \xrightarrow{\partial_2} D_\phi \xrightarrow{\partial_1} \vec{\mathbb{Z}}G$$

such that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_n \arrow[r, "\delta_n"] \arrow[d, "1"'] & C_{n-1} \arrow[r] \arrow[d, "1"'] & \cdots \arrow[r] & C_3 \arrow[r, "\delta_3"] \arrow[d, "1"'] & C_2 \arrow[r, "\delta_2"] \arrow[d, "\alpha_2"] & C_1 \arrow[r, "\phi"] \arrow[d, "\alpha_1"] & G \arrow[d, "\alpha_0"] \\
\cdots \arrow[r] & C_n \arrow[r, "\partial_n"'] & C_{n-1} \arrow[r] & \cdots \arrow[r] & C_3 \arrow[r, "\partial_3"'] & C_2^{\mathrm{ab}} \arrow[r, "\partial_2"'] & D_\phi \arrow[r, "\partial_1"'] & \vec{\mathbb{Z}}G
\end{tikzcd} \tag{7.4.1}$$

commutes and the lower line is a chain complex over G. Here α_1 is the universal ϕ-derivation, α_0 is the G-derivation $x \mapsto x - 1_q$ for $x \in G(p,q)$, as a composition $G \to \vec{I}G \to \vec{\mathbb{Z}}G$, and $\partial_n = \delta_n$ for $n \geqslant 4$.

Proof. Let $X = G_0$, and let X also denote the discrete groupoid on X. The functor $D\colon \mathsf{Gpds}^2 \to \mathsf{Mod}$, applied to the sequence of morphisms

$$\begin{tikzcd}
\cdots \arrow[r] & C_3 \arrow[r, "\delta_3"] \arrow[d, "\zeta_3"] & C_2 \arrow[r, "\delta_2"] \arrow[d, "\zeta_2"] & C_1 \arrow[r, "\phi"] \arrow[d, "\phi"] & G \arrow[d, "1"] \\
\cdots \arrow[r] & X \arrow[r] & X \arrow[r] & G \arrow[r] & G
\end{tikzcd}$$

gives a sequence of module morphisms

$$\cdots \to (D_{\zeta_3}, X) \to (D_{\zeta_2}, X) \to (D_\phi, G) \to (\vec{I}G, G).$$

Since a derivation $C_n \to M$ over a null map $\zeta_n\colon C_n \to X$ is just a morphism to an abelian groupoid, we may identify D_{ζ_n} with C_n^{ab} and its universal derivation with the abelianisation map. The map ∂_1 of the diagram in the theorem is to be the composition $D_\theta \to \vec{I}G \to \vec{\mathbb{Z}}G$. Thus we obtain the stated commutative diagram in which the vertical maps are the corresponding universal derivations (followed by an inclusion, in the case of α_0).

This establishes all the stated properties except the G-invariance of ∂_2 and the relations $\partial_2\partial_3 = 0$, $\partial_1\partial_2 = 0$.

Clearly $\partial_2\partial_3 = \alpha_1\delta_2\delta_3 = 0$.

Also $\partial_1\partial_2\alpha_2 = \alpha_0\phi\delta_2 = 0$ and since α_2 is surjective, this implies $\partial_1\partial_2 = 0$.

Finally, if $x \in C_2^{\mathrm{ab}}$, $g \in G$ and x^g is defined, choose $a \in C_2$, $b \in C_1$ such that $\alpha_2 a = x$, $\phi b = g$. Then

$$\begin{aligned}
\partial_2(x^g) &= \alpha_1\delta_2(a^b) && \\
&= \alpha_1(b^{-1}cb), && \text{where } c = \delta_2 a,\\
&= [(\alpha_1(b^{-1}))^{\phi c} + \alpha_1 c]^{\phi b} + \alpha_1 b, && \text{since } \alpha_1 \text{ is a } \phi\text{-derivation},\\
&= (\alpha_1(b^{-1}))^{\phi b} + [\alpha_1 c]^{\phi b} + \alpha_1 b, && \text{since } \phi c = 1,\\
&= -\alpha_1 b + (\alpha_1 c)^{\phi b} + \alpha_1 b && \text{since } \alpha_1 \text{ is a } \phi\text{-derivation},\\
&= (\alpha_1 c)^{\phi b} && \text{since } D_\phi \text{ is abelian},\\
&= (\partial_2 x)^g, && \text{as required.}
\end{aligned}$$

□

Remark 7.4.19. Suppose $\delta_2\colon C_2 \to C_1$ is a crossed module such that C_2 is the free crossed module on R and C_1 is the free groupoid on X. Let $\phi\colon C_1 \to G$ be the cokernel of δ_2. Then the corresponding G-module morphism $\partial_2\colon C_2^{\mathrm{ab}} \to D_\phi$ may by the above results be interpreted as the Fox derivative $(\partial r/\partial x)$.[106] □

Definition 7.4.20. For any crossed complex C, ∇C is the chain complex given in the bottom row of the main diagram of Theorem 7.4.18. This gives the *derived chain complex functor*

$$\nabla\colon \mathsf{Crs} \to \mathsf{Chn}.$$

□

7.4.iv Exactness and lifting properties of the derived functor

The following is a basic exactness result.[107]

Proposition 7.4.21. *Let $C = \{C_r\}$ be a crossed complex and suppose that the sequence of groupoids*

$$C_3 \xrightarrow{\delta} C_2 \xrightarrow{\delta} C_1 \xrightarrow{\phi} G \to 1$$

is exact. Then the sequence of G-modules in $\nabla' C$:

$$C_3 \xrightarrow{\partial} C_2^{\mathrm{ab}} \xrightarrow{\partial} D_\phi \xrightarrow{\partial'} \vec{I}G \to 0$$

is exact.

Proof. The exactness of $C_2 \to C_1 \xrightarrow{\phi} G \to 1$ implies that

$$\begin{tikzcd}
(C_2 \to 1) \arrow[r] \arrow[d] & (C_1 \to G) \arrow[d] \\
(1 \to 1) \arrow[r] & (G \to G)
\end{tikzcd}$$

is a pushout square in the morphism category Gpds^2. Applying $D\colon \mathsf{Gpds}^2 \to \mathsf{Mod}$, as in the proof of Theorem 7.4.18, and noting that D preserves colimits by Proposition 7.4.16, we obtain a pushout square

$$\begin{tikzcd}
(C_2^{\mathrm{ab}}, 1) \arrow[r, "\partial"] \arrow[d] & (D_\phi, G) \arrow[d] \\
(0, 1) \arrow[r] & (\vec{I}G, G)
\end{tikzcd}$$

in Mod. Since $\partial\colon C_2^{\mathrm{ab}} \to D_\phi$ is in fact a G-morphism, it follows that

$$C_2^{\mathrm{ab}} \to D_\phi \to \vec{I}G \to 0$$

is an exact sequence of G-modules.

To prove exactness of $C_3 \to C_2^{\mathrm{ab}} \to D_\phi$, write $N = \operatorname{Ker}\phi = \delta C_2$ and note that the exactness of

$$C_3 \to C_2 \to N \to 1$$

implies the exactness of

$$C_3 \to C_2^{\mathrm{ab}} \to N^{\mathrm{ab}} \to 1.$$

It remains, therefore, to show that the map $\gamma\colon N^{\mathrm{ab}} \to D_\phi$ induced by $\partial\colon C_2^{\mathrm{ab}} \to D_\phi$ is injective.

Now $\phi\colon C_1 \to G$ is a quotient morphism of groupoids with totally intransitive kernel N. In these circumstances the additive groupoid structure of D_ϕ is given by generators $[c] \in D_\phi(q)$ for $c \in C_1(p,q)$, with defining relations

$$[cy] = [c] + [y] \quad \text{for } c \in C_1(p,q),\ y \in N(q);$$

the groupoid C_1 acts on this additive groupoid by

$$[c]^x = [cx] - [x]$$

and N acts trivially, making D_ϕ a G-module; the canonical ϕ-derivation $\alpha_1 : C_1 \to D_\phi$ is given by $\alpha_1(c) = [c]$.

Choose coset representatives $t(c) \in cN$ of N in C_1 with $t(1_q) = 1_q$. Then for all $c \in C_1$, $c = t(c)s(c)$ where $s(c) \in N$. The map $s : C_1 \to N$ satisfies $s(y) = y$ for $y \in N$ and

$$s(cy) = s(c)y \quad \text{for all } c \in C_1(p,q),\ y \in N(q).$$

Consequently, there is an additive map $s^* : D_\phi \to N^{\mathrm{ab}}$ defined by $s^*[c] = \alpha s(c)$, where α is the canonical map $N \to N^{\mathrm{ab}}$. Since, for any $u = \alpha y$ in N^{ab},

$$s^*\gamma u = s^*\gamma\alpha y = s^*\alpha_1 y = \alpha s(y) = \alpha y = u,$$

γ is injective, as required. □

Corollary 7.4.22. *If $\delta : C_2 \to C_1$ is a crossed module with kernel K, and $\phi : C_1 \to G$ is the cokernel of δ, then the sequence $K \to C_2^{\mathrm{ab}} \to D_\phi$ is exact.*

Proof. Put $C_3 = K$ in Proposition 7.4.21. □

Definition 7.4.23. The crossed complex C (or crossed module) is *regular* if

$$K \cap [C_2, C_2] = 0,$$

where K is the kernel of $\delta : C_2 \to C_1$. □

Corollary 7.4.24. *If $C_2 \to C_1$ is a regular crossed module with kernel K, then the sequence $0 \to K \to C_2^{\mathrm{ab}} \to D_\phi$ is exact.*

Proof. This follows from Corollary 7.4.22 and the definition of regular. □

The following is a useful result for applications to free crossed resolutions and to identities among relations. It generalises to the groupoid case Proposition 2.2.6.

Proposition 7.4.25. *If in the crossed complex C, the groupoid C_1 is free, then C is regular. In particular, the fundamental crossed complex ΠX_* of a CW-complex X_* is regular.*

Proof. Since $N = \delta C_2$ is a subgroupoid of C_1, it is a free groupoid (in fact a family of free groups). Hence the map $\delta : C_2 \to N$ has a homomorphic section s. But the kernel K of δ is in the centre of C_2, since C_2 is a crossed module over C_1. Hence $C_2 = K \times_{C_0} s(N)$ is a groupoid, that is, for each $p \in C_0$, $C_2(p) = K(p) \times sN(p)$. This implies that $[C_2, C_2] = [sN, sN]$ and hence that $K \cap [C_2, C_2] = 0$. □

In the following exercise, we sketch in a special case another description of the derived module which is useful later in Section 8.4. We need the notion of universal abelianisation of a groupoid.

Exercise 7.4.26. Let $\phi\colon F \to G$ be an epimorphism of groups. Form the universal covering groupoid $p\colon \widetilde{G} \to G$, see Appendix B, Section B.7, and let $q\colon \widehat{F} \to F$ be the pullback of p by ϕ. Then q is also a covering morphism of groupoids. There is a function $\upsilon\colon F \to \widehat{F}$ which sends $a \in F$ to the unique covering element of a which ends at the object $1 \in F$. Recall the universal abelianisation of a groupoid explained in Section A.8 of Appendix A. Prove that $\widehat{F}^{\text{totab}}$ admits the structure of G-module and that the composite $F \xrightarrow{\upsilon} \widehat{F} \to \widehat{F}^{\text{totab}}$ is a ϕ-derivation. Prove that the morphism of G-modules $D_\phi \to \widehat{F}^{\text{totab}}$ given by the universal property of $F \to D_\phi$ is an isomorphism by using the 5-lemma on a map from the exact sequence of Proposition 7.4.21 to one derived from an analysis of $\widehat{F}^{\text{totab}}$ using earlier parts of this exercise. See also Remark B.7.6. $\square$

7.4.v The right adjoint of the derived functor

The main task of this section is to construct a functor $\Theta\colon \mathsf{Chn} \to \mathsf{Crs}$ and prove it is right adjoint to ∇.[108] This shows that some information on a crossed complex C can be recovered from the chain complex ∇C, and also has the important consequence that ∇ preserves colimits. We will use ∇ in Chapter 9 to give a convenient description of the tensor product of crossed complexes in dimensions > 2.

In order to construct Θ we use an intermediate functor Θ'.

Definition 7.4.27. For a chain complex A over a groupoid H, $\Theta' A = \Theta'(A, H)$ is the crossed complex

$$\Theta' A \colon = \quad \cdots \to A_n \xrightarrow{\partial_n} A_{n-1} \to \cdots \to A_3 \xrightarrow{\partial_3} A_2 \xrightarrow{(0,\partial_2)} H \ltimes A_1 .$$

Here $H \ltimes A_1$ acts on A_n ($n \geqslant 2$) via the projection $H \ltimes A_1 \to H$, so that A_1 acts trivially. $\square$

Note that $\Theta' A$ does not involve A_0. To correct this, we use another construction which brings in A_0 in an essential way. Let us begin by defining ΘA and checking that the definition works.

Definition 7.4.28. For any chain complex A, we consider the canonical covering morphism

$$\kappa\colon P(UA_0, H) \to H \ltimes A_0$$

of Proposition 7.4.11. We define

$$\Theta(A) = \kappa^* \Theta' A,$$

the pull-back along κ of the crossed complex of Definition 7.4.27.

We obtain a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \Theta(A)_3 \arrow[r] \arrow[d, "\sigma_3"] & \Theta(A)_2 \arrow[r] \arrow[d, "\sigma_2"] & \Theta(A)_1 \arrow[r] \arrow[d, "\sigma_1"] & P(A_0, H) \arrow[d, "\kappa"] \\
\cdots \arrow[r] & A_3 \arrow[r, "\partial_3"'] & A_2 \arrow[r, "(0,\partial_2)"'] & H \ltimes A_1 \arrow[r, "(1,\partial_1)"'] & H \ltimes A_0
\end{tikzcd}$$

in which each $\Theta(A)_n$ is a groupoid over $\Theta(A)_0 = A_0$, and each σ_n is a covering morphism of groupoids.

For $n \geqslant 2$, the composite map $A_n \to H \ltimes A_0$ is 0 and, since $\operatorname{Ker} \kappa$ is discrete, it follows that $\Theta(A)_n$ is just a family of groups each isomorphic to a group of A_n. There is also an action of $\Theta(A)_1$ on $\Theta(A)_n$ ($n \geqslant 2$) induced by the action of $H \ltimes A_1$ on A_n; for if $e_1 \in \Theta(A)_1(x, y)$, where $x \in A_0(p)$, $y \in A_0(q)$, and if $e_n \in \Theta(A)_n(x)$, then $\sigma_1 e_1$ acts on $\sigma_n e_n$ to give an element of $A_n(q)$ which lifts uniquely to an element of $\Theta(A)_n(y)$.

It is now easy to see that $\Theta(A) = \{\Theta(A)_n\}_{n\geqslant 0}$ is a crossed complex and that the σ_i form a morphism $\sigma\colon \Theta(A) \to \Theta' A$ of crossed complexes.

This gives a functor

$$\Theta\colon \mathsf{Chn} \to \mathsf{Crs}. \qquad \square$$

An explicit description of $\Theta(A) = \Theta(A, H)$ can be extracted from the constructions given above. Recall that A_0 is an H-module and so comes with a function $A_0 \to H_0$ making A_0 also the disjoint union of abelian groups $A_0(p)$. The set of objects of every $\Theta(A)_n$ is just A_0, regarded as a set and as a disjoint union.

An arrow of $\Theta(A)_1$ from x to y, where $x \in A_0(p)$, $y \in A_0(q)$, $p, q \in H_0$, is a triple (h, a, y), where $h \in H(p, q)$, $a \in A_1(q)$, and $x^h = y + \partial a$. Composition in $\Theta(A)_1$ is given by

$$(k, b, x)(h, a, y) = (kh, b^h + a, y)$$

whenever kh is defined in H and $x^h = y + \partial a$.

For $n \geqslant 2$, $\Theta(A)_n$ is a family of groups; the group at the object $x \in A_0(p)$ has arrows (b, x) where $b \in A_n(p)$, with composition

$$(b, x) + (c, x) = (b + c, x).$$

The boundary map $\delta\colon \Theta(A)_2 \to \Theta(A)_1$ is given by

$$\delta(b, x) = (1_p, \partial b, x) \quad \text{for } b \in A_2(p),\ x \in A_0(p).$$

The boundary map $\delta\colon \Theta(A)_n \to \Theta(A)_{n-1}$ ($n \geqslant 3$) is given by $\delta(b, x) = (\partial b, x)$ and the action of $\Theta(A)_1$ on $\Theta(A)_n$ ($n \geqslant 2$) is given by

$$(b, x)^{(h,a,y)} = (b^h, y),$$

where $h \in H(p, q)$, $a \in A_n(q)$, $y \in A_0(q)$ and $x^h = y + \partial a$.

Proposition 7.4.29. *The functor Θ is a right adjoint of ∇. Hence ∇ preserves colimits.*

Proof. A morphism $(\beta, \psi) \colon (\nabla C, G) \to (A, H)$ in Chn is equivalent to a commutative diagram in Mod:

$$\begin{tikzcd}
\cdots \arrow[r] & C_3 \arrow[r] \arrow[d, "\beta_3"] & C_2^{\mathrm{ab}} \arrow[r] \arrow[d, "\beta_2"] & D_\phi \arrow[r] \arrow[d, "\beta_1"] & \vec{I}G \arrow[r, "i"] \arrow[d, "\beta_0'"] & \vec{\mathbb{Z}}G \arrow[dl, "\beta_0"] \\
\cdots \arrow[r] & A_3 \arrow[r, "\partial_3"'] & A_2 \arrow[r, "\partial_2"'] & A_1 \arrow[r, "\partial_1"'] & A_0 &
\end{tikzcd}$$

(over some morphism $\psi \colon G \to H$) and hence, by Propositions 7.4.8, 7.4.16, to a commutative diagram in Gpds:

$$\begin{tikzcd}
\cdots \arrow[r] & C_3 \arrow[r] \arrow[d, "\beta_3"] & C_2 \arrow[r] \arrow[d, "\overline{\beta_2}"] & C_1 \arrow[r, "\phi"] \arrow[d, "\gamma_1"] & G \arrow[d, "\xi"'] \arrow[dr] & \\
\cdots \arrow[r] & A_3 \arrow[r, "\partial_3"'] & A_2 \arrow[r, "{(0,\partial_2)}"'] & H \ltimes A_1 \arrow[r, "{(1,\partial_1)}"'] & H \ltimes A_0 & P(A_0, H) \arrow[l, "\kappa"]
\end{tikzcd}$$

where $(\dots, \beta_3, \overline{\beta_2}, \gamma_1)$ is a morphism of crossed complexes, and κ is the canonical covering morphism. This in turn is equivalent to a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_3 \arrow[r] \arrow[d, "\beta_3"] & C_2 \arrow[r, "\delta"] \arrow[d, "\overline{\beta_2}"] & C_1 \arrow[r, "\omega"] \arrow[d, "\gamma_1"] & P(A_0, H) \arrow[d, "\kappa"] \\
\cdots \arrow[r] & A_3 \arrow[r, "\partial_3"'] & A_2 \arrow[r] & H \ltimes A_1 \arrow[r] & H \ltimes A_0
\end{tikzcd}$$

because, in any such diagram, $\kappa\omega\delta = 0$ and κ is a covering morphism, so $\omega\delta = 0$, that is, ω factorises through $\phi \colon C_1 \to G$.

This diagram is therefore equivalent to a morphism of crossed complexes $C \to \Theta(A)$. Hence (β, ψ) is therefore equivalent to a morphism of crossed complexes $C \to \Theta(A)$. This shows that the functor $\Theta \colon \mathsf{Chn} \to \mathsf{Crs}$ is right adjoint to ∇. $\square$

7.4.vi A colimit in chain complexes with operators

The fact that $\nabla \colon \mathsf{Crs} \to \mathsf{Chn}$ preserves all colimits implies that the Higher Homotopy Seifert–van Kampen Theorem proved in Section 8.2 for the fundamental crossed complex ΠX_* of a filtered space X_* can be converted into a similar theorem for the chain complex $CX_* = \nabla \Pi X_*$. The interpretation of this result will be discussed in Section 8.4.

The following simple example illustrates some of the interesting features that arise in computing colimits in Crs and Chn. Note that if all the crossed complexes in a diagram $\{C^\lambda\}$ are reduced then the colimit of $\{C^\lambda\}$ is reduced provided that the diagram is connected, in which case the colimit of $\{\nabla C^\lambda\}$ can be computed in the category of chain complexes over groups instead of groupoids.

Thus we consider a simple connected diagram of reduced crossed modules. Note that in the reduced case, we can abbreviate $\vec{I}, \vec{\mathbb{Z}}$ to $I, \mathbb{Z}$.

Example 7.4.30. Let $\mu\colon M \to P$, $\nu\colon N \to P$ be crossed modules over a group P. Recall from Proposition 4.3.1 that their coproduct $\gamma\colon M \circ N \to P$ in the category XMod/P is given by the pushout of the two inclusions

$$\begin{tikzcd} & (M \xrightarrow{\mu} P) \arrow[dr] & \\ (0 \to P) \arrow[ur] \arrow[dr] & & (M \circ N \xrightarrow{\gamma} P) \\ & (N \xrightarrow{\nu} P) \arrow[ur] & \end{tikzcd}$$

Let $\mathsf{Chn}^{(2)}$ be the category of chain complexes of length 2 with a groupoid of operators. In our example all operators are groups.

To find such chain complexes corresponding to the above pushout let $G = \operatorname{Cok}\mu$, $H = \operatorname{Cok}\nu$ and write ϕ, ψ for the quotient maps $P \to G$, $P \to H$. Then the corresponding derived modules are $D_\phi = IP \otimes_P \mathbb{Z}G$ and $D_\psi = IP \otimes_P \mathbb{Z}H$.

We wish to compute the pushout:

$$\begin{tikzcd} & (M^{\mathrm{ab}} \to IP \otimes_P \mathbb{Z}G \to \mathbb{Z}G, G) \arrow[dr] & \\ (0 \to IP \to \mathbb{Z}P, P) \arrow[ur] \arrow[dr] & & (X, K) \\ & (N^{\mathrm{ab}} \to IP \otimes_P \mathbb{Z}H \to \mathbb{Z}H, H) \arrow[ur] & \end{tikzcd}$$

The associated pushout of groups is

$$\begin{tikzcd} & G \arrow[dr] & \\ P \arrow[ur, "\theta"] \arrow[dr, "\psi"'] & & K \\ & H \arrow[ur] & \end{tikzcd}$$

where $K = P/(\mu M \cdot \nu N)$. This is the group acting on the pushout chain complex X.

Next we form the induced modules over K of each module in the diagram and then form pushouts of K-modules in each dimension. This gives the chain complex

$$(\dots 0 \to (M^{\mathrm{ab}} \otimes_P \mathbb{Z}K) \oplus (N^{\mathrm{ab}} \otimes_P \mathbb{Z}K) \to IP \otimes_P \mathbb{Z}K \to \mathbb{Z}K, K).$$

Since $K = P/(\mu M \nu N)$, and μM acts trivially on M^{ab}, and similarly for N, we have

$$M^{\mathrm{ab}} \otimes_P \mathbb{Z}K = \frac{M^{\mathrm{ab}}}{[M^{\mathrm{ab}}, N]}, \quad N^{\mathrm{ab}} \otimes_P \mathbb{Z}K = \frac{N^{\mathrm{ab}}}{[N^{\mathrm{ab}}, M]}$$

where $[M^{\mathrm{ab}}, N]$, $[N^{\mathrm{ab}}, M]$ denote displacement groups under actions, see Definition 4.4.1. Thus the pushout in dimension 2 is

$$\frac{M^{\mathrm{ab}}}{[M^{\mathrm{ab}}, N]} \oplus \frac{N^{\mathrm{ab}}}{[N^{\mathrm{ab}}, M]},$$

which is easily seen to be $(M \circ N)^{\mathrm{ab}}$, confirming that ∇ preserves this pushout. □

Remark 7.4.31. The fact that ∇ has a right adjoint together with the HHSvKT of the next chapter implies an HHSvKT for the composite functor

$$C = \nabla\Pi : \mathsf{FTop} \to \mathsf{Chn},$$

which seems to be a new property of this functor, and has been little applied. From Chapters 9 and 10 we will also see that there is an Eilenberg–Zilber type theorem with values in Chn.[109] □

Notes

91 p. 210 The category of modules for varying groups occurs in the theory of Mackey functors, at least for varying subgroups of a given group, see for example [TW95]. There is a curious analogy between considering the fundamental groupoid instead of the fundamental group and considering the category Mod of modules over all groupoids instead of modules over a fixed groupoid.

92 p. 211 The suggestion that the category of Hausdorff k-spaces might be ‘adequate and convenient for all purposes of topology’ was made in the Introduction to [Bro63] and the properties of ‘convenient’ were listed in [Bro64b], basically in terms of cartesian closure. The idea came from [Bro62], which used a variety of what we now call monoidal closed categories in studying the homotopy type of function spaces. The methods used there are a background to those used in this book. This notion of ‘convenient’ was taken up in [Ste67], using the now popular term ‘compactly generated’ instead of k-space. An exposition of this convenient category without the Hausdorff condition is in [Bro06], Section 5.9, using the more traditional term **k**-space. For a general discussion and more references, see for example [BT80].

93 p. 213 There are generalisations of the notion of CW-complexes, for example, the G-CW-complexes in equivariant theory, for which but one reference is [Lüc87], and other ideas in [MOt06].

94 p. 213 As one reference to modules over groups we mention [Seh03].

95 p. 214 This note gives some background to the algebraic aspects of crossed complexes over groups. The topological aspects will be discussed in the next note.

The notion of crossed module was defined by Whitehead in [Whi46]. It also appeared in work of Mac Lane in [ML49], in connection with 'abstract kernels'. The term 'crossed complex' is due to Huebschmann in his thesis, [Hue77], and in [Hue80a] and the ideas were developed in [Hue81b], [Hue81a]. The first paper contains in particular the notion of 'standard free crossed resolution' of a group. The background to the interpretation of the $H^{n+1}(G, M)$ in terms of crossed n-fold extensions is discussed in the Historical Note [ML79]. There is also a larger background here in work of Lue, [Lue71], on cohomology for algebras relative to a variety, defined in terms of such n-fold extensions, which however he did not relate to the well-known group case, where the 'variety' would be taken to be that of abelian groups, see [Lue81]. Crossed modules are essential in [Lod82]. They also occur as coefficients for cohomology of groups and of spaces in Dedecker's work on nonabelian cohomology, see for example [Ded58], [Ded60], [Ded63]. Crossed modules appear in relative cohomology in [Lod78], and in [GWL81]. The first definition of crossed complexes over groupoids was in the announcement [BH77] and the full paper [BH81]. The paper [BH81b] gives an equivalence of crossed complexes to what are there called ∞-groupoids, and are now generally called strict globular ω-groupoids.

96 p. 220 Blakers in [Bla48] defined what he calls a 'group system' associated to a (reduced) filtered space, and which we now call a reduced crossed complex. Thus he gives the definition of ΠX_* in that case, and uses this to relate homology and homotopy. Blakers attributes to S. Eilenberg the suggestion of considering the whole structure of crossed complex.

The idea was also used by J. H. C. Whitehead in his paper [Whi49b], in the case of the skeletal filtration of pointed CW-complexes, and there called the 'homotopy system' of the CW-complex. This paper contains some profound theorems, and was an inspiration for the work of Brown and Higgins. Whitehead explains that his 'homotopy systems' are a translation into relative homotopy groups of his notion of a 'natural system' on p. 1216 of [Whi41b], a truly remarkable paper, which also relies on work of Reidemeister, [Rei34], for the 'chain complex with operators' part of the concept.

Further work relating these ideas to the notion of k-invariant is given by Ando in [And57] and Huebschmann in [Hue80b]. See also [Hue07].

Crossed complexes over groups are called *crossed chain complexes* in [Bau89], [Bau91].

The many base point version of the fundamental crossed complex was defined in the announcement [BH78b] and the paper [BH81a]. We use the notation Π for the fundamental crossed complex; other notations are π in [BH81a], ρ in [Whi49b], [Bau89], [Bau91], but we use ρ in Part III for the fundamental cubical ω-groupoid of a filtered space. Some writers use ΠX to denote the singular simplicial complex of a space X, regarded as some form of 'weak ∞-groupoid', see for example [Lur09].

97 p. 221 For more on the standard theory of relative homotopy groups, see for example [Hu59], Chapter IV, Corollary 8.1. This result is also related to the methods of [Bro06], 7.2.8, and to Proposition 14.1.10.

98 p. 221 Whitehead explains in [Whi49a] that one reason for not restricting his cell complexes to the 1-vertex case is in order to include the theory of covering spaces.

99 p. 223 We use the term cotruncation for this left adjoint to sk^n, but you should be warned that this term is also used in the literature in another sense, namely for taking the part of an hierarchical structure *above* a given level.

100 p. 224 This type of definition of homotopy where both the base point and the crossed module structure are taken into account was introduced in [Whi49b]. An application to discussing automorphisms of crossed modules over groupoids, generalising work on automorphisms of crossed modules of Whitehead and Norrie in [Whi48], [Nor90] is given in [Bİ03a].

101 p. 225 This diagrammatic view of homotopies in dimension 1 is used a lot in [Bİ03b].

102 p. 228 For more on algebraic theories, see for example [Law04], [Man76].

103 p. 237 As just one paper in this area of 2-complexes and combinatorial group theory, we cite [FR05], which analyses in one case the relation between $\pi_2(K)$ and $\pi_2(L)$ when K is obtained from L by adding a 1-cell and a 2-cell.

104 p. 239 The results in this section on crossed complexes and chain complexes are largely taken from [BH90].

105 p. 246 The results of this Section 7.4 come largely from [BH90], which was intended to give a more general setting and more detailed analysis of Whitehead's results in [Whi49b]. That paper was the principal source for ∇, but Whitehead's construction requires C_1 to be a free group. (If C_1 is the free group on a set X then D_θ is just the free G-module on X.) The general construction of $(\nabla C)_1$ as

D_θ was suggested by [Cro71], although that deals only with the case of groups.

Proposition 7.4.25 for the reduced case is due to Whitehead [Whi49b]. It is interesting that the construction of ∂_2 in Theorem 7.4.18 was given by Whitehead (in [Whi49b], in the group case) well before the publication of work of Fox on his free differential calculus [Fox53], [CF63], and the relation between the two works seems not to have been generally noticed. Much of the information contained in the free differential calculus was also known to Reidemeister and his school, see [Rei34], [Rei50].

106 p. 247 See the previous note on the free differential calculus.

107 p. 247 An exact module sequence arising from an exact sequence of groups is Satz 15 of [Sch37]. This sequence is standard background to work on extensions of groups, as given for example in [ML63], p. 120. It is also developed thoroughly by Crowell in [Cro61], [Cro71]. Our Proposition 7.4.21 gives an extension of this exact sequence.

108 p. 250 These results on the relations between crossed complexes and chain complexes with a groupoid of operators come from [BH90]. The existence of an adjoint to ∇ was suggested by results in [McP69] that the Alexander module preserves colimits. Special cases of the groupoid $E_1 = (\Theta A)_1$ appear in [Cro61], [GR80], [Gro68].

109 p. 254 The category Chn has however problems in comparison with Crs in that the latter has better realisation properties, as observed in [Whi49b] for the reduced case. Thus every free crossed module over a free groupoid is isomorphic to ΠX_* for some 2-dimensional CW-complex X_*. The corresponding question for chain complexes of length 2 is known as the D_2 problem, see [Joh04], [Joh09] and the references there.

Chapter 8

The Higher Homotopy Seifert–van Kampen Theorem (HHSvKT) and its applications

Introduction

Now we turn to the first of our series of homotopical applications of crossed complexes and the functor

$$\Pi\colon \mathsf{FTop} \to \mathsf{Crs},$$

namely the consequences of a Higher Homotopy Seifert–van Kampen Theorem (HHSvKT).

The statement and many of the applications of the theorem are entirely analogous to those of the theorem in dimension 2 given in Part I. The method of proof is also analogous to that in Part I, but is much more complicated algebraically and topologically. So the proof is deferred to Part III.

There are some interesting contrasts between the results of this part and those in Part I. The applications in Part I involved crossed modules, which in dimension 2 is a generally nonabelian structure. Hence those results are largely unobtainable by traditional methods of algebraic topology.

The applications of the HHSvKT in dimensions > 2 involve modules, rather than crossed modules, over the fundamental group or groupoid, and so are much nearer to traditional results of algebraic topology. Thus, even though the Coproduct Theorem 8.3.5, and the Homotopical Excision Theorem 8.3.7, do not appear in traditional texts, or papers, they are possibly reachable by methods of singular homology and covering spaces, using the latter to bring in the operations of the fundamental group. Handling many base points is less traditional.

Our aim is to show how such results follow in a uniform way by a study of the homotopically defined functor Π. Thus the Relative Hurewicz Theorem, a key result in this borderline between homology and homotopy theory, is seen in a broader context which includes nonabelian results in dimensions 1 and 2[110].

This method gives for example some computations of the homotopy group $\pi_n(X, a)$ for $n \geqslant 2$ as a module over the fundamental group $\pi_1(X, a)$. However we would also like to take advantage of the extensions from groups to groupoids explained in Chapter 1. Thus we need to consider many base points and the extra variety of morphisms that the category of groupoids has over the category of groups, which led in Corollary 1.7.2 to the determination of the fundamental group of the circle in a way which well models the geometry.

Consider the following maps of spaces involving the n-sphere S^n, the first an inclusion and the second an identification:

$$S^n \xrightarrow{i} S^n \vee [0,2] \xrightarrow{p} S^n \vee S^1 \vee S^1. \tag{$*$}$$

Here $n \geqslant 2$, the base point $\mathbf{0}$ of S^n is in the second space identified with 0, and the map p identifies $0, 1, 2$ in the interval $[0,2]$ to give the wedge, or one point union, of two circles $S^1 \vee S^1$. The groupoid SvKT (Theorem 1.6.1) implies that $\pi_1(S^n \vee S^1 \vee S^1, 0) \cong F_2$, the free group on 2 elements, regarded as formed from the indiscrete groupoid on $\{0,1,2\}$ by identifying $\{0,1,2\}$ to a point. We will see in Section 8.3.iii that the HHSvKT implies that $\pi_n(S^n \vee S^1 \vee S^1, 0)$ is the free F_2-module on one generator.

The results of this chapter on the functor Π are crucial for later applications, such as the notion of classifying space BC of a crossed complex C and the application of this to the homotopy classification of maps of topological spaces, where the fundamental group or groupoid is involved.

In evaluating these results, and comparing with traditional expositions, it should be borne in mind that we use subdivision methods, but only cubically, as these may be modeled algebraically in higher homotopy groupoids. So simplicial approximation is not used, except where we need results from the theory of simplicial sets. Also we do not use homology theory, except to relate our results to traditional ones.

It is hoped that this re-vision of basic algebraic topology will suggest wider applications, since notions of homotopy and deformations are crucial in many areas.

Crossed complexes give in a sense a linear algebraic model of homotopy theory. This limits their rôle for many problems. On the other hand, as in many areas of mathematics, a linear approximation can be useful! More general applications also follow once the tensor product of crossed complexes has been set up and applied in later chapters.

8.1 HHSvKT for crossed complexes

The HHSvKT gives a mode of calculation of the fundamental crossed complex functor

$$\Pi \colon \mathsf{FTop} \to \mathsf{Crs}$$

from filtered topological spaces to crossed complexes. This functor is defined *homotopically*, i.e. in terms of certain homotopy classes of certain maps, and not in terms of any other combinatorial model of the filtered space. So it is remarkable that we can calculate in this way, starting with simple information on the trivial values of the functor on simple filtrations of contractible spaces.

An easy consequence of the definition of Π is that it preserves coproducts, which are in the two categories FTop, Crs just disjoint union; this is one of the advantages of the groupoid approach. Much more subtle is the application to 'gluing' spaces, and we approach this concept, as in Chapter 6, through the notion of coequaliser.

As we have seen in Chapter 1, the version of the classical Seifert–van Kampen Theorem for the fundamental groupoid rather than group gives useful results for non-connected spaces, but still requires a 'representativity' condition in dimension 0. The corresponding theorem for crossed modules, which computes certain second relative homotopy groups, as discussed in Chapter 6, also needs a '1-connected' condition. It is thus not surprising that our general theorem requires a connectivity condition in all dimensions.

Proposition 8.1.1. *For a filtered space X_* the following conditions (ϕ), (ϕ') and (ϕ'') are equivalent:*

- (ϕ) (ϕ_0): *The function $\pi_0 X_0 \to \pi_0 X_r$ induced by inclusion is surjective for all $r \geqslant 0$; and, for all $i \geqslant 1$,*
  (ϕ_i): *$\pi_i(X_r, X_i, v) = 0$ for all $r > i$ and $v \in X_0$.*
- (ϕ') (ϕ'_0): *The function $\pi_0 X_s \to \pi_0 X_r$ induced by inclusion is surjective for all $0 = s < r$ and bijective for all $1 \leqslant s \leqslant r$; and, for all $i \geqslant 1$,*
  (ϕ'_i): *$\pi_j(X_r, X_i, v) = 0$ for all $v \in X_0$ and all j, r such that $1 \leqslant j \leqslant i < r$.*
- (ϕ'') (ϕ'_0) *and, for all $i \geqslant 1$,*
  (ϕ''_i): *$\pi_j(X_{i+1}, X_i, v) = 0$ for all $j \leqslant i$, and $v \in X_0$.*

The proof is a straightforward argument on the exact homotopy sequences of various pairs and triples and is omitted.

Definition 8.1.2. A filtered space is called *connected* if it satisfies any of the equivalent conditions (ϕ), (ϕ') and (ϕ'') of the previous proposition. □

Remark 8.1.3. This condition is satisfied in many important cases. The HHSvKT will allow us to construct some new connected filtered spaces as colimits of old ones. In particular, we will prove that the skeletal filtration of a CW-complex X is a connected filtration.

Note also that the condition $\pi_1(X_r, X_1, x) = 0$ means that any path in X_r joining x to a point in X_1 is homotopic in X_r rel end points to a path in X_1. This condition is equivalent to $\pi_1(X_1, x) \to \pi_1(X_r, x)$ is surjective. □

Example 8.1.4. Clearly, a disjoint union of connected filtered spaces is connected. □

Now we have set the background to state the HHSvKT in the most general form we are going to use. Its algebraic content is that under some connectedness conditions, the fundamental crossed complex functor Π preserves certain colimits. Since Π preserves the coproducts in FTop, Crs, and colimits can be constructed from coproducts and coequalisers, the meat of the theorem is in the statement on preservation of certain coequalisers.

In order to give background to the statement of the HHSvKT, we recall that if the space X is the union of two open sets U, V then we have a pushout diagram of spaces:

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & X. \end{tikzcd}$$

If X is the union of three open sets U, V, W then we have a diagram

$$\begin{tikzcd}[row sep=small, column sep=small] & & & V \cap W \arrow[dl] \arrow[ddd] \\ U \cap V \arrow[rr] \arrow[ddd] & & V \arrow[ddd] & \\ & & & \\ & U \cap W \arrow[rr] \arrow[dl] & & W \arrow[dl] \\ U \arrow[rr] & & X & \end{tikzcd}$$

and a map $f: X \to Y$ is entirely determined by maps f_U, f_V, f_W defined on U, V, W, with values in Y, and which agree on the two fold intersections $V \cap W$, $W \cap U$, $U \cap V$.

The most general situation of this type is expressed by the notion of coequaliser, which we have used already in Chapter 6. Suppose given a cover $\mathcal{U} = \{U^\lambda\}_{\lambda \in \Lambda}$ of X such that the interiors of the sets of $\mathcal{U}$ cover X. For $\zeta = (\lambda, \mu) \in \Lambda^2$ let $U^\zeta = U^\lambda \cap U^\mu$. Then we can form the diagram

$$\bigsqcup_{\zeta \in \Lambda^2} U^\zeta \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_{\lambda \in \Lambda} U^\lambda \xrightarrow{c} X$$

where c is determined by the inclusions $U^\lambda \to X$ and a, b are determined by the inclusions $U^\zeta \to U^\lambda, U^\zeta \to U^\mu$ for $\zeta = (\lambda, \mu) \in \Lambda^2$.[111] Note that $ca = cb$, and that a map $f: \bigsqcup_{\lambda \in \Lambda} U^\lambda \to Y$ determines uniquely a map $f': X \to Y$ with $f'c = f$, if and only if $fa = fb$. Thus we say that c *is a coequaliser of* a, b in the category Top.

Now suppose further that X_* is a filtered space. For each $\zeta = (\zeta_1, \dots, \zeta_n) \in \Lambda^n$ we set

$$U^\zeta = U^{\zeta_1} \cap \cdots \cap U^{\zeta_n}$$

and consider the *induced filtration*

$$U^\zeta_* : = U^\zeta_0 \subseteq U^\zeta_1 \subseteq \cdots \subseteq U^\zeta$$

where $U_i^\zeta = U^\zeta \cap X_i$ for each $i \in \mathbb{N}$. Then we have a coequaliser diagram of filtered spaces

$$\bigsqcup_{\zeta \in \Lambda^2} U_*^\zeta \overset{a}{\underset{b}{\rightrightarrows}} \bigsqcup_{\lambda \in \Lambda} U_*^\lambda \xrightarrow{c} X_*.$$

Theorem 8.1.5 (Higher Homotopy Seifert–van Kampen Theorem). *Let X_* be a filtered space, and $\mathcal{U} = \{U^\lambda\}_{\lambda \in \Lambda}$ a family of subsets of X whose interiors cover X. Suppose that for every finite intersection U^ζ of elements of $\mathcal{U}$, the induced filtration U_*^ζ is connected. Then*

(Con) *X_* is connected, and*

(Iso) *in the following Π-diagram of the cover,*

$$\bigsqcup_{\zeta \in \Lambda^2} \Pi U_*^\zeta \overset{a}{\underset{b}{\rightrightarrows}} \bigsqcup_{\lambda \in \Lambda} \Pi U_*^\lambda \xrightarrow{c} \Pi X_*,$$

c is the coequaliser of a, b in the category Crs *of crossed complexes.*

Remark 8.1.6. The connectivity conclusion of the theorem is important and nontrivial, and can be proved by the deformation arguments given in Proposition 14.7.1 (see p. 502), without introducing the algebraic category of ω-groupoids. The isomorphism part, which determines ΠX_* in terms of the pieces ΠU_*^λ, should be seen as an *all dimensional, local-to-global result in homotopy theory, nonabelian in dimensions* 1 *and* 2. It would have been unlikely for this theorem to be conjectured in this form: it was conjectured and finally proved in cubical terms, see Chapter 14, and then found to have this consequence. □

8.2 Some immediate consequences of the HHSvKT

In the following subsections we give some results which are special cases of or consequences of the HHSvKT. Those which involve the operations of the fundamental group or groupoid are more difficult to obtain by traditional methods of algebraic topology: a common method of finding information on such operations is by means of covering spaces.

8.2.i Coproducts with amalgamation

We now consider a covering where any two elements intersect along a fixed subspace, as follows.

Theorem 8.2.1. *Let X_* be a filtered space and suppose:*

(i) $\mathcal{U} = \{U^\lambda\}_{\lambda\in\Lambda}$ *is a family of subsets of* X *whose interiors cover* X;
(ii) U^0 *is a subset of* X *such that* $U^\lambda \cap U^\mu = U^0$ *for all* $\lambda, \mu \in \Lambda$ *such that* $\lambda \neq \mu$;
(iii) U^0_* *and* $U^\lambda_*, \lambda \in \Lambda$ *are connected filtrations. Then*

(Con) *the filtration* X_* *is connected, and*
(Iso) *the following is a coequaliser diagram of crossed complexes:*

$$\Pi U^0_* \xrightarrow{(a^\lambda)} \bigsqcup_{\lambda\in\Lambda} \Pi U^\lambda_* \xrightarrow{c} \Pi X_*,$$

where a^λ, c *are induced by inclusions.*

Proof. Note that the conditions we give immediately imply the connectivity conditions on intersections required for the theorem. □

Another consequence gives the fundamental crossed complex of a wedge of filtered spaces.

Definition 8.2.2. A filtered space X_* is *reduced* if X_0 consists of a single point, i.e. $X_0 = \{*\}$; then $*$ is taken as base point of each $X_n, n \geqslant 0$, and the relative homotopy groups of X_* are abbreviated to $\pi_n(X_n, X_{n-1})$. The base point in X_0 is *nondegenerate* if each inclusion $X_0 \to X_n$, is a closed cofibration for all $n \geqslant 1$. □

Corollary 8.2.3. *Suppose, in addition to the assumptions of the Theorem, that* X_* *is a reduced filtered space, and* ΠU^0_* *is the trivial crossed complex. Then the morphisms* $\Pi U^\lambda_* \to \Pi X_*$ *induced by inclusions define an isomorphism*

$$*_\lambda \Pi U^\lambda_* \to \Pi X_*$$

from the coproduct crossed complex in Crs_* *to* ΠX_*.

8.2.ii Pushouts

In the case of a covering by two open sets we obtain the Higher Homotopy Seifert–van Kampen Theorem in the pushout form either directly from Theorem 8.1.5 or as a particular case of Theorem 8.2.1.

Theorem 8.2.4. *Let* X_* *be a filtered space and suppose:*

(i) X *is the union of the interiors of* U^1 *and* U^2;
(ii) $U^0 = U^1 \cap U^2$;
(iii) U^0_*, U^1_*, U^2_* *are connected filtrations.*

Then

(Con) X_* *is connected, and*

(Pushout) *the following diagram of morphisms of crossed complexes*

$$\begin{tikzcd} \Pi U^0_* \arrow[r] \arrow[d] & \Pi U^1_* \arrow[d] \\ \Pi U^2_* \arrow[r] & \Pi X_*, \end{tikzcd}$$

induced by inclusions, is a pushout diagram in Crs.

This pushout form of HHSvKT can be generalised to allow the case when X is the adjunction space formed from V by a map $f : A \to U$.

Theorem 8.2.5 (The pushout HHSvKT for cofibrations). *Suppose that the commutative diagram of filtered spaces*

$$\begin{tikzcd} A_* \arrow[r, "f"] \arrow[d, "i"'] & U_* \arrow[d, "\bar{\imath}"] \\ V_* \arrow[r, "\bar{f}"'] & X_* \end{tikzcd}$$

is such that for $n \geqslant 0$, the maps $i_n : A_n \to V_n$ are closed cofibrations, $A_n = A \cap V_n$, and X_n is the adjunction space $U_n \cup_{f_n} V_n$. Suppose also that the filtrations U_*, V_*, A_* are connected. Then

(Con) *X_* is connected, and*

(Iso) *the induced diagram*

$$\begin{tikzcd} \Pi A_* \arrow[r] \arrow[d] & \Pi U_* \arrow[d] \\ \Pi V_* \arrow[r] & \Pi X_* \end{tikzcd}$$

is a pushout of crossed complexes.

Proof. This is a deduction of standard kind from Theorem 8.3.5 using mapping cylinders. □

We give several illustrations of the use of Theorem 8.2.5, with first a direct application to quotient filtrations.

Theorem 8.2.6. *Let V_* be a filtered space, $A \subseteq V$, and $X = V/A$. We define the filtrations A_*, and X_* by $A_n = V_n \cap A$, and $X_n = V_n/A_n$, $n \geqslant 0$. Suppose that each $A_n \to V_n$ is a closed cofibration, and both A_*, V_* are connected. Then*

(Con) *X_* is connected, and*

(Iso) *we have a pushout of crossed complexes*

$$\begin{tikzcd} \Pi A_* \arrow[r] \arrow[d, "i_*"'] & 0 \arrow[d] \\ \Pi V_* \arrow[r] & \Pi X_*. \end{tikzcd}$$

Proof. All we have to do is to apply Π to the diagram

$$\begin{tikzcd} A_* \arrow[r, "f"] \arrow[d, "i"'] & \{*\} \arrow[d, "\bar{\imath}"] \\ V_* \arrow[r, "\bar{f}"'] & X_* \end{tikzcd}$$

that satisfies the conditions of Theorem 8.2.5. □

Applying to this result the fact that the $\mathrm{res}(-)_n$ functors of Definition 7.1.13 preserve colimits, we get some results on relative homotopy groups.

Corollary 8.2.7. *Let V_*, A_* and X_* be filtered spaces as in Theorem 8.2.6. If V_* is reduced, then we have*

$$\pi_n(X_n, X_{n-1}) \cong \pi_n(V_n, V_{n-1})/N$$

where N is the $\pi_1 V_1$-submodule of $\pi_n(V_n, V_{n-1})$ generated by $i_*\pi_n(A_n, A_{n-1})$ and all elements $\{u - u^a \mid u \in \pi_n(V_n, V_{n-1}),\ a \in i_*\pi_1 A_1\}$.

8.3 Results on pairs of spaces: induced modules and relative homotopy groups

All this section relates to the case when the filtration is essentially in two stages. The HHSvKT in this setting becomes Theorem 8.3.5 and gives quite easily some computations of relative homotopy groups of pairs of spaces and, as consequences, some classical results (e.g. the Suspension Theorem, the Brouwer Degree Theorem, and the Relative Hurewicz Theorem). These are basic theorems in homotopy theory, and it should be noted that we obtain them without the machinery of homology theory.

It will be clear from Part I that a major aspect of this work is to tie in the fundamental group and higher homotopy groups. This contrasts with previous approaches, where the action of the fundamental group is often obtained by passing to the universal covering space. It was an aesthetic objection to this detour to obtain the fundamental group of the circle which led to the groupoid work in [Bro06] and so to the present work. It is also unclear at present how to obtain the nonabelian results of Part I by covering space methods.

8.3.i Specialisation to pairs

Although there are important results on pointed pairs of spaces (X, A) we still have to use the case where A may not be path connected or at any rate has a set of base points, which we will always write A_0. Thus (X, A) with the subset A_0 of A will be called a *based pair* and will sometimes be written for extra clarity as $(X, A; A_0)$ (and a based pair (U, C) will have set of base points C_0).

To relate the homotopy groups of a pair of spaces to the fundamental crossed complex of a filtered space we associate to a based pair of spaces (X, A) a special filtration as follows:

Definition 8.3.1. For any based pair of spaces (X, A) and dimension $n \geqslant 2$, *the filtration $E_n(X, A)$ of X associated to the based pair* (X, A) is given by

$$\underset{0}{A_0} \subseteq \underset{1}{A} \subseteq \cdots \subseteq \underset{n-1}{A} \subseteq \underset{n}{X} \subseteq \cdots \subseteq \underset{r}{X} \subseteq \cdots,$$

i.e. it is A_0 in dimension 0, A in dimensions $0 < r < n$, and X in dimensions $r \geqslant n$. □

The fundamental crossed complex of the filtered space $E_n(X, A)$ has structure in 3 dimensions: the set A_0 in dimension 0, which is part of the groupoid $\pi_1(A, A_0)$ in dimension 1, and the $\pi_1(A, A_0)$-module (crossed module if $n = 2$) $\pi_n(X, A; A_0)$ in dimension n. We have the following obvious fact:

Proposition 8.3.2. *For a based pair (X, A), and $n > 2$, the fundamental crossed complex $\Pi(E_n(X, A))$ of the associated filtration $E_n(X, A)$ is the crossed complex*

$$\mathbb{K}_n(\pi_n(X, A; A_0), \pi_1(A, A_0))$$

of Definition 7.1.11 associated to the $\pi_1(A)$-module $\pi_n(X, A)$.

All we need to do to use this filtration in the HHSvKT is to translate the connectivity of $E_n(X, A)$ into conditions on the based pair (X, A) and see what form the HHSvKT takes in this case. The following is clear.

Proposition 8.3.3. *The filtration $E_n(X, A)$ associated to a based pair of spaces (X, A) is connected if and only if the induced maps*

$$\pi_0 A_0 \to \pi_0 A, \quad \pi_0 A_0 \to \pi_0 X$$

are surjective, and $\pi_i(X, A, x) = 0$ for all $x \in A_0$ and $1 \leqslant i < n$.

Note that the condition $\pi_1(X, A, x) = 0$ means that any path in X from x to a point in A is homotopic rel end points to a path in A.

Definition 8.3.4. If the conditions in the previous proposition hold, we say the based pair (X, A) is $(n - 1)$-*connected.*

8.3.ii Coproducts with amalgamation

Let us translate Theorem 8.2.1 to the case of pairs.

Theorem 8.3.5. *Let (X, A) be a based pair, and suppose:*

(i) $\mathcal{U} = \{U^\lambda\}_{\lambda\in\Lambda}$ *is a family of subsets of X whose interiors cover X;*

(ii) U^0 *is a subset of X such that $U^\lambda \cap U^\mu = U^0$ for all $\lambda, \mu \in \Lambda$ such that $\lambda \neq \mu$;*

(iii) *for $\lambda = 0$ or $\lambda \in \Lambda$, the based pairs (U^0, A^0), (U^λ, A^λ), formed by intersection with U^λ, are $(n-1)$-connected.*

Then

(Con) *the based pair (X, A) is $(n-1)$-connected, and*

(Iso) *the following is a coequaliser diagram in* XMod *if $n = 2$ and in* Mod *if $n > 2$:*

$$(\pi_n(U^0, A^0; A^0_0), \pi_1(A^0, A^0_0)) \xrightarrow{(a^\lambda)} \bigsqcup_{\lambda\in\Lambda} (\pi_n(U^\lambda, A^\lambda, A^\lambda_0), \pi_1(A^\lambda, A^\lambda_0))$$
$$\xrightarrow{c} (\pi_n(X, A, A_0), \pi_1(A, A_0)),$$

where a^λ, c are induced by inclusions.

Remark 8.3.6. In particular, when $\Lambda = \{1, 2\}$, the theorem produces a pushout diagram:

$$\begin{tikzcd}
(\pi_n(U^0, A^0; A^0_0), \pi_1(A^0, A^0_0)) \arrow[r] \arrow[d] & (\pi_n(U^1, A^1; A^1_0), \pi_1(A^1, A^1_0)) \arrow[d] \\
(\pi_n(U^2, A^2; A^2_0), \pi_1(A^2, A^2_0)) \arrow[r] & (\pi_n(X, A; A_0), \pi_1(A, A_0)).
\end{tikzcd}$$ □

We apply this result in the next subsections to deduce some classical results, including the free crossed complex description of the fundamental crossed complex of the skeletal filtration of a CW-complex.

8.3.iii Induced modules and homotopical excision

We now specialise the pushout part of Theorem 8.3.5 of the previous subsection into an excision result which has many applications:

First we recall some terminology. If $(\phi, f) : (M, G) \to (N, H)$ is a morphism of modules or crossed modules over groupoids determining N as the module or crossed module $f_*(M)$ induced by f; then we say the morphism (ϕ, f) is *cocartesian over* f, or even more succinctly that $\phi : M \to N$ is cocartesian over $f : G \to H$; for more information, see Section B.2.

Theorem 8.3.7 (Homotopical Excision Theorem). *Let the topological space X be the union of the interiors of sets U, V, and let $W = U \cap V$. Let $n \geqslant 2$. Let $W_0 \subseteq U_0 \subseteq U$ be such that the based pair $(V, W; W_0)$ is $(n-1)$-connected and W_0 meets each path component of U. Then*

(Con) *$(X, U; U_0)$ is $(n-1)$-connected, and*

(Iso) *the morphism of modules (crossed if $n = 2$)*

$$\pi_n(V, W; W_0) \to \pi_n(X, U; U_0)$$

induced by inclusions is cocartesian over the morphism of fundamental groupoids

$$\pi_1(W, W_0) \to \pi_1(U, U_0)$$

induced by inclusion.

Proof. This is just Theorem 8.3.5 applied to $E_{n-1}(X, A)$. □

Remark 8.3.8. This should be compared with the excision axiom for relative homology, which in one form simply says that if X is the union of open sets U, V then the map in homology $H_i(U, U \cap V) \to H_i(U \cup V, V)$ induced by inclusion, is an isomorphism for all $i \geqslant 0$. It is this result which makes homology calculable. By contrast, this homotopical excision result has connectivity conditions, it determines only one group, but it also links two groups in separated dimensions, where the lower one is usually nonabelian.[112] □

This last theorem applies to give a comparable result, but for closed cofibrations:

Theorem 8.3.9 (Homotopical Excision for an adjunction). *Suppose that in the commutative square of spaces*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & U \arrow[d, "\bar{\imath}"] \\ V \arrow[r, "\bar{f}"'] & X = U \cup_f V \end{tikzcd}$$

the map i is a closed cofibration and X is the adjunction space $U \cup_f V$. Let A_0 be a subset of A meeting each path component of A and V, and let U_0 be a subset of U meeting each path component of U and such that $f(A_0) \subseteq U_0$. Let $n \geqslant 2$. Suppose that the based pair (V, A) is $(n-1)$-connected. Then

(Con) *the based pair (X, U) is $(n-1)$-connected, and*

(Iso) *the induced morphism of modules (crossed if $n = 2$)*

$$\bar{f}_* : \pi_n(V, A; A_0) \to \pi_n(X, U; U_0)$$

is cocartesian over the induced morphism of fundamental groupoids

$$f_* : \pi_1(A, A_0) \to \pi_1(U, U_0).$$

Proof. This follows from Theorem 8.3.7 using mapping cylinders in a similar manner to the proof of a corresponding result for the fundamental groupoid [Bro06], 9.1.2. That is, we form the mapping cylinder $Y = M(f) \cup W$. The closed cofibration assumption ensures that the projection from Y to $X = U \cup_f V$ is a homotopy equivalence. □

Of course for $n = 2$ this result is a generalisation to crossed modules over groupoids of results of Part I, where a number of consequences of a nonabelian type were deduced.

Now we give several applications of Theorem 8.3.9, starting with the pair (CA, A) where CA denotes the cone on the space A. Since CA is contractible it has trivial homotopy groups. Hence the boundary map $\partial\colon \pi_r(CA, A, x) \to \pi_{r-1}(A, x)$ is an isomorphism for all $x \in A$ and $r \geqslant 1$ by the homotopy exact sequence of the pair. Thus the pair (CA, A) is n-connected if and only if A is $(n-1)$-connected, i.e. if A is connected and $\pi_r(A, x) = 0$ for $1 \leqslant r < n$.

First we derive the first n homotopy groups of the n-sphere S^n, using suspension and induction. Since the suspension is just a quotient of the cone, we can use Theorem 8.3.7 to relate the homotopy groups of the suspension SA to those of the base A.

Theorem 8.3.10 (Suspension Theorem). *For a space A, consider SA the (unreduced) suspension of A. If A is $(n-2)$-connected, for $n \geqslant 3$, then*

(Con) *SA is $(n-1)$-connected, and*

(Iso) $\pi_n SA \cong \pi_{n-1} A$.

Proof. We define $V = CA$ the cone on A, $U = \{*\}$ a point and f the constant map. Then $X = U \cup_f V$ is the (unreduced) suspension of A, and we can consider the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & \{*\} \arrow[d, "\bar{\imath}"] \\ CA \arrow[r, "\bar{f}"'] & X = SA. \end{tikzcd}$$

Since A is $(n-2)$-connected if and only if (CA, A) is $(n-1)$-connected, we can apply Theorem 8.3.9, getting that $(X, *)$ is $(n-1)$-connected and

$$\pi_n SA \cong \pi_n(CA, A).$$

Using again the homotopy exact sequence of this pair, we have $\pi_n(CA, A) \cong \pi_{n-1} A$. □

Corollary 8.3.11 (Brouwer Degree Theorem). *For $n \geqslant 1$, S^n is $(n-1)$-connected and*

$$\pi_n(S^n, 1) \cong \mathbb{Z}.$$

Proof. Recall that in Part I in Corollary 5.5.4 we have seen that if $n = 2$ and A is a path-connected space then SA is 1-connected and

$$\pi_2(SA, x) \cong \pi_1(A, x)^{\mathrm{ab}}.$$

Given the value of $\pi_1(S^1, 1)$ as $\mathbb{Z}$ (a result proved in Section 1.7), we deduce that $\pi_2(S^2, 1) \cong \mathbb{Z}$.

The induction step follows easily from the Suspension Theorem 8.3.10. □

This is actually a non-elementary result: that the sphere S^n is $(n-1)$-connected means that any map $S^r \to S^n$ for $r < n$ is nullhomotopic, while the determination of $\Pi(S^n, 1)$ includes the Brouwer Degree Theorem 8.3.11, that the maps $S^n \to S^n$ are classified up to homotopy by an integer, called the *degree* of the map. This was one of the early triumphs in homotopy classification results. Proofs of these results have to use some kind of subdivision argument, but we avoid completely the use of simplicial approximation.

Corollary 8.3.12. *Let E^n_* be the skeletal filtration of the n-cell with cell structure $E^0 = e^0$, $E^1 = e^0_\pm \cup e^1$, and for $n \geqslant 2$, $E^n = e_0 \cup e^{n-1} \cup e^n$. Then $\Pi E^n_* \cong \mathbb{F}(n)$, the free crossed complex on one generator of dimension n, for all $n \geqslant 0$.*

Proof. This follows from the Brouwer Degree Theorem and the homotopy exact sequence of the pair (E^n, S^{n-1}). □

Corollary 8.3.13. *Let X_* be a connected filtration, and let $Y_* = X_* \cup \{e^n_\lambda\}$ be formed by attaching n-cells by filtered maps $f_\lambda \colon S^{n-1}_* \to X_*$, $\lambda \in \Lambda$. Then Y_* is connected, and has fundamental crossed complex formed from ΠX_* by attaching free generators x^n_λ in dimension n.*

Proof. This follows from the previous corollary and the pushout version of the HHSvKT. Note that in this application, we are using many base points in the disjoint union of copies of n-cells. □

Corollary 8.3.14. *If X is a CW-complex with skeletal filtration X_*, then X_* is a connected filtration, and ΠX_* is the free crossed complex on the classes of the characteristic maps of X.*

Proof. This follows from the previous corollary by induction on the skeleta of X. □

Remark 8.3.15. We note that the use of many base points and so of groupoids rather than groups is not a luxury in these applications. Nonreduced CW-complexes such as the geometric n-simplex occur naturally, and a nontrivial covering space of a reduced CW-complex is no longer reduced. □

Remark 8.3.16. We can now deal with the example given in diagram ($*$), p. 259, of the introduction to this chapter. By previous results we know $\pi_n(S^n, 0) \cong \mathbb{Z}$. The inclusion $S^n \to Y = S^n \vee [0, 2]$ is a homotopy equivalence. Let $A_0 = \{0, 1, 2\} \subseteq [0, 2]$, $J = \pi_1([0, 2], A_0)$. Then $\pi_n(Y; A_0)$ is the free J-module on one generator. Theorem 8.3.9 now gives that $\pi_n(S^n \vee S^1 \vee S^1, 0)$ is the free F_2-module on one generator.

As another example, if Y is connected and satisfies $\pi_r(Y, a) = 0$ for $1 < r < n$ and G is a group then we can identify $\pi_n(Y \vee K(G, 1), a)$ as the module *induced* from the $\pi_1(Y, a)$-module $\pi_n(Y, a)$ by the morphism of groups $\pi_1(Y, a) \to \pi_1(Y, a) * G$. You should try to give interesting groupoid examples of this method, using the variety of types of groupoid morphisms. [113] $\square$

8.3.iv Attaching a cone and the Relative Hurewicz Theorem

We explain what Theorem 8.3.9 implies in the case when we are attaching a cone CA via a map of the space A.

Proposition 8.3.17. *Let $X = U \cup_f CA$ for some map $f : A \to U$. For any $n \geqslant 3$, if U is path connected and A is $(n-2)$-connected, then*

(Con) *(X, U) is $(n-1)$-connected, and*

(Iso) *the $\pi_1(U)$-module $\pi_n(X, U)$ is isomorphic to the induced module $\lambda_*(\pi_{n-1}(A))$, i.e.*

$$\pi_n(X, U) \cong \pi_{n-1}A \otimes \mathbb{Z}(\pi_1 U). \qquad \square$$

From this proposition we can determine the effect of attaching n-cells on some of the homotopy groups of a space.

Exercise 8.3.18. Let A, B, U be path-connected, based spaces. Let $X = U \cup_f (CA \times B)$ where CA is the (unreduced) cone on A and f is a map $A \times B \to U$. The homotopy exact sequence of $(CA \times B, A \times B)$ gives

$$\pi_i(CA \times B, A \times B) \cong \pi_{i-1}A, \quad i \geqslant 2, \qquad \text{and} \qquad \pi_1(CA \times B, A \times B) = 0.$$

Suppose now that $n > 2$ and A is $(n-2)$-connected. Then $\pi_1 A = 0$. We conclude from Theorem 8.3.7 that (X, U) is $(n-1)$-connected and $\pi_n(X, U)$ is the $\pi_1 U$-module induced from $\pi_{n-1}A$, considered as trivial $\pi_1 B$-module, by $\lambda = f_* \colon \pi_1 B \to \pi_1 U$. Hence $\pi_n(X, U)$ is the $\pi_1 U$-module

$$\pi_{n-1}A \otimes_{\mathbb{Z}(\pi_1 B)} \mathbb{Z}(\pi_1 U). \qquad \square$$

Now we deduce a version of the classical Relative Hurewicz Theorem. The proof is analogous to the proof of the theorem in dimension 2 given in Theorem 5.5.2.

Theorem 8.3.19 (Relative Hurewicz Theorem). *Let (V, A) be a pair of spaces. Suppose $n \geqslant 3$, A and V are path connected and (V, A) is $(n-1)$-connected. Then*

(Con) *$V \cup CA$ is $(n-1)$-connected, and*

(Iso) *the natural map*

$$\pi_n(V, A, x) \to \pi_n(V \cup CA, CA, x) \xrightarrow{\cong} \pi_n(V \cup CA, x)$$

presents $\pi_n(V \cup CA, x)$ as $\pi_n(V, A, x)$ factored by the action of $\pi_1(A, x)$.

Proof. Let $X = V \cup CA$. We would like to apply Theorem 8.3.5 to the diagram of inclusions

$$\begin{tikzcd} A \arrow[r] \arrow[d] & CA \arrow[d, "\bar{\imath}"] \\ V \arrow[r, "\bar{f}"'] & X = V \cup CA \end{tikzcd}$$

but the subspaces do not satisfy the interior condition. We change the subspaces to $A' = A \times [0, \frac{1}{2}[\subseteq CA$ and $V' = V \cup A'$. Those subspaces have the same homotopy type as A and V (moreover the pair (V', A') has the homotopy type of (V, A)) and we can apply Theorem 8.3.5 to the diagram of inclusions

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & CA \arrow[d, "\bar{\imath}"] \\ V' \arrow[r, "\bar{f}"'] & X = V \cup CA. \end{tikzcd}$$

This yields that X is $(n-1)$-connected, and that

$$\pi_n(X, CA) = \lambda_* \pi_n(V, A),$$

the module induced from the $\pi_1 A$-module $\pi_n(V, A)$ by $\lambda = f_* \colon \pi_1 A \to \pi_1 CA = 0$.

It follows, since $\pi_1 CA = 0$, that $\lambda_* \pi_n(V, A)$ is obtained from $\pi_n(V, A)$ by killing the $\pi_1 A$-action. (If $n = 2$ that would give the abelianisation.)

To finish, note that using that CA is contractible, we get that $\pi_r(X, CA, x)$ is isomorphic to $\pi_r(X, x)$, by the homotopy exact sequence of the pair. □

The usual forms of the Hurewicz Theorem involve homology groups, which lie outside the main scope of this book, although many readers may be well familiar with them. Here we make a few remarks to give a brief account of the relation between the two approaches, and there is more information in Section 14.7.

The homology functors H_n, $n \geqslant 0$, assign to any topological space A or pair of spaces (V, A) abelian groups $H_n(A)$, $H_n(V, A)$ such that: there is a natural exact sequence

$$\cdots \to H_{n+1}(V, A) \xrightarrow{\partial} H_n(A) \to H_n(V) \to H_n(V, A) \to H_{n-1}(A) \to \cdots;$$

if A is a point then $H_0(A) \cong \mathbb{Z}$; and the excision and homotopy axioms, which we do not state here, hold. These axioms imply that the boundary

$$\partial \colon H_{n+1}(\mathsf{E}^{n+1}, \mathsf{S}^n) \to H_n(\mathsf{S}^n)$$

is an isomorphism; it follows by induction that $H_n(\mathsf{S}^n) \cong \mathbb{Z}$ for $n > 0$. Choose a generator ι^n of this group, giving a generator, also written ι^{n+1}, of $H_{n+1}(\mathsf{E}^{n+1}, \mathsf{S}^n)$. The Hurewicz morphisms

$$\omega_n \colon \pi_n(A, x) \to H_n(A), \quad \omega_{n+1} \colon \pi_{n+1}(V, A, x) \to H_n(V, A)$$

are then defined by sending the class of a map f in such a homotopy group to $f_*(\iota^n)$, $f_*(\iota^{n+1})$ respectively where f_* denotes the induced maps in homology. This leads to a morphism from the exact homotopy sequence of a pair to the exact homology sequence, which we use in the next theorem.

We have the following:

Theorem 8.3.20 (Absolute Hurewicz Theorem). *If X is an $(n-1)$-connected space, then the Hurewicz morphism $\omega_i \colon \pi_i(X, x) \to H_i(X)$ is an isomorphism for $0 \leqslant i \leqslant n$ and an epimorphism for $i = n + 1$.*

We will outline a proof of this result in Theorem 14.7.8. The use of filtered spaces is quite appropriate for this proof, and follows the lines of some classical papers.

The usual version of the *Relative Hurewicz Theorem* involves not $\pi_n(V \cup CA, x)$ but the homology $H_n(V, A)$. It is possible to get this more usual version from the one we have just proved in a three stage process.

First, the conclusion of our theorem implies that $V \cup CA$ is $(n-1)$-connected, and so $\pi_n(V \cup CA, x)$ is isomorphic to $H_n(V \cup CA)$ by the Absolute Hurewicz Theorem.

Then it is easy to prove that $H_n(V \cup CA)$ is isomorphic to $H_n(V \cup CA, CA)$ by the homology exact sequence, using that CA is acyclic because it is contractible.

Last, we notice that by excision the morphism $H_n(V, A) \to H_n(V \cup CA, CA)$ induced by inclusion is an isomorphism.

Here is another corollary of the Relative Hurewicz Theorem, which assumes a bit more on the Hurewicz morphism from homotopy to homology. We call it Hopf's theorem, although he gave only the case $n = 2$.

Proposition 8.3.21 (Hopf's theorem). *Let (V, A) be a pair of pointed spaces such that:*

(i) $\pi_i(A) = 0$ *for* $1 < i < n$;
(ii) $\pi_i(V) = 0$ *for* $1 < i \leqslant n$;
(iii) *the inclusion $A \to V$ induces an isomorphism on fundamental groups.*

Then the pair (V, A) is n-connected, and the inclusion $A \to V$ induces an epimorphism $H_n A \to H_n V$ whose kernel consists of spherical elements, i.e. of the image of $\pi_n A$ under the Hurewicz morphism $\omega_n \colon \pi_n(A) \to H_n(A)$.

Proof. That (V, A) is n-connected follows immediately from the homotopy exact sequence of the pair (V, A) up to $\pi_n(V)$. We now consider the next part of the exact homotopy sequence and its relation to the homology exact sequence as shown in the commutative diagram:

$$\begin{tikzcd}
\pi_{n+1}(V,A) \arrow[r, "\partial"] \arrow[d, "\omega_{n+1}"'] & \pi_n(A) \arrow[r] \arrow[d, "\omega_n"'] & \pi_n(V) \arrow[r] \arrow[d] & \pi_n(V,A) \arrow[d] \\
H_{n+1}(V,A) \arrow[r, "\partial'"'] & H_n(A) \arrow[r, "i_*"'] & H_n(V) \arrow[r] & H_n(V,A).
\end{tikzcd}$$

The Relative Hurewicz Theorem implies that $H_n(V, A) = 0$, and that ω_{n+1} is surjective. Also ∂ in the top row is surjective, since $\pi_n(V) = 0$. It follows easily that the sequence $\pi_n(A) \to H_n(A) \to H_n(V) \to 0$ is exact. □

8.4 The chain complex of a filtered space and of a CW-complex

In this section we show that for certain filtered spaces X_* the chain complex $\nabla\Pi X_*$ may be identified in terms of chains of universal covers. The chain complex of cellular chains of the universal cover of a pointed CW-complex X, regarded as a chain complex of modules over $\pi_1(X, *)$, is a well-known tool in algebraic topology[114].

All spaces which arise will now be assumed to be Hausdorff and to have universal covers. Recall, [Bro06], 10.5.8, that if X is a topological space and $v \in X$ then the universal covering map $p\colon \widetilde{X}(v) \to X$, can be constructed by topologising the fundamental groupoid $\pi_1(X)$ and considering the final point map $t\colon \pi_1 X \to X$, writing

$$\widetilde{X}(v) = t^{-1}(v)$$

and identifying p with the initial point map s. This space has a canonical base point, $1_v \in \pi_1 X$. These spaces form a bundle over X on which $\pi_1 X$ operates by composition, but not preserving the base point.

Let X_* be a filtered space. For $v \in X_0$, $i \geqslant 0$, let $\widehat{X}_*(v)$ denote the filtered space consisting of $\widetilde{X}(v)$ and the family of subspaces

$$\widehat{X}_i(v) = p^{-1}(X_i).$$

Definition 8.4.1. We define for a filtered space X_* the chain complex with operators CX_* to have for all $v \in X_0, i \geqslant 1$,

$$C_0 X_* = H_0(\widehat{X}_0(v)), \quad C_i X_*(v) = H_i(\widehat{X}_i(v), \widehat{X}_{i-1}(v)),$$

and to have groupoid of operators $\pi_1(X, X_0)$ with operation induced by the bundle operations given above. This defines the functor *fundamental chain complex of a filtered space*

$$C\colon \mathsf{FTop} \to \mathsf{Chn}.$$ □

Proposition 8.4.2. *If X_* is a connected filtered space, for example a CW-filtration, then there is a natural isomorphism*

$$CX_* \cong \nabla \Pi X_*.$$

Proof. Notice that, in this case, $\widehat{X}_i(v)$ is the universal cover of X_i based at v for $i \geqslant 2$.

We will use the Relative Hurewicz Theorem 8.3.19.

Let $v \in X_0$ and let $i \geqslant 3$. The pair $(\widehat{X}_i(v), \widehat{X}_{i-1}(v))$ is $(i-1)$-connected, and $\widehat{X}_{i-1}(v))$ is simply connected, and so

$$\begin{aligned} \pi_i(X_i(v), X_{i-1}, v) &\cong \pi_i(\widehat{X}_i(v), \widehat{X}_{i-1}(v), 1_v) && \text{since } p \text{ is a covering} \\ &\cong H_i(\widehat{X}_i(v), \widehat{X}_{i-1}(v)) && \text{by the Relative Hurewicz Theorem,} \end{aligned}$$

since $\widehat{X}_i(v)$ and $\widehat{X}_{i-1}(v)$ are in fact the universal covers at v of X_i and X_{i-1} respectively. If $i = 2$, a similar argument applies but in this case $\pi_1(\widehat{X}_1, v) = \delta\pi_2(\widehat{X}_2(v), \widehat{X}_1(v), 1_v)$. So the Relative Hurewicz Theorem in dimension 2 (Theorem 5.5.2) now gives

$$\begin{aligned} H_2(\widehat{X}_2(v), \widehat{X}_1(v)) &\cong \pi_2(\widehat{X}_2(v), \widehat{X}_1, 1_v)^{\mathrm{ab}} \\ &\cong \pi_2(X_2, X_1, v)^{\mathrm{ab}} \\ &= (\nabla \Pi X_*)_2. \end{aligned}$$

For the case $i = 1$ we can use the result of Theorem 14.7.5 on the abelianisation of $\pi_1(X, X_0)$, and the result of our Exercise 7.4.26 giving a description of the derived module in terms of an abelianisation of a groupoid.[115] □

The following corollary of these results does not seem generally known, nor to have been proved by other methods.[116]

Corollary 8.4.3. *Let X_* be a filtered space and suppose that X is the union of a family $\mathcal{U} = \{U^\lambda\}_{\lambda\in\Lambda}$ of open sets such that $\mathcal{U}$ is closed under finite intersection. Let U^λ_* be the filtered space obtained from X_* by intersection with U^λ. Suppose that each U^λ_* is a connected filtered space. Then X_* is connected and the natural morphism in* Chn

$$\operatorname{colim}^\lambda CU^\lambda_* \to CX_*$$

is an isomorphism.

Proof. The HHSvKT 8.1.5 gives a similar result for Π rather than C. Then we apply ∇ which has a right adjoint and so preserves colimits. □

Notes

110 p. 258 This insight that the Relative Hurewicz Theorem followed from a Seifert-van Kampen Theorem was a stimulus to the work on an extension of the Relative Hurewicz Theorem to a triadic Hurewicz Theorem in [BL87], [BL87a], [Bro89] for which some background is given in Section B.4: it was felt that the more general theorem could be easier to prove, and so it turned out.

111 p. 261 Notice that there are duplications of the indexing in Λ^2, and unnecessary pairs (λ, λ) but this does not matter.

112 p. 268 More powerful connectivity results due initially to Blakers and Massey [BM49] may be found under the label 'Excision Theorem' in [Hat02], Theorem 4.23, p. 360, and in [tD08], Theorem 6.4.1, p. 133. The results of Blakers–Massey led to a search for the calculation of the critical triad homotopy group, see for example [Ada72], though the results given there have problems with cases where this group may be nonabelian, which were resolved in [BL87] with the notion of nonabelian tensor product of groups with actions. These early results on triads were generalised to n-ads, and a relevant paper is [BW56], though this seems to lack a proof of the connectivity result. The most general results of this kind is in [ES87], Theorems 3.7, 3.8; it uses the main result of [BL87] to prove the n-ad connectivity theorem and compute the first critical group, even when it may be nonabelian, and so unreachable by homological techniques. Another proof of the n-ad connectivity result uses general position arguments, [Goo92]; such arguments have not been modelled algebraically and so usually do not directly yield algebraic results. You should find it worthwhile to compare the methodologies of all these results and proofs.

113 p. 271 There is much work on 2-complexes which uses combinatorial group theory or module theory, but which does not use results on crossed modules, for example [FR05], [Joh03], [Joh09], so it would be interesting to know if the two kinds of methods could be combined. The paper [Pap63] does apply Whitehead's theorem on free crossed modules, but our more general results on $\pi_2(X \cup CA, X, x)$ have not so far been applied generally in low dimensional topology.

114 p. 274 These cellular chains with operators are used in simple homotopy theory, see for example [Coh73].

115 p. 275 The case $i = 1$ is essentially the result of Crowell [Cro71], Section 5. Full details of our argument for this are given in [Bro11], and are also related to [Whi49b], Section 11. Whitehead used the chains of the universal cover in a number of papers, and in this was strongly influenced by Reidemeister's work in [Rei49]. That work developed also into work of Eilenberg–Mac Lane on homology of spaces with operators, [Eil47], [EML49].

116 p. 275 We note that colimit results such as this have been used by various workers ([Lom81], [PS85]) in the case X_* is the skeletal filtration of a CW-complex and the family $\mathcal{U}$ is a family of subcomplexes, although usually in simple cases. The general form of this 'Seifert–van Kampen Theorem' for CX_* does not seem to have been noticed. This may be due to the unfamiliar form of colimits in the category Chn of chain complexes over varying groupoids, which even in the group case are not quite what might be expected, see Example 7.4.30. Of course one of the aims of this book is to make such algebraic colimit arguments and results more familiar in algebraic topology, so that they can be used there and hopefully more widely.

Chapter 9
Tensor products and homotopies of crossed complexes

This chapter is built around the notion of *monoidal closed category*, whose definition may be found in Section C.7 of Appendix C, and on the use of such a structure on the category Crs of crossed complexes.[117]

This will raise conceptual difficulties for those not used to the ideas so we shall give some background and introduction in Section 9.1.

The monoidal closed structure for the category Crs gives a natural 'exponential law', i.e. a natural isomorphism

$$e\colon \mathsf{Crs}(C \otimes D, E) \cong \mathsf{Crs}(C, \mathsf{CRS}(D, E)),$$

for crossed complexes C, D, E. Here 'monoidal' refers to the 'tensor product' $C \otimes D$ and 'closed' refers to the 'internal hom' $\mathsf{CRS}(D, E)$.

The elements of $\mathsf{CRS}(D, E)$ may be written out explicitly – they are morphisms $D \to E$ in dimension 0, homotopies of morphisms in dimension 1, and 'higher homotopies' for $n > 1$. One advantage of this procedure is that we can use crossed complex techniques not only on filtered spaces but also on maps and homotopies of filtered spaces.

The case of the 'tensor product' raises technical difficulties. For $C \otimes D$ we can give in the first instance only generators $c \otimes d$, $c \in C_m, d \in D_n$, $m, n \geqslant 0$ and the structure and axioms on these.

Sometimes we use a formal description of this monoidal closed structure on the category of crossed complexes, but the fact that we can if necessary get our hands dirty, that is write down some complex formulae and rules and calculate with them, is one of the aspects of the theory that gives power to the category of crossed complexes.

The complication of the rules for the tensor product is due to their modeling the geometry of the product of cells. It is important to get familiar with these formulae for the tensor product as they will be used frequently in the applications of this and the remaining chapters of this Part II.

The natural way to be sure we have the right formulation of this structure is not to define it directly, but through the equivalence with the category of ω-groupoids and the natural definition of tensor product and internal hom in that category. This we do in Chapter 15 and so ensure that the definitions for the category Crs will work. Here, we state directly the definition that results from this detour, risking that this could make the rules for the tensor product in Crs seem too awkward.

As an introduction to the crossed complex case, and because we need this example, we describe in Section 9.2 the structure of monoidal closed category for the category

Mod of modules over groupoids. This gives a natural equivalence in Mod

$$\mathsf{Mod}((M \otimes N, G \times H), (P, K)) \cong \mathsf{Mod}((M, G), \mathsf{MOD}((N, H), (P, K)).$$

Later in Section 9.3 we define $\mathsf{CRS}(D, E)$ for crossed complexes D, E by: the elements of $\mathsf{CRS}_0(D, E)$ are to be just the morphisms $C \to D$; the elements of $\mathsf{CRS}_1(D, E)$ are to be the homotopies defined in Section 7.1.vii; and for $m \geqslant 2$ we give explicitly the elements of $\mathsf{CRS}_m(D, E)$. The rules for addition, action and boundaries are related to the geometry associated to the free crossed complexes on products $E^m \times E^n$ of cells.

Analogously to the development of the tensor product for R-modules indicated above, the set of morphisms of crossed complexes

$$C \to \mathsf{CRS}(D, E)$$

is bijective to the set of 'bimorphisms'

$$(C, D) \to E,$$

where these bimorphisms play for crossed complexes the same role that bilinear maps play for modules.

Then, as in the case of R-modules, we can form the tensor product $C \otimes D$ of crossed complexes by taking free objects and quotienting out by the appropriate relations.[118]

We give a direct description of $C \otimes D$ first of all in dimensions 1 and 2, in Sections 9.4.i, 9.4.ii. Then we use the monoidal closed structure on the category Chn of chain complexes with a groupoid of operators, in order to get a clearer description of $(C \otimes D)_n$ for $n > 2$ as $(\nabla C \otimes \nabla D)_n$.

We end the algebraic part of this chapter by proving in Section 9.6 the important result that the tensor product preserves freeness. This uses crucially the adjoint relation of the tensor product to the internal hom.

The second part of the chapter deals with the topological applications, relating the monoidal closed category of Crs and that of FTop via the fundamental crossed complex functor

$$\Pi \colon \mathsf{FTop} \to \mathsf{Crs}.$$

We start by giving in Section 9.7 a structure of monoidal closed category to the category FTop of filtered topological spaces and filtered maps; this structure is a straightforward generalisation of the structure of cartesian closed category for Top already mentioned in this introduction.

The relationship between the structures of monoidal closed category on FTop and on Crs is explained in Section 9.8. Again, we leave the proofs of these results to Chapter 15 of Part III. The main result is Theorem 9.8.1 stating how the functor Π behaves with respect to tensor products. In particular, if X_*, Y_* are filtered spaces, then there is a natural transformation

$$\zeta \colon \Pi(X_*) \otimes \Pi(Y_*) \to \Pi(X_* \otimes Y_*)$$

which is an isomorphism if X_*, Y_* are CW-complexes.

The tensor product in the categories FTop and Crs allows homotopies to be interpreted in these categories as maps from a 'cylinder functor' which in FTop is of the form $I_* \otimes -$. Thus an immediate consequence of Theorem 9.8.1 is that the fundamental crossed complex functor Π is a homotopy functor. This, and the analysis of the cone of a crossed complex, leads in Section 9.9 to computations on the fundamental crossed complex of an n-simplex providing a version of the simplicial Homotopy Addition Lemma (Theorem 9.9.4). A similar result is true for n-cubes giving a cubical Homotopy Addition Lemma (Proposition 9.9.6).

9.1 Some exponential laws in topology and algebra

The start of the idea of a monoidal closed category is that a function of two variables $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ can also be regarded as a variable function of one variable. This is the basis of partial differentiation. In general, this transforms into the idea that if Z^Y denotes the set of functions from the set Y to the set Z, then we have a bijection of sets

$$e: Z^{X\times Y} \to (Z^Y)^X$$

given by

$$e(f)(x)(y) = f(x, y), \quad x \in X,\ y \in Y.$$

This corresponds to the exponential law for numbers $m^{np} = (m^n)^p$, and so the previous law is called the *exponential law for sets*.

Because there is a bijection $X \times Y \to Y \times X$ this also means we can set up bijections between the three sets of functions

$$X \to Z^Y, \quad Y \to Z^X, \quad X \times Y \to Z.$$

This becomes particularly interesting in its interpretation when $Y = I = [0, 1]$, the unit interval, since the functions $I \to Z$ can be thought of as paths in Z, and so the set of these functions is a kind of space of paths; in practice we will want to have topologies on these sets and speak only of continuous functions, but let us elide over that for the moment.

The functions $X \to Z$ we can intuitively call 'configurations of X in Z'. A function $X \times I \to Z$ we can think of as a deformation of configurations. This can be seen alternatively as a path in the configuration space Z^X, or as a configuration $X \to Z^I$ in the path space of Z. These alternative points of view have proved strongly useful in mathematics.

It is useful to rephrase the exponential law slightly more categorically, so as make analogies for other categories, so we write it also as a bijection

$$e: \mathsf{Set}(X \times Y, Z) \cong \mathsf{Set}(X, \mathsf{SET}(Y, Z)).$$

Here the distinction between Set and SET, i.e. between external and internal to the category, is less clear than it will be in the other examples in this section. For further examples see Appendix C, in particular the categories Cat in Section C.2 and Gpds in Section C.3.

Now suppose that X, Y, Z are topological spaces, and $\mathsf{Top}(Y, Z)$ denotes the set of continuous maps $Y \to Z$. We would like to make this set into a topological space $\mathsf{TOP}(Y, Z)$ so that the exponential correspondence gives a natural bijection

$$\mathsf{Top}(X \times Y, Z) \cong \mathsf{Top}(X, \mathsf{TOP}(Y, Z)).$$

However this turned out not to be possible for all topological spaces, and in the end a reasonable solution was found by restricting to what are called 'compactly generated spaces', and working entirely in the category of these spaces. In this book Top will mean the category of compactly generated spaces. An account of this category is given in [Bro06], Section 5.9, and we assume this to be known. The existence of the exponential law as above is summarised by saying that the category Top is a *cartesian closed category*.[119] Here 'cartesian' refers to the fact that we use the categorical product in the category, and 'closed' means that there is a space $\mathsf{TOP}(Y, Z)$ for all spaces Y, Z in the category Top. The space $\mathsf{TOP}(Y, Z)$ is also called the *internal hom* in Top.

It is a deduction from the exponential law and associativity of the cartesian product that there is also a natural homeomorphism

$$\mathsf{TOP}(X \times Y, Z) \cong \mathsf{TOP}(X, \mathsf{TOP}(Y, Z)).$$

We leave the proof of this to the reader.

There are a couple of special characteristics to this example. First, the underlying set of the space $\mathsf{TOP}(Y, Z)$ is the set $\mathsf{Top}(Y, Z)$, but for other exponential laws there is no reason why this should be so. We shall come later to this point (see Section 9.7).

Second, the product we are using above is the categorically defined cartesian product in the category. There are analogous laws in other categories involving other types of product than the cartesian product.

For example, if Mod_R denotes the category of left modules over a commutative ring R with morphisms the R-linear maps, then we can for R-modules N, P form an R-module structure on the set $\mathsf{Mod}_R(N, P)$ to give an R-module which we write $\mathsf{MOD}_R(N, P)$. This we call the *internal hom* in the category Mod_R.

Recall that a *bilinear map* $(M, N) \to P$ of R-modules is a function $f : M \times N \to P$ which satisfies

$$\begin{aligned} f(rm + m', n) &= rf(m, n) + f(m', n), \\ f(m, rn + n') &= rf(m, n) + f(m, n') \end{aligned}$$

for all $m, m' \in M$, $n, n' \in N$ and $r \in R$. The set of all of these functions is written $\mathsf{BiLin}_R((M, N); P)$.

A standard construction is the 'universal bilinear map' $(M, N) \to M \otimes_R N$ so as to obtain a natural bijection

$$\mathsf{BiLin}_R((M, N); P) \cong \mathsf{Mod}_R(M \otimes_R N, P).$$

If M, N are left R-modules, then $M \otimes_R N$ can be constructed as the free R-module on elements $m \otimes n$ for $m \in M, n \in N$ factored by the relations

$$(rm + m') \otimes n = r(m \otimes n) + (m' \otimes n),$$
$$m \otimes (rn + n') = r(m \otimes n) + (m \otimes n')$$

for all $m, m' \in M$, $n, n' \in N$, $r \in R$.

Notice that the two first families of relations have consequences such as

$$m \otimes 0 = 0 \otimes n = 0,$$

$$(-m) \otimes n = m \otimes (-n) = -(m \otimes n)$$

while the third family of relations can be used to define a structure of R-module by the action

$$r(m \otimes n) = rm \otimes n = m \otimes rn.$$

This gives as a consequence the linearity on both variables of the tensor product

$$(rm + r'm') \otimes n = r(m \otimes n) + r'(m' \otimes n),$$

$$m \otimes (sn + s'n') = s(m \otimes n) + s'(m \otimes n').$$

By the given construction, an element of $M \otimes_R N$ is an R-linear combination of *decomposable elements* of the form $m \otimes n$. In general, it is not quite so obvious what are the actual elements of $M \otimes_R N$ for specific R, M, N. Nonetheless, the tensor product of R-modules plays an important role in module theory. One reason is that whereas a bilinear map does not have a defined notion of kernel, a morphism $M \otimes_R N \to P$ to an R-module P does have a kernel. This process of using a universal property to replace a function with complicated properties by a morphism is a powerful procedure in mathematics.

The *tensor product* construction, i.e. the bifunctor,

$$- \otimes_R -\colon \mathsf{Mod}_R \times \mathsf{Mod}_R \to \mathsf{Mod}_R,$$

does not give a categorical product in the category Mod_R. To describe this situation, category theorists have developed the notion of *monoidal category*, and to give a general setting for the exponential law

$$\mathsf{Mod}_R(M \otimes_R N, P) \cong \mathsf{Mod}_R(M, \mathsf{MOD}_R(N, P))$$

have developed the notion of *monoidal closed category*. This law can also stated as: for all R-modules N the functor $- \otimes_R N$ is left adjoint to $\mathsf{MOD}_R(N, -)$. This has some immediate and valuable consequences on the preservation of colimits and limits by these functors.

Both as a reminder of and as an introduction to the more involved crossed complex case, this process is described in the next section for the category Mod of modules over groupoids. There are some key difference between the above discussion and that for Mod, namely that the operator domain varies. This would be analogous to considering modules over arbitrary rings, when the tensor product of modules (M, R), (N, S) over possibly noncommutative rings R, S would be a module $(M \otimes N, R \otimes S)$.[120]

9.2 Monoidal closed structure on the category of modules over groupoids

As a special case of the previous discussion, there are well-known definitions of tensor product and internal hom functor for abelian groups without operators. If one allows operators from arbitrary groups the tensor product is easily generalised, with the tensor product of a G-module and an H-module being a $(G \times H)$-module. However, the adjoint construction of internal hom functor does not exist, basically because the group morphisms from G to H do not form a group. To rectify this situation we allow operators from arbitrary groupoids, rather than groups, and we give a discussion of the monoidal closed category structure of Mod the category of modules over groupoids introduced in Definition 7.1.7.

As is customary, we write M for the G-module (M, G) when the operating groupoid G is clear from the context. Also, to simplify notation, we will assume throughout this chapter that the abelian groups $M(x)$ for $x \in G_0$ are all disjoint; any G-module is isomorphic to one of this type.

In many of our categories, it will be easier to describe internal homs explicitly, than the corresponding tensor product. We illustrate this by describing the internal hom structure in the category Mod.[121]

An internal hom groupoid $\mathsf{GPDS}(G, H)$ in the category Gpds is described in Section C.3 of Appendix C: its objects are morphisms $\theta\colon G \to H$ of groupoids and its morphisms are natural transformations $\alpha\colon \theta \to \theta'$. Recall that such a natural transformations α is given by a family $\{\alpha(x)\}_{x\in G_0}$ where $\alpha(x) \in H(\theta(x), \theta'(x))$ and the diagram

$$\begin{tikzcd}
\theta(x) \arrow[r, "\alpha(x)"] \arrow[d, "\theta(g)"'] & \theta'(x) \arrow[d, "\theta'(g)"] \\
\theta(y) \arrow[r, "\alpha(y)"'] & \theta'(y)
\end{tikzcd}$$

commutes for all $g \in G(x, y)$. Because of our convention for composition in a groupoid, this commutativity is read as

$$\alpha(x)\theta'(g) = \theta(g)\alpha(x).$$

Definition 9.2.1. Let (M, G), (N, H) be modules. To construct

$$\mathsf{MOD}((M, G), (N, H)),$$

the *internal hom* in MOD, we have to give the set

$$\mathsf{Mod}((M, G), (N, H))$$

of morphisms of modules $(f, \theta)\colon (M, G) \to (N, H)$ the structure of module over a groupoid. Notice that f is given by a family $\{f(x)\}_{x \in G_0}$ where $f(x)\colon M(x) \to N(\theta(x))$ are group morphisms satisfying

$$f(y)(m^g) = f(x)(m)^{\theta(g)} \quad \text{for } m \in M(x),\ g \in G(x, y).$$

For a fixed morphism $\theta\colon G \to H$ of groupoids, we define the set

$$\mathsf{Mod}((M, G), (N, H))(\theta)$$

of module morphisms over θ to be the set of morphisms of the form

$$(f, \theta) \in \mathsf{Mod}((M, G), (N, H)).$$

It is easy to see that each $\mathsf{Mod}((M, G), (N, H))(\theta)$ forms an abelian group under element-wise addition, so that $\mathsf{Mod}((M, G), (N, H))$ for all morphisms $\theta\colon G \to H$ forms a family of abelian groups indexed by the set of objects of the groupoid $\mathsf{GPDS}(G, H)$. Thus

$$\mathsf{MOD}((M, G), (N, H)) = \{\mathsf{Mod}((M, G), (N, H))(\theta) \mid \theta \in \mathsf{Gpds}(G, H)\}.$$

It remains to describe the action of $\mathsf{GPDS}(G, H)$ on $\mathsf{MOD}((M, G), (N, H))$, i.e. for each $\theta, \theta' \in \mathsf{Gpds}(G, H)$ we need a map

$$\mathsf{Mod}((M, G), (N, H))(\theta) \times \mathsf{GPDS}(G, H)(\theta, \theta') \to \mathsf{Mod}((M, G), (N, H))(\theta').$$

So let f be such that (f, θ) is a morphism of modules and let $\alpha\colon \theta \to \theta'$ be a natural transformation. We define

$$(f, \theta)^\alpha = (f^\alpha, \theta')$$

where f^α is a family of morphisms

$$f^\alpha(x)\colon M(x) \to N(\theta'(x)), \quad x \in G_0$$

where $f^\alpha(x)$ is defined as the composition

$$M(x) \xrightarrow{f(x)} N(\theta(x)) \xrightarrow{(-)^{\alpha(x)}} N(\theta'(x)),$$

i.e. $f^\alpha(x)(m) = (f(x)(m))^{\alpha(x)}$. This defines a morphism of modules because if $g \in G(x,y)$, $m \in M(x)$ then

$$\begin{aligned} f^\alpha(y)(m^g) &= (f(y)(m^g))^{\alpha(y)} \\ &= (f(x)(m))^{\theta(g)\alpha(y)} \\ &= (f(x)(m))^{\alpha(x)\theta'(g)} \\ &= (f^\alpha(x)(m))^{\theta'(g)}. \end{aligned}$$

It is not difficult to prove that this definition satisfies the properties of an action giving a structure of module

$$\mathsf{MOD}(M,N) = (\mathsf{Mod}((M,G),(N,H)), \mathsf{GPDS}(G,H))$$

which we take as the internal hom functor in Mod. □

It is quite straightforward to see that, as in the group case, we can characterise the elements of this internal hom functor in terms of 'bilinear' maps.

Definition 9.2.2. A *bilinear map of modules over groupoids*

$$(M,G) \times (N,H) \to (P,K)$$

is given by a pair of maps (f,θ) where $\theta\colon G \times H \to K$ is a map of groupoids and $f : M \times N \to P$ is given by a family of bilinear maps

$$f(x,z)\colon M(x) \times N(z) \to P(\theta(x,z)), \quad x \in G_0,\ z \in H_0$$

which preserve actions, i.e. if $g\colon x \to y$, $h\colon z \to w$ then

$$f(y,w)(m^g, n^h) = (f(x,z)(m,n))^{\theta(g,h)}.$$

The set of all those bilinear maps is written

$$\mathsf{BiLin}((M,G),(N,H);(P,K)).$$

By abuse of language we shall often suppress the base groupoid. □

Proposition 9.2.3. *There is a natural bijection*

$$\mathsf{BiLin}(M,N;P) \to \mathsf{Mod}(M, \mathsf{MOD}(N,P))$$

between the set of bilinear maps $M \times N \to P$ and the set of morphisms of modules from M to $\mathsf{MOD}(N,P)$.

Proof. Given $(f,\theta) \in \mathsf{BiLin}(M,N;P)$ we define $(\hat f, \hat\theta) \in \mathsf{Mod}(M, \mathsf{MOD}(N,P))$ by

$$\hat\theta(x)(y) = \theta(x,y) \quad \text{and} \quad \hat f(m)(n) = f(m,n).$$

It is easy to see that $(\hat f, \hat\theta)$ is a module morphism and that this assignation $(f,\theta) \mapsto (\hat f, \hat\theta)$ is a natural bijection. □

Now the tensor product is constructed as to transform these bilinear maps into morphisms of modules.

Definition 9.2.4. The *tensor product* in Mod ofmodules is given by

$$(M, G) \otimes (N, H) = (M \otimes N, G \times H)$$

where, for $x \in G_0$, $z \in H_0$,

$$(M \otimes N)(x, z) = M(x) \otimes_{\mathbb{Z}} N(z)$$

and the action is given by

$$(m \otimes n)^{(g,h)} = m^g \otimes n^h. \qquad \Box$$

Remark 9.2.5. The module $M \otimes N$ is the $(G \times H)$-module generated by all elements

$$\{m \otimes n \mid m \in M, n \in N\}$$

subject to the relations

$$\begin{aligned} (m + m') \otimes n &= (m \otimes n) + (m' \otimes n), \\ m \otimes (n + n') &= (m \otimes n) + (m \otimes n'), \\ (m \otimes n)^{(g,h)} &= m^g \otimes n^h. \end{aligned}$$

The consequence of this is that is not so easy to give an explicit description of the elements of $M \otimes N$, but, as we see in the next proposition, to define a morphism $M \otimes N \to P$ all we need is a bilinear map $M \times N \to P$. $\Box$

Proposition 9.2.6. *There is a natural bijection*

$$\mathsf{BiLin}(M, N; P) \to \mathsf{Mod}(M \otimes N, P)$$

between the set of bilinear maps $M \times N \to P$ and the set of morphisms of modules $M \otimes N \to P$.

Proof. Given a bilinear map $(f, \theta)\colon M \times N \to P$ we define

$$\hat{\theta}(g, h) = \theta(g)(h) \quad \text{and} \quad \hat{f}(m \otimes n) = f(m, n).$$

It is easy to see that $(\hat{f}, \hat{\theta})$ is a morphism of modules and that this assignment $(f, \theta) \mapsto (\hat{f}, \hat{\theta})$ gives a natural bijection as required. $\Box$

The tensor product and the internal morphisms just defined give Mod the structure of symmetric monoidal closed category with unit object the module $(\mathbb{Z}, 1)$. We leave to the reader to check the symmetric monoidal part (that the tensor product is associative, commutative, that $(\mathbb{Z}, 1)$ is the unit) and to explain the exponential law.

Proposition 9.2.7. *There is a natural bijection*

$$\mathsf{MOD}(M \otimes N, P) \cong \mathsf{MOD}(M, \mathsf{MOD}(N, P)).$$

Proof. It is straightforward to verify the natural bijection

$$\mathsf{Mod}(M \otimes N, P) \cong \mathsf{Mod}(M, \mathsf{MOD}(N, P)),$$

where P is a G-module.

These families of groups are modules over

$$\mathsf{GPDS}(G \times H, K) \cong \mathsf{GPDS}(G, \mathsf{GPDS}(H, K))$$

and the actions agree, giving a natural isomorphism of modules

$$\mathsf{MOD}(M \otimes N, P) \cong \mathsf{MOD}(M, \mathsf{MOD}(N, P)). \qquad \square$$

9.3 Monoidal closed structure on the category of crossed complexes

Analogously to the way the internal morphisms gave a correspondence from morphisms in the internal hom construction to bilinear maps and then to morphisms of the tensor product, as in

$$\mathsf{Mod}_R(C \otimes D, E) \cong \mathsf{Mod}_R(C, \mathsf{MOD}_R(D, E)),$$

so we obtain an internal hom $\mathsf{CRS}(D, E)$ for crossed complexes D, E as part of an exponential law

$$\mathsf{Crs}(C \otimes D, E) \cong \mathsf{Crs}(C, \mathsf{CRS}(D, E)), \tag{9.3.1}$$

for crossed complexes C, D, E where the 'internal hom' $\mathsf{CRS}(D, E)$ is of course again a crossed complex. Crossed complexes have structure in a range of dimensions, whereas R-modules have structure in just one dimension, so the description of the internal hom in Crs has to be much more complicated than that in Mod_R, and indeed this complication is part of its value in modeling complicated geometry.

We define the internal hom for crossed complexes as giving a 'home' for the notion of 'higher dimensional homotopy' as follows. We have already observed that in any crossed complex C, the set of m-dimensional elements C_m is bijective with the set of morphisms of crossed complexes $\mathsf{Crs}(\mathbb{F}(m), C)$, where $\mathbb{F}(m)$ is the free crossed complex on one generator of dimension m (see Proposition 7.3.14).

When, in the exponential law, we take $C = \mathbb{F}(m)$, we have

$$\mathsf{Crs}(\mathbb{F}(m) \otimes D, E) \cong \mathsf{Crs}(\mathbb{F}(m), \mathsf{CRS}(D, E)) \cong \mathsf{CRS}_m(D, E). \tag{9.3.2}$$

So, the elements of $\mathsf{CRS}_m(D, E)$ can be seen as 'm-fold left homotopies' $D \to E$ and that description is useful to understand the definition and properties of Section 9.3.i.

To proceed to the tensor product necessitates defining the notion of *bimorphism*

$$\xi\colon (C, D) \to E$$

for crossed complexes C, D, E, so that such bimorphisms correspond exactly to morphisms $C \to \mathsf{CRS}(D, E)$.

The algebraic properties of bimorphisms are quite complicated, but also reflect some important geometric properties, namely the cellular subdivision of products $E^m \times E^n$ of cells E^m. We have seen (Example 7.1.5):

$$E^0 = \{1\}, \quad E^1 = e^0_{\pm} \cup e^1, \quad E^m = e^0 \cup e^{m-1} \cup e^m, \;\; m \geqslant 2,$$

where $e^0_- = -1, e^0_+ = 1$. Thus in general the product of these cells has a cell structure with 9 cells.

The structure for the cylinder $E^1 \times E^2$ may be seen in the next two figures. The first one shows the cells in $E^1 \times S^1$, and the second one shows the three cells we have to add to get $E^1 \times E^2$.

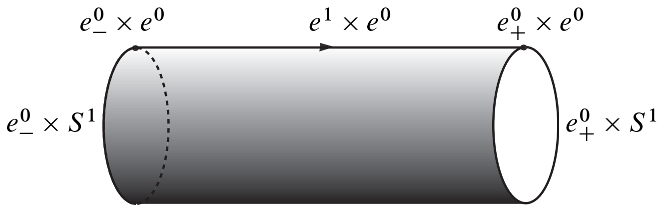


Figure 9.1. Hollow cylinder picture, horizontally for E^1 direction.

Figure 9.2. Solid cylinder picture, horizontally for E^1 direction.

We cannot draw the picture for $E^2 \times E^2$, but that structure contains two solid tori, one of which is pictured as the cylinder with both ends glued in Figure 9.3.

Note that the boundary of $E^2 \times E^2$ is homeomorphic to a 3-sphere, which can be represented as the set of points $(x, y, z, w) \in \mathbb{R}^4$ such that $x^2 + y^2 + z^2 + w^2 = 1$; one of the solid tori is represented by the subset of S^3 of points such that

$$x^2 + y^2 \leqslant 1/2, \quad \text{whence } z^2 + w^2 \geqslant 1/2.$$

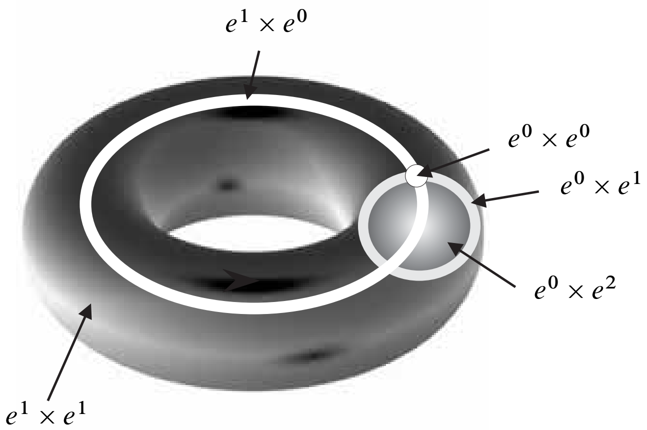


Figure 9.3. Solid torus picture.

In terms of crossed complexes, a corresponding algebraic expression for the boundary of the solid cylinder $e^1 \times e^2$ should involve the cells $e^1 \times e^1$, $e^0_- \times e^2$, $e^0_+ \times e^2$. Our conventions set the base point of the cylinder at $(1, 1)$, i.e. at the 'top' end of the cylinder. In the end we take the boundary to be

$$\delta(e^1 \times e^2) = -(e^1 \times e^1) - (1 \times e^2) + (-1 \times e^2)^{e^1 \times 1},$$

where the conventions as to sign and order of the terms come from some other considerations which we explain later. When we come to take the boundary in the solid torus in $E^2 \times E^2$ we get a similar formula, except that now $-1 \times e^2$, $1 \times e^2$ are identified to $1 \times e^2$ and so the formula becomes

$$\delta(e^1 \times e^2) = -(e^1 \times e^1) - (1 \times e^2) + (1 \times e^2)^{e^1 \times 1},$$

which relates to our picture of the solid torus.

Another complication is when we glue two cylinders together as in Figure 9.4. The base point of the whole cylinder is at the right-hand end of the picture, but the base point of the first cylinder is half way along. Thus the algebraic formulae have to reflect this.

Finally, we have to distinguish the formulae for the boundary of $E^m \times E^n$ for m, n odd, even, and equal to 0, 1, or $\geqslant 2$.

All these complications are reflected in the notion of a *bimorphism* given in Section 9.3.ii.

We then internalise the bimorphism concept by introducing the tensor product for crossed complexes in Section 9.3.iii. The mentioned complications for bimorphisms give quite different formulae for the operations on $c \otimes d$ for the dimensions of c or d being $0, 1$ or $\geqslant 2$. We stress that the specific rules we use are dictated by the clear

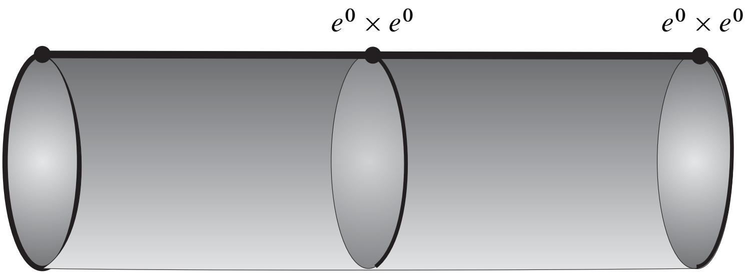


Figure 9.4. Two cylinders glued.

and intuitive rules for the monoidal closed structure on cubical ω-groupoids given in Chapter 15 and the specific conventions chosen in Chapter 13 for the equivalence between these ω-groupoids and crossed complexes.

9.3.i The internal hom structure

Recall from the introduction to this chapter that the elements of $\mathsf{CRS}_m(C, D)$ can be seen as m-fold homotopies

$$\mathbb{F}(m) \otimes C \to D$$

reflecting the geometry of $\mathbb{F}(m)$. In particular, $\mathbb{F}(1) = \mathcal{I}$ is the unit interval, which will give a cylinder construction.

An advantage of this viewpoint is that the elements of the internal hom crossed complex $\mathsf{CRS}(C, D)$ in dimension m have a precise and clear interpretation. What is not so clear is that these elements taken altogether can be given the structure of crossed complex.

This difficulty is overcome in Chapter 15 by working with a different but equivalent structure, that of ω-groupoids, which is based on cubes.

So in this section, our aim is not to give the full justification of the results, but to give full definitions and to explain their intuitive content. We begin the definition of $\mathsf{CRS}(C, D)$ from the bottom dimension upwards.

In *dimension* 0, $\mathsf{CRS}(C, D)_0$ is the set of crossed complex morphisms $C \to D$, that is,

$$\mathsf{CRS}_0(C, D) = \mathsf{Crs}(C, D).$$

For *dimension* 1, we use the concept of homotopy which has been given in Section 7.1.vii. Hence the groupoid $\mathsf{CRS}_1(C, D)$ is defined as having $\mathsf{Crs}_0(C, D) = \mathsf{Crs}(C, D)$ as objects, the morphisms from $f^-: C \to D$ to $f^+: C \to D$ are the homotopies $f^- \simeq f^+$, i.e.

$$\mathsf{CRS}_1(C, D)(f^-, f^+) = \{(H, f^+): f^- \simeq f^+ \mid \text{homotopies from } f^- \text{ to } f^+\}.$$

The composition of such homotopies is given as follows:

Definition 9.3.1. Let $H\colon f^- \simeq f^+$ and $K\colon f^+ \simeq f'$ be left homotopies, then we define $H + K\colon f^- \simeq f'$ by

$$(H+K)_n(c) = \begin{cases} K_n(c) + H_n(c)^{K_0(tc)} & \text{if } c \in C_n,\ n \geqslant 1, \\ H_0(c) + K_0(c) & \text{if } c \in C_0. \end{cases}$$ □

Exercise 9.3.2. Prove that $\mathsf{CRS}_1(C, D)$ is, with this addition, a groupoid. Deduce that homotopy between morphisms of crossed complexes is an equivalence relation. The quotient set is written $[C, D]$. □

Now we turn to the general structure, defining $\mathsf{CRS}_m(C, D)(f)$ for dimension $m \geqslant 2$ and $f \in \mathsf{Crs}(C, D)$; providing it with an action of $\mathsf{CRS}_1(C, D)$ and defining the 'boundary' maps.

Definition 9.3.3. Let C, D be crossed complexes and let $m \geqslant 2$. Then an *m-fold homotopy from C to D over a morphism $f: C \to D$*, also called the *base morphism of the homotopy*, is a pair (H, f) where H is a map of degree m from C to D given by functions $H_n\colon C_n \to D_{n+m}$ for each $n \geqslant 0$ and which satisfy the following conditions.

• Relations with *actions* for $n \geqslant 2$:

– for $n \geqslant 2$, the H_n preserve the action from dimension 1, i.e. if $c \in C_n$ and $c_1 \in C_1$, then

$$H_n(c^{c_1}) = H_n(c)^{f_1(c_1)}.$$

• Relations with *compositions* for $n \geqslant 1$:

– H_1 is a derivation over f, i.e. if $c, c' \in C_1$ and $c + c'$ is defined, then

$$H_1(c + c') = H_1(c)^{f_1(c')} + H_1(c');$$

– for $n \geqslant 2$ the H_n are morphisms, i.e. if $c, c' \in C_n$ and $c + c'$ is defined, then

$$H_n(c + c') = H_n(c) + H_n(c').$$

Thus, in each dimension, H with f preserves structure in the only reasonable way. However, there is no requirement that H should be compatible with the boundary maps $\delta_n\colon C_n \to C_{n-1}$ and $\delta_n\colon D_n \to D_{n-1}$. We define $\mathsf{CRS}(C, D)_m(f)$ to be the set of m-fold homotopies $(H, f)\colon C \to D$. □

Remark 9.3.4. For $m \geqslant 2$ we have just m-homotopies (H, f) rather than left or right m-homotopies because then H_n has images in abelian groupoids for $n \geqslant 1$. □

Let us see how this family of homotopies (H, f) may be given the structure of a crossed complex.

Definition 9.3.5. The additions, action and boundary maps on $\mathsf{CRS}_m(C, D)$ are given by:

1. *Addition on* $\mathsf{CRS}_m(C, D)$. If $m \geqslant 2$ and $(H, f), (K, f)$ are m-fold homotopies $C \to D$ over the same base morphism f we define

$$(H + K)(c) = H(c) + K(c)$$

for all $c \in C$.

2. *Actions on* $\mathsf{CRS}_m(C, D)$.- If (H, f^-) is an m-fold homotopy $C \to D$ for $m \geqslant 2$ and if $K : f^- \sim f^+$ is a left homotopy, then we define

$$H^K(c) = H(c)^{K(tc)}$$

for all $c \in C$. Then (H^K, f^+) is an m-fold homotopy.

3. *Boundaries on* $\mathsf{CRS}_m(C, D)$. If $m \geqslant 2$ and (H, f) is an m-fold homotopy we define the boundary

$$\delta_m(H, f) = (\delta_m H, f)$$

where $\delta_m H$ is the $(m-1)$-fold homotopy given by

$$(\delta_m H)(c) = \begin{cases} \delta_{m+n}(H(c)) + (-1)^{m+1} H(\delta_n c) & \text{if } c \in C_n (n \geqslant 2), \\ (-1)^{m+1} H(sc)^{f(c)} + (-1)^m H(tc) + \delta_{m+1}(H(c)) & \text{if } c \in C_1, \\ \delta_m(H(c)) & \text{if } c \in C_0. \end{cases}$$

The source and target of 1-homotopies have been given in Definition 7.1.38 as the initial and final morphisms of the homotopy. □

Theorem 9.3.6. *The above operations give* $\mathsf{CRS}(C, D)$ *the structure of crossed complex.*

Proof. This would be somewhat tedious to verify directly, and instead we rely on the fact that this internal hom structure for Crs is derived from the more easily verified internal hom structure on the category of ω-groupoids, given in Chapter 15, and the equivalence between the two categories given in Chapter 13. □

The specific conventions used for constructing the equivalence between crossed complexes and ω-groupoids impose the conventions we use for the internal hom structure on Crs, and hence for the tensor product. The fact that $\mathsf{CRS}(C, D)$ is a crossed complex contains a lot of information.

The formulae for m-fold homotopies are exactly what is needed to express the geometry of the product $E^m_* \otimes \mathbf{E}^n_*$ because an m-homotopy can be seen as a morphism $\Pi E^m_* \otimes C \to D$.

Remark 9.3.7. We will discuss the notion of 'model category' for the homotopy of crossed complexes in Sections 12.1 and B.8. However our theory and its applications to spaces of functions needs the full monoidal closed structure on crossed complexes, i.e. the structure and algebra of higher homotopies; for these the notion of model category for homotopy is not entirely adequate. □

We now give as an exercise an illustrative example of the above construction.

Example 9.3.8 (Free loop space model[122]). Let $\mu\colon M \to P$ be a crossed module with P a group and regard this crossed module as a reduced crossed complex $\mathcal{M}$ trivial above dimension 2. Our aim is to describe explicitly the crossed complex

$$L\mathcal{M} = \mathsf{CRS}(\mathbb{K}(\mathbb{Z},1),\mathcal{M}).$$

We have the following formulae, with additive notation for (noncommutative) groups:

(i) $(L\mathcal{M})_0 = P$;

(ii) $(L\mathcal{M})_1 = M \times P \times P$ with source and target given by

$$s(m,p,a) = p + a + \mu m - p, \quad t(m,p,a) = a$$

for $a, p \in P, m \in M$;

(iii) the composition of such triples is given by

$$(n,q,b) + (m,p,a) = (m + n^p, q + p, a)$$

which of course is defined under the condition that

$$b = p + a + \mu m - p$$

or, equivalently, $b^p = a + \mu m$;

(iv) if $a \in P$ then $(L\mathcal{M})_2(a)$ consists of pairs (m,a) for all $m \in M$, with addition and boundary

$$(m,a) + (n,a) = (m+n,a), \quad \mu(m,a) = (\mu m, -m^a + m, a);$$

(v) the action of $(L\mathcal{M})_1$ on $(L\mathcal{M})_2$ is given by: $(n,b)^{(m,p,a)}$ is defined if and only if $b^p = a + \mu m$ and then its value is (n^p, a).

The crunch is that the crossed complex $\mathbb{K}(\mathbb{Z},1)$ has a base point z_0 say and one free generator say z in dimension 1. Thus a morphism $f\colon \mathbb{K}(\mathbb{Z},1) \to \mathcal{M}$ is entirely determined by its value on z, and so gives an element of P. A homotopy $(h, f^+)\colon f^- \simeq f^+$ is given by f^+ and the values of h on z_0 and on z: one sets $f^+(z) = a, h(z_0) = b, h(z) = m$ and this with the formulae in Section 7.1.vii gives the description of $(L\mathcal{M})_1$. A 2-homotopy (H, f^+) is given by $f^+(z) = c$, say, and $H(z_0) = u$, say, but $\delta_2(H, f^+)$ is a homotopy $(\mu H, f^+)$ and μH has to be evaluated on z_0 and on z. We leave further details to the reader, including determining the fundamental groups and second homology groups of $L\mathcal{M}$ at various base points. □

Exercise 9.3.9. Write down explicitly, in a format similar to that of the previous example, the crossed complex $\mathsf{CRS}(\mathbb{K}(\mathbb{Z},1),\mathbb{K}_n(M,G))$ where (M,G) is a module and $n \geqslant 3$. □

9.3.ii The bimorphisms as an intermediate step

With the structure of crossed complex on $\mathsf{CRS}(C,D)$ just described, we may study the crossed complex morphisms $\mathsf{Crs}(C,\mathsf{CRS}(D,E))$, see how they are defined and reorganise the data. Such a morphism is given by a family of maps $\xi_m\colon C_m \to \mathsf{CRS}_m(D,E)$ commuting with the boundary maps. For each $c \in C_m$, $\xi_m(c)$ is a homotopy, i.e. a family of maps $\xi_m(c)_n\colon D_n \to E_{m+n}$ satisfying some conditions.

We can reorganise these maps, getting a family

$$\xi_{m,n}\colon C_m \times D_n \to E_{m+n}$$

and see what the different conditions mean for these maps. That gives the notion of bimorphism.

For the rest of this section we use additive notation in all dimensions (including 1 and 2) to reduce the number of cases in formulae but of course when values lie in dimension 1 then care has to be taken on the order in which terms are written.

Definition 9.3.10. A *bimorphism* $\xi\colon (C,D) \to E$ of crossed complexes C, D, E is a family of maps

$$xi_{mn}\colon C_m \times D_n \to E_{m+n}$$

such that, for every $c \in C_m$, the map $\xi_m(c) = \{\xi_{mn}(c,-)\}_{n\in\mathbb{N}}$ is an m-homotopy. This means that the ξ_{mn} have to satisfy the following conditions, where $c, c' \in C_m$, $d, d' \in D_n$, $c_1 \in C_1$, $d_1 \in D_1$.

• *Source and target.* Preservation of target and, whenever appropriate, source:

$$\begin{aligned} t(\xi(c,d)) &= \xi(tc,td) && \text{for all } m,\ n,\\ s(\xi(c,d)) &= \xi(c,sd) && \text{if } m=0,\ n=1,\\ s(\xi(c,d)) &= \xi(sc,d) && \text{if } m=1,\ n=0. \end{aligned}$$

• *Actions.* Preservation of the action in dimensions $\geqslant 2$:

$$\begin{aligned} \xi(c,d^{d_1}) &= \xi(c,d)^{\xi(tc,d_1)} && \text{if } m \geqslant 0,\ n \geqslant 2,\\ \xi(c^{c_1},d) &= \xi(c,d)^{\xi(c_1,td)} && \text{if } m \geqslant 2,\ n \geqslant 0. \end{aligned}$$

• *Compositions.* Preservation of compositions in C and D where possible:
– If $m=1$ or $n=1$ and both are $\geqslant 1$ there are derivation conditions:

$$\begin{aligned} \xi(c,d+d') &= \xi(c,d)^{\xi(tc,d')} + \xi(c,d') && \text{if } m \geqslant 1,\ n=1,\\ \xi(c+c',d) &= \xi(c',d) + \xi(c,d)^{\xi(c',td)} && \text{if } m=1,\ n \geqslant 1. \end{aligned}$$

– For other cases, the ξ_{mn} are bimorphisms:

$$\begin{aligned}
\xi(c, d+d') &= \xi(c,d) + \xi(c,d') \qquad \text{if } m \geqslant 1,\ n \geqslant 2 \text{ or } m = 0, n \geqslant 1,\\
\xi(c+c', d) &= \xi(c,d) + \xi(c',d) \qquad \text{if } m \geqslant 2,\ n \geqslant 1 \ \text{ or } m \geqslant 1,\ n = 0.
\end{aligned}$$

• *Boundaries.*

– In high dimensions, the boundary is analogous to that in chain complexes:

$$\delta_{m+n}(\xi(c,d)) = \xi(\delta_m c, d) + (-1)^m \xi(c, \delta_n d) \quad \text{if } m \geqslant 2,\ n \geqslant 2.$$

– When one of the elements has dimension 1, we have to take account of the action to put elements at the correct base point:

$$\begin{aligned}
&\delta_{m+n}(\xi(c,d))\\
&\quad = \begin{cases} -\xi(c, \delta_n d) - \xi(tc, d) + \xi(sc, d)^{\xi(c,td)} & \text{if } m = 1,\ n \geqslant 2,\\ (-1)^{m+1}\xi(c, td) + (-1)^m \xi(c, sd)^{\xi(tc,d)} + \xi(\delta_m c, d) & \text{if } m \geqslant 2,\ n = 1,\\ -\xi(tc, d) - \xi(c, sd) + \xi(sc, d) + \xi(c, td) & \text{if } m = n = 1. \end{cases}
\end{aligned}$$

– And, whenever one of the elements has dimension 0, we operate only on the other part:

$$\delta_{m+n}(\xi(c,d)) = \begin{cases} \xi(c, \delta_n d) & \text{if } m = 0,\ n \geqslant 2,\\ \xi(\delta_m c, d) & \text{if } m \geqslant 2,\ n = 0. \end{cases} \qquad \square$$

You should look at these carefully and note (but not necessarily learn!) the way these formulae reflect the geometry and algebra of crossed complexes, which allow for differences between the various dimensions, and also for change of base point.

The bimorphisms are used as an intermediate step in the construction of the tensor product due to the following property:

Theorem 9.3.11. *For crossed complexes C, D, E, there is a natural bijection from*

$$\mathsf{Crs}(C, \mathsf{CRS}(D, E))$$

to the set of bimorphisms $(C, D) \to E$.

9.3.iii The tensor product of two crossed complexes

Following the pattern in the tensor product of R-modules, we now 'internalise' the concept of bimorphism. That is, we construct a crossed complex, the tensor product $C \otimes D$ of two crossed complexes, and a universal bimorphism

$$\Upsilon\colon (C, D) \to C \otimes D,$$

so that the bimorphisms $(C, D) \to E$ correspond exactly to the morphisms $C \otimes D \to E$. In effect, this implies that $C \otimes D$ is generated by elements $c \otimes d$, with $c \in C_m$ and $d \in D_n$, $m, n \geqslant 0$, subject to the relations given by the rules of bimorphisms with $\xi(c, d)$ replaced by $c \otimes d$.

We shall also describe $(C \otimes D)_p$ in terms of pieces $(C \otimes D)_{m,n}$ with $m + n = p$, which, from the rules for ξ_{mn}, can be given more explicitly in terms of the structures on C_m, D_n.

Let us start by making clear the implication for the groupoid part of $C \otimes D$.

For $p = 0$, we define

$$(C \otimes D)_0 = C_0 \times D_0$$

as sets.

For $p = 1$, the groupoid $(C \otimes D)_1$ over $(C \otimes D)_0$ is determined by two parts, namely $(C \otimes D)_{1,0} = C_1 \times D_0$ and $(C \otimes D)_{0,1} = C_0 \times D_1$. Then $(C \otimes D)_1$ is their coproduct as groupoids over $(C \otimes D)_0$, we write

$$(C \otimes D)_1 = C_1 \# D_1.$$

This groupoid may be seen also as generated by the symbols

$$\{c \otimes y \mid c \in C_1\} \cup \{x \otimes d \mid d \in D_1\}$$

for all $x \in C_0$ and $y \in D_0$ subject to the relations given by the products in C_1 and on D_1. We shall return to this in Section 9.4.i.

Also, we shall prove in Section 9.4.ii that the image of $\delta_2(C \otimes D)_2$ in $(C \otimes D)_1$ is generated by all the elements of the three sets

– $\{\delta c \otimes y \mid c \in C_2,\ y \in D_0\}$,

– $\{x \otimes \delta d \mid d \in D_2,\ x \in C_0\}$,

– $\{(c \otimes y)(x \otimes d)(c \otimes y)^{-1}(x \otimes d)^{-1} \mid c \in C_1(x),\ d \in D_1(y),\ x \in C_0,\ y \in D_0\}$.

(Notice that the last set consists of the commutators of the inverses of the generators of $(C \otimes D)_{1,0} = C_1 \times D_0$ and $(C \otimes D)_{0,1} = C_0 \times D_1$).

Now this has been recorded, we can proceed with the definition of $(C \otimes D)_p$ for $p \geqslant 2$.

Definition 9.3.12. Let C, D be crossed complexes. For any $c \in C_m$, $d \in D_n$ we consider the symbol $c \otimes d$. Whenever $m + n \geqslant 2$, we define its source and target

$$s(c \otimes d) = t(c \otimes d) = tc \otimes td.$$

(Notice that for elements of dimension 0 we define $t(x) = x$ and $t(y) = y$.)

For $p \geqslant 2$, we consider F_p the free $(C \otimes D)_1$-module (or crossed module if $p = 2$) on

$$\{c \otimes d \mid c \in C_m,\ d \in D_n,\ m, n \in \mathbb{N},\ m + n = p\}.$$

To get $(C \otimes D)_p$ we have to quotient out by some relations with respect to the additions and actions. Notice that all relations are 'dimension preserving'. There are several cases.

• When both $m, n \neq 1$, we do not have to worry about source and target (both are the same), and the relations are easier:

– Additions: The relations to make $\otimes$ compatible with additions are

$$\begin{aligned} c \otimes (d + d') &= c \otimes d + c \otimes d' \quad \text{if } n \geqslant 2, \\ (c + c') \otimes d &= c \otimes d + c' \otimes d \quad \text{if } m \geqslant 2. \end{aligned}$$

– Action: The relations to make $\otimes$ compatible with the actions are

$$\begin{aligned} (c \otimes d)^{(tc \otimes d_1)} &= c \otimes d^{d_1} \quad \text{if } m \geqslant 0,\ n \geqslant 2, \\ (c \otimes d)^{(c_1 \otimes td)} &= c^{c_1} \otimes d \quad \text{if } m \geqslant 2,\ n \geqslant 0. \end{aligned}$$

– Cokernel. When $m+n \geqslant 3$, we have to kill the action of $\delta_2(C \otimes D)_2 \subseteq (C \otimes D)_1$.

• When one element has dimension 1:

– Then the operation has to be related with the action because the groupoid part acts on itself by conjugation.

$$\begin{aligned} c \otimes dd' &= (c \otimes d)^{(tc \otimes d')} + c \otimes d' \quad \text{if } m \geqslant 1,\ n = 1, \\ cc' \otimes d &= c' \otimes d + (c \otimes d)^{(c' \otimes td)} \quad \text{if } m = 1,\ n \geqslant 1. \end{aligned}$$

– Cokernel. When $m+n \geqslant 3$, we have to kill the action of $\delta_2(C \otimes D)_2 \subseteq (C \otimes D)_1$.

With this, we get $(C \otimes D)_p$ as the quotient of F_p by all these relations.

To finish the structure of $C \otimes D$ as a crossed complex, the *boundaries* are defined on generators with formulae varying according to dimensions.

• When both have dimension $\geqslant 2$:

$$\delta_{m+n}(c \otimes d) = \delta_m c \otimes d + (-1)^m (c \otimes \delta_n d).$$

• When one has dimension 1 and the other one has dimension $\geqslant 1$:

$$\begin{aligned} &\delta_{m+n}(c \otimes d) \\ &= \begin{cases} -(c \otimes \delta_n d) - (tc \otimes d) + (sc \otimes d)^{(c \otimes td)} & \text{if } m = 1,\ n \geqslant 2, \\ (-1)^{m+1}(c \otimes td) + (-1)^m (c \otimes sd)^{(tc \otimes d)} + (\delta_m c \otimes d) & \text{if } m \geqslant 2,\ n = 1, \\ -(tc \otimes d) - (c \otimes sd) + (sc \otimes d) + (c \otimes td) & \text{if } m = n = 1. \end{cases} \end{aligned}$$

• When one has dimension 0:

$$\delta_{m+n}(c \otimes d) = \begin{cases} (c \otimes \delta_n d) & \text{if } m = 0,\ n \geqslant 2, \\ (\delta_m c \otimes d) & \text{if } m \geqslant 2,\ n = 0, \end{cases}$$

and these definitions are compatible with the relations. □

Remark 9.3.13. Notice that if we denote by $F_{m,n}$ the free $(C \otimes D)_1$-module on $\{c \otimes d \mid c \in C_m, d \in D_n\}$ for some fixed $m, n \in \mathbb{N}$, F_p is the coproduct of $\{F_{m,n}\}_{m+n=p}$.

Since the relations with respect to the additions and actions we are using to get $(C \otimes D)_p$ preserve the decomposition of F_p as the coproduct of $F_{m,n}$, $(C \otimes D)_p$ also decomposes as coproduct of the quotient of $F_{m,n}$ respect to the corresponding relations. We shall call $(C \otimes D)_{m,n}$ this quotient. □

Remark 9.3.14. There is an alternative way of defining $F_{m,n}$ that works when $m, n \neq 0$, $m + n \geqslant 3$.

We could define $F'_{m,n}$ as the free abelian groupoid on $\{c \otimes d \mid c \in C_m, d \in D_n\}$ and quotient out by the relations on operations included in the previous definition getting an abelian groupoid $C'_{m,n}$. This quotient is isomorphic to $(C \otimes D)_{m,n}$ as abelian groupoid.

Next we define the $(C \otimes D)_1$-action on $C'_{m,n}$ by the formulae in the previous definition (notice that the definition is different when $m = 1$ or $n = 1$). It is not difficult to prove that this is an action and that $C'_{m,n}$ is isomorphic to $(C \otimes D)_{m,n}$ as $(C \otimes D)_1$-modules □

Exercise 9.3.15. Check the rule $\delta\delta(c \otimes d) = 0$ for some low dimensional cases, such as $\dim(c \otimes d) = 3, 4$, seeing how the crossed module rules come into play. □

Those familiar with the tensor product of chain complexes may note that in that theory the single and simple formula we need is

$$\partial(c \otimes d) = (\partial c) \otimes d + (-1)^m c \otimes (\partial d)$$

where $\dim c = m$. So it is not surprising that the tensor product of crossed complexes has much more power than that of chain complexes, and can handle more complex geometry.

The specific conventions in writing down the formulae for this tensor product of crossed complexes come from another direction, which is explained fully in Chapter 15, namely the relation with cubical ω-groupoids with connection. The tensor product there comes out simply, because it is based on the formula $I^m \times I^n \cong I^{m+n}$. The distinction between that formula and that for the product of cells as above lies at the heart of many difficulties in basic homotopy theory. The relation between ω-groupoids and crossed complexes gives an algebraic expression of these geometric relationships.

Using this definition, it can be proved that the tensor product gives a symmetric monoidal structure to Crs the category of crossed modules by defining the maps on generators and checking that they preserve the relations.

Theorem 9.3.16. *With the bifunctor $-\otimes-$, the category* Crs *of crossed complexes has a structure of a symmetric monoidal category, i.e.*

i) *For crossed complexes C, D, E, there are natural isomorphisms of crossed complexes*

$$(C \otimes D) \otimes E \cong C \otimes (D \otimes E),$$

ii) *for all crossed complexes C, D there is a natural isomorphism of crossed complexes*

$$T : C \otimes D \to D \otimes C$$

satisfying the appropriate axioms.

Proof. The existence of both isomorphisms could be established directly, giving the values on generators in the obvious way by:

i) is given by $(c \otimes d) \otimes e \mapsto c \otimes (d \otimes e)$, and

ii) is given by $T(c \otimes d) = (-1)^{mn} d \otimes c$ if $c \in C_m$ and $d \in D_n$,

and then checking that the relations on generators $c \otimes d$ in Definition 9.3.12 are preserved by both maps. The necessary coherence and naturality conditions are obviously satisfied.

But to check all the cases even for such simple maps seems tedious. An alternative approach is to go via ω-groupoids where the tensor product fits more closely to the cubical context. This will be done in Chapter 15. □

This proof of commutativity is somehow unsatisfactory because, although it is clear that $c \otimes d \mapsto d \otimes c$ does not preserve the relations in Definition 9.3.12, the fact that $c \otimes d \mapsto (-1)^{mn} d \otimes c$ does preserve them seems like a happy accident. A better explanation is provided by the transposing functor T (see Section 15.4).

Note that while the tensor product can be defined directly in terms of generators and relations and this can sometimes prove useful, such a definition may make it difficult to verify essential properties of the tensor product, such as that the tensor product of free crossed complexes is free. We shall prove that later (Section 9.6), using the adjointness of $\otimes$ and the internal hom functor as a necessary step to prove that $- \otimes C$ preserves colimits.

Nevertheless, this definition is interesting for its relation to the tensor product of filtered spaces which we shall study in Section 9.8.

Theorem 9.3.17. *For crossed complexes C, D, E, there is a natural exponential law giving a natural isomorphism*

$$\mathsf{Crs}(C \otimes D, E) \cong \mathsf{Crs}(C, \mathsf{CRS}(D, E)).$$

This gives the category Crs *of crossed complexes a structure of monoidal closed category. Moreover, this implies natural isomorphisms of crossed complexes*

$$\mathsf{CRS}((C \otimes D), E) \cong \mathsf{CRS}(C, \mathsf{CRS}(D, E)).$$

It is also important that we have to use crossed complexes of groupoids to make sense of the exponential law in Crs. This is analogous to the fact that the category of groups has no internal hom, while that of groupoids does.

Remark 9.3.18. The 'unit interval' groupoid $\mathcal{I}$, see page 26, with two objects 0,1 and exactly one morphism $\iota\colon 0 \to 1$ so that $t(\iota) = 1$, will also be considered as a crossed complex and as such is $\Pi(E^1)$. A '1-fold left homotopy' of morphisms $f^-, f^+\colon C \to D$ is seen to be a morphism $\mathcal{I} \otimes C \to D$ which takes the values of f^- on $0 \otimes C$ and f^+ on $1 \otimes C$. The existence of this 'cylinder object' $\mathcal{I} \otimes C$ allows abstract homotopy theory to be applied to crossed complexes: see 'model categories' in Section B.8 of Appendix B.[123] □

9.4 Analysis of the tensor product of crossed complexes

The definition of the tensor product of two crossed complexes $C \otimes D$ is quite complex. We devote this section to giving other descriptions of the tensor product in dimensions 1 and 2. In Section 9.5 we prove that in dimensions $n \geqslant 3$ the tensor product of crossed complexes can be completely described in terms of the tensor product of chain complexes with operators, which is a more familiar type of construction.

9.4.i The groupoid part of the tensor product

In order to become more familiar with the definition of the tensor product of crossed complexes, in this section we are going to do the computations with some detail in low dimensions.

Notice first that it is clear from the definition that to construct $(C \otimes D)_p$ we only need to know $\{C_m\}_{m \leqslant p}$ and $\{D_n\}_{n \leqslant p}$ since there are no relations among the generators in $(C \otimes D)_p$ coming from higher dimensions. Let us see what this means for low dimensions.

The case $p = 0$ is immediate. Let us start with the case $p = 1$.

Proposition 9.4.1. *For any pair of crossed complexes $C, D \in \mathsf{Crs}$ the groupoid $(C \otimes D)_1$ of their tensor product is the following pushout in the category of groupoids:*

$$\begin{tikzcd}
0_{C_1} \times 0_{D_1} \arrow[r] \arrow[d] & C_1 \times 0_{D_1} \arrow[d] \\
0_{C_1} \times D_1 \arrow[r] & (C \otimes D)_1
\end{tikzcd}$$

where, for any groupoid G, 0_G denotes the trivial sub-groupoid consisting of all identity elements of G. It is easy to see that this pushout is the coproduct of $C_1 \times 0_{D_1}$ and $0_{C_1} \times D_1$ in the category of groupoids over $C_0 \times D_0$

We give a description of this groupoid. By the previous proposition, it is actually a construction in the category of groupoids. Given a pair of groupoids G and H we form their coproduct in the category of groupoids over $G_0 \times H_0$

$$G \# H = (G \times 0_H) * (0_G \times H).$$

This groupoid $G \# H$ is generated by all elements $(1_x, h)$, $(g, 1_y)$ where $g \in G$, $h \in H$, $x \in G_0$, $y \in H_0$. We will sometimes write g for $(g, 1_y)$ and h for $(1_x, h)$. This may seem to be willful ambiguity, but when composites are specified in $G \# H$, the ambiguity is resolved; for example, if gh is defined in $G \# H$, then g must refer to $(g, 1_y)$, where $y = sh$, and h must refer to $(1_x, h)$, where $x = tg$. This convention simplifies the notation and there is an easily stated solution to the word problem for $G \# H$. Every element of $G \# H$ is uniquely expressible in one of the following forms:

(i) an identity element $(1_x, 1_y)$;
(ii) a generating element $(g, 1_y)$ or $(1_x, h)$, where $x \in G_0$, $y \in H_0$, $g \in G$, $h \in H$ and g, h are not identities;
(iii) a composite $k_1 k_2 \dots k_n$ $(n \geqslant 2)$ of non-identity elements of G or H in which the k_i lie alternately in G and H, and the odd and even products $k_1 k_3 k_5 \dots$ and $k_2 k_4 k_6 \dots$ are defined in G or H.

For example, if $g_1 \colon x \to y$, $g_2 \colon y \to z$ in G, and $h_1 \colon u \to v$, $h_2 \colon v \to w$ in H, then the word $g_1 h_1 g_2 h_2 g_2^{-1}$ represents an element of $G \# H$ from (x, u) to (y, w). Note that the two occurrences of g_2 refer to different elements of $G \# H$, namely $(g_2, 1_v)$ and $(g_2, 1_w)$. This can be represented as a path in a 2-dimensional grid as follows:

$$\begin{tikzcd}
(x,u) \arrow[d, "g_1"] & (x,v) & (x,w) \\
(y,u) \arrow[r, "h_1"] & (y,v) \arrow[d, "g_2"] & (y,w) \\
(z,u) & (z,v) \arrow[r, "h_2"] & (z,w) \arrow[u, "g_2^{-1}"']
\end{tikzcd}$$

The similarity with the free product of groups is obvious and the normal form can be verified in the same way; for example, one can use ‘van der Waerden’s trick’. Examples of this method may be found in [Hig71], [Coh89]).

9.4.ii The crossed module part of the tensor product

To identify the crossed module in the title for crossed complexes C, D, we need to use two constructions from the theory of crossed modules: the coproduct of crossed modules over the same base and the induced crossed module.

In the case when G is a group, the construction of the coproduct $M \circ_G N$ of crossed G-modules M and N has been studied in Section 4.2. This construction works equally well when G is a groupoid. The family of groups M acts on N via G, so one can form the semidirect product $M \ltimes N$. It consists of a semidirect product of groups $M_p \ltimes N_p$ at each vertex p of G and it is a pre-crossed module over G. One then obtains the crossed G-module $M \circ_G N$ from $M \ltimes N$ by factoring out its Peiffer groupoid.

Now, recall that $(C \otimes D)_2$ as $(C_1 \# D_1)$-crossed module is the coproduct

$$(C \otimes D)_2 = (C \otimes D)_{2,0} \circ (C \otimes D)_{1,1} \circ (C \otimes D)_{0,2}$$

where these last crossed modules have been defined in Remark 9.3.13.

Since C_2 is a crossed module over the groupoid C_1, $C_2 \times D_0$ is a crossed module over $C_1 \times D_0$. Using the embedding

$$\mu_1 \colon C_1 \times D_0 \to C_1 \# D_1$$

we get an induced crossed module

$$\widehat{C_2} = \mu_{1*}(C_2 \times D_0).$$

It is not difficult to see that

$$(C \otimes D)_{2,0} \cong \widehat{C_2}$$

as $(C_1 \# D_1)$-crossed modules. In the same way, we identify

$$(C \otimes D)_{0,2} \cong \widehat{D_2}$$

where $\widehat{D_2} = \mu_{2*}(C_0 \times D_2)$.

It remains to identify $(C \otimes D)_{1,1}$.

We restrict ourselves to crossed complexes associated to groupoids since the higher dimensional part does not intervene. So we assume $C_n = D_n = \{0\}$ for all $n \geqslant 2$. Then we know $(C \otimes D)_p = \{0\}$ for all $p \geqslant 3$ and we have computed that $(C \otimes D)_0 = C_0 \times D_0$ and $(C \otimes D)_1 = C_1 \# D_1$. Also, to make notation easier, we write G and H for the groupoids C_1 and D_1.

Notice that there is a canonical morphism

$$\sigma \colon G \# H \to G \times H$$

induced by the inclusions $1_G \times H \to G \times H$ and $G \times 1_H \to G \times H$. This morphism is defined on a word $k_1 k_2 k_3 \dots$, by separating the odd and even parts, i.e.

$$\sigma(k_1 k_2 k_3 \dots) = (k_1 k_3 \dots, k_2 k_4 \dots).$$

That is, the map σ introduces a sort of commutativity between G and H.

The kernel of σ will be called the Cartesian subgroupoid of $G \# H$ and will be denoted by $G \square H$, i.e.

$$G \square H = \operatorname{Ker} \sigma.$$

It consists of all identities and all words $k_1k_2\dots k_n$ for which both odd and even products are trivial. Clearly, it is generated by all 'commutators' $[g,h] = g^{-1}h^{-1}gh$, where $g \in G, h \in H$ and g,h are not identities. (Note that $[g,h]$ is uniquely defined in $G \# H$ for any such pair of elements g,h, but the two occurrences of g (or of h) do not refer to the same element of $G \# H$.)

Proposition 9.4.2. *The Cartesian subgroupoid $G \square H$ of $G \# H$ is freely generated, as a groupoid, by all elements $[g,h]$ where g, h are non-identity elements of G, H, respectively. Thus, $G \square H$ is the disjoint union of free groups, one at each vertex, and the group at vertex (x,y) has a basis consisting of all $[g,h]$ with $tg = x$ and $th = y$ (g and h not identity elements).*

Proof. In the notation introduced above the 'commutators' $[h,g]$ satisfy the same formal identities as in the group case:

$$\begin{aligned}
[h,g] &= [g,h]^{-1}, \\
[hh_1,g] &= [h,g]^{h_1}[h_1,g], \\
[h,gg_1] &= [h,g_1][h,g]^{g_1}
\end{aligned}$$

whenever gg_1 and hh_1 are defined in G, H. These identities are to be interpreted as equations in $G \# H$, with the obvious meaning for conjugates: $[h,g]^{h_1}$ means $h_1^{-1}h^{-1}g^{-1}hgh_1$, which represents a unique element of $G \# H$.

Now $G \square H$ is an intransitive free groupoid with basis consisting of all $[g,h]$ ($g \in G$, $h \in H$, $g,h \neq 1$) (see Gruenberg [Gru57], Levi [Lev40]). □

Theorem 9.4.3. *The tensor product of the groupoids G and H, considered as crossed complexes of rank 1, is the crossed complex*

$$G \otimes H = (\cdots \to 0 \to \cdots \to 0 \to G \square H \to G \# H)$$

with $g \otimes h = [h,g]$, $x \otimes h = (1_x,h)$, $g \otimes y = (g,1_y)$ for $g \in G$, $h \in H$, $x \in G_0$, $y \in H_0$.

Proof. $G \square H$ is a normal subgroupoid of $G \# H$, so

$$\delta \colon G \square H \to G \# H$$

is a crossed module which we view as a crossed complex C, trivial in dimension $\geqslant 3$. One verifies easily that the equations $\xi(g,h) = [h,g], \xi(g,\cdot) = g, \xi(\cdot,h) = h$ define a bimorphism of crossed complexes $\xi\colon (G,H) \to C$; the equations on operations in Definition 9.3.10 reduce to the standard commutator identities

$$[hh_1,g] = [h,g]^{h_1}[h_1,g],$$

$$[h,gg_1] = [h,g_1][h,g]^{g_1},$$

and the rest are trivial.

It follows that a bimorphism of crossed complexes

$$\phi\colon (G, H) \to D$$

determines a unique morphism of groupoids $\phi_2\colon G \,\Box\, H \to D_2$ such that $\phi_2([h, g]) = \phi(g, h)$ for all $g \in G, h \in H$. (Note that the definition of bimorphism implies that $\phi(g, h) = 1$ if either $g = 1$ or $h = 1$.) There is also a unique morphism $\phi_1\colon G \,\#\, H \to D_1$ such that $\phi_1(g) = \phi(g, \cdot)$ and $\phi_1(h) = \phi(\cdot, h)$ for all $g \in G$, $h \in H$. These morphisms combine to give a morphism

$$\phi\colon C \to D$$

of crossed complexes as we show below, and this proves the universal property making C the tensor product of G and H, with $g \otimes h = [h, g]$.

We need to verify that $\phi\colon C \to D$ is a morphism of crossed modules. This amounts to

(i) ϕ is compatible with $\delta\colon G \,\Box\, H \hookrightarrow G \,\#\, H$. Now

$$\begin{aligned}\delta\phi_2([h, g]) &= \delta\phi(g, h)\\ &= -\phi(\cdot, h) - \phi(g, \cdot) + \phi(\cdot, h) + \phi(g, \cdot) \quad \text{by (9.3.10)}\\ &= [\phi(\cdot, h), \phi(g, \cdot)] = [\phi_1(h), \phi_1(g)] = \phi_1[h, g]\end{aligned}$$

and

(ii) ϕ preserves the actions of $G \,\#\, H$ and D_1. Now

$$\begin{aligned}\phi_2([h, g]^{g_1}) &= \phi_2([h, g_1]^{-1}[h, gg_1])\\ &= -\phi(g_1, h) + \phi(gg_1, h)\\ &= \phi(g, h)^{\phi(g_1, \cdot)} \quad \text{by (9.3.10)}\\ &= \phi_2([h, g])^{\phi_1(g_1)}.\end{aligned}$$

There is a similar calculation for the action of $h_1 \in H$, and the result follows. □

Corollary 9.4.4. *The crossed complex $\mathcal{I} \otimes \mathcal{I}$ is a free crossed complex.*

The description in Theorem 9.4.3 is much easier for the case of groups. Any group G can be viewed as a crossed complex $\mathbb{K}_1(G)$ with $\mathbb{K}_1(G)_0 = \{\cdot\}$, $\mathbb{K}_1(G)_1 = G$, $\mathbb{K}_1(G)_n = 0$ for $n \geqslant 2$. The tensor product of two such crossed complexes will have one vertex and will be trivial in dimension $\geqslant 3$, that is, it will be a crossed module.[124] We use multiplicative notation for G for reasons which will appear later.

Proposition 9.4.5. *The tensor product of groups G, H, viewed as crossed complexes of rank 1, is the crossed module $G \,\Box\, H \to G * H$, where $G \,\Box\, H$ denotes the Cartesian subgroup of the free product $G * H$, that is, the kernel of the map $G * H \to G \times H$. If $g \in G, h \in H$, then $g \otimes h$ is the commutator $[h, g] = h^{-1}g^{-1}hg = [g, h]^{-1}$ in $G * H$.*

We can also give a useful description of the crossed module part of the tensor product of two crossed complexes C and D.

Theorem 9.4.6. *There is an isomorphism of $(C_1 \# D_1)$-crossed modules*

$$(C \otimes D)_2 \cong \mu_{1*}(C_2 \otimes D_0) \circ G \,\square\, H \circ \mu_{2*}(C_0 \otimes D_2).$$

This isomorphism maps $c \otimes y$ and $x \otimes d$ to the corresponding generators in $\mu_{1*}(C_2 \otimes D_0)$ and $\mu_{2*}(C_0 \otimes D_2)$ and $c \otimes d$ to the commutator $(c \otimes y)(x' \otimes d)(c \otimes y')^{-1}(x \otimes d)^{-1}$. So the subgroupoid $\delta_2(C \otimes D)_2$ is generated as a groupoid by the elements

$$\begin{aligned}\{\delta c \otimes y \mid c \in C_2\} &\cup \{x \otimes \delta d \mid d \in D_1\} \\ &\cup \{(c \otimes y)(x \otimes d)(c \otimes y')^{-1}(x' \otimes d)^{-1} \mid c \in C_1(x, x'),\ d \in D_1(y, y')\}\end{aligned}$$

where $x, x' \in C_0$, $y, y' \in D_0$.

Corollary 9.4.7. *If $C_2 \to C_1$ and $D_2 \to D_1$ are free crossed modules, regarded as crossed complexes C, D, then the crossed module $(C \otimes D)_2 \to (C \otimes D)_1$ is also a free crossed module.*

Proof. This follows from Theorem 9.4.6 and Proposition 9.4.2, and the facts that induced crossed modules, and coproducts, of free crossed modules are also free. □

Corollary 9.4.8. *The crossed module $(\mathcal{I} \otimes \mathbb{F}(2))_2 \to (\mathcal{I} \otimes \mathbb{F}(2))_1$ is a free crossed module over a free groupoid.*

9.5 Tensor products and chain complexes

In this section we relate the tensor product $C \otimes D$ of crossed complexes to a tensor product of chain complexes with a groupoid of operators. This continues the work of the last section by describing the structure of $(C \otimes D)_n$ in dimensions $n \geqslant 3$ in terms the tensor product $\nabla C \otimes \nabla D$ where $\nabla\colon \mathsf{Crs} \to \mathsf{Chn}$ is defined in Section 7.4.

9.5.i Monoidal closed structure on chain complexes

The closed structure on the category Crs of crossed complexes, constructed in Section 9.3 from homotopies and higher homotopies, relied crucially on the consideration of crossed complexes over groupoids as well as over groups. The same is true for the chain complexes with operators which we introduced in Section 7.4.

There are well-known definitions of tensor product and internal hom functor for chain complexes of abelian groups (without operators). If one allows operators from arbitrary groups the tensor product is easily generalised (the tensor product of a G-module and an H-module being a $(G \times H)$-module) but the adjoint construction of internal hom functor does not exist, basically because the group morphisms from G to H do not form a group. To rectify this situation we allow operators from arbitrary groupoids and we start with a discussion of the monoidal closed category structure of Mod the category of modules over groupoids given in Definition 7.1.7.

The ideas for the monoidal closed category Mod can be extended with little extra trouble to chain complexes over groupoids.

Definition 9.5.1. The *tensor product* of chain complexes A, B over groupoids G, H respectively is the chain complex $A \otimes B$ over $G \times H$ where

$$(A \otimes B)_n = \bigoplus_{i+j=n} (A_i \otimes B_j).$$

Here, the direct sum of modules over a groupoid G is defined by taking the direct sum of the abelian groups at each object of G. The boundary map

$$\partial \colon (A \otimes B)_n \to (A \otimes B)_{n-1}$$

is defined on the generators $a \otimes b$ of $(A \otimes B)_n$ by

$$\partial(a \otimes b) = \partial a \otimes b + (-1)^i a \otimes \partial b,$$

where $a \in A_i, b \in B_j, i + j = n$. □

This tensor product gives a symmetric monoidal structure to the category Chn, with symmetry map $C \otimes D \to D \otimes C$ given by

$$a \otimes b \mapsto (-1)^{ij} b \otimes a$$

for $a \in A_i$, $b \in B_j$. There is also a unit object for the tensor product given by the complex over the trivial group:

$$C(\mathbb{Z}, 0) = \cdots \to 0 \to \cdots \to 0 \to \mathbb{Z}.$$

Definition 9.5.2. The internal hom functor $\mathsf{CHN}(-,-)$ for the category Chn is defined as follows. Let A, B be chain complexes over the groupoids G, H respectively. As in the case of morphisms of modules, it is easy to see that the morphisms of chain complexes $\mathsf{Chn}(A, B)$ form an $\mathsf{GPDS}(G, H)$-module. We write

$$S_0 = \mathsf{Chn}(A, B)$$

for this module and take it as the 0-dimensional part of the chain complex $S = \mathsf{CHN}(A, B)$.

The higher-dimensional elements of S are chain homotopies of various degrees. An *i-fold chain homotopy* $(i \geqslant 1)$ from A to B is a pair (s, f) where $s \colon A \to B$ is a map of degree i (that is, a family of maps $s\colon A_n \to B_{n+i}$ for all $n \geqslant 0$) which in each dimension is a morphism of modules over $f: G \to H$.

Again the set $S_i(A, B)$ of i-fold homotopies $A \to B$ has the structure of a $\mathsf{GPDS}(G, H)$-module and we define the boundary map

$$\partial\colon S_i(A, B) \to S_{i-1}(A, B) \quad (i \geqslant 1)$$

by $\partial(s, \theta) = (\partial s, \theta)$ where

$$(\partial s)(a) = \partial(s(a)) + (-1)^{i+1} s(\partial a).$$

We observe that ∂s is of degree $i - 1$. Also ∂s commutes or anticommutes with ∂, namely

$$\partial((\partial s)(a)) = (-1)^{i+1}(\partial s)(\partial a).$$

It follows that $\partial\partial\colon S_i \to S_{i-2}$ is 0 for $i \geqslant 2$. We define $\mathsf{CHN}(A, B)$ to be the chain complex

$$\mathsf{CHN}(A, B) = \cdots \longrightarrow S_i \xrightarrow{\partial} S_{i-1} \longrightarrow \cdots \longrightarrow S_0$$

over $F = \mathsf{GPDS}(G, H)$. □

Proposition 9.5.3. *The functors* $\otimes$ *and* CHN *give* Chn *the structure of symmetric monoidal closed category.*

Proof. Again, if A, B, C are chain complex over G, H, K, there is an exponential law giving a natural bijection

$$\mathsf{Chn}(A \otimes B, C) \cong \mathsf{Chn}(A, \mathsf{CHN}(B, C))$$

which extends to a natural isomorphism of chain complexes

$$\mathsf{CHN}(A \otimes B, C) \cong \mathsf{CHN}(A, \mathsf{CHN}(B, C))$$

over $\mathsf{GPDS}(G \times H, K) \cong \mathsf{GPDS}(G, \mathsf{GPDS}(H, K))$. □

9.5.ii Crossed complexes and chain complexes: relations between the internal homs

In Section 9.3.i an internal hom functor $\mathsf{CRS}(-, -)$ was defined for crossed complexes; now we use the adjoint pair

$$\nabla\colon \mathsf{Crs} \rightleftarrows \mathsf{Chn}\colon \Theta$$

defined in Sections 7.4.iii and 7.4.v to relate the internal homs for crossed complexes and chain complexes.

Theorem 9.5.4. *For crossed complexes C, D and chain complex A there are natural isomorphisms*

(i) $\mathsf{CRS}(D, \Theta A) \cong \Theta\,\mathsf{CHN}(\nabla D, A)$,
(ii) $\nabla(C \otimes D) \cong \nabla C \otimes \nabla D$.

Proof. The two natural isomorphisms are equivalent because

$$\begin{aligned}\mathsf{CHN}(\nabla(C \otimes D), A) &\cong \mathsf{Crs}(C \otimes D, \Theta A)\\ &\cong \mathsf{Crs}(C, \mathsf{CRS}(D, \Theta A)),\end{aligned}$$

while

$$\begin{aligned}\mathsf{Chn}(\nabla C \otimes \nabla D, A) &\cong \mathsf{Chn}(\nabla C, \mathsf{CHN}(\nabla D, A))\\ &\cong \mathsf{Crs}(C, \Theta\,\mathsf{CHN}(\nabla D, A)).\end{aligned}$$

The isomorphism (i) is easier to verify than (ii) because we have explicit descriptions of the elements of both sides, whereas in (ii) we have only presentations.

In dimension 0 we have on the left of (i) the set $\mathsf{Crs}(D, \Theta A)$ of morphisms $\hat{f} \colon D \to \Theta A$; on the right we have the set $\mathsf{Chn}(\nabla D, A)$ of morphisms $(\tilde{f}, \theta) \colon \nabla D \to A$, where θ is a morphism of groupoids from $G = \pi_1 D$ to H, the operator groupoid for A. These sets are in one-one correspondence, by adjointness, and their elements are also equivalent to pairs (f, θ) where $\theta \colon G \to H$ and f is a family

$$\begin{tikzcd}
\cdots \arrow[r, "\delta"] & D_2 \arrow[r, "\delta"] \arrow[d, "f_2"'] & D_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "f_1"] & D_0 \arrow[d, "f_0"] \\
\cdots \arrow[r, "\partial"'] & A_2 \arrow[r, "\partial"'] & A_1 \arrow[r, "\partial"'] & A_0
\end{tikzcd} \tag{9.5.1}$$

such that:

(i) $f_0(d) \in A_0(\theta(d))$ for $d \in D_0$,
(ii) f_1 is a $\theta\phi$-derivation, where ϕ is the quotient map $D_1 \to G$,
(iii) f_n is a θ-morphism for $n \geqslant 2$,
(iv) $\partial f_{n+1} = f_n \delta$ for $n \geqslant 1$,
(v) $\partial f_1(d) = (f_0 \delta^0 d)^{\theta\phi c} - (f_0 \delta^1_d)$ for $d \in D_1$.

Such a family will be called a θ-derivation $f \colon D \to A$.

We recall from Definition 9.3.3 that an element in $\mathsf{CRS}_i(D, E)$ is an i-fold homotopy $(\hat{h}, \hat{f}) : D \to E$, where $\hat{f}$ is a morphism $D \to E$ and $\hat{h}$ is a family of maps

$$\begin{tikzcd}
\cdots \arrow[r] & D_2 \arrow[r] \arrow[d, "\hat{h}_2"] & D_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\hat{h}_1"] & D_0 \arrow[d, "\hat{h}_0"] \\
\cdots & E_{i+2} & E_{i+1} & E_i
\end{tikzcd} \tag{9.5.2}$$

satisfying

(i) $\hat{h}_0(d) \in E_i(\hat{f}_0(d))$, for $d \in D_0$;

(ii) $\hat{h}_1$ is a $\hat{f}_1$-derivation;

(iii) $\hat{h}_n$ is a $\hat{f}_1$-morphism for $n \geqslant 2$.

In the case $E = \Theta A$, where A is a chain complex over H, it is easy to see that, if $i \geqslant 2$, such a homotopy is equivalent to the following data: a morphism of groupoids $\theta \colon G \to H$; a θ-derivation $f \colon D \to A$ as in diagram (9.5.1); and a family h of maps

$$\begin{tikzcd}
\cdots \arrow[r] & D_2 \arrow[r] \arrow[d, "h_2"] & D_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "h_1"] & D_0 \arrow[d, "h_0"] \\
\cdots & A_{i+2} & A_{i+1} & A_i
\end{tikzcd}$$

satisfying

(i) $h_0(d) \in A_i(\theta d)$ for $d \in D_0$;

(ii) h_1 is a $\theta\phi$-derivation;

(iii) h_j is a θ-morphism for $j \geqslant 2$.

The maps $\hat{h}_j$ of diagram (9.5.2) are then given by

$$\begin{aligned}
\hat{h}_j(d) &= (h_j(d), f_0(q)) && \text{if } d \in D_j(q),\ j \geqslant 2,\\
\hat{h}_1(d) &= (h_1(d), f_0(q)) && \text{if } d \in D_1(p, q),\\
\hat{h}_0(q) &= (h_0(d), f_0(q)) && \text{if } q \in D_0.
\end{aligned}$$

In the case $i = 1$, because of the special form of E_1, we also need a map $\tau \colon D_0 \to H$ satisfying

(iv) $\tau(q) \in H(\theta'(q), \theta(q))$ for some $\theta'(q) \in \operatorname{Ob} H$,

and in this case $\hat{h}_0(q) = (\tau(q), h_0(q), f_0(q))$.

It is now an easy matter to see that these data are equivalent to an element of dimension i in $\Theta\,\mathsf{CHN}(\nabla D, A)$. In the case $i = 1$, the map τ defines a natural transformation $\tilde{\tau} \colon \theta' \to \theta$, where $\theta'(g) = \tau(p)\theta(g)\tau(q)^{-1}$ for $g \in G(p, q)$. This $\tilde{\tau}$ is an element of the groupoid $\mathsf{GPDS}(G, H)$ (the operator groupoid for $\mathsf{CHN}(\nabla D, A)$) and provides the first component of the triple $(\tilde{\tau}, \tilde{h}, \tilde{f})$ which is the required element of $\Theta_1\,\mathsf{CHN}(\nabla D, A)$; the other components are $\tilde{f} \colon \nabla D \to A$, the morphism of chain complexes induced by f, and $\tilde{h}$, the 1-fold homotopy $\nabla D \to A$ induced by h. Here $\tilde{h}_0(1_p) = h_0(p)$ and $\tilde{h}_n\alpha_n = h_n$ for $n \geqslant 1$, where the α_i are as in diagram (7.4.1) in Theorem 7.4.18. The rest of the proof is straightforward. □

9.6 The tensor product of free crossed complexes is free

The exponential law in Crs of Theorem 9.3.17 has as a consequence that the tensor product of free crossed complexes is a free crossed complex.

We will prove the following theorem.

Theorem 9.6.1. *If $C' \to C$ and $D' \to D$ are morphisms of relative free type then so also is $C' \otimes D \cup C \otimes D' \to C \otimes D$, where $C' \otimes D \cup C \otimes D'$ denotes the pushout of the pair of morphisms*

$$C' \otimes D \leftarrow C' \otimes D' \to C \otimes D'.$$

Proof. Since the tensor product $- \otimes D'$ has a right adjoint, it preserves colimits. Also, since $- \otimes -$ is symmetric, the functor $C \otimes -$ preserves colimits.

We use Propositions 7.3.20 and 7.3.21 from Section 7.3.iii, as well as the following lemma:

Lemma 9.6.2. *If the following squares are pushouts*

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & C' \arrow[d] \\ A \arrow[r] & C, \end{tikzcd} \qquad \begin{tikzcd} B' \arrow[r] \arrow[d] & D' \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

then so also is the induced square

$$\begin{tikzcd} A' \otimes B \cup A \otimes B' \arrow[r] \arrow[d] & C' \otimes D \cup C \otimes D' \arrow[d] \\ A \otimes B \arrow[r] & C \otimes D. \end{tikzcd}$$

□

The proof of the general theorem builds inductively on the previous lemma and the following special case.

Lemma 9.6.3. *The theorem is true when $C' \to C$ and $D' \to D$ are of the type $\mathbb{S}(m-1) \to \mathbb{F}(m)$ and $\mathbb{S}(n-1) \to \mathbb{F}(n)$ respectively.* □

Since $Y \otimes -$ and $- \otimes Z$ preserve coproducts, we deduce the result in the case when $C' \to C$, $D' \to D$ are of the type $\coprod_\lambda \mathbb{S}(n-1) \to \coprod_\lambda \mathbb{F}(n)$ and $\coprod_\lambda \mathbb{S}(m-1) \to \coprod_\lambda \mathbb{F}(m)$.

Putting morphisms of this type in Lemma 9.6.2, and using Proposition 7.3.21, it easily follows that the theorem is true for morphisms of simple relative free type, that is for morphisms $C' \to C$, $D' \to D$ obtained as pushouts

$$\begin{tikzcd} \coprod_\lambda \mathbb{S}(n-1) \arrow[r] \arrow[d] & C' \arrow[d] \\ \coprod_\lambda \mathbb{F}(n) \arrow[r] & C, \end{tikzcd} \qquad \begin{tikzcd} \coprod_\mu \mathbb{S}(m-1) \arrow[r] \arrow[d] & D' \arrow[d] \\ \coprod_\mu \mathbb{F}(m) \arrow[r] & D. \end{tikzcd}$$

Next, using Propositions 7.3.20, 7.3.21, and Lemma 9.6.2 we can prove the result for composites of morphisms of relative free type. A general morphism of relative free type is a colimit of simple ones, as in Corollary 7.3.23, and the full result now follows from Proposition 7.3.20 and Lemma 9.6.2. $\square$

Remark 9.6.4. Another way of seeing that the tensor product $T = C \otimes D$ of free crossed complexes is free is to use the analysis of the tensor product in previous sections: the groupoid T_1 is a free groupoid in view of its description in Proposition 9.4.1; in dimension 2 the freeness is Corollary 9.4.7; in dimensions > 2 Theorem 9.5.4 gives T_n as the dimension n part of a tensor product of chain complexes with operator groupoids, and here the freeness follows by traditional methods. $\square$

Corollary 9.6.5. *If $C' \to C$ is a morphism of relative free type and W is a crossed complex of free type, then $C' \otimes W \to C \otimes W$ is of relative free type.*

Corollary 9.6.6. *If C is a free crossed complex and $f^+ \colon C \to D$ is a morphism of crossed complexes, then a homotopy $H \colon f^- \simeq f^+$ of morphisms is entirely determined by the values of f^+ and H on the free basis of C.*

9.7 The monoidal closed category of filtered spaces

We proceed a step further and consider the category FTop of filtered spaces and look for a natural structure of closed category.

The categorical product in FTop is given by

$$(X_* \times Y_*)_n = X_n \times Y_n, \quad n \geqslant 0.$$

This product is convenient for maps into it, as for any categorical product. However our main example of filtered spaces, that of CW-complexes, suggests a different product as worth consideration, and this will turn out to be convenient for maps from it, to other filtered spaces.

If X_*, Y_* are CW-filtrations, then the product $X \times Y$ of the spaces (in the category of compactly generated spaces) has a natural and convenient CW-structure in which the n-dimensional cells are all products $e^p \times e^q$ of cells of X_*, Y_* respectively where $p + q = n$. This suggests the following definition.

Definition 9.7.1. If X_*, Y_* are filtered spaces, their *tensor product* $X_* \otimes Y_*$ is the filtered space given on $X \times Y$ by the family of subspaces

$$(X \otimes Y)_n = \bigcup_{p+q=n} X_p \times Y_q$$

where the union is simply the union of subspaces of $X \times Y$. $\square$

Exercise 9.7.2. 1. We have said that the filtration $X_* \otimes Y_*$ is not the product in the category FTop. Verify that the filtration $(X_* \times Y_*)_n = X_n \times Y_n$ for all $n \geqslant 0$ is the product $X_* \times Y_*$ in the category FTop.

2. Is there a structure of cartesian closed category on FTop? i.e. is there an internal hom that is adjoint to the cartesian product? □

Notice that, for example, I_*^n is the n-fold tensor product of I_* with itself because I_*^n is the CW-filtered space of the standard n-cube.

With the product $\otimes$, FTop is a monoidal category. It is also symmetric, i.e. the tensor product is also commutative.

We now show how to define an internal hom $\mathsf{FTOP}(Y_*, Z_*)$ in the category FTop so as to make that category a monoidal closed category with an exponential law giving a natural bijection

$$e\colon \mathsf{FTop}(X_* \otimes Y_*, Z_*) \cong \mathsf{FTop}(X_*, \mathsf{FTOP}(Y_*, Z_*)).$$

To see how this comes about, note that a filtered map $f\colon X_* \otimes Y_* \to Z_*$ will map $X_p \times Y_q$ to Z_{p+q}, by definition of the filtration on the tensor product of filtered spaces. Under the exponential law for topological spaces we have

$$\mathsf{Top}(X_p \times Y_q, Z_{p+q}) \cong \mathsf{Top}(X_p, \mathsf{TOP}(Y_q, Z_{p+q})).$$

This suggests the definition:

$$\mathsf{FTOP}(Y_*, Z_*)_p = \{g \in \mathsf{Top}(Y, Z) \mid g(Y_q) \subseteq Z_{p+q} \text{ for all } q \geqslant 0\}.$$

This gives a filtration on the topological space $\mathsf{TOP}(Y, Z)$ and so defines the filtered space $\mathsf{FTOP}(Y_*, Z_*)_p$. The exponential law in the category Top now gives the exponential law

$$e\colon \mathsf{FTop}(X_* \otimes Y_*, Z_*) \cong \mathsf{FTop}(X_*, \mathsf{FTOP}(Y_*, Z_*)),$$

from which one can deduce the exponential law

$$e\colon\ \mathsf{FTOP}(X_* \otimes Y_*, Z_*) \cong \mathsf{FTOP}(X_*, \mathsf{FTOP}(Y_*, Z_*)),$$

either using the general result in Appendix C, Section C.7 (see Equation (C.7.3)) or directly as an exercise.

An advantage of having this internal hom for filtered spaces is that we can apply our fundamental crossed complex functor Π to it. To say more on this, we first discuss the notion of homotopy in FTop.

The convenient definition of homotopy $H\colon f^- \simeq f^+\colon Y_* \to Z_*$ of maps f^-, f^+ of filtered spaces is that H is a map $I \times Y \to Z$ which is a homotopy $f^- \simeq f^+$ such that $H(I \times Y_q) \subseteq Z_{q+1}$ for all $q \geqslant 0$. This last condition is equivalent to H being a filtered map $I_* \otimes Y_* \to Z_*$. Equivalently, we can regard H also as a map

$$I_* \to \mathsf{FTOP}(Y_*, Z_*), \text{ or } Y_* \to \mathsf{FTOP}(I_*, Z_*),$$

although the latter interpretation involves the twisting map $I_* \otimes Y_* \to Y_* \otimes I_*$.

It is also possible to consider 'higher filtered homotopies' as filtered maps

$$E^n_* \otimes Y_* \to Z_*$$

or equivalently as maps

$$E^n_* \to \mathsf{FTOP}(Y_*, Z_*).$$

This fits with the exponential law for crossed complexes, see Equation (9.3.2).

9.8 Tensor products and the fundamental crossed complex

In order to obtain the Homotopy Classification Theorem 11.4.19, we need to use tensor products and homotopies of crossed complexes and their relation to homotopies of filtered maps.

We have defined the notion of homotopies for maps of filtered spaces. They give rise to 1-homotopies between the induced morphisms of fundamental crossed complexes. It is possible to prove this directly, but it follows elegantly from more general results. In particular we need:

Theorem 9.8.1. *If X_* and Y_* are filtered spaces, then there is a natural morphism*

$$\zeta\colon \Pi X_* \otimes \Pi Y_* \to \Pi(X_* \otimes Y_*)$$

such that:

i) *ζ is associative;*

ii) *if $*$ denotes a singleton space or crossed complex, then the following diagrams are commutative:*

$$\begin{tikzcd} \Pi X_* & (\Pi X_*) \otimes * \arrow[l, "\cong"'] \arrow[d, "\zeta"] \\ & \Pi(X_* \otimes *), \arrow[from=1-1, "\cong"'] \end{tikzcd} \qquad \begin{tikzcd} * \otimes \Pi X_* \arrow[r, "\cong"] \arrow[d, "\zeta"'] & \Pi X_* \arrow[dl, "\cong"] \\ \Pi(* \otimes X_*); & \end{tikzcd}$$

iii) *ζ is commutative in the sense that if $T_c\colon C \otimes D \to D \otimes C$ is the natural isomorphism of crossed complexes described in Theorem 9.3.16, and $T_t\colon X_* \otimes Y_* \to Y_* \otimes X_*$ is the twisting map, then the following diagram is commutative:*

$$\begin{tikzcd} \Pi X_* \otimes \Pi Y_* \arrow[r, "\zeta"] \arrow[d, "T_c"'] & \Pi(X_* \otimes Y_*) \arrow[d, "\Pi(T_t)"] \\ \Pi Y_* \otimes \Pi X_* \arrow[r, "\zeta"] & \Pi(Y_* \otimes X_*); \end{tikzcd}$$

iv) *if X_*, Y_* are the skeletal filtrations of CW-complexes, then ζ is an isomorphism.*

The proof is deferred to Chapter 15 where we can use the techniques of ω-groupoids. Note that the construction of the natural transformation ζ could in principle be proved directly, but this would be technically difficult because of the complications of the relations for the tensor product of crossed complexes.

In fact ζ is an isomorphism under more general conditions.[125]

We now prove that the functor $\Pi\colon \mathsf{FTop} \to \mathsf{Crs}$ is a homotopy functor.

Corollary 9.8.2. *A homotopy $H\colon f^- \simeq f^+\colon X_* \to Y_*$ in* FTop *induces a (left) homotopy $\Pi H\colon \Pi f^- \simeq \Pi f^+\colon \Pi X_* \to \Pi Y_*$ in* Crs.

Proof. Let $H\colon I_* \otimes X_* \to Y_*$. Note that we identify ΠI_* with the groupoid $\mathcal{I}$. The composition

$$\mathcal{I} \otimes \Pi X_* \xrightarrow{\cong} \Pi I_* \otimes \Pi X_* \xrightarrow{\zeta} \Pi(I_* \otimes X_*) \xrightarrow{\Pi(H)} \Pi(Y_*)$$

is a homotopy $\Pi(f^-) \simeq \Pi(f^+)$. □

Similar statements hold for right homotopies of crossed complexes. A right homotopy $C \to D$ is a morphism $C \otimes \mathcal{I} \to D$, or, equivalently, a morphism $C \to \mathsf{CRS}(\mathcal{I}, D)$. We may also define a right homotopy in FTop to be a map $Y_* \otimes I_* \to Z_*$. By Theorem 9.8.1, such a map gives rise to a right homotopy $\Pi Y_* \otimes \mathcal{I} \to \Pi Z_*$.

More generally we prove that Π takes higher homotopies in FTop to higher homotopies in Crs:

Proposition 9.8.3. *There is a natural morphism of crossed complexes*

$$\psi\colon \Pi(\mathsf{FTOP}(X_*, Y_*)) \to \mathsf{CRS}(\Pi X_*, \Pi Y_*)$$

which is Π in dimension 0.

Proof. It is sufficient to construct the morphism $\hat{\psi}$ as the composition in the following commutative diagram

$$\begin{tikzcd}
\Pi(\mathsf{FTOP}(X_*, Y_*)) \otimes \Pi X_* \arrow[rr, "\hat{\psi}"] \arrow[dr, "\zeta"'] & & \Pi Y_* \\
& \Pi(\mathsf{FTOP}(X_*, Y_*) \otimes X_*) \arrow[ur, "\Pi e"'] &
\end{tikzcd}$$

where $e\colon \mathsf{FTOP}(X_*, Y_*) \otimes X_* \to Y_*$ is the evaluation morphism, i.e. the adjoint to the identity on $\mathsf{FTOP}(X_*, Y_*)$. □

9.9 The Homotopy Addition Lemma for a simplex

In this section we describe explicitly and algebraically a free basis and the boundary for the crossed complex $\Pi\Delta^n$ where Δ^n is the topological n-simplex with its standard filtration by dimension. This formula is called the Homotopy Addition Lemma for a simplex.[126] We call $\Pi\Delta^n$ the 'n-simplex crossed complex', and its description is used in several places later (see in particular Section 12.5.i and Section 10.4.ii).

It is a feature of our exposition using crossed complexes that the Homotopy Addition Lemma can be seen as an algebraic fact which models accurately the geometry. That happens because crossed complexes model well the geometry, and a key aspect of that is the use of groupoids to handle all the vertices of the simplex.

Definition 9.9.1. First it is useful to write out the rules for the cylinder $\mathrm{Cyl}\,(C) = \mathcal{I} \otimes C$, as a reference. Let C be a crossed complex. We apply the relations in the definition of tensor product of crossed complexes (Definition 9.3.12) to this case.

For all $n \geqslant 0$ and $c \in C_n$, $\mathcal{I} \otimes C$ is generated by elements $0 \otimes c$, $1 \otimes c$ of dimension n and $\iota \otimes c$, $\iota^{-1} \otimes c$ of dimension $(n+1)$ with the following defining relations for $a = 0, 1, \iota$:

• *Source and target*.

$$\begin{aligned} t(a \otimes c) &= ta \otimes tc &&\text{for all } a \in \mathcal{I},\ c \in C, \\ s(a \otimes c) &= a \otimes sc &&\text{if } a = 0, 1,\ n = 1, \\ s(a \otimes c) &= sa \otimes c &&\text{if } a = \iota, \iota^{-1},\ n = 0. \end{aligned}$$

• *Relations with operations*.

$$a \otimes c^{c'} = (a \otimes c)^{ta \otimes c'} \quad \text{if } n \geqslant 2,\ c' \in C_1.$$

• *Relations with additions*.

$$a \otimes (c + c') = \begin{cases} a \otimes c + a \otimes c' & \text{if } a = 0, 1,\ n \geqslant 1 \text{ or if } a = \iota, \iota^{-1},\ n \geqslant 2, \\ (a \otimes c)^{ta \otimes c'} + a \otimes c' & \text{if } a = \iota, \iota^{-1}, n = 1, \end{cases}$$

$$(\iota^{-1}) \otimes c = \begin{cases} -(\iota \otimes c) & \text{if } n = 0, \\ -(\iota \otimes c)^{(\iota^{-1}) \otimes tc} & \text{if } n \geqslant 1. \end{cases}$$

• *Boundaries*.

$$\delta(a \otimes c) = \begin{cases} -(a \otimes \delta c) - (ta \otimes c) + (sa \otimes c)^{a \otimes tc} & \text{if } a = \iota, \iota^{-1},\ n \geqslant 2, \\ -ta \otimes c - a \otimes sc + sa \otimes c + a \otimes tc & \text{if } a = \iota, \iota^{-1},\ n = 1, \\ a \otimes \delta c & \text{if } a = 0, 1,\ n \geqslant 2. \end{cases}$$

□

These rules simplify if instead of the cylinder, we analyse the cone.

Definition 9.9.2. Let C be a crossed complex. The *cone* $\mathrm{Cone}(C)$ is defined by

$$\mathrm{Cone}\,(C) = (\mathcal{I} \otimes C)/(\{1\} \otimes C),$$

which can alternatively be seen as a pushout

$$\begin{tikzcd} \{1\} \otimes C \arrow[r] \arrow[d] & \{v\} \arrow[d] \\ \mathcal{I} \otimes C \arrow[r] & \mathrm{Cone}\,(C). \end{tikzcd}$$

We call v the *vertex* of the cone. □

Proposition 9.9.3. *For a crossed complex C the cone* $\mathrm{Cone}\,(C)$ *is generated by elements $0 \otimes c$, $\iota \otimes c$ of dimensions n, $n+1$ respectively, and v of dimension 0 with the rules*

Source and target.

$$t(a \otimes c) = \begin{cases} 0 \otimes tc & \text{if } a = 0, \\ v & \text{otherwise.} \end{cases}$$

Relations with operations.

$$a \otimes c^{c'} = a \otimes c \quad \text{if } n \geqslant 2,\ c' \in C_1.$$

Relations with additions.

$$a \otimes (c + c') = a \otimes c + a \otimes c'$$

and

$$(\iota^{-1}) \otimes c = \begin{cases} -(\iota \otimes c) & \text{if } n = 0, \\ -(\iota \otimes c)^{(\iota^{-1}) \otimes tc} & \text{if } n \geqslant 1. \end{cases}$$

Boundaries.

$$\delta(\iota \otimes c) = \begin{cases} -(\iota \otimes \delta c) + (0 \otimes c)^{\iota \otimes tc} & \text{if } n \geqslant 2, \\ -\iota \otimes sc + 0 \otimes c + \iota \otimes tc & \text{if } n = 1, \end{cases}$$
$$\delta(0 \otimes c) = 0 \otimes \delta c \qquad \text{if } n \geqslant 2.$$

The simplicity of the rules for operations and additions is one of the advantages of the form of our definition of the cone, in which the end at 1 *is shrunk to a point.*

We use the above to work out the fundamental crossed complex of the simplex Δ^n in an algebraic fashion. We regard Δ^n topologically as the topological cone

$$\mathrm{Cone}\,(\Delta^{n-1}) = (I \times \Delta^{n-1})/(\{1\} \times \Delta^{n-1}).$$

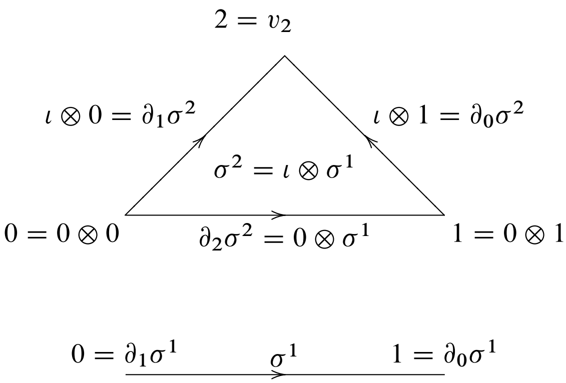
The vertices of $\Delta^1 = I$ are ordered as $0 < 1$. Inductively, we get vertices $v_0, \dots, v_n$ of Δ^n with $v_n = v$ being the last introduced in the cone construction, the other vertices v_i being $(0, v_i)$. The fact that our algebraic formula corresponds to the topological one follows from facts stated earlier on the tensor product and on the HHSvKT stated in the next section.

We now define inductively top dimensional generators of the crossed complex $\Pi\Delta^n$ by, in the cone complex:

$$\sigma^0 = v, \quad \sigma^1 = \iota, \quad \sigma^n = (\iota \otimes \sigma^{n-1}), \quad n \geqslant 2$$

with σ^0 being the vertex of $\Pi\Delta^0$.

We give conventions for the faces of σ^n, as illustrated in the following diagram:



We define inductively

$$\partial_i\sigma^n = \begin{cases} \iota \otimes \partial_i\sigma^{n-1} & \text{if } i < n, \\ 0 \otimes \sigma^{n-1} & \text{if } i = n. \end{cases}$$

Theorem 9.9.4 (Simplicial Homotopy Addition Lemma). *In the simplex Δ^n the following formulae hold, where $u_n = \iota \otimes v_{n-1}$:*

$$\delta_2\sigma^2 = -\partial_1\sigma^2 + \partial_2\sigma^2 + \partial_0\sigma^2, \tag{9.9.1}$$

$$\delta_3\sigma^3 = (\partial_3\sigma^3)^{u_3} - \partial_0\sigma^3 - \partial_2\sigma^3 + \partial_1\sigma^3, \tag{9.9.2}$$

while for $n \geqslant 4$

$$\delta_n\sigma^n = (\partial_n\sigma^n)^{u_n} + \sum_{i=0}^{n-1}(-1)^{n-i}\partial_i\sigma^n. \tag{9.9.3}$$

Proof. For the case $n = 2$ we have

$$\begin{aligned}\delta_2\sigma^2 &= \delta_2((\iota \otimes \iota)) \\ &= -\iota \otimes 0 + 0 \otimes \iota + \iota \otimes 1 \\ &= -\partial_1\sigma^2 + \partial_2\sigma^2 + \partial_0\sigma^2.\end{aligned}$$

For $n = 3$ we have

$$\begin{aligned}\delta_3\sigma^3 &= \delta_3(\iota \otimes \sigma^2) \\ &= (0 \otimes \sigma^2)^{\iota\otimes v_2} - \iota \otimes \delta_2\sigma^2 \\ &= (0 \otimes \sigma^2)^{u_3} - \iota \otimes (-\partial_1\sigma^2 + \partial_2\sigma^2 + \partial_0\sigma^2) \\ &= (\partial_3\sigma^3)^{u_3} - \partial_0\sigma^3 - \partial_2\sigma^3 + \partial_1\sigma^3.\end{aligned}$$

We leave the general case to the reader. The key points that make it easy are the rules on operations and additions of Proposition 9.9.3. □

Remark 9.9.5. (i) Notice that the formula of δ_2 has values in groupoids, and that for δ_3 has values in the top part of a crossed module which is in this case nonabelian.

(ii) There are many possible conventions for the Homotopy Addition Lemma, and that given here is unusual. However, our formula follows naturally from the geometry of the cone and our algebra for the tensor product. A more traditional formula is obtained by replacing ∂_i in the above formula in dimension n by ∂_{n-i}, that is by applying the formula to the 'transpose' of the simplex.[127]

(iii) It is a good exercise to prove that $\delta_2\delta_3 = 0$. It is not so easy to prove directly from the formula that $\delta_3\delta_4 = 0$.[128] Of course we know these composites are 0 since we are working in the category of crossed complexes. □

The representation $\mathrm{Cone}\,(\Delta^{n-1}) = \Delta^n$ gives a cellular contracting homotopy of Δ^{n-1}, and so $\Pi\Delta^n$ is a contractible crossed complex. We shall use this fact later.

We can now state the formula in terms of free generators and boundaries for the whole crossed complex $\Pi\Delta^n$. It has a free generator σ^n in dimension n and also free generators $\alpha\sigma^m$ in dimension m for all $0 \leqslant m < n$ and all increasing functions $\alpha\colon [m] \to [n]$. The boundary of such a $\alpha\sigma^m$ is given by the simplicial Homotopy Addition Lemma in dimension m.

This Homotopy Addition Lemma for the simplex will be related to the theory of simplicial sets and the notion of simplicial nerve of a crossed complex in Section 9.10.

We can also obtain a cubical Homotopy Addition Lemma using the cube crossed complex $\Pi I^n = \mathcal{I}^{\otimes n}$. In this crossed complex, let $c^n = \iota \otimes \cdots \otimes \iota$ be the n-fold tensor product of ι with itself, and for $\alpha = 0, 1$ let $c_i^\alpha = \partial_i^\alpha c^n$ be the element of dimension $(n-1)$ obtained by replacing in c^n the ι in the i-th place by α. The formulae for the boundary in the tensor product then yield by induction, using $\mathcal{I}^{\otimes n} = \mathcal{I} \otimes \mathcal{I}^{\otimes(n-1)}$:

Proposition 9.9.6 (Cubical Homotopy Addition Lemma).

$$\delta_n c^n = \begin{cases} -c_1^1 - c_2^0 + c_1^0 + c_2^1 & \text{if } n = 2, \\ -c_3^1 - (c_2^0)^{u_2 c} - c_1^1 + (c_3^0)^{u_3 c} + c_2^1 + (c_1^0)^{u_1 c} & \text{if } n = 3, \\ \sum_{i=1}^{n} (-1)^i \{c_i^1 - (c_i^0)^{u_i c}\} & \text{if } n \geqslant 4 \end{cases}$$

(*where* $c = c^n$ *and* $u_i = \partial_1^1 \partial_2^1 \ldots \hat{\imath} \ldots \partial_{n+1}^1$).

Remark 9.9.7. It should be said that this suggested 'proof' is not quite fair, since we are using for crossed complexes a lot of results the proofs of some of which rely on the cubical Homotopy Addition Lemma established independently. However, this calculation shows how the results tie together, and that once we have these results established they give powerful means of calculation, some of which are inherently nonabelian, and which usually involve module operations not so easily handled by traditional methods. □

9.10 Simplicial sets and crossed complexes

In earlier sections of this chapter we have given an account of the cubical nerve and classifying space of a crossed complex. In this section, we give a brief account of some of the corresponding simplicial theory, as this is used in discussing the standard free crossed resolution of a groupoid.

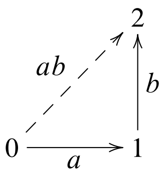
In Theorem 9.9.4 we gave an explicit description of $\Pi\Delta^n_*$, the fundamental crossed complex of the standard geometric n-simplex Δ^n. We now relate this to the theory of simplicial sets described briefly in Section A.10 of Appendix A.

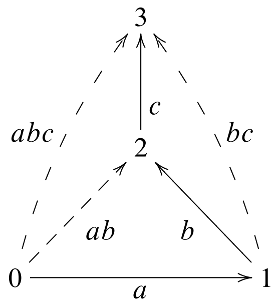
Definition 9.10.1 (The fundamental crossed complex of a simplicial set). Let K be a simplicial set. The fundamental crossed complex ΠK is to have free generators in dimension n given by the elements of K_n and the boundary δk for $k \in K_n$ is given by the Homotopy Addition Lemma in dimension n. □

Definition 9.10.2. Let C be a crossed complex. Its *simplicial nerve* is the simplicial set $N^\Delta C$ which in dimension n is given by

$$N^\Delta(C)_n = \mathsf{Crs}(\Pi\Delta^n, C). \qquad \square$$

Example 9.10.3 (The simplicial nerve of a groupoid). Let P be a groupoid. Then P can be regarded as a crossed complex and then its *simplicial nerve* $N^\Delta P$ as given by Definition 9.10.2 can be interpreted as follows: it is the simplicial set which in dimension 0 consists of the objects of P and in dimension $n > 0$ consists of the *composable sequences* of arrows of P, i.e. sequences $[a_1, \ldots, a_n]$ such that the target of a_i is the source of a_{i+1} for $1 \leqslant i < n$. The face operators ∂_i are defined on these elements so that each face of dimension 2 is commutative. This leads to the following pictures in dimensions 2 and 3:





with face operators in which ∂_i gives the face opposite to the vertex i:

$$\begin{aligned} \partial_0[a,b,c] &= [b,c], & \partial_1[a,b,c] &= [ab,c], \\ \partial_2[a,b,c] &= [a,bc], & \partial_3[a,b,c] &= [a,b]. \end{aligned}$$

This tetrahedral picture shows the relation of this construction to associativity of the groupoid operation.

The general formulae are $\partial_0[a] = ta$, $\partial_1[a] = sa$ and for $n \geqslant 2$:

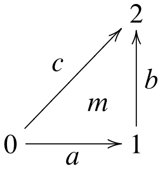
$$\partial_i[a_1,\dots,a_n] = \begin{cases} [a_2,\dots,a_n] & \text{if } i = 0, \\ [a_1,\dots,a_i a_{i+1},\dots,a_n] & \text{if } 1 < i < n, \\ [a_1,\dots,a_{n-1}] & \text{if } i = n. \end{cases}$$

We can also define degeneracy operators for $i = 0,\dots,n$ by

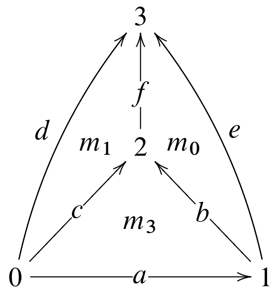
$$\varepsilon_i[a_1,\dots,a_n] = [a_1,\dots,a_{i-1},1_i,a_i,\dots,a_n]$$

where 1_i denotes uniquely the identity at the object i for which 1_i gives a composable sequence of length $n+1$. In terms of the notation of the Homotopy Addition Lemma in which $u_n = \partial_0^{n-1}$ we also have $u_n[a_1,\dots,a_n] = a_n$. So we have a formula for $\delta_n[a_1,\dots,a_n]$ which we shall use in the Definition 10.2.7 of the standard free crossed resolution of a groupoid. □

Example 9.10.4 (The simplicial nerve of a crossed module). It is useful to indicate how the classifying space $B\mathcal{M}$ for a crossed module $\mathcal{M} = (\mu\colon M \to P)$ over a groupoid is constructed as the geometric realisation of a simplicial set $K = N^{\Delta}(\mathcal{M})$. For $n \geqslant 2$, an n-simplex of K is an $(n+1)$-tuple of $(n-1)$-simplices of K which match up appropriately on their faces of dimension $(n-2)$. This gives the following pictures for $n = 2, 3$. The first of the pictures shows an element of K_2 and the second, which does not show m_2, shows an element of K_3:



$$\mu m = -c + a + b$$



provided we have the rules

$$\mu m_0 = -e + b + f, \quad \mu m_1 = -d + c + f,$$
$$\mu m_2 = -d + a + e, \quad \mu m_3 = -c + a + b,$$

together with the rule

$$(m_3)^f - m_0 - m_2 + m_1 = 0.$$

You might like to verify that these rules are consistent.

The geometric realisation of $N^\Delta(\mathcal{M})$ gives the simplicial classifying space of the crossed module.[129]

Remark 9.10.5. From the simplicial nerve $N^\Delta(P)$ of a groupoid P it is natural to form the crossed complex $\Pi N^\Delta(P)$; this has a free generator for each simplex of $N^\Delta(P)$ and boundary given by the simplicial Homotopy Addition Lemma. Then there is a natural isomorphism $\pi_1 \Pi N^\Delta(P) \cong P$, and also $\Pi N^\Delta(P)$ is aspherical, i.e. all homology in dimensions > 1 vanishes. When we discuss resolutions in Chapter 10 we will see $\Pi N^\Delta(P)$ as the standard free crossed resolution of P. However the proof of asphericity is best done using the notion of universal covering crossed complex, which we introduce in Section 10.1. $\Box$

This result can be related to the case of local coefficients (see Section 12.4).

Remark 9.10.6. Note also that if A is a chain complex with a trivial group of operators, then

$$\begin{aligned}(N^\Delta \Theta A)_r &= \mathsf{Crs}(\Pi \Delta^r, \Theta A)\\ &= \mathsf{Chn}(\nabla \Pi \Delta^r, A).\end{aligned}$$

In this last formula, $\nabla \Pi \Delta^r$ consists by Proposition 8.4.2 of the chain complex $C_* \widetilde{\Delta}^r$ of cellular chains of the universal covers of Δ^r based at the vertices of Δ^r, with the action of the groupoid $\pi_1 \Delta^r$. Since A has trivial group acting, it follows that

$$\mathsf{Chn}(\nabla \Pi \Delta^r, A) = \mathsf{Chn}(C_* \Delta^r, A),$$

where $C_* \Delta^r$ is the usual chain complex of cellular chains of Δ^r.

This shows that $N^{\Delta}\Theta A$ coincides with the simplicial abelian group of the Dold–Kan Theorem which gives an equivalence between chain complexes and simplicial abelian groups. It also explains why the work of Ashley in [Ash88], which gives an equivalence between crossed complexes and simplicial T-complexes, is subtitled 'A nonabelian version of the Dold–Kan Theorem'. See also Section 14.8 for the analogous cubical version and further references. □

Notes

117 p. 278 The notion of monoidal and monoidal closed category can be seen as central to many parts of mathematics, and for the general theory we refer the reader to [ML71]. A full exposition on monoidal categories requires the notion of *coherence*; we avoid dealing with this here because all of the conditions such as associativity on the tensor products with which we deal in the end reduce to associativity of a cartesian product, through the notion of bimorphisms, and so the coherence properties needed follow from the universal properties of categorical products.

118 p. 279 The category of crossed complexes is shown in [BH81b] to be equivalent to the category of ∞-groupoids, which are also called globular ω-groupoids. Thus the monoidal closed structure on crossed complexes yields a monoidal closed structure on the latter category, which generalises studies of such structures on 2-groupoids, for example [KP02]. However in our scheme the tensor product of 2-groupoids is naturally a 4-groupoid, from which a 2-groupoid would be obtained by cotruncation. This leads also to the notion of enrichment of the category FTop over the category Crs which requires further study. Other relevant work for such monoidal closed structures on higher categories rather than groupoids is in [Cra99], [Cra99] and [AABS02].

119 p. 281 A further reference on compactly generated spaces is [HO09].

120 p. 283 In this case to obtain a monoidal closed structure, i.e. an internal hom, one has to move to the world of rings with several objects, also known as additive categories, see [Mit72]. This is analogous to the move from groups to groupoids in order to obtain a cartesian closed category.

121 p. 283 This description of the internal hom structure in the category Mod appeared in [BH90].

122 p. 293 The example was suggested by a question of N. Ramachandran on the fundamental groupoid of a free loop space. For further details see [Bro10a].

123 p. 300 This can be useful in constructing homotopy equivalences of crossed complexes, using for example the gluing lemma [KP97], Lemma 7.3. There is probably more to be applied here than has been done so far.

124 p. 304 This tensor product of (nonabelian) groups is related to, but not the same as, the tensor product defined by Brown and Loday and used in their construction of universal crossed squares of groups in [BL87a], see also Section B.4. The Brown–Loday product is defined for two groups acting 'compatibly' on each other; it also satisfies the standard commutator identities displayed above. The relation between the two tensor products is clarified by Gilbert and Higgins in [GH89]. Baues and Conduché in [BC92] apply the tensor product to define the 'tensor algebra of a group', and this is generalised in [BB93].

125 p. 314 It is proved in [BB93] that ζ is an isomorphism if the filtered spaces X_*, Y_* are connected and cofibred: the latter condition on X_* is that each inclusion $X_n \to X_{n+1}$ is closed cofibration. These are useful conditions since they are satisfied not only by CW-filtrations but also by the filtration BC^* of the classifying space of a crossed complex by the levels BC^n where C^n is the n-skeleton of C, compare Section 13.5.

126 p. 315 The Homotopy Addition Lemma for a simplex is used in Blakers' 1948 work [Bla48] and was known to be an essential part of proofs of the absolute and relative Hurewicz theorems. However a proof appeared only in [Hu53] in 1955. The proof in G. W. Whitehead's text [Whi78] is by induction proving at the same time both Hurewicz theorems. It is clear that the algebra of crossed complexes is an essential part of the expression of this lemma. The algebraic derivation given here comes from [BS07].

127 p. 318 The traditional formula for the simplex HAL may be found in [Bla48], [Hue80a], [Whi78], [Ton03], [Bro99] and elsewhere. The story is that this formula was used in relative homotopy theory, and then it was realised in the early 1950s that no proof had been given. Who to ask to prove it? The answer is given in [Hu53]. The proof given in [Whi78] is by a combined circular induction with the absolute and relative Hurewicz theorems, following notes of J. F. Adams. The proof given here follows [BS07] in using the tensor product of crossed complexes.

128 p. 318 A direct proof that $\delta_3\delta_4 = 0$ is given for example by G. W. Whitehead in his book [Whi78]. It applies the second law for a crossed module.

129 p. 321 There are other methods of constructing the classifying space of a crossed module in the literature, often restricted to crossed modules over groups. The construction of the classifying space in [Lod82], [BS09] is in terms of bisimplicial groups, but this is more difficult to use for homotopy classification results. For a good survey of the links of crossed complexes with simplicial theory, see [Por11].

Chapter 10
Resolutions

The notion of ‘resolution’ of an algebraic object is one way of trying to describe an infinite object and its properties in finitary terms, or in some way other than attempting to list its elements, which can be a foolhardy endeavour. We also need ways of describing very large objects in manageable ways.

In Chapter 3 we showed how the notions of ‘syzygy’ and of ‘resolution by free modules’ arose from invariant theory, in trying to deal with algebras of polynomials. There we also showed how the analogous notion of ‘identity among relations’ for a presentation of a group led to the notion of free crossed module, which was first used to describe a topological situation, the structure of second relative homotopy groups.

In this chapter, we extend the notion of syzygy to all dimensions using crossed complexes, with the definition of *free crossed resolution* of a group, or groupoid. Surprisingly, the extension to groupoids rather than just groups turns out also to be useful for the purposes of calculation, as we shall see in Section 10.3. Our method there is to construct what we call

a home for a contracting homotopy

and to this end we need to pass to the universal covering groupoid of a group. The notion of ‘free crossed resolution of a groupoid’ also includes such resolutions for group actions, equivalence relations, and bundles of groups.[130]

In Section 10.4 we give an account of how the classical theory of acyclic models may be adapted to its use with crossed complexes instead of chain complexes.

10.1 Covering morphisms of crossed complexes

The results of this section are used twice in this chapter, but are of a different character to the rest, so you may wish to skip this section until it is needed.

The main results say that the theory of covering morphisms of crossed complexes is analogous to that of covering maps of spaces, in that the category of covering morphisms of a crossed complex C is equivalent to the category of covering morphisms of the groupoid $\pi_1 C$. So in this section we assume as known the notions of covering morphisms of groupoids dealt with in [Bro06], [Hig71]; some details and the notation and conventions as used here are given in Section B.7 of Appendix B.

Definition 10.1.1. A morphism $p\colon \widetilde{C} \to C$ of crossed complexes is a *covering morphism* if

(i) the morphism $p_1 \colon \widetilde{C}_1 \to C_1$ is a covering morphism of groupoids;

(ii) for each $n \geqslant 2$ and $\tilde{x} \in \widetilde{C}_0$, the morphism of groups $p_n \colon \widetilde{C}_n(\tilde{x}) \to C_n(p_0\tilde{x})$ is an isomorphism.

In such case we call $\widetilde{C}$ a *covering crossed complex* of C. □

This definition may also be expressed in terms of the unique covering homotopy property similar to the one given for fibrations in Section 12.1. Actually, coverings are fibrations with discrete fibre. So we can use the long exact sequence of a fibration given in the next chapter as Theorem 12.1.15.

Proposition 10.1.2. *Let $p \colon \widetilde{C} \to C$ be a covering morphism of crossed complexes and let $\tilde{a} \in \mathrm{Ob}(\widetilde{C})$. Let $a = p\tilde{a}$, and let $K = p_0^{-1}(a) \subseteq \mathrm{Ob}(\widetilde{C})$. Then p induces isomorphisms $\pi_n(\widetilde{C}, \tilde{a}) \to \pi_n(C, a)$ for $n \geqslant 2$ and a sequence*

$$1 \to \pi_1(\widetilde{C}, \tilde{a}) \to \pi_1(C, a) \to K \to \pi_0(\widetilde{C}) \to \pi_0(C)$$

which is exact in the sense of the exact sequence of a fibration of groupoids.

The comment about exactness has to do with operations on the pointed sets, see Theorem 12.1.15.

The following result gives a basic homotopical example of a covering morphism of crossed complexes.

Theorem 10.1.3. *Let X_* and Y_* be filtered spaces and let*

$$f \colon X \to Y$$

be a covering map of spaces such that for each $n \geqslant 0$, $f_n \colon X_n \to Y_n$ is also a covering map with $X_n = f^{-1}(Y_n)$. Then

$$\Pi f \colon \Pi X_* \to \Pi Y_*$$

is a covering morphism of crossed complexes.

Proof. By [Bro06], 10.2.1,

$$\pi_1 f_1 \colon \pi_1(X_1, X_0) \to \pi_1(Y_1, Y_0)$$

is a covering morphism of groupoids.

Now for each $n \geqslant 2$ and for each $x_0 \in X_0$, it is a standard result in homotopy theory that

$$f_* \colon \pi_n(X_n, X_{n-1}, x_0) \to \pi_n(Y_n, Y_{n-1}, p(x_0))$$

is an isomorphism (see for example, [Hu59]). □

Proposition 10.1.4. *Suppose given a pullback diagram of crossed complexes*

$$\begin{tikzcd} \widetilde{C} \arrow[r, "\bar{f}"] \arrow[d, "\bar{q}"'] & \widetilde{E} \arrow[d, "q"] \\ C \arrow[r, "f"'] & E \end{tikzcd}$$

in which q is a covering morphism. Then $\bar{q}$ is a covering morphism.

Proof. The groupoid case is [Bro06], 9.7.6. We leave the rest of the proof to the reader. □

Now we discuss the relation with π_1.

Proposition 10.1.5. *Let $p\colon \widetilde{C} \to C$ be a covering morphism of crossed complexes. Then the induced morphism $\pi_1(p)\colon \pi_1\widetilde{C} \to \pi_1 C$ is a covering morphism of groupoids. Moreover the following diagram is a pullback of groupoids:*

$$\begin{tikzcd} \widetilde{C} \arrow[r, "\bar{\phi}"] \arrow[d, "p"'] & \mathrm{sk}^1\, \pi_1\widetilde{C} \arrow[d, "\pi_1 p"] \\ C \arrow[r, "\phi"'] & \mathrm{sk}^1\, \pi_1 C. \end{tikzcd}$$

Proof. Let $\tilde{x} \in \widetilde{C}_0$, $p\tilde{x} = x$. We will show that $p'_{\tilde{x}}\colon \mathrm{Cost}_{\pi_1\widetilde{C}}\, \tilde{x} \to \mathrm{Cost}_{\pi_1 C}\, x$ is bijective.

Surjectivity follows easily from the surjectivity of $\mathrm{Cost}_{\widetilde{C}_1}\, \tilde{x} \to \mathrm{Cost}_{C_1}\, x$.

For injectivity, suppose that $[p\tilde{a}] = [p\tilde{b}]$, where $[\tilde{a}], [\tilde{b}] \in \mathrm{Cost}_{\pi_1\widetilde{C}}$. Let $p\tilde{a} = a, p\tilde{b} = b$. Then $[a] = [b] \in \pi_1 C$. Hence $sa = sb$ and $b^{-1}a = \delta_2 c$ for some $c \in C_2(x)$. Since p is a covering morphism, there is a $\tilde{c} \in \widetilde{C}_2$ such that $p\tilde{c} = c$. Since p has discrete kernel, $s\tilde{a} = s\tilde{b}$ and $\tilde{b}^{-1}\tilde{a} = \delta_2\tilde{c}$. Hence $[\tilde{a}] = [\tilde{b}]$ as required.

Finally, the fact that the diagram is a pullback of groupoids is clear from the conditions for a covering morphism, since an element of $\widetilde{C}$ is completely determined by its projection to C and its final point. □

Our next result is the analogue for covering morphisms of crossed complexes of a classical result for covering maps of spaces (see, for example, [Bro06]), 9.6.1. It gives a complete classification of covering morphisms of crossed complexes.

Let C be a crossed complex. We write CrsCov/C for the full subcategory of the slice category Crs/C whose objects are the covering morphisms of C.

Theorem 10.1.6. *If C is a crossed complex, then the functor $\pi_1\colon \mathsf{Crs} \to \mathsf{Gpds}$ induces an equivalence of categories*

$$\pi_1'\colon \mathsf{CrsCov}/C \to \mathsf{GpdsCov}/(\pi_1 C).$$

Proof. The inverse functor is constructed using the pullback diagram of Proposition 10.1.5. □

In Section 15.7 we will need the following results, which are analogues for crossed complexes of known results for groupoids ([Bro06], 10.3.3) and spaces.

Proposition 10.1.7. *Let $p\colon \widetilde{C} \to C$ be a covering morphism of crossed complexes, and let $y \in \widetilde{C}_0$. Let F be a connected crossed complex, let $x \in F_0$, and let $f\colon F \to C$ be a morphism of crossed complexes such that $f(x) = p(y)$. Then the following are equivalent:*

(i) *f lifts to a morphism $\tilde{f}\colon F \to \widetilde{C}$ such that $\tilde{f}(x) = y$ and $p\tilde{f} = f$;*

(ii) *$f(F_1(x)) \subseteq p(\widetilde{C}_1(y))$;*

(iii) *$f_*(\pi_1(F, x)) \subseteq p_*(\pi_1(\widetilde{C}, y))$.*

Further, if the lifted morphism as above exists, then it is unique.

Proof. That (i)$\Rightarrow$ (ii) $\Rightarrow$ (iii) is clear.

So we assume (iii) and prove (i).

We first assume F_0 consists only of x. Then the value of $\tilde{f}$ on x is by assumption defined to be y.

Next let $a \in F_1(x)$. By the assumption (iii) there is $c \in C_2(py)$ and $b \in \widetilde{C}(y)$ such that $f(a) = p(b) + \delta_2(c)$. Since p is a covering morphism there is a unique $d \in \widetilde{C}_2(y)$ such that $p(d) = c$. Thus $f(a) = p(b + \delta_2(d))$. So we define $\tilde{f}(a) = b + \delta_2(d) \in \widetilde{C}_2(y)$. It is easy to prove from the definition of covering morphism of groupoids that this makes $\tilde{f}$ a morphism $F_1(x) \to \widetilde{C}_1(y)$ such that $p\tilde{f} = f$.

For $n \geqslant 2$ we define $\tilde{f}\colon F_n(x) \to \widetilde{C}_n(y)$ to be the composition of f in dimension n and the inverse of the bijection $p\colon \widetilde{C}_n(y) \to C_n(py)$.

It is now straightforward to check that this defines a morphism $\tilde{f}\colon F \to \widetilde{C}$ as required.

If F_0 has more than one point, then we choose a tree groupoid in F_1 in the usual way as in the argument in [Bro06], 10.3.3, in order to extend over all of F. □

We will use the above result in the following form.

Corollary 10.1.8. *Let $p\colon \widetilde{C} \to C$ be a covering morphism of crossed complexes, and let F be a connected and simply connected crossed complex. Then the following diagram is a pullback in the category of crossed complexes:*

$$\begin{tikzcd} \mathsf{Crs}(F, \widetilde{C}) \times F \arrow[r, "\varepsilon"] \arrow[d, "p_* \times 1"'] & \widetilde{C} \arrow[d, "p"] \\ \mathsf{Crs}(F, C) \times F \arrow[r, "\varepsilon"'] & C, \end{tikzcd}$$

where the set of morphisms of crossed complexes has the discrete crossed complex structure, and the ε are evaluation maps.

Proof. This is simply a restatement of the existence and uniqueness of liftings of morphisms. □

Remark 10.1.9. It is possible that the covering morphisms are part of a factorization system as are the discrete fibrations in the contexts of [Bou87] and [SV10]. □

10.1.i Coverings of free crossed complexes

Recall that the utility of a free crossed complex is that if C is a free crossed complex on X_*, then a morphism $f : C \to D$ can be constructed inductively provided one is given the values $f_n x \in D_n$, $x \in X_n$, $n \geqslant 0$ and provided the following geometric conditions are satisfied: (i) $\delta^\alpha f_1 x = f_0 \delta^\alpha x$, $x \in X_1$, $\alpha = 0, 1$; (ii) $t f_n(x) = f_0(tx)$, $x \in X_n$, $n \geqslant 2$; (iii) $\delta_n f_n(x) = f_{n-1}\delta_n(x)$, $x \in X_n$, $n \geqslant 2$.

Notice that in (iii), f_{n-1} has to be defined on all of C_{n-1} before this condition can be verified.

We now show that freeness can be lifted to covering crossed complexes, using the following result of Howie ([How79], Theorem 5.1).

Theorem 10.1.10. *Let $p\colon A \to B$ be a morphism of crossed complexes. Then p is a fibration if and only if the pullback functor $p^*\colon \mathsf{Crs}/B \to \mathsf{Crs}/A$ has a right adjoint.*

As a consequence we get the following:[131]

Corollary 10.1.11. *If $p\colon A \to B$ is a covering morphism of crossed complexes, then $p^*\colon \mathsf{Crs}/B \to \mathsf{Crs}/A$ preserves all colimits.*

We shall use this last result to prove that coverings of free crossed complexes are free. For the proof, we need the simple crossed complexes $\mathbb{F}(n)$, $\mathbb{S}(n-1)$ from Definitions 7.1.11, 7.1.12.

Theorem 10.1.12. *Suppose given a pullback square of crossed complexes*

$$\begin{tikzcd} \widetilde{A} \arrow[r, "\tilde{j}"] \arrow[d, "p'"'] & \widetilde{C} \arrow[d, "p"] \\ A \arrow[r, "j"'] & C \end{tikzcd}$$

in which p is a covering morphism and $j\colon A \to C$ is relatively free. Then $\tilde{j}\colon \widetilde{A} \to \widetilde{C}$ is relatively free.

Proof. We suppose given the sequence of pushout diagrams

$$\begin{tikzcd} \coprod_{\lambda\in\Lambda_n} \mathbb{S}(n-1) \arrow[r] \arrow[d] & C^{n-1} \arrow[d] \\ \coprod_{\lambda\in\Lambda_n} \mathbb{F}(n) \arrow[r] & C^n \end{tikzcd}$$

defining C as relatively free. Let $\widehat{C}^n = p^{-1}(C^n)$. By Corollary 10.1.11, the following diagram is also a pushout:

$$\begin{tikzcd} p^*\left(\coprod_{\lambda\in\Lambda_n}\mathbb{S}(n-1)\right) \arrow[r] \arrow[d] & \widehat{C}^{n-1} \arrow[d] \\ p^*\left(\coprod_{\lambda\in\Lambda_n}\mathbb{F}(n)\right) \arrow[r] & \widehat{C}^n. \end{tikzcd}$$

Since p is a covering morphism, we can write $p^*\left(\coprod_{\lambda\in\Lambda_n}\mathbb{F}(n)\right)$ as $\coprod_{\lambda\in\widetilde{\Lambda}_n}\mathbb{F}(n)$ for a suitable $\widetilde{\Lambda}_n$. This completes the proof. □

Corollary 10.1.13. *Let $p\colon \widetilde{C}\to C$ be a covering morphism of crossed complexes. If C is free on X_*, then $\widetilde{C}$ is free on $p^{-1}(X_*)$.*

A similar result to Corollary 10.1.13 applies in the m-truncated case.

The significance of these results is as follows. We start with an m-truncated free crossed resolution C of a group G, so that we are given $\phi\colon C_1\to G$, and C is free on X_*, where X_n is defined only for $n\leqslant m$. Our extension process of Section 10.3.ii will start by constructing the universal cover $p\colon \widetilde{C}\to C$ of C; this is the covering crossed complex corresponding to the universal covering groupoid $p_0\colon \widetilde{G}\to G$. By the results above, $\widetilde{C}$ is the free crossed complex on $p^{-1}(X_*)$. It also follows from Proposition 10.1.2 that the induced morphism $\widetilde{\phi}\colon \widetilde{C}\to\widetilde{G}$ makes $\widetilde{C}$ a free crossed resolution of the contractible groupoid $\widetilde{G}$. Hence $\widetilde{C}$ is acyclic and hence, since it is free, also a contractible crossed complex.

We will use covering morphisms of crossed complexes with the corresponding version for ω-groupoids to prove in Section 15.8 that the tensor product of free aspherical crossed complexes is aspherical.

10.2 Free crossed resolutions

In this section we introduce the concept of free crossed resolution of a groupoid G, prove that any two resolutions of the same groupoid are homotopy equivalent and give some explicit examples.[132] We then study some more complex examples requiring extra theoretical background.

10.2.i Existence, examples

Definition 10.2.1. A crossed complex C is called *aspherical* if for all $n\geqslant 2$ and $x\in C_0$, we have $H_n(C,x)=0$. It is *acyclic* if it is aspherical, connected and in addition $\pi_1(C,x)=0$ for all $x\in C_0$.

An *augmented crossed complex* (C, ϕ) is a crossed complex C together with a morphism $\phi\colon C_1 \to G$ to a groupoid G such that $G_0 = C_0$ and ϕ induces an isomorphism $\pi_1 C \to G$. We also say C is *augmented by* G or by $\phi\colon C_1 \to G$. Such a complex with augmentation may be written as:

$$\cdots \longrightarrow C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots\cdots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\phi} G.$$

A *crossed resolution of a groupoid* G is an aspherical crossed complex C augmented over G; and this is called a *free crossed resolution* if C is also a free crossed complex. See Definition 7.3.13. □

Theorem 10.2.2. *Any group(oid) G admits a free crossed resolution.*

Proof. We first choose a presentation $\mathcal{P} = \langle X \mid R \rangle$, of G and so we get a free crossed module over a free groupoid

$$C(R) \xrightarrow{\delta_2} F(X)$$

together with $\phi\colon F(X) \to G$ inducing an isomorphism $\operatorname{Cok}(\delta_2) \to G$. Now $A = \operatorname{Ker}\delta_2$ is a G-module and we proceed as in classical homological algebra as outlined in Chapter 3. □

This follows a traditional method of constructing complexes, either CW-complexes or forms of resolutions, by 'killing kernels'; at stage 1 this requires a free crossed module to map onto the normal subgroupoid of a free groupoid normally generated by the relations; at stage $n \geqslant 2$ the kernel of δ_n is a G-module and we choose a graph X_n of generators for this and map the free G-module on X_n onto this kernel.

Of course what this outline construction does not show is how to get hold of a convenient graph of generators X_n for the kernel; some such graph exists, for instance we could take $X_n = \operatorname{Ker}\delta_{n-1}$, but this is not at all constructive or convenient. This problem of construction is addressed in Section 10.3, in the case G is a group, using the idea of constructing inductively a free crossed resolution with a contracting homotopy not of G, but of the universal covering groupoid $\widetilde{G}$ of G.

If G is itself free, we need go no further.

Example 10.2.3. If G is a free groupoid $F(X_1)$, then G has a free crossed resolution which is $F(X_1)$ in dimension 1 and is trivial in higher dimensions. □

We can also state a small free crossed resolution of finite cyclic groups, which is a modification in dimensions $\leqslant 2$ of a classical chain complex resolution for these groups.

Example 10.2.4 (Finite cyclic groups). A cyclic group C_q of order q with generator c has a free crossed resolution $F = F(\mathsf{C}_q)$ as follows:

$$F(\mathsf{C}_q)\colon = \quad \cdots \longrightarrow \mathbb{Z}[\mathsf{C}_q] \xrightarrow{\delta_4} \mathbb{Z}[\mathsf{C}_q] \xrightarrow{\delta_3} \mathbb{Z}[\mathsf{C}_q] \xrightarrow{\delta_2} \mathsf{C}_\infty \xrightarrow{\phi} \mathsf{C}_q$$

where C_∞ is the infinite cyclic group with free generator x_1, and $\phi(x_1) = c$; in dimension $n \geqslant 2$ the term $\mathbb{Z}[\mathsf{C}_q]$ is the free C_q-module on one generator x_n; and the boundary maps are defined by $\delta_2(x_2) = x_1^q$ and for $n > 2$

$$\delta_n(x_n) = \begin{cases} x_{n-1}\,(1-c) & \text{if } n \text{ is odd,} \\ x_{n-1}\,(1+c+c^2+\cdots+c^{q-1}) & \text{if } n \text{ is even.} \end{cases}$$ □

Exercise 10.2.5. Prove directly that the preceding example gives a free crossed resolution of C_q.[133] □

Example 10.2.6. A presentation $\langle X \mid R\rangle$ of a group G is a *one relator presentation* if R consists of a single element. Suppose this element r is not a proper power, i.e. $r = z^q$ for some z implies $q = \pm 1$. It is then a theorem that the kernel of the free crossed module $C(r) \to F(X)$ is trivial, so that this crossed module itself is in essence a free crossed resolution of G. However, the proof of this triviality is by no means easy.[134] □

10.2.ii The standard free crossed resolution of a groupoid

Definition 10.2.7. Let G be a groupoid. By Example 9.10.3 we can form its simplicial nerve $N^\Delta G$, and then the crossed complex $\Pi N^\Delta G$: this we call the *standard free crossed resolution of* G.

The standard free crossed resolution of a groupoid G is of the form

$$\cdots \longrightarrow F_*^{\mathrm{st}}(G)_3 \xrightarrow{\delta_3} F_*^{\mathrm{st}}(G)_2 \xrightarrow{\delta_2} F_*^{\mathrm{st}}(G)_1 \xrightarrow{\phi} G$$

in which by Example 9.10.3 $F_n^{\mathrm{st}}(G)$ is free on the set

$$(N^\Delta G)_n = \{\,[a_1, a_2, \ldots, a_n] \mid a_i \in G\}$$

of composable sequences of elements of G, where the base point $t[a_1, a_2, \ldots, a_n]$ is the final point ta_n of a_n. For $n \geqslant 2$ the boundary

$$\delta_n \colon F_n^{\mathrm{st}}(G) \to F_{n-1}^{\mathrm{st}}(G)$$

is given by

$$\begin{aligned} \delta_2[a,b] &= [ab]^{-1}[a][b], \\ \delta_3[a,b,c] &= [a,b]^c[b,c]^{-1}[a,bc]^{-1}[ab,c], \end{aligned}$$

and for $n \geqslant 4$

$$\begin{aligned} \delta_n[a_1, a_2, \ldots, a_n] = {} & [a_1, \ldots, a_{n-1}]^{a_n} + (-1)^n[a_2, \ldots, a_n] \\ & + \sum_{i=1}^{n-1}(-1)^{n-i}[a_1, a_2, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_n]. \end{aligned}$$

See also the pictures in Example 9.10.3. □

Proposition 10.2.8. *The standard free crossed resolution of a group is aspherical.*

Proof. Let G be a group, and let $p\colon \widetilde{G} \to G$ be the universal covering morphism. Then the induced morphism of standard resolutions

$$p_*\colon F_*^{\mathrm{st}}\widetilde{G} \to F_*^{\mathrm{st}}G$$

is also a covering morphism, this time of crossed complexes.

Now a contracting homotopy of $F_*^{\mathrm{st}}\widetilde{G}$ is given on free generators by

$$([a_1, a_2, \dots, a_n], a) \mapsto ([a_1, a_2, \dots, a_n, a], 1_{x_0}).$$

We leave it as an exercise for you to use the simplicial Homotopy Addition Lema and the formulae given for a contracting homotopy in Example 7.1.44 to verify that this gives a contracting homotopy. □

Remark 10.2.9. In Remark 12.5.4 we will extend this last proposition to the case of a groupoid: the proof we give there needs information on homotopies which is conveniently given later. □

Remark 10.2.10. An exact sequence of groups $1 \to M \to E \to G \to 1$ is called in the literature *an extension of M by G*.[135] One also finds in the literature on extensions of a group M by a group G the notion of *factor set* of G in M. This consists of a pair of functions

$$k^1\colon G \to \mathrm{Aut}(M), \quad k^2\colon G \times G \to M$$

satisfying a number of conditions. We will see in Example 12.5.2 that these conditions can be interpreted as saying that a factor set is equivalent to a morphism of crossed complexes from $F_*^{\mathrm{st}}(G)$ to the trivial crossed complex extension of the crossed module $\chi\colon M \to \mathrm{Aut}(M)$, i.e. to $\mathrm{sk}^2(\chi\colon M \to \mathrm{Aut}(M))$. An *equivalence of factor sets* is then just homotopy of morphisms from $F_*^{\mathrm{st}}(G)$ to the latter crossed complex. Thus all the complications necessary to describe a factor set and their equivalences are embedded in the standard free crossed resolution of G. □

10.2.iii Uniqueness of free crossed resolutions up to homotopy

The following two theorems imply that free crossed resolutions of a groupoid are determined up to homotopy; this motivates the desire to find those free crossed resolutions useful for various aims. The result and the method of proof should be compared with those in Proposition 12.1.10.

Theorem 10.2.11. *Let C, D be crossed complexes such that C is free and D is aspherical. Let $\alpha\colon \pi_1 C \to \pi_1 D$ be a morphism of groupoids. Then there is a morphism $f\colon C \to D$ of crossed complexes such that $\pi_1(f) = \alpha$.*

Such a morphism f is said to be a *lift* of α.

Proof. We consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\delta_{n+1}"] & C_n \arrow[r, "\delta_n"] & C_{n-1} \arrow[r] & \cdots \arrow[r] & C_2 \arrow[r, "\delta_2"] & C_1 \arrow[r, Rightarrow, "\phi"] \arrow[d, dashed, "f_1"] & \pi_1 C \arrow[d, "\alpha"] \\
\cdots \arrow[r, "\delta_{n+1}"'] & D_n \arrow[r, "\delta_n"'] & D_{n-1} \arrow[r] & \cdots \arrow[r] & D_2 \arrow[r, "\delta_2"'] & D_1 \arrow[r, Rightarrow, "\psi"'] & \pi_1 D
\end{tikzcd}$$

in which ϕ, ψ are the quotient morphisms.

Let the free basis of C be denoted by X_*, where $X_0 = C_0$, and we assume X_n is a subgraph of C_n.

For $x \in X_1$ we choose $f_1(x) \in D_1$ such that $\psi f_1(x) = \alpha\phi(x)$. This is possible because ψ is surjective. Since X_1 is a free basis of C_1, this choice extends uniquely to a morphism

$$f_1 : C_1 \to D_1.$$

Since $\psi f_1 = \alpha\phi$ on the generating set X_1, it follows that

$$\psi f_1 = \alpha\phi$$

on C_1. Note also that

$$\psi f_1 \delta_2 = \alpha\phi\delta_2 = 0.$$

Since $\operatorname{Ker}\psi = \operatorname{Im}\delta_2$, it follows that $\operatorname{Im} f_1\delta_2 \subseteq \operatorname{Im}\delta_2$. For all $x \in X_2$, we choose $f_2(x) \in D_2$ so that

$$\delta_2 f_2(x) = f_1\delta_2(x).$$

Now we proceed inductively. Suppose that

$$f_{n-1} : C_{n-1} \to D_{n-1}$$

has been defined so that

$$\delta_{n-1} f_{n-1} = f_{n-2}\delta_{n-1}.$$

Then

$$\delta_{n-1} f_{n-1}\delta_n = f_{n-2}\delta_{n-1}\delta_n = 0.$$

By asphericity of D, $\operatorname{Im}(f_{n-1}\delta_n) \subseteq \operatorname{Im}\delta_n$. So for all x in the free basis X_n, there is an $f_n(x) \in D_n$ such that $\delta_n f_n(x) = f_{n-1}\delta_n(x)$. This defines a morphism

$$f_n : C_n \to D_n$$

such that $\delta_n f_n = f_{n-1}\delta_n$. □

Exercise 10.2.12. Let C_q and C_{qr} be cyclic groups of order q and qr with generators c and c_1 respectively. Consider their free crossed complex resolutions $F(\mathsf{C}_q)$ and $F(\mathsf{C}_{qr})$ studied in Example 10.2.4. Given the morphism $\alpha\colon \mathsf{C}_q \to \mathsf{C}_{qr}$ which sends c to c_1^r, find a morphism $F(\mathsf{C}_q) \to F(\mathsf{C}_{qr})$ which lifts α. □

Theorem 10.2.13. *Let C, D be crossed complexes such that C is free and D is aspherical. Let $\alpha\colon \pi_1 C \to \pi_1 D$ be a morphism of groupoids and $f^-, f^+\colon C \to D$ morphisms of crossed complexes such that $\pi_1(f^-) = \pi_1(f^+) = \alpha$. Then there is a homotopy $h\colon f^- \simeq f^+$.*

Proof. We proceed as before to define the homotopy (see Definition 9.3.3) starting with

$$h_0\colon C_0 \to D_1.$$

Since $\pi_1(f^-) = \pi_1(f^+) = \alpha$, we have $\psi f_1^- = \alpha\phi = \psi f_1^+$. We set $h_0(c) = 1_{\alpha c} \in D_1$, for $c \in C_0$.

We have to define a map

$$h_1\colon C_1 \to D_2$$

such that for every $c \in C_1$,

$$f_1^-(c) = (h_0 s(c))(f_1^+ c)(\delta_2 h_1 c)(h_0 t(c))^{-1}$$

which because of our definition of h_0 reduces to

$$f_1^-(c) = f_1^+(c)(\delta_2 h_1 c)$$

or

$$\delta_2 h_1 c = f_1^+(c)^{-1} f_1^-(c).$$

But

$$\psi(f_1^+(c)^{-1} f_1^-(c)) = 1.$$

Hence for each $x \in X_1$ we can choose an $h_1(x)$ such that $\delta_2 h_1 x = f_1^+(x)^{-1} f_1^-(x)$. This extends to an f_1^+- derivation $h_1\colon C_1 \to D_2$, as explained in Remark 7.1.42.

At the next level, for $c \in C_2$, we note that $f_1^+\delta_2 = \delta_2 f_2^+$, $f_1^-\delta_2 = \delta_2 f_2^-$ and we require h_2 such that

$$f_2^-(c) = f_2^+(c) h_1\delta_2(c)\delta_3 h_2(c). \tag{$*$}$$

But

$$\delta_2(h_1\delta_2(c)^{-1} f_2^+(c)^{-1} f_2^-(c)) = (f_1^+(\delta_2 c)^{-1} f_1^-(\delta_2 c)^{-1}\delta_2 f_2^+(c)^{-1}\delta_2 f_2^-(c) = 1.$$

So again, we can choose $h_2(x)$ for $x \in X_2$ so that $(*)$ holds for $c = x$. This extends to an f_1^+-morphism $h_2\colon C_2 \to D_3$ as required.

We now look at the situation around dimension n.

$$\begin{tikzcd}
{} \arrow[r] & C_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, shift right, "f^-_{n+1}"'] \arrow[d, shift left, "f^+_{n+1}"] & C_n \arrow[r, "\delta_n"] \arrow[d, shift right, "f^-_n"'] \arrow[d, shift left, "f^+_n"] \arrow[dl, dashed, "h_n"'] & C_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[d, shift right, "f^-_{n-1}"'] \arrow[d, shift left, "f^+_{n-1}"] \arrow[dl, "h_{n-1}"'] & C_{n-2} \arrow[r] \arrow[d, shift right, "f^-_{n-2}"'] \arrow[d, shift left, "f^+_{n-2}"] \arrow[dl, "h_{n-2}"'] & {} \\
{} \arrow[r] & D_{n+1} \arrow[r, "\delta_{n+1}"'] & D_n \arrow[r, "\delta_n"'] & D_{n-1} \arrow[r, "\delta_{n-1}"'] & D_{n-2} \arrow[r] & {}
\end{tikzcd}$$

We suppose given the morphisms f^-, f^+ and also the h_{n-2}, h_{n-1} such that

$$f^-_{n-1} = f^+_{n-1} + h_{n-2}\delta_{n-1} + \delta_n h_{n-1}.$$

But for $c \in C_n$

$$\begin{aligned}
&\delta_n(f^-_n c - f^+_n c - h_{n-1}\delta_n c) \\
&\quad = f^-_{n-1}\delta_n c - f^+_{n-1}\delta_n c - \delta_n h_{n-1}\delta_n c \\
&\quad = f^-_{n-1}\delta_n c - f^+_{n-1}\delta_n c - (f^-_{n-1}\delta_n c - f^+_{n-1}\delta_n c - h_{n-2}\delta_{n-1}\delta_n c) \\
&\quad = 0 \qquad\quad \text{since } \delta_{n-1}\delta_n = 0.
\end{aligned}$$

By asphericity of D, for each x in the basis X_n we can find an $h_n x$ in D_{n+1} such that

$$h_n x = f^-_n x - f^+_n x - h_{n-1}\delta_n x.$$

This extends to an operator morphism $h_n \colon C_n \to D_{n+1}$ with the required properties for the next stage of the induction. □

This proof is typical of one method of constructing homotopies which will be useful again in later sections.

Corollary 10.2.14. *Any two free crossed resolutions of a group G are homotopy equivalent.*

Remark 10.2.15. A refinement of Theorem 10.2.13 is to assume that we have two morphisms $\alpha^-, \alpha^+ \colon G \to H$, that the morphisms f^-, f^+ lift α^-, α^+ respectively and that η is a homotopy (or natural transformation) $\alpha^- \simeq \alpha^+$. Then we assert that under the same conditions of freeness and asphericity, η lifts to a homotopy $h \colon f^- \simeq f^+$. Let us assume ϕ, ψ are the identity on objects. Here η_0 assigns to each p in C_0 an element $\eta(p) \in H(\alpha^- p, \alpha^+ p)$ such that the usual naturality condition holds: if $g \in G(p,q)$ then $\alpha^-(g)\eta(q) = \eta(p)\alpha^+(g)$. For each $p \in C_0$ choose an $h_0(p) \in D_1(p,q)$ such that $\psi(h_0(p)) = \eta(p)$. Now we repeat the arguments of the proof of Theorem 10.2.13 but using the more complicated formulae for homotopies which involve h_0. We leave the details as an exercise for the reader. □

However we still want a method of constructing a free crossed resolution in a more or less algorithmic way, or at least in terms of data specifying a group or groupoid. Such a method is given in Section 10.3 for a finite group defined by a presentation by generators and relations. The next section describes some other ways of constructing resolutions related to particular constructions of groups.[136]

10.2.iv Some more complex examples: Free products with amalgamation and HNN-extensions

We will prove the following theorem in Corollary 15.8.5, using cubical methods, covering crossed complexes, and the notion of dense subcategory. This result, combined with the fact that the tensor product of free crossed complexes is free, gives one method of making new free crossed resolutions from old ones.[137]

Theorem 10.2.16. *If C, D are aspherical free crossed complexes, then their tensor product $C \otimes D$ is also aspherical.*

Example 10.2.17. Let $\mathcal{P}_G = \langle X_G \mid R_G \rangle$ and $\mathcal{P}_H = \langle X_H \mid R_H \rangle$ be presentations of groups G, H, respectively, and let $\mathcal{F}(\mathcal{P}_G)$ and $\mathcal{F}(\mathcal{P}_H)$ be the corresponding free crossed modules, regarded as 2-truncated crossed complexes. The tensor product $T = \mathcal{F}(\mathcal{P}_G) \otimes \mathcal{F}(\mathcal{P}_H)$ is 4-truncated and is given as follows (where we now use additive notation in dimensions 3, 4 and multiplicative notation in dimensions 1, 2):

- T_1 is the free group on the generating set $X_G \sqcup X_H$;
- T_2 is the free crossed T_1-module on $R_G \sqcup (X_G \otimes X_H) \sqcup R_H$ with the boundaries on R_G, R_H as given before and
$$\delta_2(g \otimes h) = h^{-1}g^{-1}hg \quad \text{for all } g \in X_G,\ h \in X_H;$$
- T_3 is the free $(G \times H)$-module on generators $r \otimes h$, $g \otimes s$, $r \in R_G$, $s \in R_H$ with boundaries
$$\delta_3(r \otimes h) = r^{-1}r^h(\delta_2 r \otimes h), \quad \delta_3(g \otimes s) = (g \otimes \delta_2 s)^{-1}s^{-1}s^g;$$
- T_4 is the free $(G \times H)$-module on generators $r \otimes s$, with boundaries
$$\delta_4(r \otimes s) = (\delta_2 r \otimes s) + (r \otimes \delta_2 s).$$

The important point is that we can if necessary calculate with these formulae, because elements such as $\delta_2 r \otimes h$ may be expanded using the rules for the tensor product. Alternatively, the forms $\delta_2 r \otimes h$, $g \otimes \delta_2 s$ may be left as they are since they naturally represent subdivided cylinders. □

We next illustrate the use of crossed complexes of groupoids, rather than just of groups, by the construction of a free crossed resolution of a free product with amalgamation, and a similar result for HNN-extensions, given free crossed resolutions of the individual groups.[138]

Suppose the group G is given as a free product with amalgamation

$$G = A *_C B,$$

which we can alternatively describe as a pushout of groups

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[d, "i"'] & B \arrow[d, "i'"] \\ A \arrow[r, "j'"'] & G. \end{tikzcd}$$

We are assuming the morphisms i, j are injective so that, by standard results, i', j' are injective.[139] Suppose we are given free crossed resolutions

$$\mathcal{A} = \mathcal{F}(A), \quad \mathcal{B} = \mathcal{F}(B), \quad \mathsf{C} = \mathcal{F}(C).$$

The morphisms i, j may then be lifted (not uniquely) to morphisms

$$i'' \colon \mathsf{C} \to \mathcal{A}, \quad j'' \colon \mathsf{C} \to \mathcal{B}.$$

However we cannot expect that the pushout of these morphisms in the category Crs gives a free crossed resolution of G.

To see this, suppose that for $Q \in \{A, B, C\}$ the CW-filtrations $K(Q)$ realise the crossed resolutions of Q, i.e. $\Pi K(Q)_* \cong \mathcal{F}(Q)$, and that i'', j'' are realised by cellular maps $K(i) \colon K(C) \to K(A)$, $K(j) \colon K(C) \to K(B)$. However, the pushout in topological spaces of cellular maps does not in general yield a CW-complex – for this it is required that one of the maps is an inclusion of a subcomplex, and there is no reason why this should be true in this case. The standard way of dealing with this problem is to form the double mapping cylinder $M(i, j)$ given by the *homotopy pushout*

$$\begin{tikzcd} K(C) \arrow[r, "K(j)"] \arrow[d, "K(i)"'] \arrow[dr, phantom, "\simeq"] & K(B) \arrow[d] \\ K(A) \arrow[r] & M(i, j) \end{tikzcd}$$

where $M(i, j)$ is obtained from $K(A) \sqcup (I \times K(C)) \sqcup K(B)$ by identifying

$$(0, x) \sim K(i)(x), \ (1, x) \sim K(j)(x)$$

for $x \in K(C)$. This ensures that $M(i, j)$ is a CW-complex containing $K(A)$, $K(B)$ and $\{\frac{1}{2}\} \times K(C)$ as subcomplexes and that the composite maps $K(C) \to M(i, j)$ given by the two ways round the square are homotopic cellular maps.

It is therefore reasonable to assume that for crossed complexes the appropriate algebraic construction is also a homotopy pushout, this time in Crs, obtained by applying

Π to this homotopy pushout: this yields a diagram

$$\begin{tikzcd} \mathsf{C} \arrow[r, "j''"] \arrow[d, "i''"'] \arrow[dr, phantom, "\simeq"] & \mathcal{B} \arrow[d] \\ \mathcal{A} \arrow[r] & \mathcal{F}(i,j). \end{tikzcd}$$

Since $M(i,j)$ is aspherical we know that $\mathcal{F}(i,j)$ is aspherical and so is a free crossed resolution. Of course $\mathcal{F}(i,j)$ has two vertices 0, 1. Thus it is not a free crossed resolution of G but is a *free crossed resolution of the homotopy pushout* in the category Gpds:

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[d, "i"'] \arrow[dr, phantom, "\simeq"] & B \arrow[d] \\ A \arrow[r] & G(i,j). \end{tikzcd}$$

The groupoid $G(i,j)$ is obtained from the disjoint union of the groupoids A, B, $\mathcal{I}\times C$ by adding the relations $(0,c)\sim i(c)$, $(1,c)\sim j(c)$ for $c\in C$; thus $G(i,j)$ has two objects 0, 1 and each of its object groups is isomorphic to the amalgamated product group G, but we need to keep its two object groups distinct.[140]

The two crossed complexes of groups $\mathcal{F}(i,j)(0)$, $\mathcal{F}(i,j)(1)$, which are the parts of $\mathcal{F}(i,j)$ lying over 0, 1 respectively, are free crossed resolutions of the groups $G(i,j)(0)$, $G(i,j)(1)$. From the formulae for the tensor product of crossed complexes we can identify free generators for $\mathcal{F}(i,j)$: in dimension n we get

- free generators a_n at 0 where a_n runs through the free generators of A_n;
- free generators b_n at 1 where b_n runs through the free generators of B_n;
- free generators $\iota\otimes c_{n-1}$ at 1 where c_{n-1} runs through the free generators of C_{n-1}.

Example 10.2.18. Let A, B, C be infinite cyclic groups, written multiplicatively. The trefoil group T can be presented as a free product with amalgamation $A *_C B$ where the morphisms $C\to A$, $C\to B$ have cokernels of orders 3 and 2 respectively. The resulting homotopy pushout we call the *trefoil groupoid*. We immediately get a free crossed resolution of length 2 for the trefoil groupoid, whence we can by a retraction argument deduce the free crossed resolution $F(T)$ of the trefoil group T with presentation $\mathcal{P}_T=\langle a,b\mid a^3b^{-2}\rangle$. By the construction in Section 10.3, there is a free crossed resolution of T of the form

$$F(T):\quad \cdots \longrightarrow 0 \longrightarrow C(r) \xrightarrow{\delta_2} F\{a,b\} \xRightarrow{\phi} T$$

where $\delta_2 r = a^3b^{-2}$. Hence a 2-cocycle on T with values in K can also be specified totally by elements $s(c,d)\in K$, $c,d\in \mathrm{Aut}(K)$ such that $\partial s(c,d)=c^3d^{-2}$; this is a finite description. It is also easy to specify equivalence of cocycles.[141] □

Now we consider HNN-extensions. Let A, B be subgroups of a group G and let $k\colon A \to B$ be an isomorphism. Then we can form a pushout of groupoids

$$\begin{tikzcd}
\{0,1\} \times A \arrow[r, "(k_0,k_1)"] \arrow[d, "i"'] & G \arrow[d, "j"] \\
\mathcal{I} \times A \arrow[r, "f"'] & *_k\ G
\end{tikzcd}$$

where

$$k_0(0,a) = ka, \quad k_1(1,a) = a, \quad \text{and} \quad i \text{ is the inclusion.}$$

In this case of course $*_k\ G$ is a group, known as the HNN-extension. It can also be described as the factor group

$$(\mathsf{C}_\infty * G) \,/\, \{c^{-1}a^{-1}c\,(ka) \mid a \in A\}$$

of the free product, where C_∞ is the infinite cyclic group generated by c.

Now suppose we have chosen free crossed resolutions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{G}$ of A, B, G respectively. Then we may lift k to a crossed complex morphism $k''\colon \mathcal{A} \to \mathcal{B}$ and k_0, k_1 to

$$k_0'', k_1'' : \{0,1\} \times \mathcal{A} \to \mathcal{G}.$$

Next we form the pushout in the category of crossed complexes:

$$\begin{tikzcd}
\{0,1\} \otimes \mathcal{A} \arrow[r, "(k_0'',k_1'')"] \arrow[d, "i''"'] & \mathcal{G} \arrow[d, "j''"] \\
\mathcal{I} \otimes \mathcal{A} \arrow[r, "f''"'] & \otimes_{k''}\ \mathcal{G}.
\end{tikzcd}$$

Theorem 10.2.19. *The crossed complex $\otimes_{k''}\ \mathcal{G}$ is a free crossed resolution of the group $*_k\ G$.*

The proof is given in [BMPW02] as a special case of a theorem on the free crossed resolutions of the fundamental groupoid of a graph of groups. Here we show that Theorem 10.2.19 gives a means of calculation. Part of the reason for this success is that we do not need to know in detail the definition of free crossed resolution and of tensor products, we just need free generators, boundary maps, values of morphisms on free generators, and how to calculate in the tensor product with $\mathcal{I}$ using the rules given previously.

Example 10.2.20. The Klein Bottle group K has presentation $\langle c, z \mid z^{-1}c^{-1}z\,c^{-1} \rangle$. Thus $K = *_k \mathsf{C}_\infty$ where C_∞ is infinite cyclic generated by c and $kc = c^{-1}$. This yields a free crossed resolution

$$F(K): \quad \cdots \longrightarrow 1 \longrightarrow C(r) \xrightarrow{\delta_2} F\{c, z\} \overset{\phi}{\Longrightarrow} K$$

where $\phi_2\, r = z^{-1}c^{-1}z\,c^{-1}$. Of course this was already known since K is a surface group, and so is aspherical, and also because it is a one relator group whose relator is not a proper power. □

Example 10.2.21. Developing the previous example, let $\langle c, z \mid c^q,\ z^{-1}c^{-1}z\,c^{-1} \rangle$ be a presentation of the group L. Then $L = *_k \mathsf{C}_q$ where C_q is the cyclic group of order q generated by c and $k\colon \mathsf{C}_q \to \mathsf{C}_q$ is the isomorphism $c \mapsto c^{-1}$. A small free crossed resolution of C_q is given in Example 10.2.4 as

$$F(\mathsf{C}_q): \quad \cdots \longrightarrow \mathbb{Z}[\mathsf{C}_q] \xrightarrow{\delta_n} \mathbb{Z}[\mathsf{C}_q] \longrightarrow \cdots \longrightarrow \mathbb{Z}[\mathsf{C}_q] \xrightarrow{\delta_2} \mathsf{C}_\infty \overset{\phi}{\Longrightarrow} \mathsf{C}_q$$

with a free generator x_1 as a group of C_∞ in dimension 1; free generators x_n as C_q-modules in dimension $n \geqslant 2$; with $\phi\, x_1 = c$; $\delta_2(x_2) = x_1^q$ and

$$\delta_n\, x_n = \begin{cases} x_{n-1}\,(1 - c) & \text{if } n \text{ is odd,} \\ x_{n-1}\,(1 + c + c^2 + \cdots + c^{q-1}) & \text{otherwise.} \end{cases}$$

The isomorphism k lifts to a morphism $k''\colon F(\mathsf{C}_q) \to F(\mathsf{C}_q)$ which is also inversion in each dimension. Hence L has a free crossed resolution $\otimes_{k''} \mathsf{C}_q$ given by

$$\cdots \xrightarrow{\lambda_{n+1}} L_n \xrightarrow{\lambda_n} \cdots\cdots \xrightarrow{\lambda_3} L_2 \xrightarrow{\lambda_2} L_1 \overset{\phi'}{\Longrightarrow} G$$

having free generators x_1, z in dimension 1; generators $x_2, z \otimes x_1$ in dimension 2; and generators $x_n, z \otimes x_{n-1}$ in dimension $n \geqslant 3$. The extra boundary rules are

$$\begin{aligned} \lambda_2(z \otimes x_1) &= z^{-1}x_1^{-1}z\,x_1^{-1}, \\ \lambda_3(z \otimes x_2) &= (z \otimes x_1^q)^{-1}x_2^{-1}(x_2^{-1})^z, \\ \lambda_{n+1}(z \otimes x_n) &= -(z \otimes \delta_n x_n) - x_n - {x_n}^z \quad \text{for } n \geqslant 3. \end{aligned}$$

In particular, the identities among relations for this presentation of L are generated by

$$x_2 \quad \text{and} \quad \lambda_3(z \otimes x_2) = (z \otimes \delta_2 x_2)^{-1}{x_2}^{-1}({x_2}^{-1})^z.$$

Similarly, relations for the module of identities are generated by

$$x_3 \quad \text{and} \quad \lambda_4(z \otimes x_3) = -(z \otimes x_2(1 - c)) - x_3 - {x_3}^z.$$

Of course we can expand expressions such as $(z \otimes \delta_n x_n)$ using the rules for the cylinder given in Example 9.3.18.[142] □

10.3 From presentations to free crossed resolutions

In this section we address the problem of constructing a resolution for a group G defined by a presentation $\langle X \mid R\rangle$. A theoretical solution is suggested by Theorem 10.2.2. First we construct the free crossed module augmented to G

$$C(R) \xrightarrow{\delta_2} F(X) \overset{\phi}{\Longrightarrow} G.$$

Now we take a free resolution of the G-module $A = \operatorname{Ker}\delta_2$ in the 'usual way' of constructing 'chains of syzygies'.

That means that at each step we have to choose a set of generators X_n of the kernel A_n of $\delta_n \colon F_n \to F_{n-1}$ in order to define $\delta_{n+1} \colon F_{n+1} \to F_n$ where F_{n+1} is the free G-module on X_n. This can be called the process of 'killing kernels'.

The problem is how to choose X_n. Theoretically the answer is easy: if in doubt, take $X_n = A_n$. Obviously this is not in any way an algorithmic answer, and indeed A_n could be infinite, so we would like to have a way of constructing smaller resolutions which has the possibility of realisation as an algorithm for a reasonable class of cases, for example if G is finite.

10.3.i Home for a contracting homotopy: chain complexes

The answer turns out to be what we call 'constructing a home for a contracting homotopy'. To motivate the idea, we consider first the case of chain complexes of R-modules. For these we show there is an easily described way of constructing inductively at the same time the resolution and the contracting homotopy (in chain complexes resolutions are contractible). The method uses homotopy information in dimensions $\leqslant n$ to construct C_{n+1}, h_n and then ∂_{n+1}, from free generators of C_n and lower dimensional homotopy information. For precision, we recall the standard definition.

Let C be a positive, augmented, chain complex of R-modules, where R is a ring with identity. Thus C_n is defined for $n \geqslant 0$ and the augmentation is a module morphism $\varepsilon\colon C_0 \to A$ where A is an R-module, regarded as a chain complex concentrated at dimension 0, such that $\varepsilon\partial_1 = 0\colon C_1 \to A$. A *contracting homotopy* for this situation is chain mapping $f\colon A \to C$ such that $\varepsilon f = 1_A$ together with a chain homotopy $h\colon 1 \simeq f\varepsilon$, i.e. a family of module morphisms $h_r\colon C_r \to C_{r+1}, r \geqslant 0$ such that

$$\partial_1 h_0 + f\varepsilon = 1_C, \quad \partial_{r+1}h_r + h_{r-1}\partial_r = 1_C, \quad r > 0.$$

Assume that we have constructed such data for $r < n$ and C_n is free on say X_n. We want a free G-module C_{n+1} and a morphism $h_n\colon C_n \to C_{n+1}$ satisfying

$$\partial_{n+1}h_n + h_{n-1}\partial_{n-1} = 1.$$

We construct C_{n+1} as 'a home for h_n' as follows. We consider a set X'_{n+1} in one-to-one correspondence with X_n by $x' \mapsto x$ and define C_{n+1} to be the free R-module on the set X'_{n+1}. Let

$$h_n : C_n \to C_{n+1}, \quad \partial_{n+1} : C_{n+1} \to C_n$$

be given by: h_n is the unique morphism extending the bijection $X_n \to X'_{n+1}$ and ∂_{n+1} is the unique morphism extending

$$\partial_{n+1}(x') = x - h_{n-1}\partial_n(x), \quad x' \in X'_{n+1}.$$

By definition, $\partial_{n+1}h_n + h_{n-1}\partial_n = 1_C$ on elements of X_n, and one checks that again for $x' \in X'_{n+1}$

$$\begin{aligned} \partial_n\partial_{n+1}x' &= \partial_n(x - h_{n-1}\partial_n x) \\ &= \partial_n x - (1 - h_{n-2}\partial_{n-1})\partial_n x && \text{by the inductive assumption} \\ &= 0 && \text{since } \partial_{n-1}\partial_n = 0. \end{aligned}$$

In practice for some $x \in X_n$ we may have $\partial_{n+1}h_n x = 0$; we can define $h_n x = 0$ on such x, and eliminate the corresponding x' from X_{n+1}. One can usually find a subset X_{n+1} of X'_{n+1} such that $\partial_{n+1}X_{n+1}$ also generates $\partial_{n+1}C_{n+1}$. This enables one to find a smaller candidate for the next step. We shall see this in practice in Section 10.3.ii.[143]

By iterating we get a free chain complex and a contracting homotopy, so the resulting chain complex is a free resolution. This method we call 'constructing a home for a contracting homotopy', in contrast to the traditional method of 'killing kernels'.

The immediate problem with repeating this process for crossed resolutions of a group G is that such resolutions are not contractible, since their fundamental group is isomorphic to G! We resolve this by passing to the 'universal covering groupoid' $p: \widetilde{G} \to G$ which we set up in the next sections, and construct a free crossed resolution of $\widetilde{G}$, by essentially the above process, taking care of the extra complications of homotopies for crossed complexes as against chain complexes. It is then easy to pass from the free crossed resolution of $\widetilde{G}$ to one for G.

We will see that there are many choices involved in this process. The process deals with Cayley graphs, a standard tool in combinatorial group theory, and we can start by choosing a maximal tree in the Cayley graph. The theory well reflects the geometry of covering spaces and extends the notion of Cayley graph to include higher dimensional information, i.e. Cayley graphs with relations, and so on.

One of the reasons for our developing the machinery of covering morphisms of crossed complexes in Section 10.1 is to be able to describe this computational process of constructing free crossed resolutions.

10.3.ii Computing a free crossed resolution

The initial motivation for the work of this section was to determine in an algorithmic mode generators and relations for the G-module $\pi(\mathcal{P})$ of identities among relations for a presentation $\mathcal{P} = \langle X \mid \omega \rangle$ of a group G.[144] Here $\omega\colon R \to F(X)$ is a function and we regard R as a set disjoint from $F(X)$. The advantages of using the function ω are (i) to allow for the possibility of repeated relations; (ii) to distinguish between an element $r \in R$ and the corresponding element $\omega(r) \in F(X)$; (iii) to model the procedure of attaching 2-cells to a space.

Associated to this presentation of G, we will construct by induction on dimension a free crossed resolution $\widetilde{F}$ for the universal covering groupoid $\widetilde{G}$ of G such that $\widetilde{F}$ projects to a free crossed resolution of G; further, the construction of a contracting homotopy of $\widetilde{F}$ is part of this process. If G is finite, and the presentation is finite, then this free crossed resolution will have a finite number of free generators in each dimension.

Let us start in low dimensions.

2a. Resolution of G up to dimension 2. In Chapter 3 we proved that a presentation $\mathcal{P} = \langle \omega\colon R \to F(X) \mid X \rangle$ of a group G gives the beginning of a free crossed resolution

$$C(R) \xrightarrow{\delta_2} F(X) \xRightarrow{\phi} G \qquad (10.3.1)$$

where δ_2 is the free crossed module associated to ω. Then $\pi(\mathcal{P})$ is defined to be $\operatorname{Ker} \delta_2$.

The elements of $C(R)$ are 'formal consequences'

$$c = \prod_{i=1}^{n} (r_i^{\varepsilon_i})^{u_i}$$

where $n \geqslant 0$, $r_i \in R$, $\varepsilon_i = \pm 1$, $u_i \in F(X)$, $\delta_2(r^{\varepsilon})^u = u^{-1}(\omega r)^{\varepsilon} u$, subject to the crossed module rule $ab = ba^{\delta_2 b}$, $a, b \in C(R)$.

Remark 10.3.1. It follows from the Higher Homotopy Seifert–van Kampen Theorem for crossed modules (as in the proof of Theorem 5.4.8) that $\pi(\mathcal{P})$ is given geometrically as the second homotopy group $\pi_2(K(\mathcal{P}))$ of the cell complex of the presentation. This result is not necessary for the work of this chapter, but it does emphasise the topological relevance of our methods.[145] □

Actually Diagram (10.3.1) is equivalent to the more general situation

$$F_2 \to F_1 \to G$$

where F_1 is a free groupoid, F_2 is a free crossed module over F_1, G is a groupoid and ϕ induces an isomorphism $\operatorname{Cok} \delta_2 \cong G$; this is because the free generators of F_1 have to map to a set of generators X of G so $F_1 = F(X)$ and F_2 has to be free on some map $\omega\colon R \to F_1 = F(X)$.

Thus, we want to extend (10.3.1) to a crossed resolution of G. To do this we require algebraic analogues of methods of covering spaces, as developed in the preceding sections. We are following the method outlined in the introduction to Section 10.3 (p. 341) for the case of chain complexes. The crucial point is that the algorithmic nature of the argument derives from the construction of homotopies: the fact that these homotopies give strong deformation retractions also simplifies the conditions on the homotopies, as shown in Example 7.1.44.[146]

2b. Resolution of the covering $\widetilde{G}$ up to dimension 2. First we construct a covering of part of diagram (10.3.1) by a pullback diagram

$$\begin{tikzcd} & \widehat{F} \arrow[r, Rightarrow, "\widehat{\phi}"] \arrow[d, "p_1"'] & \widetilde{G} \arrow[d, "p_0"] \\ C(R) \arrow[r, "\delta_2"'] & F(X) \arrow[r, Rightarrow, "\phi"'] & G \end{tikzcd} \tag{10.3.2}$$

with the following properties.

(i) The morphism $p_0 \colon \widetilde{G} \to G$ is the universal covering groupoid of the group G. The objects of $\widetilde{G}$ are the elements of G, and an arrow of $\widetilde{G}$ is a pair

$$(h, g) \colon hg \to g \quad \text{for } (h, g) \in G \times G.$$

The composition in $\widetilde{G}$ is $(k, hg)(h, g) = (kh, g)$ for $k, h, g \in G$. The projection morphism p_0 is given by $(h, g) \mapsto h$. For more details on this and the following, see Example B.7.3.

(ii) Since the diagram is a pullback, and G is a group, we can assume $\operatorname{Ob} \widehat{F} = G$ and $\widehat{F}$ consists of triples (u, h, g) such that $u \in F(X)$, $h, g \in G$, $(u, h, g) \colon hg \to g$ and, by the pullback condition, $\phi(u) = h$. Hence we can abbreviate the notation and write an element of $\widehat{F}$ as $(u, g) \colon (\phi u)g \to g$. Then $p_1(u, g) = u$, $\widehat{\phi}(u, g) = (\phi u, g)$, and the composition in $\widehat{F}$ is

$$(v, (\phi u)g)(u, g) = (vu, g).$$

(iii) By [Bro06], 10.8.1 (Corollary 1), $\widehat{F}$ is the free groupoid on the graph $\widehat{X} = p_1^{-1}(X)$.[147] But we can write elements of $\widehat{X}$ as pairs $(x, g) \colon (\phi x)g \to g$ for $x \in X$, $g \in G$. Thus $\widehat{X}$ is the well-known *Cayley graph* of the pair (G, X). We write now $F(\widehat{X})$ for $\widehat{F}$.

As is explained in Section B.7, $\widetilde{G} \to G$ is the covering morphism corresponding to the trivial subgroup of G, and $F(\widehat{X}) \to F(X)$ is the covering morphism corresponding to the normal subgroup $N = \operatorname{Ker} \phi$ of $F(X)$.

The next step is to take diagram (10.3.2) one dimension higher, getting

$$\begin{tikzcd} C(\widehat{R}) \arrow[r, "\hat{\delta}_2"] \arrow[d, "p_2"'] & F(\widehat{X}) \arrow[r, Rightarrow, "\hat{\phi}"] \arrow[d, "p_1"'] & \widehat{G} \arrow[d, "p_0"'] \\ C(R) \arrow[r, "\delta_2"'] & F(X) \arrow[r, Rightarrow, "\phi"'] & G \end{tikzcd} \tag{10.3.3}$$

where the left-hand square of the diagram is again a pullback of groupoids, $\widehat{R} = R \times G$ and $\hat{\delta}_2 : C(\widehat{R}) \to F(\widehat{X})$ is the free crossed $F(\widehat{X})$-module on $\hat{\omega}\colon \widehat{R} \to F(\widehat{X})$, $(r, g) \mapsto (\omega(r), g)$. This is the free crossed module of the *covering presentation* $\langle \widehat{X} \mid \widehat{R} \rangle$ of the universal covering groupoid $\widetilde{G}$ of the group G.

Thus $C(\widehat{R})$ is the disjoint union of groups $C(\widehat{R})(g)$, $g \in G$, all mapped by p_2 isomorphically to $C(R)$. Elements of $C(\widehat{R})(g)$ are pairs $(c, g) \in C(R) \times \{g\}$, with multiplication $(c, g)(c', g) = (cc', g)$. The action of $F(\widehat{X})$ is given by

$$(c, (\phi u)g)^{(u,g)} = (c^u, g).$$

The boundary $\hat{\delta}_2$ is given by $(c, g) \mapsto (\delta_2 c, g)$. The morphism $p_2\colon C(\widehat{R}) \to C(R)$ is given by $(c, g) \mapsto c$.

The elements of $C(\widehat{R})(g)$ are also all 'formal consequences'

$$(c, g) = \prod_{i=1}^{n} ((r_i, (\phi u_i)g_i)^{\varepsilon_i})^{(u_i, g_i)} = \Big(\prod_{i=1}^{n} (r_i^{\varepsilon_i})^{u_i}, g\Big)$$

where $n \geqslant 0$, $r_i \in R$, $\varepsilon_i = \pm 1$, $u_i \in F(X)$, $g_i \in G$, $(\phi u_i)g_i = g$, subject to the crossed module rule $ab = ba^{\hat{\delta}_2 b}$, $a, b \in C(\widehat{R})$.

It is useful to think of these formulae topologically in terms of CW-complexes. The generating set X should be thought of as a set of loops giving the 1-cells of a reduced CW-complex Y, so that we identify $F(X)$ with $\pi_1(Y^1, *)$. The elements $r \in R$ can be thought of as defining the 2-cells of Y, each attached according to the formula for ωr, so that $G = \pi_1(Y, *)$. The element (r, g) for $r \in R$, $g \in G$ then corresponds to the covering cell of the cell r at the point g, considered as a vertex of $\widetilde{Y}$, and (r, g) is also a relator for the 'covering presentation' of $\widetilde{G}$. Let $\widehat{Y}^n$ be the n-skeleton of $\widetilde{Y}$; then $\pi_1(\widehat{Y}^1, \widehat{Y}^0)$ may be identified with the groupoid $F(\widehat{X})$. If $(u, g)\colon (\phi u)g \to g$ is a path in $F(\widehat{X})$, and $(r, (\phi u)g)$ is a free generator corresponding to a 2-cell of the universal cover, then this generator also contributes to the *group* $C(\widehat{R})(g)$ with the element $(c, (\phi u)g)^{(u,g)} = (c^u, g)$.

In effect, we are giving:

1) a presentation $\langle \widehat{X} \mid \hat{\omega} \rangle$ of the groupoid $\widetilde{G}$, and

2) the free crossed module corresponding to this presentation.

That this construction gives a free crossed module is thanks to Theorem 10.1.12.

2c. Contractibility of the covering up to dimension 2. So we have started the construction of a crossed complex that we want to be acyclic. To prove this acyclicity we construct a contracting homotopy at the same time as we are constructing the crossed complex. So we need to construct $h_0\colon \operatorname{Ob}\widetilde{G} \to F(\widehat{X})$ and $h_1\colon F(\widehat{X}) \to C(\widehat{R})$ as in the following diagram:

$$\begin{tikzcd}
C(\widehat{R}) \arrow[r, "\hat{\delta}_2"] \arrow[d, "1"] & F(\widehat{X}) \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "1"] \arrow[dl, "h_1"'] & G \arrow[d, "1"] \arrow[dl, "h_0"'] \\
C(\widehat{R}) \arrow[r, "\hat{\delta}_2"'] & F(\widehat{X}) \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G.
\end{tikzcd} \tag{10.3.4}$$

Remark 10.3.2. We note for the record that h_1 is to be a morphism, by Example 7.1.44 on the conditions for a contracting homotopy. Later we will use that for $n \geqslant 2$, h_n is to be a morphism killing the operation of the groupoid $F(\widehat{X})$. □

For h_0, choose a section $\sigma\colon G \to F(X)$ of ϕ such that $\sigma(1) = 1$. Then σ determines

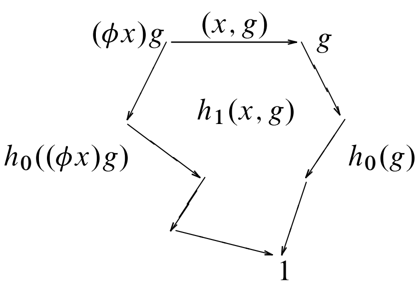
$$h_0\colon G \to F(\widehat{X}), \quad g \mapsto (\sigma g, 1). \tag{10.3.5}$$

Thus $p_1 h_0(g) = \sigma(g)$, and $h_0(1) = (1,1)$ and by unique path lifting for covering morphisms, $h_0(g)$ for $g \in G$ can, if expressed as a word in the free generators, be thought of as a path $g \to 1$ in the Cayley graph $\widehat{X}$.

Remark 10.3.3. Such a choice σg writing g as a word in the generators is called a 'normal form' for the element g of G; even for a finite presentation, σ cannot always be found over all of G by a finite algorithm. The usual way of finding it is by a 'rewriting' process, which may or may not terminate in finite time.[148]

The choice of h_0 is often, but not always, made by choosing a maximal tree in the graph $\widehat{X}$ – such a choice is equivalent to a choice of what is called in group theory a Schreier transversal for the subgroup $N = \operatorname{Ker}\phi$ of the free group $F(X)$. □

In the following picture, $(x, g)\colon (\phi x)g \to g$ is an arrow in $F(\widehat{X})$; $h_0((\phi x)g)$ represents a path in $\widehat{X}$ from $(\phi x)g$ to 1, thought of as an element of $F(\widehat{X})$; and $h_0(g)$ represents an path in $\widehat{X}$ from g to 1, again thought of as an element of $F(\widehat{X})$:



We now construct an element $h_1(x,g) \in C(\widehat{R})$ which fills the middle, as follows. Let

$$\ell(x,g) = (h_0(\phi x)g)^{-1}(x,g)h_0(g) = ((\sigma(\phi x)g)^{-1}x\sigma(g), 1)$$

which is a loop at 1 in $F(\widehat{X})$; so $\ell(x,g)$ maps to 1 in the singleton $\widetilde{G}(1,1)$. Hence $\ell(x,g)$ is in the image of $\hat{\delta}_2$. For each arrow (x,g) of $\widehat{X}$ choose an element $h_1(x,g) \in C(\widehat{R})(1)$ such that

$$\hat{\delta}_2(h_1(x,g)) = \ell(x,g) = (h_0(\phi x)g)^{-1}(x,g)h_0(g). \tag{10.3.6}$$

Then, recalling Remark 10.3.2, and because $F(\widehat{X})$ is free on these generators (x,g), h_1 extends uniquely to a morphism

$$h_1 \colon F(\widehat{X}) \to C(\widehat{R})(1) \tag{10.3.7}$$

which, because it is a morphism, see again Remark B.7.9, satisfies

$$\hat{\delta}_2(h_1(u,g)) = h_0((\phi u)g)^{-1}(u,g)h_0(g) \tag{10.3.8}$$

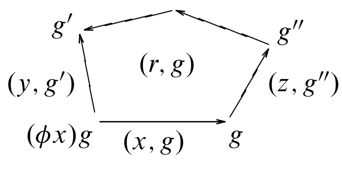
for all arrows (u,g) of $F(\widehat{X})$. In particular for $r \in R$,

$$\hat{\delta}_2(h_1(\omega r,g)) = h_0(g)^{-1}(\omega r,g)h_0(g). \tag{10.3.9}$$

It follows also that $\hat{\delta}_2 h_1(h_0(g)) = (1,1)$ for all $g \in G$. Further, if h_0 is determined by a choice of maximal tree T in the Cayley graph, then for each (x,g) in T we may choose $h_1(x,g) = (1,1)$.

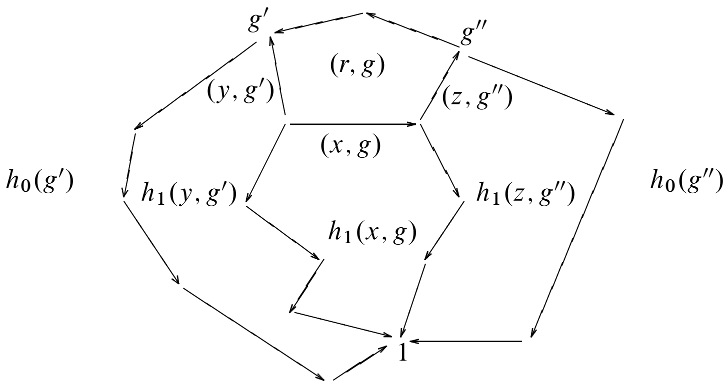
Remark 10.3.4. The specification of h_1 is equivalent to choosing for each $(x,g) \in X \times G$ a representation of $(\sigma(\phi x)g)^{-1}x\sigma(g)$ as a consequence of the relations R. There is in general no algorithm for such a choice, but standard rewriting procedures have been enhanced to give these choices.[149] □

Here are pictures of what we have so far. For each $g \in G, r \in R$ we have a 2-cell (r,g) in the Cayley graph with relations, where the boundary of (r,g) in $F(\widehat{X})$, illustrated in the following picture, is $(\omega r, g)$.



In this situation we have of course $g = (\phi z)g''$ and $(\phi x)g = (\phi y)g'$.

The $h_1(e)$ for all edges e of (r,g) together form a kind of cone $h_1(\omega r,g)$ on the boundary of (r,g), see Equation (10.3.9); gluing this cone to (r,g) along the common boundary forms what is known as a 'separation element', giving a polygonally subdivided 2-sphere as partially shown in the following picture:



This 'separation element' defines geometrically an element of the module of identities among relations $\pi(\mathcal{P})$. We now show that these separation elements form a set of generators of $\pi(\mathcal{P})$ as a G-module; they are determined by h_0 and h_1, but the proof that they generate uses h_2.

This gives all the maps shown in diagram (10.3.3) necessary to give a contracting homotopy up to dimension 2. We now extend these to dimension 3, by constructing elements which 'fill' our separation elements.

3a. Resolution of the covering up to dimension 3. Let I be a set in one-to-one correspondence with $R \times G$ with elements written $[r, g]$, $r \in R$, $g \in G$. Let $C_3(I)$ be the free G-module on I. For any $[r, g] \in I$ we define

$$\delta_3[r, g] = p_2\left((h_1(\omega r, g))^{-1}\right) r^{\sigma g}.$$

This definition on the free generators extends uniquely to an operator morphism

$$\delta_3 \colon C_3(I) \to C(R).$$

It follows from Equation (10.3.7) that $\delta_2\delta_3[r, g] = 1$, and so the given values $\delta_3[r, g]$ lie in $\pi(\mathcal{P}) = \operatorname{Ker}\delta_2$, the G-module of identities among relations. Hence we have a truncated crossed complex:

$$C_3(I) \xrightarrow{\delta_3} C(R) \xrightarrow{\delta_2} F(X) \xRightarrow{\phi} G \qquad (10.3.10)$$

and we now extend our previous covering truncated crossed complex by including $C_3(\widehat{I})$, defined to be the free $\widetilde{G}$-module on the projection $\widehat{I} = I \times G \to G$. This implies that $C_3(\widehat{I})$ is the disjoint union of abelian groups $C(\widehat{I})(g)$, $g \in G$, all mapped by p_3 isomorphically to $C_3(I)$. Elements of $C_3(\widehat{I})(g)$ are pairs $(i, g) \in C_3(I) \times \{g\}$ with addition $(i, g) + (i', g) = (i + i', g)$. The action of $\widetilde{G}$ on $C_3(\widehat{I})$ is given by $(i, gg')^{(g,g')} = (i, g')$; note that this makes sense since in $\widetilde{G}$ $(g, g') \colon gg' \to g'$.

Let $\hat{\delta}_3 \colon C_3(\widehat{I}) \to C(\widehat{R})$ be the $\widetilde{G}$-morphism given for $c \in C_3(I)$, $g \in G$ by $\hat{\delta}_3(c, g) = (\delta_3 c, g)$.

These definitions give the morphism of truncated augmented crossed complexes:

$$\begin{tikzcd}
C_3(\widehat{I}) \arrow[r, "\hat{\delta}_3"] \arrow[d, "p_3"'] & C(\widehat{R}) \arrow[r, "\hat{\delta}_2"] \arrow[d, "p_2"'] & F(\widehat{X}) \arrow[r, Rightarrow, "\hat{\phi}"] \arrow[d, "p_1"'] & \widetilde{G} \arrow[d, "p_0"'] \\
C_3(I) \arrow[r, "\delta_3"'] & C(R) \arrow[r, "\delta_2"'] & F(X) \arrow[r, Rightarrow, "\phi"'] & G
\end{tikzcd} \tag{10.3.11}$$

where the upper row is proven acyclic up to dimension 1.

3b. Contractibility of the covering up to dimension 2. To construct the next part of the homotopy, and again recalling Remark 10.3.2, we define $h_2 \colon C(\widehat{R}) \to C_3(\widehat{I})(1)$ to be the groupoid morphism given on generators by $(r, g) \mapsto ([r, g], 1)$, $(r, g) \in R \times G$, and killing the operation of $F(\widehat{X})$, i.e. it satisfies $h_2((c, g)^{(u,g)}) = h_2(c, g)$ for all $(c, g) \in C(\widehat{R}), u \in F(X)$.

Then from the definition of $\hat{\delta}_3$ we deduce that

$$\hat{\delta}_3 h_2(c, g) = (h_1(\delta_2 c, g))^{-1}\ (c^{\sigma g}, 1)$$

for all $g \in G, c \in C(R)$ and we have got a contracting homotopy up to dimension 2:

$$\begin{tikzcd}
C_3(\widehat{I}) \arrow[r, "\hat{\delta}_3"] \arrow[d, "1"] & C(\widehat{R}) \arrow[r, "\hat{\delta}_2"] \arrow[d, "1"] \arrow[dl, "h_2"'] & F(\widehat{X}) \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "1"] \arrow[dl, "h_1"'] & G \arrow[d, "1"] \arrow[dl, "h_0"'] \\
C_3(\widehat{I}) \arrow[r, "\hat{\delta}_3"'] & C(\widehat{R}) \arrow[r, "\hat{\delta}_2"'] & F(\widehat{X}) \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G.
\end{tikzcd} \tag{10.3.12}$$

We use h_2 to prove that h_0, h_1, which were constructed from the presentation by certain choices, give all identities among relations.[150]

Theorem 10.3.5. *The module $\pi(\mathcal{P})$ of identities among relations is generated as G-module by the elements*

$$\delta_3[r, g] = (p_2 h_1(\omega r, g))^{-1}\ r^{\sigma g}$$

for all $g \in G, r \in R$.

Proof. Since h_2 and h_1 give a contracting homotopy, we have $\hat{\delta}_2\hat{\delta}_3 = 0$, and so the elements $p_2(\hat{\delta}_3 h_2(c, g))$ do give identities. On the other hand, if $c \in C(R)$ and $\delta_2 c = 1$, then $(c, 1) = \hat{\delta}_3 h_2(c, 1)$, and so $c = \delta_3(d)$ for some d. $\square$

We note also that our algebraic setup is rich enough so that for specific presentations the elements given in this theorem may often be computed.

4. Dimension 4 and higher. However some of the elements of $\delta_3(I)$ may be trivial, and others may depend $\mathbb{Z}G$-linearly on a smaller subset. That is, there may be a proper subset J of I such that $\delta_3(J)$ also generates the module $\pi(\mathscr{P})$. Then for each element $i \in I \setminus J$ there is a formula expressing $\delta_3 i$ as a $\mathbb{Z}G$-linear combination of the elements of $\delta_3(J)$. These formulae determine a $\mathbb{Z}G$-retraction $\rho\colon C_3(I) \to C_3(J)$ such that for all $d \in C_3(I)$, $\delta_3(\rho d) = \delta_3(d)$. So we replace I in the above diagram by J, replacing the boundaries by their restrictions. Further, and this is the crucial step, we replace h_2 by $h_2' = \rho' h_2$ where $\rho'\colon C_3(\widehat{I})(1) \to C_3(\widehat{J})(1)$ is mapped by p_3 to ρ.

This $h_2'\colon C(\widehat{R}) \to C_3(\widehat{J})(1)$ is now used to continue the above construction. We define $C_4(\overline{J})$ to be the free G-module on elements written $[d, g] \in \overline{J} = G \times J$, with

$$\delta_4[d, g] = -p_3(h_2'(\delta_3 d, g)) + dg^{-1}.$$

These boundary elements give generators for the relations among the generators $\delta_3(J)$ of $\pi(\mathscr{P})$.

Theorem 10.3.6. *A G-module generating set of relations among these generators $\delta_3(J)$ of $\pi(\mathscr{P})$ is given by*

$$\delta_4[\gamma, g] = -k_2(\delta_3\gamma, g) + \gamma g^{-1}$$

for all $g \in G$, $\gamma \in J$, where $k_2\colon C(\widehat{R}) \to C_3(J)$ is a morphism from the free crossed $F(\widehat{X})$-module on $\hat{\delta}_2\colon G \times R \to F(\widehat{X})$ such that k_2 kills the operation of $F(\widehat{X})$ and is determined by a choice of writing the generators $\delta_3[r, g] \in \delta_3(I)$ for $\pi(\mathscr{P})$ in terms of the elements of $\delta_3(J)$.

Proof. This is a similar argument to the proof of Theorem 10.3.5, using the definition of δ_4 and setting $k_2 = p_3 h_2'$. □

From here onwards we proceed as indicated for the chain complex case in the introductory paragraphs of this section (p. 341).

Remark 10.3.7. In the above we have defined morphisms and homotopies by their values on certain generators, and so it is important for this that the structures of the potential domains for these morphisms and homotopies are free on these generators. For example, h_2' is defined by its values on the elements $(r, g) \in R \times G$. So, noting that h_2 kills the operation of $F(\widehat{X})$, we calculate for example

$$h_2'(r^u s^v, g) = h_2'(r, g(\phi u)^{-1}) + h_2'(s, g(\phi v)^{-1}).$$

In this way the formulae reflect the choices made at different parts of the Cayley graph in order to obtain a contracting homotopy. □

Remark 10.3.8. The determination of minimal subsets J of I such that $\delta_3 J$ also generates $\pi(\mathscr{P})$ is again not straightforward. Some dependencies are easy to find, and others are not. A basic result given in Corollary 7.4.24 is that the abelianisation map $C(R) \to (\mathbb{Z}G)^R$ maps $\pi(\mathcal{P})$ isomorphically to the kernel of the Reidemeister–Whitehead–Fox derivative $(\partial r/\partial x)\colon (\mathbb{Z}G)^R \to (\mathbb{Z}G)^X$. Hence we can test for dependency among identities by passing to the free $\mathbb{Z}G$-module $(\mathbb{Z}G)^R$, and we use this in the next section. For bigger examples, this testing can be a formidable task by hand.[151] □

Exercise 10.3.9. Use the above procedure to calculate identities among relations for the presentation $\langle a, b \mid a^2,\ b^2,\ a^{-1}b^{-1}ab\rangle$ of the Klein four group $\mathbb{Z}_2 \times \mathbb{Z}_2$. □

As an example of these techniques, we give the universal cover and contracting homotopy for an earlier example.

Example 10.3.10. Here we shall prove that the free crossed resolution of a finite cyclic group given in Example 10.2.4 is a resolution by describing its universal cover and a contracting homotopy.

We write C_∞ for the (multiplicative) infinite cyclic group with generator x, and C_q for the finite cyclic group of order q with generator c. Let $\phi\colon \mathsf{C}_\infty \to \mathsf{C}_q$ be the morphism sending x to c. We show how the inductive procedure given earlier recovers the small free crossed resolution of C_q together with a contracting homotopy of the universal cover.

Let $p_0\colon \widetilde{\mathsf{C}}_q \to \mathsf{C}_q$ be the universal covering morphism, and let $p_1\colon \widehat{\mathsf{C}}_\infty \to \mathsf{C}_\infty$ be the induced cover of C_∞. Then $\widehat{\mathsf{C}}_\infty$ is the free groupoid on the Cayley graph $\widehat{X}$ pictured as follows:

$$\begin{tikzcd}
1 & c \arrow[l, "(x,1)"'] & c^2 \arrow[l, "(x,c)"'] & \cdots \arrow[l, "(x,c^2)"'] & c^{q-2} \arrow[l] & c^{q-1} \arrow[l, "(x,c^{q-2})"'] \arrow[lllll, bend right, "(x^{q-1},1)"']
\end{tikzcd}$$

A section

$$\sigma\colon \mathsf{C}_q \to \mathsf{C}$$

of ϕ is given by $c^i \mapsto x^i$, $i = 0, \dots, q-1$, and this defines

$$h_0\colon \mathsf{C}_q \to \widehat{F}_1$$

by $c^i \mapsto (x^i, 1)$. It follows that for $i = 0, \dots, q-1$ we have

$$h_0(c^{i+1})^{-1}(x, c^i)h_0(c^i) = \begin{cases} (1,1) & \text{if } i \neq q-1, \\ (x^q, 1) & \text{if } i = q-1. \end{cases}$$

So we take a new generator x_2 for F_2 with $\delta_2 x_2 = x^q$ and set

$$h_1(x, c^i) = \begin{cases} (1,1) & \text{if } i \neq q-1, \\ (x_2, 1) & \text{if } i = q-1. \end{cases}$$

Hence

$$h_1(x^q, c^i) = h_1((x, c^i)(x, c^{i+1}) \dots (x, c^{i+q-1})) = (x_2, 1).$$

Then for all $i = 0, \dots, q-1$ we have

$$\tilde{\delta}_2 h_1(x, c^i) = h_0(c^i)^{-1}(x, c^i)h_0(c^{i+1}).$$

Hence

$$-h_1\tilde{\delta}_2(x_2, c^i) + (x_2, c^i)x^{-i} = (-x_2, 1) + (x_2 c^{-i}, 1) = (x_2(c^{q-i} - 1), 1).$$

This gives $(0, 1)$ for $i = 0$, and $(x_2(c-1), 1)$ for $i = q-1$. Let $N(i) = 1 + c + \dots + c^{i-1}$, so that $c^{q-i} - 1 = (c-1)N(q-i)$ for $i = 1, \dots, q-1$. Hence we can take a new generator x_3 for F_3 with $\delta_3 x_3 = x_2(c-1)$ and define

$$h_2(x_2, c^i) = \begin{cases} (0, 1) & \text{if } i = 0, \\ (x_3 N(q-i), 1) & \text{if } 0 < i \leqslant q-1. \end{cases}$$

Now we find that if we evaluate

$$-h_2\tilde{\delta}_2(x_3, c^i) + (x_3 c^{-i}, 1) = -h_2((x_2, c^{i-1})c + (x_2, c^i)) + (x_3 c^{-i}, 1)$$

we obtain for $i = 0$

$$-h_2(x_2, c^{q-1}) + (x_3, 1) = (0, 1),$$

for $i = 1$

$$0 + h_2(x_2, c) + (x_3 c^{q-1}, 1) = (x_3(N(q-1) + c^{q-1}), 1) = (x_3 N(q), 1)$$

and otherwise

$$(x_3(-N(q-i+1) + N(q-i) + c^{q-i}), 1) = (0, 1).$$

Thus we take a new generator x_4 for F_4 with $\delta_4 x_4 = x_3 N(q)$ and

$$h_3(x_3, c^i) = \begin{cases} (x_4, 1) & \text{if } i = 1, \\ (0, 1) & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} -h_3\tilde{\delta}_4(x_4, c^i) + (x_4 c^{-i}, 1) &= -h_3(x_3 N(q), c^i) + (x_4 c^{-i}, 1) \\ &= -h_3(x_3 N(q) c^{-i}, 1) + (x_4 c^{-i}, 1) \\ &= (x_4(c^{q-i} - 1), 1). \end{aligned}$$

Thus we are now in a periodic situation and we have the result given earlier as Example 10.2.4:

Theorem 10.3.11. *A free crossed resolution F_* of C_q may be taken to have single free generators x_n in dimension $n \geqslant 1$ with $\phi(x_1) = c$, $\delta_2(x_2) = x_1^q$ and when $n \geqslant 3$*

$$\delta_n(x_n) = \begin{cases} x_{n-1}(c-1) & \text{if } n \text{ is odd,} \\ x_{n-1}(1 + c + \cdots + c^{q-1}) & \text{if } n \text{ is even.} \end{cases}$$

Remark 10.3.12. These methods may also be used to derive the standard free crossed resolution of a group or groupoid which we have given in Definition 10.2.7. □

Some fundamental results relating free crossed complexes to CW-complexes are the following, for which we cannot give the proofs here.[152]

Theorem 10.3.13. *Let X_* be a CW-filtered space, and let $\phi\colon \Pi X_* \to C$ be a homotopy equivalence to a free crossed complex with a preferred free basis. Then there is a CW-filtered space Y_*, and an isomorphism $\alpha\colon C \cong \Pi Y_*$ of crossed complexes with preferred basis, such that $\alpha\phi$ is realised by a homotopy equivalence $f\colon X_* \to Y_*$, i.e. $\alpha\phi = \Pi(f)$.*[153]

Corollary 10.3.14. *If A is a free crossed resolution of a group G, then A is realised as free crossed complex with preferred basis by some CW-filtered space Y_*.*

Proof. We only have to note that the group G has a classifying CW-space BG whose fundamental crossed complex $\Pi(BG)$ is homotopy equivalent to A. □

Baues also points out in [Bau89], p. 357 an extension of these results which we can apply to the realisation of morphisms of free crossed resolutions. A new proof of this extension is given by Faria Martins in [FM07], using methods of Ashley [Ash88].

Proposition 10.3.15. *Let $X = K(G,1)$, $Y = K(H,1)$ be CW-models of Eilenberg–Mac Lane spaces and let $h\colon \Pi X_* \to \Pi(Y_*)$ be a morphism of their fundamental crossed complexes with the preferred bases given by skeletal filtrations. Then $h = \Pi(g)$ for some cellular $g\colon X \to Y$.*

Proof. Certainly h is homotopic to $\Pi(f)$ for some $f\colon X \to Y$ since the set of pointed homotopy classes $X \to Y$ is bijective with the morphisms of groups $A \to B$. The result follows from [Bau89], p. 357, (**), ('if f is Π-realisable, then each element in the homotopy class of f is Π-realisable'). □

These results are exploited in [Moo01], [BMPW02] to calculate free crossed resolutions of the fundamental groupoid of a graph of groups.

10.4 Acyclic models

The classical theory of acyclic models is a powerful tool for comparing different representations of homology by chain complexes.[154] It has also been useful for comparing cohomology theories of algebraic structures. The traditional method applied to examples of spaces gives isomorphisms of homology and so does not directly give, even with an isomorphism of fundamental groups, an isomorphism of homotopy groups. The same sort of technique works for crossed complexes, but with some technical differences. Thus the advantage of the theorem for crossed complexes is that to get a homotopy equivalence does not require a detour, sometimes mistakenly omitted, via universal covering spaces. The method of proof is closely related to those of Theorem 10.2.11 and to that of a version of the traditional theorem, but the special features of crossed complexes in dimensions $\leqslant 2$ have to be taken into account.

10.4.i The Acyclic Model Theorem

Definition 10.4.1. Let C be a category and let $F : \mathsf{C} \to \mathsf{Crs}$ be a functor. A *base* $\mathcal{B}$ for F is in the first instance a family of elements $b_j \in F(B_j)_n$ where the B_j are objects of C and the j come from a family J^n, $n \geqslant 0$, of indexing sets. Thus $\mathcal{B}$ is to consist of the whole structure of the J^n, B_j, b_j. The B_j are called the *objects* of the base and the b_j the *elements* of the base. The requirement is that for all $n \geqslant 0$ $F(X)_n$ is 'free' on the elements $F(\sigma)(b_j)$ for all $j \in J^n$ and $\sigma : B_j \to X$ in C. This means that for each $n \geqslant 0$ the elements $F(\sigma)(b_j)$:

(i) if $n = 0$, are distinct and give all elements of $F(X)_0$;
(ii) if $n = 1$, freely generate the groupoid $F(X)_1$;
(iii) if $n = 2$, freely generate the crossed module $F(X)_2 = (\delta_2 : C_2 \to C_1)$ as C_1-module;
(iv) if $n \geqslant 3$, freely generate the $\pi_1 F(X)$-module $F(X)_n$.

If $A \subseteq \mathrm{Ob}(\mathsf{C})$ is a class of objects containing all B_j, $j \in J^n$, $n \geqslant 0$ then we also say *F has a base in A*. □

Example 10.4.2. Let $\mathsf{C} = \mathsf{Top}$ and let $\Pi^\Upsilon : \mathsf{Top} \to \mathsf{Crs}$ be given by

$$\Pi^\Upsilon(X) = \Pi(\|S^\Delta(X)\|_*)$$

where $S^\Delta(X)$ is the simplicial singular complex of X consisting of all continuous maps $\Delta^n \to X$ with its natural structure as simplicial set, and $\|K\|$ denotes the thick geometric realisation of a simplicial set as explained in Section A.10 of Appendix A. Then Π^Υ has a base where J^n consists solely of n, $B_n = \Delta^n$ and b_n is the identity map $\iota_n : \Delta^n \to \Delta^n$. The point is that a singular simplex $\sigma : \Delta^n \to X$ is induced from the identity map $1 : \Delta^n \to \Delta^n$ by σ. Note that this does not give a base for $\Pi : \mathsf{Simp} \to \mathsf{Crs}$ which is why we need the following definition. □

Definition 10.4.3. Let C be a category. A functor $P : \mathsf{C} \to \mathsf{Crs}$ is said to be *projective* if there is a functor $F : \mathsf{C} \to \mathsf{Crs}$ which has a base and such that for each $n \geqslant 1$:[155]

(i) $\mathrm{tr}^2 P$ is a natural retract of $\mathrm{tr}^2 F$ considered as functors to XMod;

(ii) if $n \geqslant 3$ then P_n is a natural retract of F_n, considered as functors to Mod, and over the natural retraction $\pi_1 P$ of $\pi_1 F$ induced by that given by (i). □

Remark 10.4.4. It is because the operations in a crossed complex are so much part of their structure, in a sense are intrinsic, that we have to make this careful definition of projective; as we shall see, this condition is satisfied in useful circumstances, namely the functors Π from simplicial and cubical sets to crossed complexes. □

Definition 10.4.5. Let $Q : \mathsf{C} \to \mathsf{Crs}$ be a functor. We say Q is *acyclic* on the base $\mathcal{B}$ if $Q(B_j)$ is an acyclic crossed complex for all objects B_j of the base. □

Theorem 10.4.6 (Acyclic Model Theorem). *Let $P, Q : \mathsf{C} \to \mathsf{Crs}$ be functors such that Q is acyclic and suppose there is a base $\mathcal{B}$ and functor $F : \mathsf{C} \to \mathsf{Crs}$ free on $\mathcal{B}$, and for which P is projective. Then any natural transformation $\tau : P_0 \to Q_0$ is realised by a natural transformation $T : P \to Q$ and any two such realisations are naturally homotopic.*

Proof. Let $\eta_n : P_n \to F_n$, $\mu_n : F_n \to P_n$ be the family of natural transformations supplied by the definition of projective, so that $\mu_n \eta_n = 1$. We will often drop the suffix n when it can be understood from the context.

We consider first the right-hand part of the following diagram:

$$\begin{tikzcd}
& F_2 \arrow[rr, "\delta_2"] \arrow[dl, shift left, "\mu"] \arrow[dd, dashed, "S_2"] & & F_1 \arrow[rr, shift left, "\delta^-"] \arrow[rr, shift right, "\delta^+"'] \arrow[dl, shift left, "\mu"] \arrow[dd, dashed, "S_1"] & & F_0 \arrow[rr, "\phi_F"] \arrow[dl, shift left, "\mu"] \arrow[dd, dashed, "S_0"] & & \pi_0 F \arrow[dl, shift left, "\mu"] \\
P_2 \arrow[ur, shift left, "\eta"] \arrow[rr, "\delta_2"] \arrow[dr, dashed, "T_2"'] & & P_1 \arrow[ur, shift left, "\eta"] \arrow[rr, shift left, "\delta^-"] \arrow[rr, shift right, "\delta^+"'] \arrow[dr, dashed, "T_1"'] & & P_0 \arrow[ur, shift left, "\eta"] \arrow[rr, "\phi_P"] \arrow[dr, dashed, "T_0"'] & & \pi_0 P \arrow[ur, shift left, "\eta"] \arrow[dr, "\tau"] & \\
& Q_2 \arrow[rr, "\delta_2"] & & Q_1 \arrow[rr, shift left, "\delta^-"] \arrow[rr, shift right, "\delta^+"'] & & Q_0 \arrow[rr, "\phi_Q"'] & & \pi_0 Q .
\end{tikzcd}$$

Notice that by our assumptions, η, μ give natural morphisms of the crossed module parts $P \to F$, $F \to P$ respectively. Our method is to construct $S_n : F_n \to Q_n$, then define $T_n = S_n \eta$, and find this is the appropriate extension.

We are trying to find a natural crossed complex morphism $T : P \to Q$ which induces τ. The general plan is shown by the diagram

$$\begin{tikzcd}
& b \arrow[dl, shift left, "\mu"] \arrow[dd, dashed] & & \\
{} \arrow[ur, shift left, "\eta"] \arrow[rr, "\phi_P"] & & {} \arrow[dr, "\tau"] & \\
& q_b \arrow[rr, dotted, "\phi_Q", "\text{choose}"'] & & {}
\end{tikzcd}$$

where b is a free generator.

We first define T in dimension 0.

The points of $F(X)_0$ are of the form $F(\sigma)(b)$ for $b \in F(B)_0$, $B \in \mathcal{B}$ and all morphisms $\sigma\colon B \to X$. Choose a point $q_b \in Q(B)_0$ such that $\phi_Q q_b = \tau\phi_P \mu b$, and define $S_0^X(F(\sigma)(b)) = Q(\sigma)(q_b)$. This defines S_0^X and we set $T_0^X = S_0^X\eta$. Then

$$\phi_Q T_0^X = \phi_Q S_0^X \eta = \tau\phi_P\mu\eta = \tau\phi_P.$$

We next verify naturality of S_0, and so of T_0. Let $f\colon X \to Y$ be a morphism in C. Then we check the naturality condition on the basis elements $F(\sigma)(b)$ of $F_0(X)$. Then

$$\begin{aligned} Q_0(f)S_0(X)(F(\sigma)(b)) &= Q_0(f)(Q_0(\sigma)S_0(B)(b)) \\ &= Q_0(f\sigma)(S_0(B)(b)) \\ &= S_0(Y)F_0(f)(S_0(B)(b)). \end{aligned}$$

Thus naturality is automatic from the construction, and we will not repeat this proof in higher dimensions.

We next define a morphism of groupoids $S_1\colon F(X)_1 \to Q(X)_1$.

Let $X \in \mathsf{C}$. We know the groupoid $F_1(X)$ has a free basis of elements $F(\sigma)(b)$ for $b \in F_1(B)$, $B \in \mathcal{B}$ and all morphisms $\sigma\colon B \to X$. Consider $q_b^- = T_0\delta^-\mu b, q_b^+ = T_0\delta^+\mu b \in Q(B)_0$. By acyclicity of $Q(B)_1$ there is an element $q_b\colon q_b^- \to q_b^+ \in Q(B)_1$. We therefore define $S_1^X\colon F_1(X) \to Q_1(X)$ to have value $Q(\sigma)(q_b)$ on the basis element $F(\sigma)(b)$. We then set $T_1 = S_1\eta$. By naturality, $\delta^\pm S_1 = S_0\delta^\pm = T_0\delta^\pm\mu$. Hence $T_0\delta^\pm = \delta^\pm T_1$.

At the next stage we consider the above diagram and a basis element $F(\sigma)(b)$ for $b \in F(B)_2$. Then from the commutativity of the diagram of solid arrows, $T_1\delta_2\mu b$ is a loop in $Q_2(B)$. By acyclicity of $Q(B)$ we can find $q_b \in Q_2(B)$ such that $\delta_2 q_b = T_1\delta_2\mu b$, and we set $S_2^X(F(\sigma)(b)) = Q_2(\sigma)(q_b)$. This defines S_2 and we set $T_2 = S_2\eta$. Since η, μ in dimensions 1 and 2 give crossed module morphisms, because $\mu\eta = 1$, and by the above argument on naturality, T_2 gives the required extension of T_1.

Continuing this argument, gives the natural transformation T as required.

We now have to show that any two such natural transformations, say T, U are naturally homotopic. For this, we replace F by $F' = \mathcal{I} \otimes F$ and use analogous arguments to extend the natural transformation defined by T, U on $\{0,1\} \otimes F$ to $\mathcal{I} \otimes F$ on the extra basis elements $\iota \otimes (F(\sigma)(b))$. We omit further details. □

Corollary 10.4.7. *Let $P, Q\colon \mathsf{C} \to \mathsf{Crs}$ be functors such that P is projective with respect to a free functor with base $\mathcal{B}$ on which Q is acyclic, and Q is projective with respect to a free functor with base $\mathcal{B}'$ on which P is acyclic. Then any natural equivalence $\pi_0 P \to \pi_0 Q$ extends to a natural homotopy equivalence $P \to Q$.*

Exercise 10.4.8. Develop a version of the Acyclic Model Theorem in which the notion of free is replaced by relatively free. □

10.4.ii Simplicial sets and normalisation

An introduction to the theory of simplicial objects setting out our notation is given in Appendix A, Section A.10. In particular we need the notion of simplicial set without degeneracies which we call a presimplicial set (it might also be called an Υ-set, since it is defined by a subcategory which we write Υ of the usual simplicial site $\mathbf{\Delta}$).[156]

The singular simplicial set $S^{\Delta}X$ of a topological space X has a geometric realisation as a simplicial set which is written $|S^{\Delta}X|$ and also a realisation as a presimplicial set which is written $\|S^{\Delta}X\|$, and often called the thick realisation. It is proved in Section A.10 that the natural projection

$$\|S^{\Delta}X\| \to |S^{\Delta}X|$$

is a homotopy equivalence, and it follows that the corresponding morphism of free crossed complexes using their skeletal filtrations is also a homotopy equivalence.

As explained in Section 9.9, there is for each $n \geqslant 0$ a crossed complex $a\Delta^n$ which is a crossed complex model of the n-dimensional simplex; thus the boundary is given by the Homotopy Addition Lemma, Theorem 9.9.4. This family of crossed complexes $a\Delta^n$ can be regarded as a cosimplicial set $a\Delta\colon \mathbf{\Delta} \to \mathsf{Crs}$ so that we obtain the *fundamental crossed complex of a simplicial set* K as a coend

$$\Pi K = \int^{\mathbf{\Delta},n} K_n \times a\Delta^n. \tag{10.4.1}$$

Also we have the *unnormalised crossed complex of a simplicial set*

$$\Pi^{\Upsilon} K = \int^{\Upsilon,n} K_n \times a\Delta^n. \tag{10.4.2}$$

These crossed complexes are homotopy equivalent; this is the normalisation theorem for which we will give an acyclic model proof here.[157] Also both crossed complexes are needed for the purposes of acyclic models. First we prove:

Theorem 10.4.9. *For all $q \geqslant 0$ the functors $(\Pi K)_q$, $(\Pi^{\Upsilon} K)_q$ on simplicial sets K have the property that the first is a natural retract of the second.*

Proof. We first construct an intermediate 'reduced' functor $\Pi^{\mathrm{red}}K$.

A simplicial set K contains its subsimplicial set generated by the elements of K_0: we write this as $\bar{K}_0$. It is the disjoint union of the simplicial sets generated by the elements of K_0. We form the crossed complex $\Pi^{\mathrm{red}}K$ by the pushout in the category Crs:

$$\begin{tikzcd} \Pi^{\Upsilon}\bar{K}_0 \arrow[r, "\xi"] \arrow[d] & K_0 \arrow[d] \\ \Pi^{\Upsilon}K \arrow[r, "\bar{\xi}"'] & \Pi^{\mathrm{red}}K \end{tikzcd} \tag{10.4.3}$$

where K_0 denotes here also the trivial crossed complex on the set K_0. Because as a crossed complex K_0 is a natural retract of $\Pi^{\Upsilon}\overline{K}_0$ it follows that $\Pi^{\mathrm{red}}K$ is also a natural retract of $\Pi^{\Upsilon}K$. In fact ξ is a homotopy equivalence of crossed complexes and so it follows from the gluing theorem for homotopy equivalences in the category Crs (see Remark B.8.2) that $\bar{\xi}$ is also a homotopy equivalence of crossed complexes.

We have to consider the dimensions 1, 2 and $q \geqslant 3$.

The groupoid $C_1 = (\Pi^{\mathrm{red}}K)_1$ is the free groupoid on the elements of K_1, but with the elements $\varepsilon_0 v$ equated to identities for each $v \in K_0$. Thus $(\Pi^{\mathrm{red}}K)_1 = (\Pi K)_1$.

In dimension 2, we note that if $x \in K_1$, then in $(\Pi^{\mathrm{red}}K)_2$, $\delta_2(\varepsilon_i x) = 1_{tx}$ for $i = 0, 1$: this is a reason for constructing $(\Pi^{\mathrm{red}}K)$. For $i = 0, 1$, let $\Phi_i : (\Pi^{\mathrm{red}}K)_2 \to (\Pi^{\mathrm{red}}K)_2$ be given on the basis elements by $\Phi_i k = k(\varepsilon_i \partial_{i+1} k)^{-1}$. Then

$$\Phi_i \varepsilon_i x = 1, \quad \Phi_1 \varepsilon_0 x = (\varepsilon_0 x)(\varepsilon_0^2 tx)^{-1} = \varepsilon_0 x$$

in $(\Pi^{\mathrm{red}}K)_2$, so that $\Phi = \Phi_0\Phi_1$ vanishes on degenerate elements of K_2. Further, $\delta_2\Phi = \delta_2$. So Φ defines a morphism $(\Pi^{\mathrm{red}}K)_2 \to (\Pi^{\mathrm{red}}K)_2$ of crossed $(\Pi K)_1$-modules which vanishes on degenerate elements and hence defines in dimension 2 a section of the projection $\Pi^{\mathrm{red}}K \to \Pi K$.

In dimensions $q \geqslant 3$ and for $0 \leqslant j < q$ we define $\Phi_j : (\Pi^{\mathrm{red}}K)_q \to (\Pi^{\mathrm{red}}K)_q$ by $\Phi_j k = k - \varepsilon_j \partial_{j+1} k$ on the free basis of elements k of K_q not degeneracies of the vertices, and set $\Phi = \Phi_0 \dots \Phi_{q-1}$. Then $\Phi_j \varepsilon_j x = 0$ and for $i < j$ we have

$$\Phi_j \varepsilon_i x = \varepsilon_i x - \varepsilon_i \varepsilon_{j-1} \partial_{j+1} x.$$

Hence Φ is trivial on degeneracies and so determines a section of $(\Pi^{\mathrm{red}}K)_q \to (\Pi K)_q$ which is also natural for maps of K. □

Theorem 10.4.10 (Simplicial Normalisation Theorem). *The natural map $\Pi^{\Upsilon}K \to \Pi K$ is a natural homotopy equivalence of crossed complexes.*

Proof. This is a consequence of the Acyclic Model Theorem. □

10.4.iii Cubical sets and normalisation

The reader should leave this section till after studying cubical sets and their geometric realisations Chapter 11, but the material here is part of the subject of Acyclic Models. So in this section we use notions given later in Definition 11.1.4, namely that a cubical set is a functor $K : \square^{\mathrm{op}} \to \mathsf{Set}$, where $\square$ is the category called the cubical site. Alternatively, K is defined by a family $K = \{K_n\}_{n \geqslant 0}$ of sets together with face operations $\partial_i^{\pm} : K_n \to K_{n-1}$ and degeneracy operations $\varepsilon_i : K_{n-1} \to K_n$ for $i = 1, \dots, n$ and $n \geqslant 1$, satisfying the usual cubical relations.

We shall also need the notion of cubical set without degeneracies, which we call a *precubical set*; this is given by a functor $\Xi^{\mathrm{op}} \to \mathsf{Set}$ where Ξ is the appropriate subcategory of $\Box$. Clearly any cubical set determines a precubical set by means of the inclusion $\Xi \to \Box$.

The family of crossed complexes $\mathcal{I}^n$ can be regarded as a cocubical set $\mathcal{I}^\bullet : \Box \to$ Crs so that the fundamental crossed complex of a cubical set K given in Definition 11.4.3 may be obtained as a coend

$$\Pi K = \int^{\Box,n} K_n \times \mathcal{I}^n. \tag{10.4.4}$$

Also we may define the *unnormalised crossed complex of a cubical set* K as the coend

$$\Pi^\Xi K = \int^{\Xi,n} K_n \times \mathcal{I}^n. \tag{10.4.5}$$

These crossed complexes are not homotopy equivalent, but we need both for the purposes of acyclic models.

Theorem 10.4.11. *For all $q \geqslant 0$ the functors $(\Pi K)_q$, $(\Pi^\Xi K)_q$ on cubical sets K have the property that the first is a natural retract of the second.*

Proof. Again we work through an intermediate step. A cubical set K contains its subcubical set generated by the elements of K_0: we write this as $\bar{K}_0$. It is the disjoint union of the cubical sets generated by the elements of K_0. We form the crossed complex $\Pi^{\mathrm{red}} K$ by the pushout in the category Crs:

$$\begin{tikzcd} \Pi^\Xi \bar{K}_0 \arrow[r] \arrow[d] & K_0 \arrow[d] \\ \Pi^\Xi K \arrow[r] & \Pi^{\mathrm{red}} K \end{tikzcd} \tag{10.4.6}$$

where K_0 denotes here also the trivial crossed complex on the set K_0. Because as a crossed complex K_0 is a natural retract of $\Pi^\Xi \bar{K}_0$ it follows that $\Pi^{\mathrm{red}} K$ is also a natural retract of $\Pi^\Xi K$.

We have to consider the dimensions 1, 2 and $q \geqslant 3$.

The groupoid $C_1 = (\Pi^{\mathrm{red}} K)_1$ is the free groupoid on the elements of K_1, but with the elements $\varepsilon_1 v$ equated to identities for each $v \in K_O$. Thus $(\Pi^{\mathrm{red}} K)_1 = (\Pi K)_1$.

In dimension 2, we note that if $x \in K_1$, then in $(\Pi^{\mathrm{red}} K)_2$, $\delta_2(\varepsilon_i x) = 1_{tx}$ for $i = 1, 2$: this is a reason for constructing $(\Pi^{\mathrm{red}} K)$. Let $\Phi_i : (\Pi^{\mathrm{red}} K)_2 \to (\Pi^{\mathrm{red}} K)_2$ be given on the basis elements by $\Phi_i k = k(\varepsilon_i k \partial_i^+ k)^{-1}$. Then

$$\Phi_i \varepsilon_i x = 1, \quad \Phi_2 \varepsilon_1 x = (\varepsilon_1 x)(\varepsilon_1^2 t x)^{-1} = \varepsilon_1 x$$

in $(\Pi^{\mathrm{red}} K)_2$, so that $\Phi = \Phi_2 \Phi_1$ vanishes on degenerate elements of K_2. Further, $\delta_2 \Phi = \delta_2$. So Φ defines a morphism of crossed $(\Pi K)_1$-modules $(\Pi^{\mathrm{red}} K)_2 \to (\Pi^{\mathrm{red}} K)_2$ which vanishes on degenerate elements and hence defines in dimension 2 a section of the projection $\Pi^{\mathrm{red}} K \to \Pi K$.

In dimensions $q \geqslant 3$ we define $\Phi_i : (\Pi^{\text{red}} K)_q \to (\Pi^{\text{red}} K)_q$ on the free basis of elements of K_q which are not degeneracies of the vertices by $\Phi_i k = k - \varepsilon_i \partial_i^+ k$, and set $\Phi = \Phi_1 \dots \Phi_q$. Then Φ is trivial on degeneracies and so determines a section of $(\Pi^{\text{red}} K)_q \to (\Pi K)^q$ which is also natural for maps of K. □

10.4.iv Relating simplicial and cubical by acyclic models

In this section we relate simplicial and cubical singular theories on spaces.[158]

Lemma 10.4.12. *Let X be a contractible space. Then the crossed complexes $\Pi|S^{\Delta}(X)|$ and $\Pi|S^{\square}(X)|$ are both contractible.*

Proof. Let $\eta\colon I \times X \to X$ be a contracting homotopy. This can also be regarded as a map $\eta'\colon CX \to X$ where $CX = (I \times X)/(\{1\} \times X)$. If $f : \Delta^n \to X$ is a singular simplex, define $h'(f)\colon \Delta^{n+1} \to X$ to be the composite

$$\Delta^{n+1} = C\Delta^n \xrightarrow{Cf} CX \xrightarrow{\eta'} X.$$

This defines a contracting homotopy on the free basis of $\Pi|S^{\Delta}(X)|$.

In the cubical case, we get the homotopy by taking $f : I^n \to X$ to the composite

$$I^{n+1} = I \times I^n \xrightarrow{1 \times f} I \times X \xrightarrow{\eta} X.$$ □

Theorem 10.4.13. *For any space X, there is a natural crossed complex homotopy equivalence*

$$\Pi|S^{\Delta}(X)| \simeq \Pi|S^{\square}(X)|.$$

Proof. This homotopy equivalence extends the identity in dimension 0. □

10.4.v The Eilenberg–Zilber–Tonks Theorem

We have now set up enough machinery to prove this theorem by an acyclic model argument, but in fact a more precise result has been proved and so it is this that we state, referring the reader to [Ton03] for the proof and more detail.[159]

Here we give a statement of Tonks' result.

Theorem 10.4.14 (Eilenberg–Zilber–Tonks). *Let K, L be simplicial sets. Then there are natural morphisms of crossed complexes*

$$a\colon \Pi(K \times L) \to \Pi K \otimes \Pi L$$

natural in simplicial sets K, L and which are associative in the sense that the following diagram commutes:

$$\begin{tikzcd}
\Pi(K \times L \times M) \arrow[r, "a"] \arrow[d, "a"'] & \Pi(K \times L) \otimes \Pi M \arrow[d, "a\otimes \mathrm{id}"] \\
\Pi K \otimes \Pi(L \times M) \arrow[r, "\mathrm{id}\otimes a"'] & \Pi K \otimes \Pi L \otimes \Pi M .
\end{tikzcd}$$

for simplicial sets K, L, M. Further, there is a homotopy inverse b to a, also associative, such that $ab = 1$, and making $b(\Pi K \otimes \Pi L)$ a strong natural deformation retract of $\Pi(K \times L)$.

Remark 10.4.15. There should be enough information in this chapter and in standard facts in simplicial theory on anodyne extensions for you to complete the acyclic model proof.[160] □

Remark 10.4.16. The formula given by Tonks for a is

$$a_n(x, y) = \begin{cases} x \otimes y & \text{for } n = 0, \\ x \otimes y_1 + x_0 \otimes y & \text{for } n = 1, \\ (x \otimes y_2)^{x_0} \otimes y_{01} + x_0 \otimes y + (x_{01} \otimes y_{12})^{x_0} \otimes y_{01} & \text{for } n = 2, \\ x_0 \otimes y + \sum_{i=1}^{n} (x_{0\dots i} \otimes y_{i\dots n})^{x_0 \otimes y_{01}} & \text{for } n \geqslant 3. \end{cases}$$

In this a subscript notation is used for face of simplices:

$$x_{i_0 \dots i_k} = \partial_{j_1} \dots \partial_{j_{r-k}} x \in K_k$$

if $\{i_0 < \cdots < i_k\}$, $\{j_1 < \cdots < j_{r-k}\}$ is a partition of $[r] = \{0, 1, \dots, r\}$ and $x \in K_r$. In particular x_i is the i'th vertex of x and x_{01} is the 1-simplex between the vertices x_0 and x_1 of x. Further, in this theorem the following more traditional formula is used for the boundary operator in ΠK on $x \in K_n$:

$$\delta_n x = \begin{cases} -\partial_1 x + \partial_0 x + \partial_2 x & \text{if } n = 2, \\ \partial_2 x + (\partial_0 x)^{x_{01}} - \partial_3 x - \partial_1 x & \text{if } n = 3, \\ (\partial_0 x)^{x_{01}} + \sum_{i=1}^{n} (-1)^i \partial_i x & \text{if } n \geqslant 4. \end{cases}$$

The formula given for b is in terms of shuffles, as in the chain complex approach.[161] □

Remark 10.4.17. Theorem 9.5.4 proves that $\nabla\colon \mathsf{Crs} \to \mathsf{Chn}$ preserves tensor products. So we also obtain an Eilenberg–Zilber–Tonks Theorem with values in chain complexes with a groupoid of operators. □

10.4.vi Excision

We give here a crossed complex version of a result which in work on singular homology is commonly called an Excision Theorem.[162]

We write TopCov for the category of pairs $(X, \mathcal{U})$ where X is a topological space and $\mathcal{U}$ is a family of subsets of X whose interiors cover X. If Y is a subspace of X and $\mathcal{U}$ is such a cover of X then $Y \cap \mathcal{U}$ consists of the sets $Y \cap U$ for all $U \in \mathcal{U}$. A *morphism* $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is a map $f : X \to Y$ such that for every $U \in \mathcal{U}$ there is a $V \in \mathcal{V}$ such that $f(U) \subseteq V$. Two maps $f, g : (X, \mathcal{U}) \to (Y, \mathcal{V})$ are called *homotopic*, $f \simeq g$, if there exists a homotopy $H : I \times X \to Y$ from f to g such that for every $U \in \mathcal{U}$ there is $V \in \mathcal{V}$ such that $H(I \times U) \subseteq V$. The definitions of a *homotopy equivalence* and of *strong deformation retract* are the obvious ones. The *trivial pair* $(X, \mathcal{T})$ has $\mathcal{T}$ consisting solely of X.

Definition 10.4.18. We write $S^{\Delta}(X, \mathcal{U})$ for the subsimplicial set of $S^{\Delta}(X)$ of simplices $\sigma : \Delta^n \to X$ such that $\sigma(\Delta^n) \subseteq U$ for some $U \in \mathcal{U}$. □

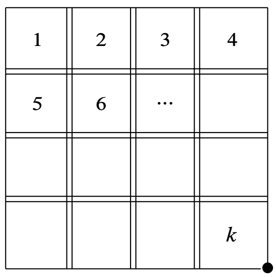
Although in this section we are working on simplicial singular complexes, it is convenient in the following argument to use some notions of *collapsing* of cubes which are developed in Section 11.3.i, and to which you may need to refer. You will see that in the argument of the proof of the next lemma, the geometry of the cube is more convenient than that of the simplex.

Lemma 10.4.19. *Every object* $(\Delta^n, \mathcal{U})$ *in* TopCov *is contractible.*

Proof. Because of the homeomorphism of Δ^n with I^n it is sufficient to prove that any $(I^n, \mathcal{U})$ is contractible in TopCov.

We construct a finite sequence $(X_i, \mathcal{U}_i)$, $i = 0, \dots, k$, of objects in TopCov such that $(X_0, \mathcal{U}_0) = (I^n, \mathcal{U})$ and $(X_k, \mathcal{U}_k) = (*, \mathcal{T})$ with $*$ a singleton, and $(X_i, \mathcal{U}_i)$ is a strong deformation retract of $(X_{i-1}, \mathcal{U}_{i-1})$.

By the Lebesgue covering lemma, the cube I^n may be subdivided by hyperplanes parallel to its faces into a finite number say k of subcubes each of which is contained in some $U \in \mathcal{U}$. Now beginning in one corner we collapse one subcube after another into that part of its boundary which is in common with the remaining ones, shown in the diagram by double lines. The last cube is retracted onto the corner •. (An analogous argument is used in the proof of Proposition 14.2.8.)



(collapse)

So we define X_i as X_{i-1} with the i-th cube retracted off and $\mathcal{U}_i$ as $X_i \cap \mathcal{U}_{i-1}$. Obviously $(X_i, \mathcal{U}_i)$ is a strong deformation retract of $(X_{i-1}, \mathcal{U}_{i-1})$. □

Theorem 10.4.20. *The inclusion*

$$i: S^{\Delta}(X, \mathcal{U}) \to S^{\Delta}(X) \tag{$*$}$$

is a homotopy equivalence.

Proof. We actually prove that the induced morphism

$$i': \Pi S^{\Delta}(X, \mathcal{U}) \to \Pi S^{\Delta}(X) \tag{$**$}$$

is a homotopy equivalence of crossed complexes.

We consider both sides of $(**)$ as functors P, Q from TopCov to the category of crossed complexes with

$$P(X, \mathcal{U}) = \Pi S^{\Delta}(X, \mathcal{U}), \quad Q(X, \mathcal{U}) = \Pi S^{\Delta}(X).$$

We use the Acyclic Model Theorem. As models in TopCov we choose all pairs $(\Delta^n, \mathcal{V})$, $n \geqslant 0$, with Δ^n a standard simplex and $\mathcal{V}$ a covering of Δ^n having an open refinement. Both functors are acyclic on models, by Lemma 10.4.19.

Let F: TopCov $\to$ Crs be given by $\Pi \bar{S}^{\Delta}(X, \mathcal{U})$ where this singular complex has n-simplices the maps $(\Delta^n, \mathcal{V}) \to (X, \mathcal{U})$ in TopCov. Then F has a base the identities $(\Delta^n, \mathcal{V}) \to (\Delta^n, \mathcal{V})$. The inclusion $i: Q(X, \mathcal{U}) \to F(X, \mathcal{U})$ is given by considering $\sigma: \Delta^n \to X$ as $\sigma: (\Delta^n, \sigma^{-1}\mathcal{U}) \to (X, \mathcal{U})$, and the forgetful functor TopCov $\to$ Top defines $r: F \to Q$ such that $ri = 1$. So Q is a retract of a free functor, while P is actually free with base in dimension n the identity $(\Delta^n, \mathcal{T}) \to (\Delta^n, \mathcal{T})$. So the Acyclic Model Theorem and its Corollary 10.4.7 applies. □

Exercise 10.4.21. Develop an analogue of the above argument for the cubical singular complex. □

Notes

130 p. 324 The notion of resolution by chain complexes has led to an advanced view of homological algebra using the notion of triangulated category. A substantial reference is [Nee01]. Perhaps this idea can be usefully developed for crossed complexes, and so give a somewhat nonabelian view of that area.

131 p. 328 For more references on this kind of argument using fibred exponential laws see [BH87], [HK89]. The necessity and sufficiency of what is now called

the Giraud–Conduché condition on a functor of categories is summed up in Theorem 4.4 on page 40 of [Gir64], and was rediscovered in [Con72]. A general discussion is in [BN00].

132 p. 329 The notion of free crossed resolution was crucial in the work of Huebschmann [Hue80a], [Hue81a], [Hue81b].

133 p. 331 This free crossed resolution of finite cyclic groups was introduced by Brown and Wensley in [BW95].

134 p. 331 A proof of a generalisation of this fact to the case r is a proper power, say of order q, may be found in [DV73]. They attach to the free crossed module on r a free resolution of the cyclic group of order q.

135 p. 332 An exact sequence of groups $1 \to M \to E \to G \to 1$ is also in the literature called an extension of G by M, see for example [ML63], a terminology seen to agree with the notation $H^2(G, M)$ for the case M is a G-module. However there is a good case for saying an extension of M should be bigger than M, as in field extensions. Generalising the following remarks on factor sets from groups to groupoids is done in [BH82].

136 p. 335 Another method of giving a group is in terms of polynomial laws, and for this other methods are appropriate, such as homological perturbation theory, see for example [Hue91], [GL01].

137 p. 336 Tonks proved in [Ton94], Theorem 3.1.5, that the tensor product of free crossed resolutions of a group is a free crossed resolution: his proof used the crossed complex Eilenberg–Zilber Theorem, [Ton94], Theorem 2.3.1, which was published in [Ton03].

138 p. 336 These are special cases of results on graphs of groups which are given in [Moo01], [BMPW02], but these cases nicely show the advantage of the method and in particular the necessary use of groupoids. See also [Hor79].

139 p. 337 The injectivity of i', j' follows from the normal form for this construction, which may be found in books on combinatorial group theory. This injectivity result is a special case of the normal form for the fundamental groupoid of a graph of groups given in [Hig76], [Moo01].

140 p. 338 This idea of forming a fundamental groupoid is due to Higgins in the case of a graph of groups [Hig76], where it is shown that it leads to convenient normal forms for elements of this fundamental groupoid. This view is pursued in [Moo01], see also [BMPW02], from which this section is largely taken. The book [Koz08] is a handy reference for homotopy colimits from a practical viewpoint.

141 p. 338 For more elaborate examples and discussion see [Moo01], [BMPW02].

142 p. 340 Further examples are developed in [Moo01].

143 p. 342 This process is carried out for the example of the group S_3 in the setting in [BRS99] for the setting described below. In modules over commutative rings these ideas involve usually Gröbner basis calculations. These ideas have been developed in [Ell04] to compute free resolutions of groups.

144 p. 343 The methods of Section 10.3.ii were published in [BRS99]. They have been developed by Ellis in [Ell04] and in subsequent GAP programs, [Ell08]. He works by constructing a universal covering CW-complex rather than the corresponding crossed complex.

145 p. 343 The earlier results of Peiffer, Reidemeister and Whitehead, [Pei49], [Rei49], [Whi49b] on the relations between identities among relations and second homotopy groups of 2-complexes were given an exposition in [BH82], written in memory of Peter Stefan who died in a climbing accident in 1979. The notion of calculating using pictures explained there was developed by a number of authors, see for example [CCH81], and the survey in [HAMS93]. The paper [BRS99] gave the calculation method explained here, which has the advantage over methods in [HAMS93] of being able to calculate higher syzygies (identities among identities, and so on). The ideas have been developed in [HW03] and implemented in GAP4, [WA97].

146 p. 344 This use of homotopies was inspired by work on the Homological Perturbation Lemma, for example [BL91], where the construction of homotopies is crucial. Such use also agrees with the general groupoid philosophy, in which an arrow $g\colon a \to b$ in a groupoid may be thought of as a 'proof that a and b are equivalent'. The $(n+1)$-st stage of the resolution is constructed from algebraic information on the contraction of the resolution at lower levels. Such a process is more difficult to carry out geometrically: for example the method of pictures used in [HAMS93] uses deformations of pictures to obtain generators of second homotopy groups, but it is not so obvious how to record the information on these deformations for use at the next stage of syzygies.

147 p. 344 The free groupoid on a graph was introduced in [Hig71], and a recent use is in [CP01].

148 p. 346 See works on 'rewriting', for example [BO93], [EW07], and references in, say, Wikipedia.

149 p. 347 It is shown in [HW03] how a 'logged rewriting procedure' will give such a choice if the monoid rewrite system determined by R may be completed, and that this allows for an implementation of the determination of h_1.

150 p. 349 The other method well known in the literature for constructing identities among relations is that of 'pictures', see for example [BH82], [HAMS93]. However these methods have difficulties with finding the next level of syzygies, since these should be constructed from deformations of pictures. Such deformations are described by 3-dimensional information, which is more difficult to record.

151 p. 351 An implementation of Gröbner basis procedures for finding minimal subsets which still generate is described in [HR99]. The above methods are used in [BRS99] to construct levels 1, 2 and 3 of a free crossed resolution for the symmetric group S_3. It would take us too long to give more details. These methods have been developed by Ellis in [Ell04], [Ell08], phrased in terms of constructing the universal covering CW-complex, but successful in obtaining results.

152 p. 353 The close relation between free crossed resolutions and CW-complexes shown in the following results is illustrated by the successful translation of these methods to the construction of cell complexes in [Ell04] and subsequent implementations of homological calculations in [Ell08].

153 p. 353 This result goes back to Whitehead [Whi50b], Theorem 17, and Wall [Wal65], and is discussed further by Baues in [Bau89], Chapter VI, §7. Baues points out that Wall states his result in terms of chain complexes, but that the crossed complex formulation seems more natural, and avoids questions of realisability in dimension 2, which are unsolved for chain complexes. Also Baues' results are more general since they deal with complexes under a given complex D.

154 p. 353 The theory of acyclic models was founded in the classic paper [EML53a], and was in the adjacent paper used to prove what is now called the Eilenberg–Zilber Theorem in [EZ53], determining the chain complex of a product of simplicial sets as a tensor product of chain complexes (compare Section 10.4.v). Many other applications were subsequently developed, for example [GM57]. The work of Barr which is given an exposition in [Bar02] gives an advanced view of the traditional chain complex theory, using double chain complexes, and it would be interesting to know if analogues can be usefully developed for crossed complexes. Our method is analogous to the exposition in [Dol95]. It is hoped that this version of the Acyclic Model Theorem will find new applications. Another route to new work could be relating this theorem to the monoidal structure, see [GSNPR08].

155 p. 354 We refer the reader to texts or web information for further information on the term 'projective', particularly projective module.

156 p. 357 Presimplicial sets were called Δ-sets in [RS71].

157 p. 357 The crossed complex simplicial normalisation theorem given here was first proved directly in [BS07], following the chain complex model in [ML63].

158 p. 360 This relation between simplicial and cubical theory follows the method given in [EML53a], but obtains a more powerful result.

159 p. 360 This is a theorem that in its chain complex version was first proved by the method of acyclic models, [EZ53]. See also [Eps66]. The corresponding theorem for crossed complexes was asserted by an acyclic model proof in [BH91], and proved directly and in detail in [Ton94], [Ton03]. The theorem and the subtle extra rules proved by Tonks were used in [BGPT97], [BGPT01]. Tonks' work was also inspired by showing him a 10-page letter from Michael Barratt to Brown written in 1959, reworking in detail the Eilenberg–Mac Lane proof in [EML53b]; the letter ended: 'Dawn breaks; I hope nothing else does!'.

160 p. 361 For the concept of anodyne extension in the simplicial context, see for example [GZ67], [GJ99].

161 p. 361 For more on shuffles, see Section 4 in [Gug57]. The extra relations between a, b, ϕ were first written down in [Shi62], and proved in the crossed complex case in [Ton03]. These properties of this deformation retraction in the chain complex case have been used in what is now called homological perturbation theory, see for example [Bro65b], [LS87], [JL01], [Hue89], but a corresponding crossed complex version of that theory has not been developed. Indeed, in the traditional chain theory a considerable importance is attached to twisting cochains, first introduced in [Bro59]. In later work, a key role is also played by the homotopy $\phi\colon ba \simeq 1$ and the 'side conditions' this satisfies, which enable specific formulae for twisted differentials. These conditions are valid in the crossed complex situation, as shown by Tonks, but it has not been seen how to make the perturbation theory work in the crossed complex situation. These conditions on the maps of the Eilenberg–Zilber Theorem are also used in two papers on equivariant crossed complexes, [BGPT97], [BGPT01]. Another application of the EZT Theorem is in [MT07].

162 p. 362 The work for this section is a modified version of [Sch76]. The same retraction argument on a subdivision of the n-cube is used in our proof of Proposition 14.2.8. For the standard proof of this excision result involving singular simplicial chain complexes see for example Theorem 9.4.5 in [tD08].

Chapter 11

The cubical classifying space of a crossed complex

Introduction

The homotopy classification of maps between topological spaces is among the most difficult areas in homotopy theory, and so this chapter is one of the most important in this book. We define for a crossed complex C, and in a functorial way, a topological space BC, called the *classifying space* of C. The main property is the following Homotopy Classification Theorem which generalises classical theorems of Eilenberg–Mac Lane:

$$[X, BC] \cong [\Pi X_*, C]$$

for a CW-complex X with skeletal filtration X_*, and where ΠX_* is the fundamental crossed complex of the filtered space X_*. Here the left-hand side gives continuous homotopy classes of maps of spaces and the right-hand side gives algebraic homotopy classes of morphisms of crossed complexes. The proof uses a considerable part of the technology of crossed complexes developed in the rest of this book, and the result is a special case of a description of the weak homotopy type of the space of maps $X \to BC$.

Because the crossed complex ΠX_* is free, the right-hand side of the above equation can be quite explicit, particularly if X_* is a finite cell complex. A morphism $\Pi X_* \to C$ is determined by a list of elements of C in various dimensions, subject to boundary conditions. The homotopy classification of these is then an explicit equivalence relation. Of course, because of the nonabelian nature of some of the information in a crossed complex, there are computability questions, and there are also questions of how to analyse this information. We shall find the notion of *fibration of crossed complexes*, and some associated exact sequences, useful in this respect in the later Section 12.1.ii.

Because of the central nature of cubical methods for some of our major results on crossed complexes, it is convenient to define the classifying space BC cubically.[163] So the first sections of this chapter are devoted to an account of cubical sets and related results. Cubical sets have an advantage over simplicial sets in the easier account of homotopies and higher homotopies.

The main applications of the classifying space of a crossed complex follow as for the simplicial version in [BH91], and are given in Chapter 12.

11.1 The cubical site

This section contains an introductory account of the category Cub of cubical sets and its relationship with the category Top of topological spaces.[164]

11.1.i The box category

The usual definition of cubical set is as a functor from a small category which we call the *site* for cubical sets. We begin by defining this category.

Definition 11.1.1. The *box category* $\square$ is the subcategory of Top having as objects the standard n-cubes $I^n = [0,1]^n$ for $n \geqslant 0$ and the morphisms $\square(I^n, I^m)$ are the maps that can be got by composition of the face inclusions and of projections

$$\delta_i^\alpha : I^n \to I^{n+1}, \quad \sigma_i : I^{n+1} \to I^n$$

defined respectively by

$$\delta_i^\alpha(x_1, \dots, x_{i-1}, x_i, \dots, x_n) = (x_1, \dots, x_{i-1}, l(\alpha), x_i, \dots, x_n)$$

for $i = 1, 2, \dots, n$, $\alpha = +, -$ where $l(+) = 1$, $l(-) = 0$; and

$$\sigma_i(x_1, \dots, x_{i-1}, x_i, \dots, x_{n+1}) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1})$$

for $i = 1, 2, \dots, n+1$. $\square$

Proposition 11.1.2. *The morphisms in the category $\square$ are given by all possible compositions of inclusions of faces and of projections subject to the relations*

$$\delta_j^\beta \delta_i^\alpha = \delta_i^\alpha \delta_{j-1}^\beta \qquad (i < j), \tag{A.1.i}$$

$$\sigma_j \sigma_i = \sigma_i \sigma_{j+1} \qquad (i \leqslant j), \tag{A.1.ii}$$

$$\sigma_j \delta_i^\alpha = \begin{cases} \delta_i^\alpha \sigma_{j-1} & (i < j), \\ \delta_{i-1}^\alpha \sigma_j & (i > j), \\ \mathrm{id} & (i = j). \end{cases} \tag{A.1.iii}$$

Remark 11.1.3. Using these relations we can easily check that any morphism in $\square$ has a unique expression as $\delta_{i_1}^{\alpha_1} \dots \delta_{i_k}^{\alpha_k} \sigma_{j_1} \dots \sigma_{j_l}$ with $i_1 \leqslant \dots \leqslant i_k$ and $j_1 < \dots < j_l$. $\square$

11.1.ii The category of cubical sets

Now cubical sets are just covariant functors from the $\square$ category to the category of sets.

Definition 11.1.4. The category Cub of *cubical sets* is defined to be the functor category $\mathsf{Cat}(\square^{\mathrm{op}}, \mathsf{Set})$. Thus a cubical set is a functor

$$K \colon \square^{\mathrm{op}} \to \mathsf{Set},$$

and a map of cubical sets is a natural transformation of functors. □

Remark 11.1.5. A cubical set K is defined by the family of sets $\{K_n = K(I^n)\}_{n \geqslant 0}$, the face maps $\partial_i^\alpha = K(\delta_n^\alpha) \colon K_n \to K_{n-1}$ $(i = 1, 2, \dots, n,\ \alpha = +, -)$ and the degeneracy maps $\varepsilon_i = K(\sigma_i) \colon K_{n-1} \to K_n$ $(i = 1, 2, \dots, n)$ satisfying the usual cubical relations:

$$\partial_i^\alpha \partial_j^\beta = \partial_{j-1}^\beta \partial_i^\alpha \qquad (i < j), \tag{B.1.i}$$

$$\varepsilon_i \varepsilon_j = \varepsilon_{j+1} \varepsilon_i \qquad (i \leqslant j), \tag{B.1.ii}$$

$$\partial_i^\alpha \varepsilon_j = \begin{cases} \varepsilon_{j-1} \partial_i^\alpha & (i < j), \\ \varepsilon_j \partial_{i-1}^\alpha & (i > j), \\ \mathrm{id} & (i = j). \end{cases} \tag{B.1.iii}$$

We will need in Chapter 13 the following definition.

Definition 11.1.6. A *cubical face operator* d is simply a product of various $\partial_j^\pm$s. This product may be empty, so that we allow $d = 1$. We say d *does not involve* ∂_{n+1}^τ if d cannot be written as $d' \partial_{n+1}^\tau$. □

A very important example of cubical set is the 'free cubical set on one generator in dimension n' which we denote $\mathbb{I}^n$:

Definition 11.1.7. For $n \geqslant 0$ we define $\mathbb{I}^n$ as the cubical set whose m-cells are $\square(I^n, I^m)$ for all $m \geqslant 0$ and whose morphisms are defined by composition. □

Proposition 11.1.8. *Any cubical morphism* $\hat{x} \in \mathsf{Cub}(\mathbb{I}^n, K)$ *corresponds to an element* $x = \hat{x}(1_{I^n})$ *of* K_n*, giving a natural bijection* $\mathsf{Cub}(\mathbb{I}^n, K) \to K_n$. □

Remark 11.1.9. Thus there is an embedding $\square \to \mathsf{Cub}$ which sends $I^n \mapsto \mathbb{I}^n$. This is an example of the *Yoneda embedding* $\Upsilon \colon C \to \mathsf{Cat}(C^{\mathrm{op}}, \mathsf{Set})$ for any small category C. One of the properties of this embedding is that any object of $\mathsf{Cat}(C^{\mathrm{op}}, \mathsf{Set})$ is a colimit of images under Υ of the objects of the category C. □

Another important example of cubical set is the singular cubical set of a topological space:

Definition 11.1.10. For any topological space X, its *singular cubical set* $S^\square X$ is given by all singular cubes, i.e.

$$(S^\square X)_n = \{\sigma \colon I^n \to X \mid \sigma \text{ a continuous map}\}$$

with faces and degeneracies given by composition with the maps $\delta_i^\alpha \colon I^{n-1} \to I^n$ and $\sigma_i \colon I^{n+1} \to I^n$ defined above. This gives a functor

$$S^\square \colon \mathsf{Top} \to \mathsf{Cub}. \qquad \square$$

For brevity, we will in Part III write KX for $S^\square X$.

This definition is a preliminary to the construction of cubical singular homology of a space which we outline in Section 14.7.[165]

11.1.iii Geometric realisation of a cubical set

There is a left adjoint to this singular cubical set functor:

Definition 11.1.11. For any cubical set $K \colon \square^{\mathrm{op}} \to C$, its *geometric realisation* $|K|$ is the quotient space

$$|K| = \frac{\bigsqcup_n K_n \times I^n}{\equiv}$$

where K_n is given the discrete topology, I^n its standard topology, and the equivalence relation is generated by $(\partial_i^\alpha x, u) \equiv (x, \delta_i^\alpha u)$ and $(\varepsilon_i y, u) \equiv (y, \sigma_i u)$ where $x \in K_{n+1}$, $y \in K_{n-1}$ and $u \in I^n$. $\square$

This definition comes under the general scheme of a *coend* (see Appendix A, Section A.9. The formal properties of coends and ends are useful for deriving the properties we need for the geometric realisation.

Remark 11.1.12. The realisation of a cubical set $|K|$ can also be interpreted as a coend:

$$|K| = \int^{\square, n} K_n \times I^n. \qquad \square$$

Proposition 11.1.13. *The realisation of a cubical set is a CW-complex having one n-cell for each nondegenerate n-cell.*

Remark 11.1.14. Thus each point of the realisation of a cubical set $|K|$ is an equivalence class $|x, u|$ with $x \in K_n$ and $u \in I^n$ and it has a unique representative $|x, u|$ with x a nondegenerate cube. $\square$

Using this representation it is not difficult to prove that the realisation functor $|\ \ |$ is left adjoint to the singular cubical functor $S^\square$.

Theorem 11.1.15. *The realisation functor $|\ \ |$ is left adjoint to the singular cubical functor $S^\square$. That is for each cubical set K and topological space X, there is a natural bijection*

$$\Psi \colon \mathsf{Top}(|K|, X) \to \mathsf{Cub}(K, S^\square X).$$

Proof. For any continuous map $g \colon |K| \to X$, the cubical map

$$\Psi(g) \colon K \to S^{\square}X$$

is given in dimension n by

$$\Psi_n(g)(x)(u) = g(|x, u|),$$

for any n-cube $x \in K_n$ and point $u \in I^n$. The maps Ψ defines a natural transformation, whose inverse is given by sending a cubical map f to the continuous map defined by mapping any class $|x_n, u|$ to $f_n(x_n)(u)$. $\square$

Our aim is to define homotopy theory for cubical sets and to relate this to the usual homotopy theory for topological spaces. This is essential for our main result on homotopy classification.

We first recall the following definition, which is put in a broader context in Section B.8.

Definition 11.1.16. A map $f \colon X \to Y$ of spaces is called a *weak homotopy equivalence* if it induces a bijection $\pi_0 X \to \pi_0 Y$ and an isomorphism $\pi_n(X, x) \to \pi_n(Y, fx)$ for all $x \in X$ and all $n \geqslant 1$. $\square$

The following result is part of the relation between the homotopy theories of cubical sets and topological spaces sketched in Section 11.3.iv.

Proposition 11.1.17. *For any topological space X, the counit of the adjunction between the realisation and singular functors,*

$$\varepsilon_X \colon |S^{\square}(X)| \to X,$$

is a weak homotopy equivalence.

Remark 11.1.18. A famous theorem of Whitehead, compare Theorem B.8.1, states that a weak equivalence of CW-complexes is a homotopy equivalence. So the previous proposition shows that any CW-complex is naturally of the homotopy type of the geometrical realisation of a cubical set. $\square$

11.2 Monoidal closed structure on the category of cubical sets

A *monoidal closed structure* on the category of cubical sets gives for cubical sets K, L, M natural constructions of a *tensor product* $K \otimes L$ and an *internal hom* or *morphism object* $\mathsf{CUB}(L, M)$ which satisfy an *exponential law* in the form of a natural isomorphism

$$\mathsf{Cub}(K \otimes L, M) \cong \mathsf{Cub}(K, \mathsf{CUB}(L, M)). \tag{11.2.1}$$

Since there is also a 'unit interval object' $\mathbb{I}$ as a cubical set, this enables homotopies between cubical sets L and M to be studied, as in any monoidal closed category with a unit interval object, using either maps from the product $\mathbb{I} \otimes L$ to M or maps from $\mathbb{I}$ to the internal morphism object $\mathsf{CUB}(L, M)$ from L to M. However a special condition on the cubical set M (the 'fibrant' or 'Kan extension', condition) is needed to ensure homotopy between maps $L \to M$ is an equivalence relation.

The monoidal closed structure on cubical sets is very useful for elaborating their homotopy properties, as we see in Section 11.3.ii.

These results on the monoidal closed structure for cubical sets will also be used in Chapter 15, applied to cubical sets with connection, and to cubical ω-groupoids, whose clear monoidal closed structure is used to construct the monoidal closed structure for crossed complexes.

11.2.i Tensor product of cubical sets

We first give the tensor product, which gives the monoidal structure and is an intermediate step in the construction of the internal morphisms functor. The tensor product is defined by a universal property with respect to bicubical maps, analogous to the the property the usual tensor product of modules has with respect to bilinear maps.

The tensor product is associative (Proposition 11.2.6), but not symmetric; the failure of symmetry can be controlled by a 'transposition' functor which will be given in Proposition 11.2.20 and Remark 11.2.22.

An n-cube in the tensor product $K \otimes L$ will be the 'product' of a p-cube $x \in K$ and a q-cube $y \in L$ for $p + q = n$. We just take care that the last degeneracy in the first factor agrees with the first in the second factor, the reason becomes clear in the geometric example.

Definition 11.2.1. If K, L are cubical sets, their *tensor product* $K \otimes L$ is defined by

$$(K \otimes L)_n = \frac{\left(\bigsqcup_{p+q=n} K_p \times L_q\right)}{\sim}$$

where $\sim$ is the equivalence relation generated by $(\varepsilon_{r+1} x, y) \sim (x, \varepsilon_1 y)$ for $x \in K_r$, $y \in L_s$ $(r + s = n - 1)$. We write $x \otimes y$ for the equivalence class of (x, y). The maps $\partial_i^\alpha, \varepsilon_i$ are defined for $x \in K_p$, $y \in L_q$ by

$$\partial_i^\alpha (x \otimes y) = \begin{cases} (\partial_i^\alpha x) \otimes y & \text{if } 1 \leqslant i \leqslant p, \\ x \otimes (\partial_{i-p}^\alpha y) & \text{if } p + 1 \leqslant i \leqslant p + q, \end{cases}$$

$$\varepsilon_i (x \otimes y) = \begin{cases} (\varepsilon_i x) \otimes y & \text{if } 1 \leqslant i \leqslant p + 1, \\ x \otimes (\varepsilon_{i-p} y) & \text{if } p + 1 \leqslant i \leqslant p + q + 1 \end{cases}$$

and make $K \otimes L$ a cubical set. □

Remark 11.2.2. We note that in $K \otimes L$, we have

$$(\varepsilon_{p+1}x) \otimes y = x \otimes (\varepsilon_1 y)$$

when $x \in K_p$. The intuitive reason for this is that the tensor product is analogous to adjoining lists, or words, while the degeneracy operation is analogous to inserting a 0. Thus for lists a, b, adjoining $a0$ on the left of b yields $a0b$ which is the same as adjoining a to the left of $0b$. □

The realisation functor has a strong and simple relation to the tensor product. This is one of the reasons for the utility of cubical methods in contrast to simplicial methods.

Proposition 11.2.3. *Let K, L be cubical sets. Then there is a cellular isomorphism*

$$\chi\colon |K| \otimes |L| \to |K \otimes L|.$$

Proof. The bracketing homeomorphism $I^n \cong I^r \times I^s$ whenever $r + s = n$ yields a homeomorphism

$$K_r \times L_s \times I^n \cong K_r \times I^r \times L_s \times I^s$$

whenever $r + s = n$. One now checks that the identifications to give the realisations are on both sides obtained from

$$\bigsqcup_{r+s=n} K_r \times L_s \times I^n$$

by the same identifications. □

To give Cub the structure of a monoidal closed category, we have to construct not only a tensor product, but also an internal hom functor $\mathsf{CUB}(L, M)$ for cubical sets L, M, and a natural equivalence of the form (11.2.1).

First, we are going to interpret the left part of this equivalence in terms of bicubical maps. This procedure resembles the use of bilinear maps as an intermediate step between the tensor product of R-modules and the R-module of homomorphisms.

Definition 11.2.4. A family of maps

$$f_{pq}\colon K_p \times L_q \to M_{p+q}$$

is called a *bicubical map* $f\colon (K, L) \to M$ if it satisfies for all p, q and $\alpha = \pm$:

$$\partial_i^\alpha f_{pq}(x, y) = \begin{cases} f_{p-1,q}(\partial_i^\alpha x, y) & \text{if } 1 \leqslant i \leqslant p \\ f_{p,q-1}(x, \partial_{i-p}^\alpha y) & \text{if } p+1 \leqslant i \leqslant p+q, \end{cases} \tag{i}$$

$$\varepsilon_i f_{pq}(x, y) = \begin{cases} f_{p+1,q}(\varepsilon_i x, y) & \text{if } 1 \leqslant i \leqslant p+1 \\ f_{p,q+1}(x, \varepsilon_{i-p} y) & \text{if } p+1 \leqslant i \leqslant p+q+1. \end{cases} \tag{ii}$$

(Notice that this last part gives further vindication of the rule $(\varepsilon_{p+1}x) \otimes y = x \otimes (\varepsilon_1 y)$.) □

We now check that the tensor product is the universal construction with respect to bicubical maps. This fact is used as an intermediate step in our route to the internal hom functor CUB.

Proposition 11.2.5. *The projections*

$$\chi_{pq} \colon K_p \times L_q \to (K \otimes L)_{p+q}$$

defined by $\chi_{pq}(x, y) = x \otimes y$ *form a bicubical map which is universal with respect to all bicubical maps from* (K, L).

Proof. Any cubical map $f \colon K \otimes L \to M$ defines a family of functions $\hat{f}_{pq} \colon K_p \times L_q \to M_{p+q}$ (given by $\hat{f}_{pq}(x, y) = f_{p+q}(x \otimes y)$) that clearly form a bicubical map.

Conversely, given a bicubical map $f \colon (K, L) \to M$, there is a unique cubical map $\hat{f} \colon K \otimes L \to M$ defined by $\hat{f}_{p+q}(x \otimes y) = f_{pq}(x, y)$. The uniqueness is clear. The map f is well defined because the defining equations (ii) for a bicubical map imply that, for $x \in K_p$ and $y \in L_q$

$$f_{p+1,q}(\varepsilon_{p+1} x, y) = \varepsilon_{p+1} f_{pq}(x, y) = f_{p,q+1}(x, \varepsilon_1 y).$$

It is an easy exercise to prove that the resulting map $K \otimes L \to M$ is cubical. □

Proposition 11.2.6. *For cubical sets* K, L, M *there is a natural isomorphism*

$$(K \otimes L) \otimes M \cong K \otimes (L \otimes M).$$

Proof. Both sides of the above equation may be defined as universal with respect to *tricubical maps* from (K, L, M). We leave details to the reader. □

11.2.ii Homotopies of cubical maps

Let us move on to the construction of CUB, the internal hom for cubical sets. Recall from Proposition 11.1.8 that for any cubical set K we have $K_n \cong \mathsf{Cub}(\mathbb{I}^n, K)$ where $\mathbb{I}^n$ is the cubical set freely generated by one element c_n in dimension n.

Thus the internal morphism construction $\mathsf{CUB}(K, L)$ has to be a cubical set satisfying

$$\mathsf{CUB}_n(K, L) \cong \mathsf{Cub}(\mathbb{I}^n, \mathsf{CUB}(K, L)) \cong \mathsf{Cub}(\mathbb{I}^n \otimes K, L)$$

i.e. the n-dimensional elements of $\mathsf{CUB}(K, L)$ are 'n-fold left homotopies'.

Using Proposition 11.2.5 any element $h \in \mathsf{CUB}_n(K, L)$ may be considered also as a bicubical map

$$\hat{h} \colon (\mathbb{I}^n, K) \to L.$$

Let us begin with the case $n = 1$: then $\mathbb{I}^1 = \mathbb{I}$ is the cubical set generated by c_1 in dimension 1. We denote its vertices by $0 = \partial^- c_1$ and $1 = \partial^+ c_1$. The cubical set

$\mathbb{I}$ plays a role analogous to that of the unit interval in the usual homotopy theory. It is clear that a homotopy

$$h\colon \mathbb{I} \otimes K \to L$$

would be given by the images of all $h(c_1, x) \in L_{n+1}$ for all $x \in K_n$. Essentially it should be a 'degree one' cubical morphism that forgets about the $\partial_1^{\pm}$ (which are used to give the images of 0 and 1). Let us make this precise.

Definition 11.2.7. For any cubical set K we define the *left path complex* PK to be the cubical set with

$$(PK)_r = K_{r+1}$$

and cubical operations

$$\partial_2^\alpha, \partial_3^\alpha, \dots, \partial_{r+1}^\alpha \colon (PK)_r \to (PK)_{r+1},$$

and

$$\varepsilon_2, \varepsilon_3, \dots, \varepsilon_{r+1} \colon (PK)_{r-1} \to (PK)_r$$

(that is, we ignore the first operations ∂_1^-, ∂_1^+, ε_1 in each dimension r).

This construction gives a functor

$$P : \mathsf{Cub} \to \mathsf{Cub}. \qquad \Box$$

Proposition 11.2.8. *The functor P is right adjoint to $\mathbb{I} \otimes -$, i.e. there is a natural one-one correspondence between*

1. *cubical maps $\tilde{f} : K \to PL$, and*
2. *cubical maps $f : \mathbb{I} \otimes K \to L$.*

Proof. The proposition follows because both are clearly equivalent to bicubical maps $(\mathbb{I}, K) \to L$. $\Box$

Remark 11.2.9. Here corresponding maps f, $\tilde{f}$ are related by $\tilde{f}(x) = f(c_1 \otimes x)$ and either of them is termed a *left homotopy* from f_0 to f_1, where $f_\alpha \colon K \to L$ is given by

$$f_\alpha x = f(\alpha \otimes x) = \partial_1^\alpha \tilde{f} x \quad (\alpha = 0, 1). \qquad \Box$$

The iteration of the left path complex gives a cubical set that is right adjoint to the tensor product with respect to $\mathbb{I}^n$, and thus classifies n-fold left homotopies.

Definition 11.2.10. We define the *n-fold left path complex $P^n M$* inductively by $P^n M = P(P^{n-1} M)$, so that

$$(P^n M)_r = M_{n+r}$$

with cubical operations

$$\partial_{n+1}^\alpha, \partial_{n+2}^\alpha, \dots, \partial_{n+r}^\alpha \colon (P^n M)_r \to (P^n M)_{r-1},$$
$$\varepsilon_{n+1}, \varepsilon_{n+2}, \dots, \varepsilon_{n+r} \colon (P^n M)_{r-1} \to (P^n M)_r$$

(that is, we ignore the first n operations ∂_i^α, ε_i for $i = 1, \dots, n$ in each dimension.) $\Box$

As before, this functor is a special case of the right adjoint to the tensor product.

Proposition 11.2.11. *The functor P^n is right adjoint to $\mathbb{I}^n \otimes -$, i.e. there is a natural one-one correspondence between*

1. *cubical maps $\tilde{f} : L \to P^n M$, and*
2. *cubical maps $f : \mathbb{I}^n \otimes L \to M$.*

Proof. As before, we can check that these two kinds of maps are equivalent to bicubical maps $(\mathbb{I}^n, L) \to M$. □

Remark 11.2.12. Here corresponding maps f, $\tilde{f}$ are related by $\tilde{f}(x) = f(c_n \otimes x)$ and either of them is termed a *n-fold left homotopy*. □

That gives the following relation between free cubical sets.

Corollary 11.2.13. *There are natural* (*and coherent*) *isomorphisms of cubical sets*

$$\mathbb{I}^m \otimes \mathbb{I}^n \cong \mathbb{I}^{m+n}.$$

Proof. This follows from Proposition 11.2.11 since $P^m \circ P^n = P^{m+n}$. □

11.2.iii The internal hom functor on cubical sets

Using homotopies, we have constructed the sets $\mathsf{CUB}_n(L, M)$ for cubical sets L and M and any $n \geqslant 0$. To define the cubical set $\mathsf{CUB}(L, M)$ it remains to define faces and degeneracies.

Notice that the omitted operations

$$\partial_1^\alpha, \dots, \partial_n^\alpha \quad \text{and} \quad \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$$

in each dimension induce morphisms of cubical sets

$$\partial_1^\alpha, \dots, \partial_n^\alpha : P^n M \longrightarrow P^{n-1} M, \quad \text{and} \quad \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n : P^{n-1} M \longrightarrow P^n M.$$

These morphisms satisfy the cubical laws.

Definition 11.2.14. We now define the *cubical internal hom*

$$\mathsf{CUB}_n(L, M) = \mathsf{Cub}(L, P^n M)$$

and observe that the family $\mathsf{CUB}(L, M)$ of sets $\mathsf{CUB}_n(L, M)$ for $n \geqslant 0$ gets a cubical structure. Its cubical operations

$$\partial_1^\alpha, \partial_2^\alpha, \dots, \partial_n^\alpha : \mathsf{CUB}_n(L, M) \to \mathsf{CUB}_{n-1}(L, M),$$

$$\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n : \mathsf{CUB}_{n-1}(L, M) \to \mathsf{CUB}_n(L, M)$$

are induced by those of M. □

Remark 11.2.15. Thus a typical $f \in \mathsf{CUB}_n(L, M)$ is a family of maps $f_r\colon L_r \to M_{n+r}$ satisfying

$$f_{r-1}\partial_i^\alpha = \partial_{n+i}^\alpha f_r, \quad f_r\varepsilon_j = \varepsilon_{n+j} f_{r-1} \quad (i, j = 1, 2, \dots, r)$$

and its faces and degeneracies are given by

$$(\partial_i^\alpha f)_r = \partial_i^\alpha f_r \quad (\varepsilon_j^\alpha f)_r = \varepsilon_j^\alpha f_r \quad (i, j = 1, 2, \dots, n,\ \alpha = 0, 1).$$

In geometric terms, the elements of $\mathsf{CUB}_0(L, M)$ are the cubical maps $L \to M$, the elements of $\mathsf{CUB}_1(L, M)$ are the (left) homotopies between such maps, the elements of $\mathsf{CUB}_2(L, M)$ are homotopies between homotopies, etc. □

Proposition 11.2.16. *The functor* $\mathsf{CUB}(L, -)\colon \mathsf{Cub} \to \mathsf{Cub}$ *is right adjoint to* $- \otimes L$. *Moreover, the bijections*

$$\mathsf{Cub}(K \otimes L, M) \cong \mathsf{Cub}(K, \mathsf{CUB}(L, M))$$

giving the adjointness are natural with respect to K, L, M.

Proof. As before the bijections can be obtained via bicubical maps $(K, L) \to M$. □

As a special case:

Corollary 11.2.17. *The functor* $- \otimes \mathbb{I}^n$ *is left adjoint to* $\mathsf{CUB}(\mathbb{I}^n, -)\colon \mathsf{Cub} \to \mathsf{Cub}$.

Corollary 11.2.18. *For cubical sets* K, L, M *there is a natural isomorphism of cubical sets*

$$\mathsf{CUB}(K \otimes L, M) \cong \mathsf{CUB}(K, \mathsf{CUB}(L, M)).$$

Proof. It is easy to use associativity of the tensor product and the exponential law repeatedly to give for any cubical set E a natural bijection

$$\mathsf{Cub}(E, \mathsf{CUB}(K \otimes L, M)) \cong \mathsf{Cub}(E, \mathsf{CUB}(K, \mathsf{CUB}(L, M)).$$

The result follows. □

The tensor product is not symmetric because $(x, y) \mapsto y \otimes x$ is not a bicubical map. We have also seen that the functors $- \otimes \mathbb{I}^n$ and $\mathbb{I}^n \otimes -$ have different right adjoints. Nevertheless, we can get some symmetry via a 'transposition' functor.

Definition 11.2.19. We define a 'transposition' functor

$$T\colon \mathsf{Cub} \to \mathsf{Cub},$$

where TK has the same elements as K in each dimension but has its face and degeneracy operations numbered in reverse order, that is, the cubical operations

$$d_i^\alpha\colon (TK)_n \to (TK)_{n-1} \quad \text{and} \quad e_i\colon (TK)_{n-1} \to (TK)_n$$

are defined by $d_i^\alpha = \partial_{n+1-i}^\alpha$, $e_i = \varepsilon_{n+1-i}$. □

There are some immediate consequences:

Proposition 11.2.20. *The functor T satisfies*

1. *T is an involution, i.e. T^2K is naturally isomorphic to K;*
2. *$T(K \otimes L)$ is naturally isomorphic to $T(L) \otimes T(K)$; and*
3. *there is an obvious cubical isomorphism $T\mathbb{I}^n \cong \mathbb{I}^n$.* □

Instead of an isomorphism of $\mathsf{CUB}(\mathbb{I}^n, L)$ with P^nL, we have:

Corollary 11.2.21. *There is a natural isomorphism of cubical sets*

$$\mathsf{CUB}(\mathbb{I}^n, L) \cong TP^nTL.$$

Proof. By Corollary 11.2.11, P^n is right adjoint to $\mathbb{I}^n \otimes -$, so P^nT is right adjoint to $T(\mathbb{I}^n \otimes -)$. Hence TP^nT is right adjoint to $T(\mathbb{I}^n \otimes T-) \cong (- \otimes T\mathbb{I}^n)$ that is naturally isomorphic to $(- \otimes T\mathbb{I}^n) \cong (- \otimes \mathbb{I}^n)$. Hence TP^nT is naturally isomorphic to the right adjoint $\mathsf{Cub}(\mathbb{I}^n, -)$ of $- \otimes \mathbb{I}^n$. □

Remark 11.2.22. A simpler argument shows that for any cubical complex K the functor $K \otimes -\colon \mathsf{Cub} \to \mathsf{Cub}$ has right adjoint $T(\mathsf{CUB}(TK, T-))$ and hence that the monoidal closed category Cub is *biclosed*, in the sense of Kelly [Kel82], even though it is not symmetric. □

Corollary 11.2.23. *For cubical sets K, L the functors $\mathsf{Cub} \to \mathsf{Cub}$ given by $K \otimes -$ and $- \otimes L$ preserve colimits.* □

11.3 Homotopy theory of cubical sets

In this section we sketch how to develop a homotopy theory of cubical sets directly from the cubical structure.

11.3.i Fibrant cubical sets

In order to have a useful homotopy theory for a cubical set, and in particular to define its homotopy groups in a direct way, we need an extra condition known as the Kan extension, or fibrant, condition, see Definition 11.3.8. In arguing with this condition it is often easier to work with geometric models.[166] These are easier to see as real cubes made from the geometric I^n, where $I = [0, 1]$ is the unit interval, and subcomplexes of I^n, but the same arguments can be given for the models $\mathbb{I}^n$ of these complexes in the category Cub, which we call 'formal cubes' and their subcomplexes 'formal subcomplexes'. By a 'cell' in $\mathbb{I}^n$ we mean a nondegenerate element. We have to be careful in this section because we are thinking in terms of geometric cubes and their union, but for cubical sets we have elements of various dimensions. Thus in $C \cup a$ as given below where C is a subcomplex and a is a cell, the $\cup$ means union in the sense of subcomplexes generated by C, a. We will use the results of this section in Section 14.1 and later. You should also relate these methods to various retractions used in Section 6.3.

Definition 11.3.1. Let B, C be subcomplexes of $\mathbb{I}^n$ such that $C \subseteq B$. We say that C is an *elementary collapse* of B, written $B \searrow^e C$, if for some $s \geqslant 1$ there is an s-cell a of B and $(s-1)$-face b of a such that

$$B = C \cup a, \quad C \cap a = \partial a \setminus b$$

(where $\partial a \setminus b$ denotes the set of the proper faces of a except b). The face b is called the *free face* of the collapse.

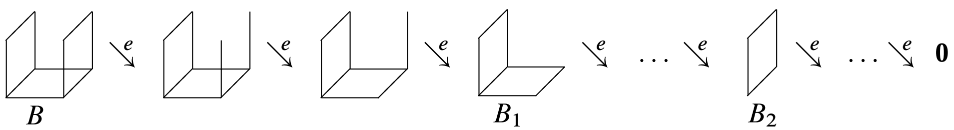
If there is a sequence

$$B_1 \searrow^e B_2 \searrow^e \dots \searrow^e B_r$$

of elementary collapses, then we write $B_1 \searrow B_r$ and say B_1 *collapses* to B_r. □

Example 11.3.2. If a is a cell then $a \otimes \mathbb{I}$ collapses to $a \otimes \{0\} \cup \partial a \otimes \mathbb{I}$. Here the free face of the collapse is $a \otimes \{1\}$.

We will next define the notion of a 'partial box'. The following picture gives three examples B, B_1, B_2 of these, as stages in a choice of a sequence of collapsings $B \searrow \mathbf{0}$ through B_1 and B_2.



□

The formal definition of 'partial box' allows us to give a more widely applicable formulation of the usual fibrant, or Kan, extension condition on a cubical set.[167]

Definition 11.3.3. Let C be an r-cell in the n-cube $\mathbb{I}^n$. Two $(r-1)$-faces of C are called *opposite* if they do not meet (except possibly in degenerate elements). A *partial $(r-1)$-box* in C is a subcomplex B of C generated by one $(r-1)$-face b of C (called a *base* of B) and a number, possibly zero, of other $(r-1)$-faces of C none of which is opposite to b. The partial box is a *box* if its $(r-1)$-cells consist of all but one of the $(r-1)$-faces of C. □

Proposition 11.3.4. *If B is a partial box in $\mathbb{I}^m$ then* (i) $B \otimes \mathbb{I}^n$, *and* (ii) $B \otimes \mathbb{I}^n \cup \mathbb{I}^m \otimes \partial \mathbb{I}^n$, *are partial boxes in* $\mathbb{I}^m \otimes \mathbb{I}^n \cong \mathbb{I}^{m+n}$.

Proof. Let b be a base for B. Then $b \otimes c^n$ is a base for $B \otimes \mathbb{I}^n$. This proves (i). Further, $\partial(\mathbb{I}^m \otimes \mathbb{I}^n) = (\partial\mathbb{I}^m) \otimes \mathbb{I}^n \cup \mathbb{I}^m \otimes \partial\mathbb{I}^n$, and so (ii) follows. □

We now come to a key theorem on the existence of chains of partial boxes; this applies to give many examples of collapsing, even as a kind of algorithm, and is also essential in the work of Chapter 14. This implies that the inclusion of partial boxes in the following theorem is an *anodyne extension* in the sense common in the literature.[168]

Theorem 11.3.5 (Chains of partial boxes). *Let B, B' be partial boxes in an r-cell C of $\mathbb{I}^n$ such that $B' \subseteq B$. Then there is a chain*

$$B = B_s \searrow B_{s-1} \searrow \cdots \searrow B_1 = B'$$

such that

(i) *each B_i is a partial box in C;*

(ii) *$B_{i+1} = B_i \cup a_i$ where a_i is an $(r-1)$-cell of C not in B_i;*

(iii) *$a_i \cap B_i$ is a partial box in a_i.*

Proof. We first show that there is a chain

$$B' = B_1 \subset \cdots \subset B_{s-1} \subset B = B_s$$

of partial boxes and a set of $(r-1)$-cells $a_1, a_2, \ldots, a_{s-1}$ such that $B_{i+1} = B_i \cup a_i$, $a_i \not\subseteq B_i$.

If B and B' have a common base this is clear, since we may adjoin to B' the $(r-1)$-cells of $B \setminus B'$ one at a time in any order. If B and B' have no common base, choose a base b for B and let b' be its opposite face in C. Then neither b nor b' is in B'. Hence $B_2 = B' \cup b$ is a partial box with base b and we are reduced to the first case.

Now consider the partial box $B_{i+1} = B_i \cup a_i$, $a \not\subseteq B_i$. We claim that $a_i \cap B_i$ is a partial box in a_i. To see this, choose a base b for B_{i+1} with $b \neq a_i$; this is possible because if a_i were the only base for B_{i+1}, then B_i would consist of a number of pairs of opposite faces of C and would not be a partial box. We now have $a_i \neq b$, $a_i \neq b'$, so $a_i \cap b$ is an $(r-2)$-face of a_i. Its opposite face in a_i is $a_i \cap b'$ and this is not in B_i because the only $(r-1)$-faces of C which contain it are a_i and b'. Hence $a_i \cap B_i$ is a partial box with base $a_i \cap b$.

The proof is now completed by induction on the dimension r of C. If $r = 1$, the theorem is trivial. If $r > 1$, choose B_i, a_i as above. Since $a_i \cap B_i$ is a partial box in a_i, there is a box J in a_i containing it. The elementary collapse $a_i \overset{e}{\searrow} J$ gives $B_{i+1} \overset{e}{\searrow} B_i \cup J$. But by the induction hypothesis, J can be collapsed to the partial box $a_i \cap B_i$ in a_i, and this implies $B_{i+1} \searrow B_i$. □

Corollary 11.3.6. *If C is a partial $(n-1)$-box in $\mathbb{I}^n$ then $\mathbb{I}^n$ collapses to C.*

Proof. We extend C to a box B. By definition, $\mathbb{I}^n$ collapses to B. By the previous theorem, B collapses to C. □

Corollary 11.3.7. *$\mathbb{I}^n$, and any box in $\mathbb{I}^n$, collapses to any of its vertices.*

Proof. It is sufficient to prove collapsing to the vertex $\mathbf{0}$. We know $\mathbb{I}^n$ collapses to the partial box $\{0\} \otimes \mathbb{I}^{n-1}$. Similarly, any partial box in $\mathbb{I}^n$ collapses to any of its faces. Now proceed by induction. □

Definition 11.3.8. Let K be a cubical set. We say K is *fibrant*, or satisfies the *Kan extension condition*, or *is Kan*, if for every $r \geqslant 1$ and any partial $(r-1)$-box in $\mathbb{I}^r$, any map $B \to K$ extends over I^r. □

Proposition 11.3.9. *A cubical set is fibrant if and only if for every $n \geqslant 1$ and any $(n-1)$-box in $\mathbb{I}^n$, any map $B \to K$ extends over $\mathbb{I}^n$.*

Proof. The implication one way is trivial. Suppose then the extension over boxes condition is fulfilled, and C is a partial $(n-1)$-box in $\mathbb{I}^n$. Then C is contained in a box B. By assumption and Theorem 11.3.5, $\mathbb{I}^n \searrow B \searrow C$. By repeated application of the fibrant condition, any map $C \to K$ extends over $\mathbb{I}^n$. □

Example 11.3.10. For any space X the singular cubical set $S^{\square}X$ is a fibrant cubical set. This is because there exists a retraction

$$I^n \to \{0\} \times I^{n-1} \cup I \times \partial I^n$$

(see Figure 1.3 p. 20 for the case $n = 2$) and indeed to any other box in a similar manner. □

11.3.ii Fibrations of cubical sets

The applications of the classifying space of a crossed complex require the notion of fibration of cubical set, so we give the theory here. This is also useful, as we shall see, in developing the homotopy theory of cubical sets.

Remark 11.3.11. Given a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow[g]{} & D \end{array}$$

in a category C, a morphism $\theta\colon C \to B$ such that $p\theta = g$ and $\theta i = f$ is called a *regular completion* of the diagram. □

Definition 11.3.12. Let $p\colon L \to M$ be a cubical map. We say p is a *fibration* if for all $n \geqslant 0$ and inclusion $i : B \to \mathbb{I}^n$ of a partial $(n-1)$-box B in $\mathbb{I}^n$, any diagram such as the following

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "i"'] & L \arrow[d, "p"] \\ \mathbb{I}^n \arrow[r, "g"'] \arrow[ur, dashed, "\theta"] & M \end{tikzcd} \tag{11.3.1}$$

has a regular completion θ. □

Example 11.3.13. A cubical set L is a fibrant cubical set if and only if the constant map $L \to *$, where $*$ is a point, is a fibration. □

Theorem 11.3.14. *Let $j : A \to K$ be an inclusion of a subcubical set and let $p\colon L \to M$ be a fibration of cubical sets. Let $m \geqslant 0$ and let B be an $(m-1)$-box in $\mathbb{I}^m$. Then any diagram in* Cub *of the form*

$$\begin{tikzcd} B \otimes K \cup \mathbb{I}^m \otimes A \arrow[r, "f"] \arrow[d, "i"'] & L \arrow[d, "p"] \\ \mathbb{I}^m \otimes K \arrow[r, "g"'] & M \end{tikzcd}$$

has a regular completion $\theta\colon \mathbb{I}^m \otimes K \to L$.

Proof. Let $P = \mathbb{I}^m \otimes K$, let c_m be the generator of $\mathbb{I}^m$ and let $P^{[n]}$ be the subcubical set of $\mathbb{I}^m \otimes K$ generated by $\mathbb{I}^m \otimes A$ and $c_m \otimes k$ for $k \in K_i$, $0 \leqslant i \leqslant n$. We construct $\theta_n : P^{[n]} \to L$ by induction on n. The case $n = 0$ is easy since $P^{[0]}$ is isomorphic to the disjoint union of $\mathbb{I}^m \otimes A$ and a disjoint union of copies of $\mathbb{I}^m$, one for each element of $K_0 \setminus A_0$.

Let $k \in K_{n+1} \setminus A$. Then $\hat{k}\colon \mathbb{I}^{n+1} \to K$ and we construct the square

$$\begin{tikzcd} B \otimes \mathbb{I}^{n+1} \cup \mathbb{I}^m \otimes \partial\mathbb{I}^{n+1} \arrow[r, "f_k"] \arrow[d, "\subseteq"'] & L \arrow[d, "p"] \\ \mathbb{I}^m \otimes \mathbb{I}^{n+1} \arrow[r, "g_k"'] & M; \end{tikzcd}$$

here $g_k = g \circ (1 \otimes \hat{k})$; $f_k | B \otimes \mathbb{I}^{n+1} = f \circ (1 \otimes \hat{k})$; and $f_k | \mathbb{I}^m \otimes \partial\mathbb{I}^{n+1}$ is determined by θ_n. Now $B \otimes \mathbb{I}^{n+1} \cup \mathbb{I}^m \otimes \partial\mathbb{I}^{n+1}$ is a partial box in $\mathbb{I}^m \otimes \mathbb{I}^{n+1}$ which has top cell $d = c_m \otimes c_{n+1}$. Since p is a fibration, this square has a regular completion $\theta'\colon \mathbb{I}^m \otimes \mathbb{I}^{n+1} \to L$. Then we can set $\theta(k) = \theta'(d)$. This completes the inductive step. □

Corollary 11.3.15. *If $p\colon L \to M$ is a fibration, then so also is*

$$\mathsf{CUB}(K,p)\colon \mathsf{CUB}(K,L) \to \mathsf{CUB}(K,M)$$

for any cubical set K. □

Proof. Let $p_* = \mathsf{CUB}(K,p)$. We have to prove that if B is a box in $\mathbb{I}^n$ then any diagram

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "i"'] & \mathsf{CUB}(K,L) \arrow[d, "p_*"] \\ \mathbb{I}^n \arrow[r, "g"'] & \mathsf{CUB}(K,M) \end{tikzcd}$$

has a regular completion $\theta\colon \mathbb{I}^n \to \mathsf{CUB}(K,L)$. By the exponential law, this is equivalent to asking for a regular completion $\bar\theta$ of the diagram

$$\begin{tikzcd} B \otimes K \arrow[r, "\bar f"] \arrow[d, "i'"'] & L \arrow[d, "p"] \\ \mathbb{I}^n \otimes K \arrow[r, "\bar g"'] & M \end{tikzcd}$$

where $\bar f$, $\bar g$ are the maps corresponding to f, g respectively by the exponential law, and $i' = i \otimes 1_K$. This follows from Theorem 11.3.14, with $A = \emptyset$. □

Corollary 11.3.16. *Let $j : A \to K$ be an inclusion of a subcubical set, and let L be a fibrant cubical set. Then $j^*\colon \mathsf{CUB}(K,L) \to \mathsf{CUB}(A,L)$ is a fibration.*

Proof. Let B be an $(m-1)$-box in $\mathbb{I}^m$. We have to prove that any commutative diagram

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "i"'] & \mathsf{CUB}(K,L) \arrow[d, "j^*"] \\ \mathbb{I}^m \arrow[r, "g"'] & \mathsf{CUB}(A,L) \end{tikzcd}$$

has a regular completion $\theta\colon \mathbb{I}^m \to \mathsf{CUB}(K,L)$. By the exponential law, the maps f, g determine a map $h\colon B \otimes K \cup \mathbb{I}^m \otimes A \to L$. Since L is fibrant, the constant map $L \to *$ is a fibration. So the corollary follows from Theorem 11.3.14. □

Corollary 11.3.17. *If K, L are cubical sets such that L is fibrant, then* $\mathsf{CUB}(K,L)$ *is also fibrant.*

Proof. This follows from Theorem 11.3.14 using the fibration $L \to *$. □

Exercise 11.3.18. If $p\colon L \to M$ is a fibration, then for each $v \in M_0$, $p^{-1}(v)$ is fibrant. More generally, the pullback of a fibration by any map is also a fibration. □

Example 11.3.19. If X_* is a filtered space then there is defined a 'filtered' cubical singular complex RX_* and a quotient morphism $p\colon RX_* \to \rho X_*$ by taking homotopy classes through filtered maps and relative to the vertices. A key result of Chapter 15, whose proof uses Theorem 11.3.5, is that p is a fibration of cubical sets. □

11.3.iii Homotopy

In this section we introduce the basic concepts of homotopy for cubical sets, and define the fundamental groupoid for a fibrant cubical set.

Definition 11.3.20. Let L be a cubical set. For $x, y \in L_0$, we say $x \sim y$ if there is an $a \in L_1$ such that $\partial_1^- a = x$, $\partial_1^+ a = y$.

Proposition 11.3.21. *If L is a fibrant cubical set, then the relation $\sim$ on L_0 is an equivalence relation.*

Proof. Reflexivity $x \sim x$ is easy by taking $a = \varepsilon_1 x$. For the other conditions we need the extension condition. In the following two diagrams

$$
\begin{tikzcd}
x \arrow[r, dashed] \arrow[d, "a"'] & z \\
y \arrow[r, "b"'] & z \arrow[u, "\varepsilon_1 z"']
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, "\varepsilon_1 x"] \arrow[d, "a"'] & x \arrow[d, "\varepsilon_1 x"] \\
y \arrow[r, dashed] & x
\end{tikzcd}
$$

the first shows the box to fill to obtain transitivity, and the second shows the box to fill to obtain symmetry. □

Definition 11.3.22. Two cubical maps $f, g\colon K \to L$ are said to be *homotopic* if $f \sim g$ as elements of $\mathsf{CUB}(K, L)_0$.

Remark 11.3.23. Clearly two maps are homotopic if and only if there exists a homotopy

$$H : \mathbb{I} \otimes K \to L$$

from one to the other. □

Proposition 11.3.24. *If L is a fibrant cubical set , then homotopy is an equivalence relation on maps $K \to L$.*

Definition 11.3.25. If K, L are cubical sets and L is fibrant, we define the set of homotopy classes of cubical maps as the quotient

$$[K, L] = \mathsf{Cub}(K, L)/\sim .$$

Let $i: A \to K$ be the inclusion of a subcubical set of the cubical set K. Then the fibre of the map

$$\mathsf{CUB}(i, L)\colon \mathsf{CUB}(K, L) \to \mathsf{CUB}(A, L)$$

over a map $u\colon A \to L$, considered as a vertex of $\mathsf{CUB}(A, L)$, is written $\mathsf{CUB}(K, L; u)$. By previous results, if L is fibrant so also is $\mathsf{CUB}(K, L; u)$.

Definition 11.3.26. If L is a fibrant cubical set, the set $\pi_0 \mathsf{CUB}(K, L; u)$ is written $[K, L; u]$ and called the set of *homotopy classes of maps* $K \to L$ *rel* u. The disjoint union of these sets is the set of homotopy classes $K \to L$ rel A and is written $[K, L; -]$. □

We need this notion to define the fundamental groupoid $\pi_1 M$ of a fibrant cubical set L.

Let L be a fibrant cubical set. We write $\pi_1 L$ for the set of homotopy classes $\mathbb{I} \to L$ rel $\{0, 1\}$. We know this is well defined. We now introduce a composition on these classes in the usual way, using the ideas of the proof of Proposition 11.3.21. This leads to:

Proposition 11.3.27. *If L is a fibrant cubical set, then $\pi_1 L$ may be given the structure of groupoid.*

11.3.iv An equivalence of cubical and topological homotopy sets

In this section we state the results that we need on the relation between the cubical and the topological homotopy theories.

A full account of this relation would take us too far afield.[169] The general theory says that the 'homotopy theories' of cubical sets and of spaces of the homotopy type of CW-complexes are equivalent. For an indication of more theory, see Appendix B, Section B.8.

The main result we need is the following fragment of the general theory:

Theorem 11.3.28. *Let K, L be cubical sets such that L is fibrant. Then geometric realisation induces a bijection of homotopy classes:*

$$|\ |: [K, L] \cong [|K|, |L|]$$

with homotopy classes of maps for cubical sets on the left-hand side, and for spaces on the right-hand side.

We can use this result to prove the following generalisation.

Theorem 11.3.29. *If K, L are cubical sets such that L is fibrant, then there is a natural map*

$$\phi\colon |\mathsf{CUB}(K, L)| \to \mathsf{TOP}(|K|, |L|)$$

and this map is a weak homotopy equivalence of spaces.

Proof. There is a continuous map

$$|\,\mathsf{CUB}(K,L)|\times|K|\to|\,\mathsf{CUB}(K,L)\otimes K|\xrightarrow{|\mathrm{eval}|}|L|.$$

The topological adjoint of this is the map ϕ.

To prove that ϕ is a weak homotopy equivalence we look at the effect of this map on homotopy classes from $|M|$ for an arbitrary cubical set M. We have natural bijections of homotopy classes:

$$\begin{aligned}
[\,|M|,|\,\mathsf{CUB}(K,L)|\,] &\cong [M,\mathsf{CUB}(K,L)] && \text{by Theorem 11.3.28}\\
&\cong [M\otimes K,L] && \text{by the exponential law in } \mathsf{Cub}\\
&\cong [\,|M\otimes K|,|L|\,] && \text{by Theorem 11.3.28}\\
&\cong [\,|M|\times|K|,|L|\,] && \text{by Proposition 11.2.3}\\
&\cong [\,|M|,\mathsf{TOP}(|K|,|L|)] && \text{by the exponential law in } \mathsf{Top}.
\end{aligned}$$

Since these maps are natural, the composite is induced by ϕ. These bijections imply that ϕ is a weak equivalence of spaces. □

We need to move from this to an equivalence of relative, and indeed filtered, theories. Thus in the standard homotopy theory of spaces the relative homotopy group $\pi_n(X,A,a)$ is defined as homotopy classes of maps $I^n,\ \partial_1^- I^n,\ J^{n-1}_{(-,1)}\to(X,A,a)$ where I^n is the standard n-cube, $\partial_1^- I^n$ is the $(-,1)$-face and $J^{n-1}_{(-,1)}$ is the union of the other faces of I^n. It is then proved that this set has for $n\geqslant 2$ a group structure induced by composition of cubes in direction 2, and that this structure is abelian for $n\geqslant 3$. For a filtered space X_* various relative homotopy groups may be combined to give a crossed complex ΠX_* where $(\Pi X_*)_n$ is the family of groups $\pi_n(X_n,X_{n-1},x)$ for $x\in X_0$, for $n\geqslant 2$, $(\Pi X_*)_1$ is the fundamental groupoid $\pi_1(X_1,X_0)$, and $(\Pi X_*)_0$ is X_0.

Such a theory can also be formulated for the relative and indeed filtered homotopy theory of fibrant cubical sets.[170] One of the facts we will use is also that in the cubical set situation we can identify the n-th relative homotopy group of a fibrant pair (K,L) as given by elements k of K_n such that $\partial_1^- k\in L_{n-1}$ and $\partial_i^\alpha k\in\operatorname{Im}\varepsilon_1^{n-1}$ for $(\alpha,i)\neq(-,1)$. This is the way we wish to define the fundamental crossed complex ΠK_* of a fibrant filtered cubical set, by which is meant a cubical set filtered by subcubical sets each of which is fibrant. In these terms we have the following corollary of the equivalence of homotopy categories:

Corollary 11.3.30. *If K_* is a filtration consisting of fibrant cubical sets, then the realisation functor gives an isomorphism*

$$\Pi K_*\to\Pi|K_*|.$$ □

In applying the Homotopy Classification Theorem of later sections, we will need to use the realisation of fibrations of cubical sets. The following result will be sufficient for these applications.

Theorem 11.3.31. *Let $p\colon L \to M$ be a fibration of cubical sets such that M, and hence also L, is fibrant. Then $|p|\colon |L| \to |M|$ is homotopy equivalent over $|M|$ to a fibration of spaces.*

Proof. Let $f = |p|$ and choose a factorisation of f

$$|L| \xrightarrow{e} E_f \xrightarrow{\psi} |M|$$

through a homotopy equivalence e and a fibration ψ. Since p is a fibration of cubical sets there is an associated family of long homotopy exact sequences, one for each base point $m \in M_0$ and for the corresponding fibre. Because of the equivalence of homotopy sets given by Theorem 11.3.28, this long exact sequence is mapped isomorphically to the long exact sequence of the fibration ψ. □

11.4 Cubical sets and crossed complexes

We proceed now a step further and relate the category Cub of cubical sets (or the equivalent of topological spaces) to that of crossed complexes. We use extensively the monoidal closed structure on the category Crs discussed in Chapter 9. The main aim is the Homotopy Classification Theorem 11.4.19, which gives a considerable generalisation of classical theorems in algebraic topology.

11.4.i The fundamental crossed complex of a cubical set

We define in this section the fundamental crossed complex ΠK of a cubical set K; it is basic to our work on the classifying space of a crossed complex. First we define $\Pi\mathbb{I}^1$ as the groupoid $\mathcal{I}$, with generator $\iota\colon 0 \to 1$, regarded as a crossed complex, or, equivalently, as the crossed complex $\mathbb{F}(1)$ of Definition 7.1.12. Then we define $\Pi\mathbb{I}^r = \mathcal{I}^{\otimes r}$, the r-fold tensor product of $\mathcal{I}$ with itself. We obtain the following:

Theorem 11.4.1 (Homotopy Addition Lemma). *In $\Pi\mathbb{I}^n$ for $n \geqslant 2$ we have a free generator $c^n = \iota^{\otimes n}$ in dimension n with boundary given by*

$$\delta(c) = \begin{cases} \sum_{i=1}^{n}(-1)^i\{(\partial_i^+ c) - (\partial_i^- c)^{(u_i c)}\}, & n \geqslant 4, \\ -(\partial_3^+ c) - (\partial_2^- c)^{(u_2 c)} - (\partial_1^+ c) + (\partial_3^- c)^{(u_3 c)} + (\partial_2^+ c) + (\partial_1^- c)^{(u_1 c)}, & n = 3, \\ -(\partial_1^+ c) - (\partial_2^- c) + (\partial_1^- c) + (\partial_2^+ c), & n = 2, \end{cases}$$

where $u_i = \partial_1^+ \partial_2^+ \dots \hat{\imath} \dots \partial_n^+$.

Proof. The proof is by induction using the explicit description of the tensor product, analogously to that for the HAL for the simplex Δ^n. □

Remark 11.4.2. The HAL shows that if a is a face of c^n then a can be expressed uniquely in terms of δc^n and the other faces of c^n. We use this fact in the proof of Proposition 12.1.13. □

Definition 11.4.3. The *fundamental crossed complex* ΠK *of a cubical set* K is defined as the coend in Crs:

$$\Pi K = \int^{\square, n} K_n \times \Pi \mathbb{I}^n. \qquad \square$$

Thus ΠK is freely generated by the nondegenerate cubes of K with boundaries given by the HAL.

Theorem 11.4.4. *For any cubical sets* K, L *there is a natural isomorphism*

$$\Pi K \otimes \Pi L \cong \Pi(K \otimes L).$$

Proof. It is immediate from the definition that there is an isomorphism

$$\Pi \mathbb{I}^n \otimes \Pi \mathbb{I}^m \cong \Pi \mathbb{I}^{n+m}.$$

Now the coend definition of ΠK yields the result, analogously to the proof of Proposition 11.2.3, using the isomorphism of crossed complexes for $p + q = n$:

$$(K_p \times L_q) \otimes \mathcal{I}^n \cong (K_p \times \mathcal{I}^p) \otimes (L_q \times \mathcal{I}^q)$$

where K_p, L_q are discrete crossed complexes. □

Remark 11.4.5. Another view of this result is given in Theorem 15.2.10, where the tensor product is set up using the properties of cubical ω-groupoids and the monoidal closed structure on those objects. □

Remark 11.4.6. An application of the Higher Homotopy Seifert–van Kampen Theorem 8.1.5 gives an isomorphism $\Pi K \cong \Pi |K|_*$ for a cubical set K where $|K|_*$ is the skeletal filtration of the realisation. □

11.4.ii The cubical nerve of a crossed complex

Let us construct an adjoint to the fundamental crossed complex of cubical sets just studied.

Definition 11.4.7. We define the *cubical nerve* NC of a crossed complex C to be in dimension n the set

$$(NC)_n = \mathsf{Crs}(\Pi \mathbb{I}^n, C).$$

Remark 11.4.8. In Remark 14.6.6 this definition is related to the fundamental algebraic equivalence between the category Crs of crossed complexes and that of cubical ω-groupoids with connections. □

Proposition 11.4.9. *For a cubical set K and crossed complex C there is a natural isomorphism*

$$\mathsf{Cub}(K, NC) \cong \mathsf{Crs}(\Pi K, C)$$

making $\Pi\colon \mathsf{Cub} \to \mathsf{Crs}$ *left adjoint to* $N\colon \mathsf{Crs} \to \mathsf{Cub}$.

Proof. The proof is based in the fact that one side of the proposed congruence may be described as a colimit and the other one as a limit:

$$\begin{aligned}\mathsf{Cub}(K, NC) &\cong \int_{\square, n} \mathsf{Set}(K_n, (NC)_n)\\ &\cong \int_{\square, n} \mathsf{Set}(K_n, \mathsf{Crs}(\Pi \mathbb{I}^n, C))\\ &\cong \int_{\square, n} \mathsf{Crs}(K_n \times \Pi \mathbb{I}^n, C)\\ &\cong \mathsf{Crs}\Big(\Big(\int^{\square, n} K_n \times \Pi \mathbb{I}^n\Big), C\Big)\\ &\cong \mathsf{Crs}(\Pi K, C).\end{aligned}$$

□

Remark 11.4.10. There is an analogous simplicial nerve N^Δ of a crossed complex C defined in dimension n by $N^\Delta(C)_n = \mathsf{Crs}(\Pi\Delta^n, C)$, and this is right adjoint to $\Pi\colon \mathsf{Simp} \to \mathsf{Crs}$.[171] □

Proposition 11.4.11. *The cubical nerve NC of a crossed complex C is a fibrant cubical set.*

Proof. Let $n \geqslant 0$ and let B be a box in $\mathbb{I}^n$. We use the last proved adjointness relation. So a map $B \to NC$ corresponds to a morphism $f\colon \Pi B \to C$. Let c^n be the top cell in $\mathbb{I}^n$. We extend f to $g\colon \Pi\mathbb{I}^n \to C$ by mapping c^n to 0, with the value of g on the omitted $(n-1)$-cell of B being given by the HAL (Theorem 11.4.1). □

Remark 11.4.12. Actually the result may be strengthened to say that NC is a cubical T-complex, see Remark 13.7.6; indeed N gives an equivalence between the category Crs and that of cubical T-complexes.[172] □

Proposition 11.4.13. *Let C, D be crossed complexes. There is a natural transformation of cubical sets*[173]

$$\eta\colon N(C) \otimes N(D) \to N(C \otimes D).$$

Proof. It is easy to verify that the function

$$\begin{aligned} b\colon N(C), N(D) &\to N(C \otimes D),\\ f, g &\mapsto f \otimes g\end{aligned}$$

is bicubical. □

Corollary 11.4.14. *The nerve functor* $N : \mathsf{Crs} \to \mathsf{Cub}$ *preserves homotopy.*

Proof. A homotopy $h : \mathcal{I} \otimes D \to E$ in Crs determines a cubical homotopy as the composition

$$\mathbb{I}^1 \otimes ND \xrightarrow{\eta} N(\mathcal{I} \otimes D) \xrightarrow{N(h)} NE. \qquad \square$$

Remark 11.4.15. The proof of Proposition 11.4.13 should be compared with the proof of Theorem 15.3.1. □

Now we come to the key theorem which is the basis of the Homotopy Classification Theorem for maps of spaces.

Theorem 11.4.16. *For a cubical set* K *and a crossed complex* C *there is a natural isomorphism of cubical sets:*

$$\mathsf{CUB}(K, NC) \cong N(\mathsf{CRS}(\Pi K, C)).$$

Proof. Let L be a cubical set. Then we have natural bijections

$$\begin{aligned}
\mathsf{Cub}(L, \mathsf{CUB}(K, NC)) &\cong \mathsf{Cub}(L \otimes K, NC) \\
&\cong \mathsf{Crs}(\Pi(L \otimes K), C) \\
&\cong \mathsf{Crs}(\Pi L \otimes \Pi K, C) \\
&\cong \mathsf{Crs}(\Pi L, \mathsf{CRS}(\Pi K, C)) \\
&\cong \mathsf{Cub}(L, N(\mathsf{CRS}(\Pi K, C)).
\end{aligned}$$

Since this natural bijection holds for all cubical sets L, the theorem follows. □

Remark 11.4.17. Notice the power of the combination of various adjunctions and the notion of representability (see Proposition A.2.1) in the proof of the last theorem. □

11.4.iii The Homotopy Classification Theorem

Definition 11.4.18. The (cubical) *classifying space* BC of a crossed complex C is defined to be the realisation $|NC|$ of the nerve of C.[174] □

Our aim is the following theorem:

Theorem 11.4.19 (Homotopy Classification Theorem). *Let* X *be a CW-complex and* C *a crossed complex. Then there is a weak equivalence*

$$\chi : B(\mathsf{CRS}(\Pi X_*, C)) \to \mathsf{TOP}(X, BC).$$

Hence there is a bijection

$$[\Pi X_*, C] \cong [X, BC],$$

where the left-hand side is homotopy classes of crossed complex maps, and the right-hand side is homotopy classes of maps of spaces.

Proof. We first assume X is $|K|$ where K is a cubical set. Then the theorem with $X = |K|$ follows from Theorems 11.3.28, 11.4.16. In particular this applies to the case $K = S^{\square}X$. Now by Remark 11.1.18 X has the homotopy type of $|S^{\square}X|$. □

Remark 11.4.20. This theorem generalises many classical results. It is important that it includes information on fundamental groups and their actions. We analyse this more closely in Section 12.2. □

Example 11.4.21. Use the Homotopy Classification Theorem and Example 9.3.8 to analyse the homotopy 2-type of the space LX of *free loops* on X, i.e. the function space $\mathsf{TOP}(S^1, X)$, when X is given by the classifying space of a crossed module of groups. □

11.5 The pointed case

We are going to consider briefly the modifications needed get a pointed, or base point, based version of Theorem 11.4.19.

First, recall that we have defined Crs_* the category of pointed crossed complexes, that has objects the crossed complexes C having a distinguished element $* \in C_0$ and only morphisms preserving this basepoint are included.

Next, we need the notions of tensor product and homotopy in Crs_*. They are the same notions that in crossed complex but adding the good behaviour with respect to the base point. let us make the conditions explicit.

For any pointed crossed complexes C and D, we define an *m-fold pointed left homotopy* from C to D to be an m-fold left homotopy (H, f) satisfying $f(*) = *$ and $H(*) = 0_* \in D_m$. The collection of all these is a sub-crossed complex $\mathsf{CRS}_*(C, D) \subseteq \mathsf{CRS}(C, D)$ with basepoint the zero morphism $c \mapsto 0_*$. This defines the *pointed internal hom* for crossed complexes.

A *pointed bimorphism* $\theta\colon (C, D) \to E$ is a bimorphism satisfying

$$\begin{cases} \theta(c, *) = 0_* & \text{for } c \in C, \\ \theta(*, d) = 0_* & \text{for } d \in D. \end{cases}$$

The *pointed tensor product* $C \otimes_* D$ is the pointed crossed complex generated by all $c \otimes_* d$ with defining relations those for the tensor product and

$$\begin{cases} c \otimes_* * = 0_* & \text{for } c \in C, \\ * \otimes_* d = 0_* & \text{for } d \in D. \end{cases}$$

It is quite clear that the associativity and the symmetry of the tensor product preserves the relations in the definition of the pointed tensor product, giving as a consequence the following theorem.

Theorem 11.5.1. *The pointed tensor products and internal hom functors described above define a symmetric monoidal closed structure on the pointed category* Crs_*. □

We denote by $[X, Y]_*$ the set of pointed homotopy classes of pointed maps $X \to Y$ of pointed spaces X, Y. Similarly, for pointed crossed complexes C, D, we denote by $[C, D]_*$ the set of pointed homotopy classes of pointed morphisms $C \to D$.

Also, notice that if C is a pointed crossed complex, then BC is naturally a pointed space.

We have all the ingredients to state the pointed version of Theorem 11.4.19.

Theorem 11.5.2. *If X is a pointed CW-complex and C is a pointed crossed complex, there is a commutative diagram*

$$\begin{tikzcd}
{[X, BC]_*} \arrow[rr, "\alpha"] \arrow[dr] & & {[\Pi X_*, C]_*} \arrow[dl] \\
& \mathsf{Hom}(\pi_1(X,*), \pi_1(C,*)), &
\end{tikzcd}$$

in which α is a bijection of sets of pointed homotopy classes, natural with respect to pointed morphisms of C and pointed, cellular maps of X, and in which we have identified $\pi_1(BC,*)$ with $\pi_1(C,*)$, $\pi_1(X,*)$ with $\pi_1(X_*,*)$.

Proof. The proof of the existence of the horizontal bijection α of sets of pointed homotopy classes follows the same pattern as the proof of Theorem 11.4.19, but using the pointed constructions $\otimes_*$ and CRS_* described before. We leave the details as an exercise.

The slanting map on the left is induced by the functor $\pi_1(-,*)$ and the first identification indicated in the statement.

The slanting map on the right comes from the second identification indicated in the statement.

To prove commutativity, it is sufficient to assume that $X = |L|$ for some fibrant cubical set L. Then we have to check that maps transformed by the following arrows induce the same map of fundamental groups:

$$\mathsf{Top}(|L|, BC) \longleftarrow \mathsf{Cub}(L, NC) \to \mathsf{Crs}(\Pi L, C).$$

But this is clear on checking the values of these maps on 1-dimensional elements. □

In the next chapter we will use methods of fibrations of crossed complexes to analyse these results further and to make specific calculations of some homotopy classes of maps.

Notes

163 p. 368 The cubical classifying space of a crossed module is used in [FRS95].

The work of this chapter on homotopy classification is generalised to the equivariant case in [BGPT97], [BGPT01], but using the simplicial classifying space, [BH89], which fits better with the published studies on homotopy coherence, [CP97], which are required for the proof. The simplicial classifying space of a crossed module is used for work on HQFTs in [PT08], [Por07].

164 p. 369 These basic facts may be found in many places, for example in [Cis06], [Jar06], [GM03], [Mal09].

165 p. 371 For more on cubical homology theory, see for example [HW60], [Mas80].

166 p. 379 This condition was introduced for cubical sets in [Kan55], and is often known as the *Kan extension condition*. It has become common to call this the fibrant condition, thus linking the idea with work on model categories, see Section B.8.

The notions of expansion and collapse that we use below were extensively studied in a simplicial context by Whitehead in [Whi39], [Whi41b] and in the first of these papers he writes that transformations of this kind were previously studied in [Joh32]. Whitehead later rewrote and extended these papers as [Whi49a], [Whi49b], [Whi50b]. It is interesting to read the earlier papers to see how the later expositions emerged.

167 p. 380 This method of partial boxes seems related to results used in [Kan55] which introduced the extension condition.

168 p. 381 This notion of anodyne extension may be found in the cubical case in [Cis06], [Jar06], and there is a well-known simplicial version going back to [GZ67].

169 p. 386 The relation between the theory of simplicial sets and topological spaces has been well worked over, see for example [GZ67], [May67], [GJ99] but published attention to cubical sets has been more recent. The paper [BH81a] relied on a Masters thesis from Warwick [Hin73]. Later work on cubical theory was done in [Ant00], and by Cisinski in [Cis06] as part of a general approach to homotopical conjectures posed by Grothendieck in [GroPS1], [GroPS2] and given a full account in [Mal05]. The papers [BJT10], [Isa11], [Pat08] show recent uses of cubical theory in various forms.

170 p. 387 The detailed use of cubical extension conditions for obtaining standard results of homotopy theory in the abstract case has been analysed by Kamps in [Kam72] and is explained in [KP97].

171 p. 390 That construction was defined in essence in [Bla48] in the reduced case, and further exploited in [And57], and in essence in [Lab99]. It was used as the basis for the simplicial classifying space of a crossed complex in [BH91], where the adjointness with Π was proved (that paper uses the notation π for what we write as Π). However this paper assumed the Eilenberg–Zilber theorem for crossed complexes, which was proved by Tonks in [Ton94], [Ton03]. The adjointness and coherence properties of the crossed complex Eilenberg–Zilber maps developed by Tonks were exploited in proving homotopy classification results in the equivariant case in [BGPT97], [BGPT01], using methods of [CP97].

172 p. 390 The relation between crossed complexes and cubical ω-groupoids with connection has been used in [BH03] to give a cubical version of the Dold–Kan Theorem, namely an equivalence between chain complexes and cubical abelian groups with connections. We give a brief account of this in Section 14.8. Compare also Remark 9.10.6.

173 p. 390 This is a result which is more difficult to prove for the simplicial nerve, see [BH91], and so shows an advantage of cubical methods. A combination of simplicial and cubical methods is given in the paper [Gug57].

174 p. 391 This classifying space was defined first in [Bro84b] and its properties extended in [BH89]. An application was given in [FRS95]. However the published version in [BH91] was a simplicial version, deemed to be more palatable. It also fitted better with the homotopy coherence theory of [CP97], allowing proofs of an equivariant version of the homotopy classification, [BGPT97], [BGPT01].

Chapter 12
Nonabelian cohomology: spaces, groupoids

Introduction

Our approach to the cohomology of topological spaces is that it is common for a space to be given not just in abstract but also with some kind of structure; for example it may be a simplicial complex, a CW-complex, or even in the case of a manifold have the so-called 'handlebody decomposition', which arises in Morse theory.[175] There is also the notion of 'stratified space', which is a filtered space X_* with special conditions on the way the 'strata' X_n are related. Thus it is quite reasonable to start with a filtered space X_*, and form its associated fundamental crossed complex ΠX_*. To obtain the cohomology of a group G one can form the simplicial nerve $N^{\Delta}G$ of G and take the fundamental crossed complex of that. One could also use the cubical nerve.

Thus in these applications one can take as the starting point a free crossed complex, say F, and consider cohomology derived from that.

We follow the tradition that cohomology of spaces may be interpreted in terms of homotopy classes of mappings. This derives from the classical theorem of Eilenberg–Mac Lane that for a CW-complex X, $n \geqslant 2$, and abelian group M there is a bijection

$$[X, K(M, n)] \cong H^n(X, M)$$

between the homotopy classes of maps from X to the Eilenberg–Mac Lane space $K(M, n)$ which has exactly one nontrivial homotopy group, namely M in dimension n and the n-th cohomology group of X with coefficients in M. This result has been generalised to the case of $K_n(M, G)$, a connected space with fundamental group G which operates on the abelian group M.

Of course our homotopy classification result for a CW-complex X and crossed complex C,

$$[X, BC] \cong [\Pi X_*, C],$$

contains both the above results, and so it is natural to consider the right-hand side of this bijection as a kind of cohomology of X with coefficients in C.[176]

Exact sequences of cohomology have a considerable use in traditional cohomology. They may be determined by a pair of spaces or by an exact sequence of coefficients; both cases arise from a short exact sequence of chain complexes. In our case Theorem 12.1.15 shows that we get rather more elaborate families of exact sequences from a fibration of crossed complexes. Such fibrations arise under circumstances given in Section 12.1. We take advantage of this in our analysis of the above stated Homotopy Classification Theorem.

Our first major result is Theorem 12.2.10 on the homotopy classification of morphisms of crossed complexes. Theorem 12.2.10 may be regarded as a generalisation of what is called 'abstract kernel theory' in the cohomology of groups; we explore this further in Section 12.6.

This theorem has applications to the homotopy classification of maps of spaces, in Theorem 12.3.12. We give one explicit example, namely the calculation of based homotopy classes of maps from $X = \mathbb{R}P^2 \times \mathbb{R}P^2$ to the space $\mathbb{R}P^3$ with homotopy groups in dimensions $\geqslant 3$ killed and which induce on fundamental groups the morphism $(1,1)\colon \mathsf{C}_2 \times \mathsf{C}_2 \to \mathsf{C}_2$ (Example 12.3.13). In this example, the operations of fundamental groups are crucial, as is the representation of the cell structure of X in terms of free crossed complexes, including the formulae for the tensor product from Chapter 9.

In Section 12.4 we show how $[\Pi X_*, C]$ can be analysed in terms of a 'local system' with values in chain complexes with a groupoid of operators, thus generalising the traditional theory involving a module over a group, and also that of coefficients in a chain complex.

In Section 12.5 we introduce the notion of cohomology of a groupoid, and apply this in Section 12.5.i to give an account of the nonabelian Čech cohomology of a cover of a space with coefficients in a crossed complex.

In Section 12.6 we give an account of the representations of nonabelian extensions of groups of the type of a crossed module.

In Section 12.7 we explain the representation of the $(n+1)$-st cohomology of a group in terms of crossed n-fold extensions. This in particular gives the notion of of k-invariant, or Postnikov invariant, of a crossed module, as an element of a third cohomology group, and our results allow us to do some calculations of these invariants.

12.1 Fibrations of crossed complexes

The notion of fibration of crossed complexes has an important role in analysing the set $[F, C]$ of homotopy classes of morphisms from a free crossed complex F to a crossed complex C. The notion also allows for relating the homotopy theory of crossed complexes to homotopy theories in other structures, for example that of cubical sets, and as briefly described in general in Section B.8. We assume the notion of *fibration of groupoid* as developed in [Bro06], §7.2, and given on p. 590.[177]

Definition 12.1.1. A morphism $p\colon E \to B$ of crossed complexes is a *fibration* if[178]

(i) the morphism $p_1\colon E_1 \to B_1$ is a *fibration* of groupoids;

(ii) for each $n \geqslant 2$ and $x \in E_0$, the morphism of groups $p_n\colon E_n(x) \to B_n(px)$ is surjective.

The morphism p is an *acyclic fibration* if it is a fibration and also a weak equivalence, by which is meant that p induces a bijection on π_0 and isomorphisms $\pi_1(E, x) \to \pi_1(B, px)$, $H_n(E, x) \to H_n(B, px)$ for all $x \in E_0$ and $n \geqslant 2$.[179] □

Example 12.1.2. Here is an example we will use later in the proof of Theorem 12.2.10. Let $n \geqslant 2$, and let C be a crossed complex. Let $M = \operatorname{Ker}\delta_n$, $Q = \pi_1 C$. Then the following diagram defines a morphism of crossed complexes $p\colon \operatorname{Cosk}^n C \to \mathbb{K}_{n+1}(M, Q)$:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "="'] & C_n \arrow[r, "\delta_n"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & C_2 \arrow[r] \arrow[d] & C_1 \arrow[d, "\phi"] \\
\cdots \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & Q.
\end{tikzcd}$$

It is also a fibration of crossed complexes, since ϕ, as a quotient morphism of groupoids, is a fibration of groupoids, and the other conditions for a fibration are clearly satisfied. □

Definition 12.1.3. Consider the following diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\
C \arrow[r] \arrow[ur, dotted] & B.
\end{tikzcd} \tag{12.1.1}$$

If given i the dotted completion exists for all morphisms p in a class F, then we say that i has the *left lifting property* (*LLP*) with respect to F. We say a morphism $i : A \to C$ is a *cofibration* if it has the LLP with respect to all acyclic fibrations. We say a crossed complex C is *cofibrant* if the inclusion $\emptyset \to C$ is a cofibration. □

Here is an important example of a cofibration. The proof is analogous to standard methods in homological algebra, as in Section 10.2.iii, and to inductive constructions of maps on CW-complexes.

Proposition 12.1.4. *Let $i : A \to F$ be a relatively free crossed complex. Then i is a cofibration.*

Proof. We consider the diagram

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "i"'] & E \arrow[d, "p"] \\
F \arrow[r, "f"'] \arrow[ur, dotted, "g"] & B
\end{tikzcd}$$

in which p is supposed an acyclic fibration, and the morphisms f, a satisfy $fi = pa$. We construct the regular completion g on a relatively free basis X of F by induction, the cases $n = 0, 1$ being easy.

Suppose $n \geqslant 2$ and g is defined on X^{n-1}. Consider an element x of the free basis in dimension n. Then $g\delta x$ is defined and $pg\delta x = f\delta x$.

By the fibration condition, we can choose $y \in E_n$ such that $py = fx$. Let $w = g\delta x - \delta y \in E_{n-1}$. Then $pw = 0$, $\delta w = 0$. By the acyclicity condition, w is a boundary, i.e. $w = \delta z$ for some $z \in E_n$. Then $\delta(z + y) = g\delta x$. So we can extend g by defining it on x to be $z + y$. □

Now we can characterise fibrations of crossed complexes in terms of a right lifting property.

Definition 12.1.5. The morphism $p\colon E \to B$ has the *right lifting property* (*RLP*) with respect to a class F if in the above diagram (12.1.1) the dotted completion exists for all i in the class F.

Recall that $\mathbb{F}(n)$ denotes the free crossed complex on one generator c^n of dimension n. Thus $\mathbb{F}(0)$ is a singleton, $\mathbb{F}(1)$ is essentially the groupoid $\mathcal{I}$ and for $n \geqslant 2$ $\mathbb{F}(n)$ consists the integers in dimensions $n, n-1$ with boundary the identity map. Also $\mathbb{S}(n-1)$ denotes the subcomplex of $\mathbb{F}(n)$ generated by the part in dimension $n-1$, and so is simply $\mathbb{K}(\mathbb{Z}, n-1)$.

Proposition 12.1.6. *Let $p\colon E \to B$ be a morphism of crossed complexes. Then the following conditions are equivalent:*

(i) *p is a fibration;*

(ii) (*covering homotopy property*) *p has the RLP with respect to the inclusion $C \otimes 1 \to C \otimes \mathbb{F}(m)$ for all cofibrant crossed complexes C and $m \geqslant 1$;*

(iii) *the covering homotopy property* (ii) *holds for $m = 1$; and*

(iv) *for any cofibrant crossed complex C, the induced morphism*

$$p_*\colon \mathsf{CRS}(C, E) \to \mathsf{CRS}(C, B)$$

is a fibration.

Proof. (i)⇒(ii) We verify the covering property by constructing a lifting in the left-hand of the following diagrams, where $1 \to \mathbb{F}(m)$ is the inclusion. Let p' in the right-hand diagram be induced by p and the inclusion $1 \to \mathbb{F}(m)$.

$$\begin{tikzcd} C \otimes 1 \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ C \otimes \mathbb{F}(m) \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \mathsf{CRS}(\mathbb{F}(m), E) \arrow[d, "p'"] \\ C \arrow[r] & \mathsf{CRS}(1, E) \times_B \mathsf{CRS}(\mathbb{F}(m), B) \end{tikzcd}$$

Then a lifting in the left-hand diagram is equivalent to a lifting in the right-hand diagram. Since C is cofibrant, such a lifting exists if p' is an acyclic fibration. But by the exponential law, for this it is sufficient to show that p has the RLP with respect to the inclusion

$$\mathbb{S}(n) \otimes \mathbb{F}(m) \cup \mathbb{F}(n+1) \otimes 1 \to \mathbb{F}(n+1) \otimes \mathbb{F}(m).$$

For $n = -1$, this corresponds precisely to the fibration property of p. In general, a lifting of the image of the top basis element of $\mathbb{F}(n+1) \otimes \mathbb{F}(m)$ is chosen, and the value of the lifting on the remaining basis element of $\mathbb{F}(n+1) \otimes \mathbb{F}(m)$, namely $c_{n+1} \otimes \delta c_m$ if $m \geqslant 2$, $c_n \otimes 0$ if $m = 1$, is determined by the boundary formula for $c_n \otimes c_m$ and the values on $\delta c_{n+1} \otimes c_m$ if $n \geqslant 1$ and $0 \otimes c_m$ and $1 \otimes c_m$ if $n = 0$.

(ii)⇒(iii) is immediate.

(iii)⇒(iv) It is sufficient to show that the diagram

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & \mathsf{CRS}(C, E) \arrow[d, "p"] \\ \mathbb{F}(m) \arrow[r] & \mathsf{CRS}(C, B) \end{tikzcd}$$

has a regular completion; this follows from (iii) an the exponential law, and symmetry of the tensor product.

(iv)⇒(i) This is easily proved on taking C to be the crossed complex $\mathbb{F}(n)$. □

The following is proved in a similar manner, which we leave to you as an exercise.

Proposition 12.1.7. *The following are equivalent for a morphism $p\colon E \to B$ in* Crs*:*

(i) *p is an acyclic fibration;*

(ii) *p_0 is surjective; if $x, y \in E_0$ and $b \in B_1(p_0 x, p_0 y)$, then there is $e \in E_1$ such that $p_1 e = y$; if $n \geqslant 1$ and $d \in E_n$ satisfies $sd = td$ for $n = 1, \delta d = 0$ for $n \geqslant 2$, and $b \in B_{n+l}$ satisfies $\delta b = p_n d$, then there is*

$$e \in E_{n+1} \quad \text{such that} \quad p_{n+1} e = b \quad \text{and} \quad \delta e = d;$$

(iii) *p has the RLP with respect to $\mathbb{S}(n-1) \to \mathbb{F}(n)$ for all $n \geqslant 0$;*

(iv) *if C is a free crossed complex then p has the* RLP *with respect to $\mathbb{S}(n-1) \otimes C \to \mathbb{F}(n) \otimes C$ for all $n \geqslant 0$;*

(v) *if C is a free crossed complex then the induced morphism*

$$p_*\colon \mathsf{CRS}(C, E) \to \mathsf{CRS}(C, B)$$

is an acyclic fibration.

Remark 12.1.8. It is proved in [BG89b], Corollary 2.9, that a cofibration of crossed complexes is a retract in the category of maps of Crs of a relatively free morphism of crossed complexes, and from this it is deduced that if $i\colon A \to D$ is a cofibration and C is cofibrant, then $i \otimes 1\colon A \otimes C \to D \otimes C$ is a cofibration. This enables one to generalise Proposition 12.1.7 (v) to the case C is cofibrant. □

We will also need a pointed version of part of the previous proposition.

Proposition 12.1.9. *If $p\colon E \to B$ is an acyclic fibration, F is a free crossed complex, and F, E, B are reduced and pointed, then the induced morphism of internal homs*

$$p_*\colon \mathsf{CRS}_*(F, E) \to \mathsf{CRS}_*(F, B)$$

is also an acyclic fibration

Proof. This relies on the pointed exponential law from Section 15.5 and the clear fact that $\mathbb{S}^{n-1} \otimes_* F \to \mathbb{F}(n) \otimes_* F$ is of relatively free type, as is easy to prove by a modification of methods of Section 9.6. □

Proposition 12.1.10. *Let F, C be reduced pointed crossed complexes with F free and C aspherical. Let $G = \pi_1 C$. Then there are bijections*

$$\pi_0 \mathsf{CRS}_*(F, C) = [F, C]_* \cong [F, \mathbb{K}(G, 1)] \cong \mathsf{Hom}(\pi_1 F, G),$$

and for all morphisms $f\colon F \to C$ and $n \geqslant 1$ we have

$$\pi_n(\mathsf{CRS}_*(F, C), f) = 0.$$

Proof. Since C is aspherical, the natural morphism $p\colon C \to \mathbb{K}(G, 1)$ is not only a fibration but also a weak equivalence of crossed complexes, and so p is an acyclic fibration. Hence so also is p_*, by Proposition 12.1.9 (v). In particular, p_* is a weak equivalence. This gives the first result.

The second result also follows, since all homotopies and higher homotopies $F \to \mathbb{K}(G, 1)$ are clearly trivial. □

Remark 12.1.11. This type of argument replaces an inductive argument of lifting morphisms and homotopies which is traditional in homological algebra, and which we gave in Section 10.2.i, see Theorem 10.2.11. Of course an inductive procedure is at the heart of the proof we have just given. □

We end this section with a statement of a theorem which relates to the notion of model category given in Section B.8.[180]

Theorem 12.1.12. *The notions of weak equivalence, fibration and cofibration for crossed complexes satisfy the axioms for a model category for homotopy.*

12.1.i Fibrations of crossed complexes and cubical nerves

There is a close relation between a map of crossed complexes being a fibration and the nerve of the map being a fibration of cubical sets.

Proposition 12.1.13. *Let $p\colon E \to D$ be a morphism of crossed complexes. Then p is a fibration if and only if the induced map of nerves $Np\colon NE \to ND$ is a fibration of cubical sets.*

Proof. Let B be an $(n-1)$-box in $\mathbb{I}^n$. To say that $Np\colon NE \to ND$ is a fibration is equivalent to saying that any diagram in Cub:

$$\begin{tikzcd} B \arrow[r] \arrow[d] & NE \arrow[d, "Np"] \\ \mathbb{I}^n \arrow[r] \arrow[ur, dotted] & ND \end{tikzcd}$$

has a regular completion given by the dotted arrow. By adjointness, this is equivalent to the existence of a regular completion in Crs of the following diagram:

$$\begin{tikzcd} \Pi B \arrow[r, "k"] \arrow[d, "j"'] & E \arrow[d, "p"] \\ \Pi(\mathbb{I}^n) \arrow[r, "k'"'] \arrow[ur, dotted, "g"] & D. \end{tikzcd} \qquad (*)$$

The argument depends on the fact that $\Pi\mathbb{I}^n$ has just one free generator c^n in dimension n, by Corollary 8.3.14, and the boundary δc^n is determined by the cubical Homotopy Addition Lemma, in terms of all the faces of c^n, see Proposition 9.9.6. But B misses one of the faces of c^n, the so called free face. The value of g on this free face is therefore determined by $g(c^n)$ and the HAL, see Remark 11.4.2.

If $n = 0$, this existence is equivalent to $E_1 \to D_1$ being a fibration of groupoids.

If $n \geqslant 2$, let us see that this existence is equivalent to each $E_n(x) \to D_n(px)$ being surjective. To see this, note that if these maps are surjective, and v is the usual base point of $\mathbb{I}^n$, then we can choose $a \in E_n(pv)$ such that $pa = k'c^n$. If we now define $g(c^n) = a$ and $g(x) = k(x)$ for each nondegenerate element x of B, then there is a unique value for g on the free face of B, determined by the Homotopy Addition Lemma, and this with the other values on B defines a morphism $g\colon \Pi(\mathbb{I}^n) \to E$. This g is a regular completion of $(*)$.

On the other hand, suppose each diagram $(*)$ has a regular completion. Let $b \in D_n(px)$. Define $k\colon \Pi B \to E$ to be the acyclic morphism with value 0_x. Define

$$k'\colon \Pi(\mathbb{I}^n) \to D \quad \text{by } k'(c^n) = b,\ k'(B) = 0_{px},$$

and k' on the free face of B is δb. Then $pk = k'j$. Let g be a regular completion. Then $pg(c^n) = b$. □

Corollary 12.1.14. *Let $p\colon E \to D$ be a fibration of crossed complexes and let $x \in D_0$. Let $F = p^{-1}(x)$. Then the sequence of classifying spaces $BF \to BE \to BD$ is homotopy equivalent to a fibration sequence.*

Proof. This follows from Theorem 11.3.31. □

12.1.ii Long exact sequences of a fibration of crossed complexes

Recall from Definition 12.1.1 that a fibration $p\colon E \to B$ of crossed complexes is a morphism of crossed complexes such that p_1 is a fibration of groupoids and for each $n \geqslant 2$ and $x \in E_0$, the morphism of groups $p_n\colon E_n(x) \to B_n(px)$ is surjective.

We state how such a fibration p yields a family of exact sequences involving the H_n, π_1 and π_0, as follows, and which is clearly related to the exact homotopy sequence for a pair stated in Equation (2.1.3).[181] Let $p\colon E \to B$ be a morphism of crossed complexes. By the *fibre* $\mathcal{F}_y = p^{-1}(y)$ *of* p *over* y we mean the sub-crossed complex of E of all elements of E_n which for $n = 0$ map by p to y and for $n > 0$ map by p to the identity at y.

Theorem 12.1.15 (Howie). *If $p\colon E \to B$ is a fibration of crossed complexes, and $y \in B_0$, then for each object x of $\mathcal{F}_y$ there is an exact sequence*

$$\begin{gathered}\cdots \to H_n(\mathcal{F}_y, x) \xrightarrow{i_n} H_n(E, x) \xrightarrow{p_n} H_n(B, y) \xrightarrow{\partial_n} \cdots\\ \cdots \to \pi_1(\mathcal{F}_y, x) \xrightarrow{i_1} \pi_1(E, x) \xrightarrow{p_1} \pi_1(B, y) \xrightarrow{\partial_1} \pi_0(\mathcal{F}_y) \xrightarrow{i_*} \pi_0(E) \xrightarrow{p_*} \pi_0(B).\end{gathered}$$

Here the terms of the sequence are all groups, except the last three which are sets with base points $x_{\mathcal{F}}$, x_E, y_B the components of x, x, y in $\mathcal{F}_y$, E, B respectively.

(i) *There is an operation of the group $\pi_1(E, x)$ on the right of the group $\pi_1(\mathcal{F}_y, x)$ making the morphism*

$$i_1\colon \pi_1(\mathcal{F}_y, x) \to \pi_1(E, x)$$

into a crossed module.

(ii) *There is an operation of the group $\pi_1(B, y)$ on the right of the set $\pi_0(\mathcal{F}_y)$, written $([u], \alpha) \mapsto [u] \cdot \alpha$, such that the boundary*

$$\pi_1(B, y) \xrightarrow{\partial_1} \pi_0(\mathcal{F}_y)$$

is given by $\partial_1(\alpha) = x_{\mathcal{F}} \cdot \alpha$.

Further we have additional exactness at the bottom end as follows:

(a) *if $\alpha, \beta \in \pi_1(B, y)$, then $\partial_1\alpha = \partial_1\beta$ if and only if there is a $\gamma \in \pi_1(E, x)$ such that $p_1\gamma = -\beta + \alpha$;*

(b) *if $u_{\mathcal{F}}$, $v_{\mathcal{F}}$ denote the components in $\mathcal{F}_y$ of objects u, v of $\mathcal{F}_y$, then $i_*u_{\mathcal{F}} = i_*v_{\mathcal{F}}$ if and only if there is an $\alpha \in \pi_1(B, y)$ such that $u_{\mathcal{F}} \cdot \alpha = v_{\mathcal{F}}$;*

(c) *if y_B denotes the component of y in B then*

$$i_*(\pi_0\mathcal{F}_y) = p_*^{-1}(y_B).$$

Proof. The bottom six terms of this sequence are discussed in detail for a fibration of groupoids in [Bro06], 7.2.9.[182] The operation is defined as follows. Let $\alpha \in \pi_1(B, y)$

have representative $a \in B(y)$, and let u be an object of $\mathcal{F}_y$. Let a lift to an element $\tilde{a}$ starting at $u \in (\mathcal{F}_y)_0$; then $[u] \cdot \alpha$ is defined to be the class $[u']$ of the final point u' of $\tilde{a}$. One checks that this definition is independent of the choices made. We leave the full proof of the theorem as an exercise. $\square$

Corollary 12.1.16. *Under the conditions of the theorem, we have:*

(i) *the mapping $\pi_1(B, y) \to \pi_0(\mathcal{F}_y)$, $\alpha \mapsto x_E \cdot \alpha$ induces a bijection from the set of cosets $\pi_1(B, y)/p_1\pi_1(E, x)$ to the set $i_*^{-1}(x_E)$;*

(ii) *the mapping $i_* \colon \pi_0(\mathcal{F}_y) \to \pi_0(E)$ induces a bijection from the orbit set of $\pi_0(\mathcal{F}_y)$ under the action of $\pi_1(B, y)$ to the set $p_*^{-1}(y_B)$.* $\square$

Remark 12.1.17. Notice that the first bijection of the corollary is dependent on the choice of object x in the fibre $\mathcal{F}_y$. If in a particular example $p_1\pi_1(E, x)$ is normal in $\pi_1(B, y)$, then the set of cosets inherits from $\pi_1(B, y)$ a group structure which is transferred to $i_*^{-1}(x_E)$ by the above bijection: but this group structure is dependent on the choice of x. Results of this type have been long extant in the literature, particularly in extension theory, but are not usually put in the context of fibrations of groupoids or crossed complexes.

The power of these results comes when we apply them in the next section to fibrations of internal homs $\mathsf{CRS}(F, E) \to \mathsf{CRS}(F, B)$ using Proposition 12.1.6 (iv).[183] $\square$

12.2 Homotopy classification of morphisms

In this section we analyse the set $[F, C]_*$ of pointed homotopy classes of morphisms from a reduced crossed complex F to a reduced crossed complex C for particular examples of C. Usually F will be free. We stick to the reduced and pointed case as in this case it is easier to relate the results to classical theorems, but the other case can be treated by the same methods.[184]

The cases we are thinking of are

- $F = \Pi X_*$ for X_* the skeletal filtration of a CW-complex X; and
- $F = F^{\mathrm{st}}(G)$, the standard free crossed resolution of a group or groupoid G.

In the first case we think of $[F, C]_*$ as a kind of 'nonabelian cohomology set of the space X with coefficients in the crossed complex C'. In the second case it is a kind of nonabelian cohomology of the group or groupoid G. In either case, our analysis of this set works by using fibrations of the 'coefficients', i.e. fibrations of crossed complexes. It turns out that this leads to some nice formulations of or generalisations of classical results.

Recall from Definition 7.1.11 that $\mathbb{K}(H, 1)$ is the crossed complex which is the group or groupoid H in dimension 1 and is otherwise trivial. In this simplest case we have:

Proposition 12.2.1. *For any $n \geqslant 2$, reduced crossed complex F with fundamental group G, and any group H, there is a natural bijection*

$$[F, \mathbb{K}(H, 1)]_* \cong \mathsf{Hom}(G, H).$$

Proof. This follows easily from the definitions. □

For analysis of our next cases we need the following definition.

Definition 12.2.2. Let C, D be crossed complexes, and let $f : C \to D$ be a morphism. Let $i : A \to C$ be a morphism of crossed complexes. A *homotopy* $h : f \simeq g$ *rel* i (or rel A) is a homotopy such that hi is a constant homotopy; this of course implies $fi = gi$. The resulting set of homotopy classes is written

$$[C, D; fi]. \qquad \square$$

We often employ this definition when A is the skeleton $\mathrm{Sk}^n C$ of C (see Definition 7.1.32) and i is the inclusion, so that we are dealing with a homotopy relative to levels $\leqslant n$.

Remark 12.2.3. The set $[C, D; fi]$ of homotopy classes will also be usefully interpreted using the internal hom in Crs: the morphism i induces a morphism of crossed complexes

$$i^* : \mathsf{CRS}(C, D) \to \mathsf{CRS}(A, D).$$

Let $\mathcal{F}_f$ be the fibre of i^* over fi: then

$$[C, D; fi] = \pi_0(\mathcal{F}_f). \qquad \square$$

Now we consider $\mathbb{K}_n(M, H)$ the crossed complex which is H in dimension 1, the H-module M (with the given action of H) in dimension $n \geqslant 2$, and is otherwise trivial; in particular, if $n = 2$ then the boundary $M \to H$ is assumed trivial (see Definition 7.1.11).

Theorem 12.2.4. *Let F be a reduced free crossed complex, let $G = \pi_1 F$ and let M be an H-module. Then there is a bijection*

$$[F, \mathbb{K}_n(M, H)]_* \cong \bigsqcup_{\theta} [F, \mathbb{K}_n(M, H); \theta\phi]_*$$

to the disjoint union of sets one for each morphism $\theta : G \to H$, namely those homotopy classes inducing θ. Further, the morphisms $F \to \mathbb{K}_n(M, H)$ inducing $\theta : G \to H$ may be given the structure of abelian group which is inherited by homotopy classes.

Proof. The morphism $q : \mathbb{K}_n(M, H) \to \mathbb{K}(H, 1)$ which is the identity in dimension 1 and 0 elsewhere is a fibration, which, by the pointed version of Proposition 12.1.6, induces a fibration of internal homs

$$q_* : \mathsf{CRS}_*(F, \mathbb{K}_n(M, H)) \to \mathsf{CRS}_*(F, \mathbb{K}(H, 1)).$$

The induced map on π_0 is surjective since every morphism $f : F \to \mathbb{K}(H, 1)$ may be lifted by 0 to a morphism $F \to \mathbb{K}_n(M, H)$. By Proposition 12.2.1,

$$\pi_0 \operatorname{CRS}_*(F, \mathbb{K}(H, 1)) \cong \operatorname{Hom}(G, H).$$

So we can write, using the exact sequence of Theorem 12.1.15,

$$\pi_0 \operatorname{CRS}_*(F, \mathbb{K}_n(M, H)) \cong \bigsqcup_{\theta : G \to H} [F, \mathbb{K}_n(M, H); \theta\phi]_*.$$

The abelian group structure induced by addition of values in dimension n on the set of morphisms $F \to \mathbb{K}_n(M, H)$ which extend $\theta\phi$ is clear from the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & F_{n+1} \arrow[r] \arrow[d] & F_n \arrow[r] \arrow[d] & F_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & F_2 \arrow[r] \arrow[d] & F_1 \arrow[r, "\phi"] \arrow[d] & G \arrow[d] \arrow[dl, "\theta"'] \\
\cdots \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & H \arrow[r, equal] & H
\end{tikzcd}$$

as is also the fact that this addition passes to homotopy classes rel $\theta\phi$. □

Remark 12.2.5. Thus the situation for crossed complexes is not quite like that for chain complexes with a group or groupoid of operators. In that category, two morphisms $C \to D$ over the same operator morphism $G \to H$ do indeed have a sum by addition of values. □

Definition 12.2.6. Let F be a reduced free crossed complex. We write $H^n_{\theta\phi}(F, M)$ for $[F, \mathbb{K}_n(M, H); \theta\phi]_*$, and call this abelian group the *n-th cohomology over $\theta\phi$ of F with coefficients in M*. Thus we can interpret $[F, \mathbb{K}_n(M, H)]_*$ as the disjoint union of the abelian groups $H^n_{\theta\phi}(F, M)$ for all morphisms $\theta : G \to H$. When convenient and clear, we abbreviate $\theta\phi$ to θ. □

Remark 12.2.7. In the case $F = \Pi X_*$ for a CW-filtration X_*, then we recover the cellular cohomology of X, while in the case $F = F^{\mathrm{st}}(G)$ for a group or groupoid G, then we recover the usual notions of cohomology of G as we shall see in Section 12.5. □

A generalisation of Theorem 12.2.4 is as follows.

Example 12.2.8. Let C be a reduced crossed complex such that $C_1 = H$, and $\delta_2 = 0 : C_2 \to C_1$. Let F be a free crossed complex. Then $\operatorname{Crs}_*(F, C)$ and $[F, C]_{\theta\phi}$ may be given the structure of abelian group by addition of values. □

We can use the previous results to analyse $[F, C]_*$ in another interesting case, given partly by the following definition.

Definition 12.2.9. A crossed complex C is called *n-aspherical* when C has trivial homology H_i for $1 < i < n$. □

Theorem 12.2.10 (Obstruction Class Theorem). *Let $n \geqslant 2$ and let F, C be reduced crossed complexes such that F is free, C is n-aspherical, and $C_i = 0$ for $i > n$. Let $G = \pi_1 F$, $H = \pi_1 C$, $M = \operatorname{Ker} \delta_n \colon C_n \to C_{n-1}$. Let $\theta \colon G \to H$ be a morphism of groups. Then there is defined an element $k_\theta \in H^{n+1}_{\theta\phi}(F, M)$, called the* ***obstruction class*** *of θ, such that the vanishing of k_θ is necessary and sufficient for θ to be realised by a morphism $F \to C$.*

If $k_\theta = 0$, then the set $[F, C; \theta\phi]$ of homotopy classes of morphisms $F \to C$ realising $\theta\phi$ is bijective with $H^n_{\theta\phi}(F, M)$.

Proof. Consider the morphisms of crossed complexes $C \xrightarrow{j} \xi C \xrightarrow{p} \mathbb{K}_{n+1}(M, H)$ as shown in the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & C_n \arrow[r, "\delta_n"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & C_2 \arrow[r] \arrow[d] & C_1 \arrow[d] \\
\cdots \arrow[r] & M \arrow[r] \arrow[d] & C_n \arrow[r, "\delta_n"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & C_2 \arrow[r] \arrow[d] & C_1 \arrow[d] \\
\cdots \arrow[r] & M \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[d, "j"] \\
\xi C \arrow[d, "p"] \\
\mathbb{K}_{n+1}(M, H)
\end{tikzcd}$$

Then ξC is aspherical, and $p \colon \xi C \to \mathbb{K}_{n+1}(M, H)$ is a fibration of crossed complexes which we have given more generally in Example 12.1.2.

Since F is a free reduced crossed complex, we have by Proposition 12.1.6 (iv) an induced fibration of crossed complexes

$$p_* \colon \mathsf{CRS}_*(F, \xi C) \to \mathsf{CRS}_*(F, \zeta C). \tag{12.2.1}$$

On applying π_0 to this we get, considering previous identifications, a map of sets

$$p_* \colon \mathsf{Hom}(G, H) \to \bigsqcup_{\theta \in \mathsf{Hom}(G,H)} H^{n+1}_{\theta\phi}(F, M), \tag{12.2.2}$$

and we may define $k_\theta = p_*(\theta)$.

• $k_\theta = 0$ if and only if θ is induced by a morphism $F \to C$.

Suppose θ is induced by a morphism $f \colon F \to C$. Then f factors through pj and is therefore 0 in $H^{n+1}_{\theta\phi}(F, M)$.

Suppose conversely that θ determines 0 in $H^{n+1}_{\theta\phi}(F, M)$. We know that θ is induced by a morphism $f' \colon F \to \xi C$. Then pf is homotopic to 0 and so by the fibration condition f' is homotopic to f'' such that $pf'' = 0$. Hence f'' determines $f \colon F \to C$ such that $jf = f''$. Then f also induces θ. This proves the lemma.

• If $k_\theta = 0$ the set $[F, C; \theta\phi]$ is bijective with $H^n_{\theta\phi}(F, M)$.

If $k_\theta = 0$, the morphism θ lifts to a morphism of crossed complexes $f \colon F \to C$. Let $\mathcal{F}(f)$ denote the fibre of $p_* \colon \mathsf{CRS}_*(F, \xi C) \to \mathsf{CRS}_*(F, \zeta C)$ over pf. Then we have an exact sequence

$$\cdots \to \pi_1(\mathsf{CRS}_*(F, \xi C), f) \to \pi_1(\mathsf{CRS}_*(F, \mathbb{K}_{n+1}(M, H)), pf)$$
$$\to \pi_0 \mathcal{F}(f) \to \pi_0 \, \mathsf{CRS}_*(F, \xi C) \to \cdots .$$

By Proposition 12.1.10, $\pi_1(\mathsf{CRS}_*(F, \xi C), f) = 0$, so Corollary 12.1.16 (i) gives a bijection

$$H^n_{\theta\phi}(F, M) \to [F, C; \theta\phi].$$

This completes the proof of the theorem. □

This result generalises the classical theory of extensions of groups and abstract kernels.[185] To apply the theory to that case, the crossed complex F is taken to be a free crossed resolution of the group G. One advantage of the approach taken here is that it is clear that the standard free crossed resolution may be replaced by any free crossed resolution of G, and in many cases it is possible to construct small such resolutions, so leading to a finite description of for example the classes of extensions.

Remark 12.2.11. These results can also be related to the question of obtaining a universal cover of a topological group X. In the case X is connected and admits a universal cover, the result is easy: any choice $\tilde{x}$ of base point in the universal cover whose projection is the identity of the group structure of X determines a topological group structure on the universal cover. The nonconnected case is not so straightforward: for the existence of such a topological structure there is in general an obstruction which lies in $H^3(\pi_0(X), \pi_1(X, 1))$ and which can be identified with the first Postnikov invariant of the classifying space BX of the topological group X.[186] □

12.3 Homotopy classification of maps of spaces

In this section we give some of the many consequences that can be drawn from the bijection

$$[\Pi X_*, C] \cong [X, BC]$$

proved in Theorem 11.4.19.[187]

We first get some results on reduced CW-complexes whose n-type can be realised by a classifying space of a crossed complex. In fact the main applications we give here are in the pointed case, but an example for the unpointed case is for free loop spaces in Example 11.4.21.

We obtain a key homotopy classification result, Corollary 12.3.7, expressing the topological homotopy set $[X, Y]$ as an algebraic homotopy set $[\Pi X_*, \Pi Y_*]$ when Y is n-aspherical and X is of dimension $\leqslant n$.

We end this section by looking at the algebraic part $[\Pi X_*, C]$ in some particular cases.

Our first applications of Theorem 11.4.19 gives sufficient conditions on a homotopy n-type to be realisable as BC for some crossed complex C.

Theorem 12.3.1. *Let $n \geqslant 1$, and let X be a reduced CW-complex with $\pi_i X = 0$, $1 < i < n$. (Notice that this condition is vacuous if $n = 1, 2$.) Then there is a crossed complex C with $C_i = 0$, for all $i > n$ together with a map*

$$f : X \to BC$$

inducing an isomorphism of homotopy groups $f_* : \pi_i X \to \pi_i BC$ for $1 \leqslant i \leqslant n$.

Proof. Let X_* be the skeletal filtration of X, let $X_0 = \{x\}$, and let $D = \Pi X_*$. We define C be the crossed complex such that

$$C_i = \begin{cases} D_i, & 0 \leqslant i < n, \\ \operatorname{Cok} \partial_{n+1}, & i = n, \\ 0, & i > n. \end{cases}$$

Then there is a unique morphism $g : D \to C$ which is the identity in dimensions $< n$ and is the quotient morphism in dimension n. Clearly, this morphism g induces an isomorphism of fundamental groupoids, and of homology groups $H_i(D, x) \to H_i(C, x)$ for $2 \leqslant i \leqslant n$.

By Theorem 11.5.2 there is a pointed morphism

$$f : X \to BC$$

whose homotopy class corresponds to $g : \Pi X_* \to C$. Without loss of generality we may assume f is cellular. Then for all $i \geqslant 1$, the following diagram is commutative, where $S^i = e^0 \cup e^i$ is the i-sphere:

$$\begin{tikzcd}
{[S^i, X]_*} \arrow[r, "f_*"] \arrow[d] & {[S^i, BC]_*} \arrow[d, "\cong"] \arrow[dr, "\cong"] & \\
{[\Pi S^i_*, \Pi X_*]_*} \arrow[r, "(\Pi f)_*"] \arrow[d, "\cong"'] & {[\Pi S^i_*, \Pi BC_*]_*} \arrow[r, "\cong"'] & {[\Pi S^i_*, C]_*} \arrow[d, "\cong"] \\
H_i(\Pi X_*, x) \arrow[rr, "\cong", "H_i g"'] & & H_i(C, x).
\end{tikzcd}$$

The assumptions on X imply that the map $[S^i, X]_* \to [\Pi S^i_*, \Pi X_*]_*$ is bijective for $1 \leqslant i \leqslant n$. So the result on Π_i follows. □

Remark 12.3.2. This theorem shows that if $\pi_i X = 0$, $1 < i < n$, then the n-type of X is described completely by a crossed complex.[188] □

Remark 12.3.3. The crossed complex C in the proof of Theorem 12.3.1 has the property that

$$H_i(C) = 0, \ 1 < i < n; \quad H_j(C) \cong \pi_j(X), \ j = 1, n; \quad C_i = 0, \ i > n.$$

It will be shown in Section 12.7 that $H^{n+1}(\pi_1 X, \pi_n X)$ can be represented by equivalence classes of such complexes In particular, the equivalence class of C is known as the first k-invariant, or Postnikov invariant, of X.

In the crossed module case, there is an additional result that is sometimes useful for giving an explicit presentation of a crossed module representing the 2-type of a space.[189]

Proposition 12.3.4. *Let X be a reduced CW-complex and let P be a group such that there is a map $f : BP \to X$ which is surjective on fundamental groups. Let $F(f)$ be the homotopy fibre of f and let $M = \pi_1 F(f)$, so that we have a crossed module $M \to P$. Then there is a map $X \to B(M \to P)$ inducing an isomorphism of π_1 and π_2.*

Proof. Let $f : BP \to X$ be a cellular map which is surjective on fundamental groups. Let Y be the reduced mapping cylinder $M(f)$ of f, and let $j : BP \to Y$ be the inclusion. Then the crossed module $\pi_2(Y, BP) \to \pi_1 BP$ is isomorphic to $\mu : M \to P$.

Also j is surjective on fundamental groups, and it follows that the inclusion $X^1 \to Y$ is deformable by a homotopy to a map g', say, with image in BP. This homotopy extends to a homotopy of the inclusion $X \to Y$ to a map $g : X \to Y$ extending g'.

Let Y_* be the filtered space in which Y_0 is the base point of Y, $Y_1 = BP$, $Y_i = Y$ for $i \geqslant 2$. Then $C = \Pi Y_*$ is $\mathrm{sk}^2(M \to P)$, the acyclic extension by zeros of the crossed module $M \to P$. The map $g : X \to Y$ induces a morphism $g_* : \Pi X_* \to \Pi Y_*$ which is realised by a map $X \to B(M \to P)$ inducing an isomorphism of π_1 and π_2. □

Example 12.3.5. We use the HHSvKT for crossed modules to give an application of the last proposition. Let X be a CW-complex which is the union of connected subcomplexes Y and Z such that $A = Y \cap Z$ is a $K(P, 1)$, i.e. is a space BP. Suppose that the inclusions of A into Y and Z induce isomorphisms of fundamental groups. Then, as in Proposition 12.3.4, the 2-types of Y and Z may be described by crossed modules $M \to P$ and $N \to P$ respectively, say.

By results of Part I, the crossed module describing the 2-type of X is the coproduct $M \circ N \to P$ of the crossed P-modules M and N. □

We now give another application to the homotopy classification of maps. It also concerns n-aspherical spaces and says that the homotopy classes of maps from a CW-complex of dimension $\leqslant n$ to an n-aspherical space are classified by the homotopy classes of morphisms of their fundamental crossed complexes.

Proposition 12.3.6. *For any CW-complex Y with skeletal filtration Y_*, there is a homotopy fibration*

$$F \to Y \to B\Pi Y_*.$$

Thus if $\pi_i(Y, y) = 0$ *for* $1 < i < n$*, then the fibre* F *is* n*-connected.*

Proof. Results of Chapter 14, particularly Theorem 14.2.7, give a Kan fibration

$$RY_* \to N\Pi Y_*.$$

Also for a CW-complex Y_* the inclusion of RY_* into the singular complex of Y is a homotopy equivalence. So when realising, we have a homotopy fibration sequence

$$F \to Y \to B\Pi Y_*.$$

The results on n-connectedness come from the homotopy exact sequence of this fibration. □

Corollary 12.3.7. *If* Y *is a connected CW-complex such that* $\pi_i Y = 0$ *for* $1 < i < n$*, and* X *is a CW-complex with* $\dim X \leqslant n$*, then there is a natural bijection of homotopy classes*[190]

$$[X, Y] \cong [\Pi X_*, \Pi Y_*].$$

Proof. The assumptions imply that the fibration

$$Y \to B\Pi Y_*$$

induces a bijection $[X, Y] \to [X, B\Pi Y_*]$.

The fact that the map $[X, B\Pi Y_*] \to [\Pi X_*, \Pi Y_*]$ is a bijection follows from Theorem 11.4.19. □

By Theorem 11.4.19, we get that the homotopy classes of maps are bijective with the set $[\Pi X_*, C]$. We are going to consider some cases where this algebraic defined set is computable.

The first case applies to the crossed complex $\mathbb{K}_1(G)$ associated to a groupoid G. Recall that $\mathbb{K}_1(G)$, which is actually just $\mathrm{sk}^1(G)$, is the crossed complex which is G in dimensions 0, 1 and trivial elsewhere. Then there is a bijection

$$\mathsf{Crs}(C, \mathbb{K}_1(G)) \cong \mathsf{Gpds}(\pi_1 C, G),$$

which carries over to homotopy classes

$$[C, \mathbb{K}_1(G)] \cong [\pi_1 C, G]$$

with crossed complexes on the left and groupoids on the right.

Example 12.3.8. Let G be a group, and let C_∞ be the infinite cyclic group on one generator. Then the components of the groupoid $\mathsf{GPDS}(\mathsf{C}_\infty, G)$ are bijective with the conjugacy classes of elements of G, and the vertex group of the groupoid at an element a is bijective with

$$C_a(G) = \{b \in G \mid a^b = a\},$$

the centraliser of a in G. □

Proposition 12.3.9. *If C is a crossed complex and G is a groupoid, then there is a homotopy equivalence of crossed complexes*

$$\mathsf{CRS}(C, \mathbb{K}_1(G)) \simeq \mathbb{K}_1(\mathsf{GPDS}(\pi_1 C, G)).$$

Proof. Let D be a crossed complex. Then there are natural bijections

$$\begin{aligned}
[D, \mathsf{CRS}(C, \mathbb{K}_1(G))] &\cong [D \otimes C, \mathbb{K}_1(G)] && \text{because } \mathsf{Crs} \text{ is a closed category}\\
&\cong [\pi_1(D \otimes C), G] && \text{as indicated above}\\
&\cong [\pi_1 D \times \pi_1 C, G] && \text{because } \pi_1 \text{ preserves products}\\
&\cong [\pi_1 D, \mathsf{GPDS}(\pi_1 C, G)] && \text{because } \mathsf{Gpds} \text{ is a closed category}\\
&\cong [D, \mathbb{K}_1(\mathsf{GPDS}(\pi_1 C, G))] && \text{as before.}
\end{aligned}$$

The result follows directly. □

If G is connected, $x \in G_0$, and $f : G \to H$ is a morphism, then the vertex group $\mathsf{GPDS}(G, H)(f)$ is isomorphic to the centraliser of $f(G(x))$ in $H(fx)$.[191] So the previous result with Theorem 11.4.19 gives results on the homotopy groups of a space of maps to an Eilenberg–Mac Lane space $K(H, 1)$.

In the pointed case the proposition gives an even simpler result.

Proposition 12.3.10. *If C is a pointed, connected crossed complex and G is a pointed groupoid, then the crossed complex $\mathsf{CRS}_*(C, \mathbb{K}_1(G))$ has its set of components bijective with $\mathsf{Gpds}(\pi_1(C, *), G(*))$ the set of morphisms of groups $\pi_1(C, *) \to G(*)$, and all components of $\mathsf{CRS}_*(C, \mathbb{K}_1(G))$ have trivial π_1 and H_i for $i \geqslant 2$.*

Proof. An argument similar to that in the proof of the previous proposition yields

$$[*, \mathsf{CRS}_*(C, \mathbb{K}_1(G))] \cong [*, \mathsf{GPDS}_*(\pi_1 C, G)],$$

which gives the first result. The second result is obtained as follows: for any pointed crossed complex Z and morphism $f : C \to \mathbb{K}_1(G)$ which we take as base point, let u be given by the on the appropriate subcomplex of $Z \otimes C$ by the morphisms $1 \otimes f : * \otimes C \to \mathbb{K}_1(G), * : Z \otimes * \to \mathbb{K}_1(G)$ and let v be given on the appropriate subgroupoid of $\pi_1 Z \times \pi_1 C$ by the morphisms $1 \times \pi_1 f : * \times \pi_1 C \to G, * : \pi_1 Z \times * \to G$. Then we have

$$\begin{aligned}
[(Z, *), (\mathsf{CRS}_*(C, \mathbb{K}_1(G)), f)] &\cong [Z \otimes C, \mathbb{K}_1(G); u]\\
&\cong [\pi_1 Z \times \pi_1 C, G; v]\\
&\cong [\pi_1 C, G; \pi_1 f]\\
&\cong *.
\end{aligned}$$
□

There is another interesting special case of the homotopy classification. Let M be an abelian group with automorphism group Aut M and let $n \geqslant 2$. Then we have the crossed complex $\mathbb{K}_n(M, \mathrm{Aut}\, M)$ which is Aut M in dimension 1, M in dimension n, has the given action of Aut M on M, and has trivial boundaries.

Let C be a crossed complex; in useful cases, C will be of free type. We suppose C reduced and pointed. Let $\alpha\colon \pi_1(C,*) \to \mathrm{Aut}\, M$ be a morphism. The set of pointed homotopy classes of morphisms $C \to \mathbb{K}_n(M, \mathrm{Aut}\, M)$ which induce α on fundamental groups is written $[C, \mathbb{K}_n(M, \mathrm{Aut}\, M)]^{\alpha}_*$. This set is easily seen to have an abelian group structure, induced by the addition on operator morphisms $C_n \to M$ over α. So we obtain the homotopy classification:

Proposition 12.3.11. *If X is a pointed reduced CW-complex, and $\alpha\colon \pi_1(X,*) \to \mathrm{Aut}\, M$, then there is a natural bijection*

$$[X, B\mathbb{K}_n(M, \mathrm{Aut}\, M)]^{\alpha}_* \cong [\Pi X_*, \mathbb{K}_n(M, \mathrm{Aut}\, M)]^{\alpha}_*$$

where the former set of homotopy classes denotes the set of pointed homotopy classes of maps inducing α on fundamental groups.

Proof. The proof is immediate from Theorem 11.5.2. □

We can also interpret the more general Obstruction Class Theorem 12.2.10 as follows.[192]

Theorem 12.3.12. *Let X be a connected CW-complex of dimension $\leqslant n$, and let Y be a connected CW-complex such that $\pi_i Y = 0$ for $1 < i < n$. Let $G = \pi_1 X$, $H = \pi_1 Y$, and let $M = \pi_n Y$ considered as an H-module. Then a morphism $\theta\colon G \to H$ determines an element $k_\theta \in H^n_{\theta\phi}(X, M)$ whose vanishing is necessary and sufficient for θ to be realised by a map $f : X \to Y$. If θ is realisable, then the homotopy classes of maps realising θ are bijective with $H^{n-1}_{\theta\phi}(X, M)$.* □

Example 12.3.13. The following example is intended to illustrate some features of this homotopy classification result, including the fact that we can sometimes do the specific group ring calculations involved, in this case easily as the main calculations are over the group C_2.

Let D be the crossed complex $\mathrm{Sk}^3\, F(\mathsf{C}_2)$; thus D is a reduced crossed complex trivial above dimension 3, and is in dimensions $\leqslant 3$ given as

$$\mathbb{Z}[\mathsf{C}_2] \xrightarrow{\delta_3} \mathbb{Z}[\mathsf{C}_2] \xrightarrow{\delta_2} \mathsf{C}$$

with free generators x_i in dimension i for $i = 1, 2, 3$ and

$$\delta_2(x_2) = x_1^2, \quad \delta_3(x_3) = x_2(1 - c)$$

where c is the nontrivial element of C_2; we also include in D the base point x_0. The classifying space $Z = B(D)$ is a model of $\mathbb{R}P^3$ with homotopy groups of dimension $\geqslant 3$ killed. The main Homotopy Classification Theorem 11.4.19 shows that if K is a CW-complex, then the homotopy classes of maps $K \to BD$ are bijective with the homotopy classes of crossed complex morphisms $\Pi K_* \to D$. Then Theorem 12.3.12 can be applied since $\pi_2(BD) = 0$.

In order to illustrate our methods and the tensor product, we choose $K = \mathbb{R}P^2 \times \mathbb{R}P^2$, and let K_* be its standard skeletal filtration. Then $C = \Pi K_*$ is isomorphic to the tensor product $\Pi \mathbb{R}P^2_* \otimes \Pi \mathbb{R}P^2_*$, and so is freely generated by $x_i \otimes x_j$ for $0 \leqslant i, j \leqslant 2$ with base point $x_0 \otimes x_0$ and boundaries given by the rules of the tensor product, i.e.

$$\begin{aligned}
\delta_2(x_2 \otimes x_0) &= x_1^2 \otimes x_0,\\
\delta_2(x_0 \otimes x_2) &= x_0 \otimes x_1^2,\\
\delta_2(x_1 \otimes x_1) &= (x_0 \otimes x_1)^{-1}(x_1 \otimes x_0)^{-1}(x_0 \otimes x_1)(x_1 \otimes x_0),\\
\delta_3(x_2 \otimes x_1) &= -x_2 \otimes x_0 + (x_2 \otimes x_0)^{x_0 \otimes x_1} + x_1^2 \otimes x_1\\
&= -x_2 \otimes x_0 + (x_2 \otimes x_0)^{x_0 \otimes x_1} + x_1 \otimes x_1 + (x_1 \otimes x_1)^{x_1 \otimes x_0},\\
\delta_3(x_1 \otimes x_2) &= -x_1 \otimes x_1^2 - x_0 \otimes x_2 + (x_0 \otimes x_2)^{x_1 \otimes x_0}\\
&= -(x_1 \otimes x_1) - (x_1 \otimes x_1)^{x_0 \otimes x_1} - (x_0 \otimes x_2) + (x_0 \otimes x_2)^{x_1 \otimes x_0},\\
\delta_4(x_2 \otimes x_2) &= x_1^2 \otimes x_2 + x_2 \otimes x_1^2\\
&= x_1 \otimes x_2 + (x_1 \otimes x_2)^{x_1 \otimes x_0} + (x_2 \otimes x_1)^{x_0 \otimes x_1} + x_2 \otimes x_1.
\end{aligned}$$

Now the group $\pi_1(C)$ is isomorphic to $\mathsf{C}_2 \times \mathsf{C}_2$ so we have to consider the morphisms $\theta\colon \mathsf{C}_2 \times \mathsf{C}_2 \to \mathsf{C}_2$ and consider which of these lift to a morphism $C \to D$. We consider only the case $\theta = (1, 1)$. Then θ lifts to a morphism $f : C \to D$ given by

$$\begin{aligned}
f_1(x_1 \otimes x_0) &= -f_1(x_0 \otimes x_1) = x_1,\\
f_2(x_2 \otimes x_0) &= -f_2(x_0 \otimes x_2) = x_2,\\
f_2(x_1 \otimes x_1) &= 0,\\
f_3(x_2 \otimes x_1) &= -f_3(x_1 \otimes x_2) = x_3,\\
f_4(x_2 \otimes x_2) &= 0.
\end{aligned}$$

We leave the checks to the reader.

So we have the following diagram:

$$\begin{tikzcd}
Y_4 \arrow[r, "\delta_4"] \arrow[d, dashed, "f_4"'] & Y_3 \arrow[r, "\delta_3"] \arrow[d, dashed, "f_3"'] & Y_2 \arrow[r, "\delta_2"] \arrow[d, dashed, "f_2"'] & Y_1 \arrow[r, Rightarrow, "\phi"] \arrow[d, dashed, "f_1"'] & \mathsf{C}_2 \times \mathsf{C}_2 \arrow[d, "\theta"] \\
0 \arrow[r] & \mathbb{Z}[\mathsf{C}_2] \arrow[r, "\delta_3"'] & \mathbb{Z}[\mathsf{C}_2] \arrow[r, "\delta_2"'] & \mathsf{C} \arrow[r, Rightarrow, "\phi"'] & \mathsf{C}_2.
\end{tikzcd} \tag{12.3.1}$$

Let $M = \operatorname{Ker}(\delta_3\colon \mathbb{Z}[\mathsf{C}_2] \to \mathbb{Z}[\mathsf{C}_2])$; then M is generated in $\mathbb{Z}[\mathsf{C}_2]$ by $x_3(1 + c)$.

The based homotopy classes of maps inducing θ are bijective with $H^3_{\theta\phi}(C, M)$. So we have to see how to calculate this.

A morphism $g\colon Y_3 \to M$ over θ is determined by its values on the free generators $x_2 \otimes x_1$, $x_1 \otimes x_2$ of Y_3 and g is a cocycle if and only if $g\delta_4 = 0$. From the formulae above we see that

$$g\delta_4(x_2 \otimes x_2) = (g(x_1 \otimes x_2) + g(x_2 \otimes x_1))(1 + c),$$

and therefore g is a cocycle if and only if $g(x_1 \otimes x_2) + g(x_2 \otimes x_1) = 0$.

Now the question is how much such a g can be altered by a homotopy, i.e. by $H\delta_3$ where $H : Y_2 \to M$ is a morphism over θ. But as H is a θ-morphism, with values in M, which is generated by $x_3(1 + c)$ we find that if $H(x_1 \otimes x_1) = ax_3(1 + c)$ where $a \in \mathbb{Z}$ then

$$\begin{aligned} H\delta_3(x_2 \otimes x_1) &= H(x_2 \otimes x_0)(c - 1) + H(x_1 \otimes x_1)(1 + c) \\ &= 0 + x_3(1 + c)^2 a \quad \text{say, where } a \in \mathbb{Z} \\ &= 2ax_3(1 + c). \end{aligned}$$

Similarly, $H\delta_3(x_1 \otimes x_2) = 2ax_3(1 + c)$. Hence $H^3_{\theta\phi}(K, M) \cong \mathbb{Z}_2$. □

12.4 Local coefficients and local systems

The homotopy classification result of Theorem 11.4.19 suggests that if A is a crossed complex, and X is a space, with singular complex SX (either simplicial or cubical), then the set $[\Pi|SX|, A]$ may be thought of as singular cohomology of X with coefficients in A, and written $H^0(X, A)$. With our present machinery, it is easy to see that this cohomology is a homotopy functor of both X and of A. We show in this section that $H^0(X, A)$ is (non-naturally) a union of abelian groups, each of which is a kind of cohomology with 'local coefficients' in a generalised local system where the coefficients are chain complexes with a groupoid of operators.[193]

Let C and A be crossed complexes. In examples, C is to be thought of as ΠX_* for some CW-complex X. Let α be a morphism of groupoids $\alpha : C_1 \to A_1$ such that $\alpha(\delta_2 C_2) \subseteq \delta_2 A_2$. This last condition ensures that α induces a morphism of groupoids $\pi_1 C \to \pi_1 A$. It is also a necessary condition for there to exist a morphism $C \to A$ extending α.

Definition 12.4.1. By a *local system of type A on C* we mean the crossed complex A together with an α satisfying the above condition. □

The morphism α induces an operation of C_1 on all the groupoids A_n for $n \geqslant 2$. By a cocycle of C with coefficients in α we mean a morphism $f : C \to A$ of crossed complexes such that $f_1 = \alpha$. By a homology of such cocycles f, g we mean a homotopy $(h, g) : f \simeq g$ of morphisms of crossed complexes such that $h_0 x$ is a zero for all $x \in C_0$, and $\delta_2 h_1 = 0$. The set of homology classes of cocycles of C with coefficients in $\mathcal{A}$ is written $[C, A]_\alpha$.

Definition 12.4.2. If A is a crossed complex and $n \geqslant 1$ we write $A^{(n)}$ for the crossed complex

$$\mathrm{sk}^{n+1}(\operatorname{Im}\delta_{n+1} \to A_n \xrightarrow{\delta_n} A_{n-1} \to \cdots \to A_1).$$

The projection $p\colon A \to A^{(n)}$ is a fibration which is an n-equivalence and its kernel is written $A^{[n]}$. This is often called the *n-th Postnikov decomposition* of the crossed complex A. □

Remark 12.4.3. You should notice in the proof of the following proposition how the choice of a cocycle f with coefficients in α, i.e. a choice of morphism $C \to A$ extending α, allows the imposition of an abelian group structure on the morphisms also extending α, with f as zero. The proof uses the axiom CM2) for crossed modules. □

Proposition 12.4.4. *Let C, A be crossed complexes and let α be a local system of type A on C. Let $Q = \pi_1 A$ and let A' be the crossed complex which is Q in dimension 1 and agrees with $A^{[n]}$ in higher dimensions, with Q as groupoid of operators and with trivial boundary from dimension 2 to dimension 1. Let α' be the composite*

$$C_1 \xrightarrow{\alpha} A_1 \to Q.$$

Then a choice of cocycle f with coefficients in α determines a bijection

$$[C, A]_\alpha \to [C, A']_{\alpha'},$$

and hence an abelian group structure on $[C, A]_\alpha$.

Proof. We are given f extending α. Let g be another morphism $C \to A$ extending α. Then $g_1 = f_1$ and $\delta g_2 = \delta f_2$. For such a g we define $rg_n = g_n - f_n$, $n \geqslant 2$. Clearly rg_n is a morphism of abelian groups for $n \geqslant 3$; we prove that it is also a morphism for $n = 2$. Let $c, d \in C_2$. Then

$$\begin{aligned} rg_2(c+d) &= g_2(c+d) - f_2(c+d) \\ &= g_2c + (g_2d - f_2d) - f_2c \\ &= g_2c - f_2c + (g_2d - f_2d) \quad \text{since } \delta_2(g_2 - f_2) = 0 \\ &= rg_2c + rg_2d. \end{aligned}$$

Clearly rg_n is a C_1-operator morphism where C_1 acts on $A^{[1]}$ via α. Also $\delta rg_2 = 0$ and for $n \geqslant 3$, $\delta rg_n = r\delta g_{n-1}$. So we may regard rg as a morphism $C \to A'$ extending α. It is clear that r defines a bijection between the morphisms $g\colon C \to A$ extending α and the morphisms $g'\colon C \to A'$ extending α'. Next suppose that (h, g) is a homology $\bar{g} \simeq g$ as defined above. Then $\delta h_1 = 0$. Hence h_1 defines uniquely $k_1\colon C_1 \to \operatorname{Ker}\delta_2$. Further we have for $n \geqslant 2$

$$\bar{g}_n = g_n + h_{n-1}\delta_n + \delta_{n+1}h_n.$$

For $n \geqslant 2$ let $k_n = h_n$. Then k_n is a C_1-operator morphism. Now for $x \in C_n$ and $n \geqslant 3$, we have $h_{n-1}\delta_n x + \delta_{n+1} h_n x$ lies in an abelian group, while for $n = 2$ it lies in the centre of A_2 and so commutes with $f_2 x$. It follows that (k, rg) is a homology $r\bar{g} \simeq rg$.

Conversely, a homology $r\bar{g} \simeq rg$ of α'-cocycles determines uniquely a homology $\bar{g} \simeq g$ of $\mathcal{A}$-cocycles. It follows that r defines a bijection $[C, A]_\alpha \to [C, A']_{\alpha'}$ as required.

Notice also that the set $[C, A']_{\alpha'}$ obtains an abelian group structure, by addition of values, and with the class of rf as zero. □

Let C be a reduced cofibrant crossed complex, and let A be a reduced crossed complex. We are interested in analysing the fibres of the function

$$\eta \colon [C, A]_* \to \mathsf{Hom}(\pi_1 C, \pi_1 A).$$

We write $[C, A]_*^\alpha$ for $\eta^{-1}(\alpha)$. This set may be empty. We have in Theorem 12.2.10 analysed the obstruction to an element α lying in the image of η. Here our aim is to show that if $f : C \to A$ is a morphism realising $\alpha \colon \pi_1 C \to \pi_1 A$ then f determines an abelian group structure on $\eta^{-1}(\alpha)$. To this end we recall from Section 7.4 the relations between crossed complexes and chain complexes with operators.

Let $Q = \pi_1 A$. If $A^{[1]}$ is as in Definition 12.4.2 we consider the pair $(A^{[1]}, Q)$ to be a chain complex with Q as groupoid of operators.

Proposition 12.4.5. *Let C, A be reduced crossed complexes such that C is free. Let $f : C \to A$ be a morphism inducing $\alpha \colon \pi_1 C \to \pi_1 A$ on fundamental groups. Then f determines a bijection*

$$[C, A]_*^\alpha \cong [\nabla C, (A^{[1]}, Q)]^\alpha,$$

where the latter term is the set of pointed homotopy classes of morphisms which are morphisms of chain complexes with operators and which induce α on operator groups.

Proof. Let $p \colon A \to A^{(1)}$ denote the Postnikov fibration so that $A^{[1]}$ is the kernel of p. Let $G = \pi_1 C$. Recall that $\mathbb{K}(Q, 1)$ denotes the crossed complex which is Q in dimension 1 and is zero elsewhere. The projection $A^{(1)} \to \mathbb{K}(Q, 1)$ is a acyclic fibration. Since C is free, Proposition 12.1.10 implies that the induced morphism $\mathsf{CRS}_*(C, A^{(1)}) \to \mathsf{CRS}_*(C, \mathbb{K}(Q, 1))$ is also a acyclic fibration. It follows from Proposition 12.2.1 that $\mathsf{CRS}_*(C, A^{(1)})$ has component set $\mathsf{Hom}(G, H)$ and has trivial fundamental and homology groups.

Suppose that $f : C \to A$ induces $\alpha \colon G \to Q$ on fundamental groups. Let $F(f)$ be the fibre of $\mathsf{CRS}_*(C, A) \to \mathsf{CRS}_*(C, A^{(1)})$ over pf. Then the exact sequence of this fibration yields an exact sequence

$$1 \to \pi_0 F(f) \to [C, A]_* \xrightarrow{\eta} \mathsf{Hom}(G, Q)$$

such that the first map is an inclusion with image $\eta^{-1}(\alpha)$.

Let $\mathcal{A} = f_1$. Then $\pi_0 F(f) = [C, A]_{\mathcal{A}}$. So by Proposition 12.4.4 $\pi_0 F(f)$ is bijective with $[C, A']_{\mathcal{A}'}$. But $A' = \Theta(A^{[1]}, Q)$. The proposition follows directly from the adjointness of ∇ and Θ. $\square$

Corollary 12.4.6. *If $\alpha \colon \pi_1 C \to \pi_1 A$ is realisable by a morphism $f : C \to A$, then a choice of such morphism determines an abelian group structure on $[C, A]^{\alpha}_{*}$.*

This result gives the expected abelian group structure on this generalised cohomology with local coefficients.

12.5 Cohomology of a groupoid

We now introduce in our terms the notion of cohomology of a groupoid. This generalisation from groups to groupoids is used in Section 12.5.i to give a formulation of the cohomology of a cover of a topological space.

Definition 12.5.1. Let G be a groupoid, and M a G-module. The *n-th cohomology of G with coefficients in M* is defined to be the set of homotopy classes

$$H^n(G, M) = [F^{\mathrm{st}}_*(G), \mathbb{K}_n(M, G); \phi] \tag{12.5.1}$$

where $F^{\mathrm{st}}_*(G)$ is the standard free crossed resolution of the groupoid G, and the map $\phi \colon F^{\mathrm{st}}_1(G) \to G$ is the standard morphism; recall that $F^{\mathrm{st}}_1(G)$ is the free groupoid on the elements of G. $\square$

Recall from Definition 10.2.7 that the standard free crossed resolution of a groupoid G is:

$$\cdots \longrightarrow F^{\mathrm{st}}_*(G)_3 \xrightarrow{\delta_3} F^{\mathrm{st}}_*(G)_2 \xrightarrow{\delta_2} F^{\mathrm{st}}_*(G)_1 \xrightarrow{\phi} G$$

in which $F^{\mathrm{st}}_n(G)$ is free on the set $(N^{\Delta}G)_n$ of composable sequences

$$[g_1, g_2, \ldots, g_n]$$

of elements g_i of G, and the base point $t[g_1, g_2, \ldots, g_n]$ is the final point tg_n of g_n. For $n \geqslant 2$ the boundary

$$\delta_n \colon F^{\mathrm{st}}_n(G) \to F^{\mathrm{st}}_{n-1}(G)$$

is given by

$$\begin{aligned} \delta_2[g, h] &= [gh]^{-1}[g][h], \\ \delta_3[g, h, k] &= [g, h]^k [h, k]^{-1} [g, hk]^{-1} [gh, k], \end{aligned}$$

and for $n \geqslant 4$

$$\delta_n[g_1, g_2, \dots, g_n] = [g_1, \dots, g_{n-1}]^{g_n} + (-1)^n[g_2, \dots, g_n] + \sum_{i=1}^{n-1}(-1)^{n-i}[g_1, g_2, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_n].$$

See also the pictures in Example 9.10.3.[194]

Example 12.5.2. Let G, M be groups, let $\mathcal{M} = (\chi\colon M \to \operatorname{Aut} M)$ be the automorphism crossed module of M and let $\kappa\colon F_*^{\mathrm{st}}(G) \to \mathrm{sk}^2\,\mathcal{M}$ be a morphism of crossed complexes. Then κ is determined by its values on the free generators of $F_*^{\mathrm{st}}(G)$ in dimensions 1 and 2 and so is equivalent to a pair of functions

$$\kappa^1\colon G \to \operatorname{Aut} M, \quad \kappa^2\colon G \times G \to M$$

satisfying

$$\chi\kappa^2(g,h) = \kappa^1(gh)^{-1}\kappa^1(g)\kappa^1(h), \tag{fset1}$$

$$1 = \kappa^2(g,h)^{\kappa^1(k)}\kappa^2(h,k)^{-1}\kappa^2(g,hk)^{-1}\kappa^2(gh,k) \tag{fset2}$$

for all $g, h, k \in G$. The last two conditions are (possibly with different conventions) the conditions for what is known in the literature as a *factor set*, usually with $\mathcal{M}$ being the crossed module $M \to \operatorname{Aut} M$. Then an extension of M by G may be constructed by giving a product structure on $E = G \times M$ by the rule

$$(g,m)(h,n) = (gh, \kappa^2(g,h)m^{\kappa^1 h}n), \tag{prod}$$

and defining $i\colon M \to E$, $p\colon E \to G$ by $i(m) = (1,m)$, $p(g,m) = g$. The condition (fset2) is then exactly the condition for the product on E to be associative. This is not surprising because of the relation to associativity of the boundary δ_3 in $F_*^{\mathrm{st}}(G)$.

Conversely, given an extension $1 \to M \xrightarrow{i} E \xrightarrow{p} G \to 1$ of M by G, then choose a section $s\colon G \to E$ of p such that $s(1) = 1$. This defines a bijection $\alpha\colon E \to G \times M$ by $e \mapsto (pe, i^{-1}((sp(e))^{-1}e))$. Note that $p((sp(e))^{-1}e)) = 1$ since $ps = 1_G$. The problem is to define a multiplication on $G \times M$ so that α is a morphism (and so an isomorphism). This choice of s is also equivalent to choosing $\kappa'\colon F_1^{\mathrm{st}}(G) \to E$ such that $p\kappa' = \phi$. But since M is normal in E the operation of E on M by conjugation gives a morphism $\chi_E\colon E \to \operatorname{Aut} M$. Let $\kappa^1 = \chi_E\kappa'$. The rule (prod) then gives the 'obstruction' to the product on E being just the semidirect product. □

Exercise 12.5.3. Verify the assertions of the last example, and relate the construction to that in Proposition 12.6.3. □

Remark 12.5.4 (Homotopies for F^{st}). We have shown in Corollary 11.4.14 that the cubical nerve $N\colon \mathsf{Crs} \to \mathsf{Cub}$ is a homotopy functor, and from this one can deduce that $\Pi \circ N\colon \mathsf{Crs} \to \mathsf{Crs}$ is also a homotopy functor.

However it is the simplicial nerve which gives rise to the more usual standard crossed resolution F^{st} of a group or groupoid, and we also want to see this resolution as a homotopy functor. This remark explains the background to this.

Methods analogous to those of Section 11.4.ii also give an adjunction for the simplicial nerve

$$\Pi\colon \mathsf{Simp} \rightleftarrows \mathsf{Crs}\colon N^{\Delta}$$

with counit $\varepsilon\colon \Pi N^{\Delta} \to 1$. If the crossed complex C is essentially a groupoid, this counit $\varepsilon_C\colon \Pi N^{\Delta}C \to C$ is a weak homotopy equivalence. The crossed complex simplicial Eilenberg–Zilber–Tonks Theorem of Section 10.4.v may be used to show that the simplicial nerve N^{Δ} of a crossed complex is a homotopy functor. For suppose given a homotopy $h\colon \mathcal{I} \otimes C \to D$ of crossed complexes. We need to construct from h a simplicial homotopy $h^{\Delta}\colon \Delta^1 \times N^{\Delta}C \to N^{\Delta}D$. Recall that $\Pi\Delta^1 \cong \mathcal{I}$. By adjointness, we need a map

$$\Pi(\Delta^1 \times N^{\Delta}C) \to D.$$

This is to be the composite

$$\Pi(\Delta^1 \times N^{\Delta}C) \xrightarrow{a} \Pi\Delta^1 \otimes \Pi N^{\Delta}C \xrightarrow{1\otimes\varepsilon} \mathcal{I} \otimes C \xrightarrow{h} D,$$

where a is the map of the Eilenberg–Zilber–Tonks Theorem 10.4.14, and ε is the counit of the adjunction. From h^{Δ} we can get a morphism $\mathcal{I} \otimes \Pi N^{\Delta}C \to \Pi N^{\Delta}D$ as the composite

$$\Pi\Delta^1 \otimes \Pi N^{\Delta}C \xrightarrow{b} \Pi(\Delta^1 \times N^{\Delta}C) \xrightarrow{\Pi(h^{\Delta})} \Pi N^{\Delta}D$$

where b is the other map of the Eilenberg–Zilber–Tonks Theorem 10.4.14.[195] □

Remark 12.5.5. It might be argued that the case of the cohomology of a groupoid is not so interesting as that of a group, since the homotopy type of a groupoid is that of a disjoint union of groups. The argument against this is that very often we are interested in structured groupoids or, as we have seen in the Seifert–van Kampen Theorem, the relations between families of groupoids, and very often there is no way to obtain this 'reduction to a family of groups' in a way respecting the structure. In the next section, we consider a groupoid arising from a cover of a space, and this groupoid has a natural topology making it a topological groupoid. This topological groupoid again does 'not reduce to a family of disjoint topological groups'. In such case it is possible to make the standard resolution into a *topological standard resolution* in such a way that it has the universal property of a free crossed resolution but for continuous maps. This allows for cohomology with coefficients in a topological module or crossed complex. □

12.5.i The cohomology of a cover of a space

In this section we show how to assign a free crossed complex to a cover $\mathcal{U}$ of a topological space X, so leading to a notion of nonabelian cohomology of the cover.

Let $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ be family of subsets of the topological space X. We define the projection map

$$p\colon E\mathcal{U} = \bigsqcup_\lambda U_\lambda \to X \tag{12.5.2}$$

to send $(x,\lambda) \mapsto x$, $x \in U_\lambda$. This projection defines an equivalence relation Equ $\mathcal{U}$ on $E\mathcal{U}$, which is of course a special kind of groupoid: the objects of Equ $\mathcal{U}$ are pairs (x,λ) such that $x \in U_\lambda$; there is a unique arrow $(x,\lambda) \to (x,\mu)$ if and only if $x \in U_\lambda \cap U_\mu$. Hence we can form

$$F_*(\mathcal{U}) = F_*^{\mathrm{st}}(\mathrm{Equ}\,\mathcal{U}), \tag{12.5.3}$$

which we call the *standard crossed resolution of the cover* $\mathcal{U}$. If C is a crossed complex, then we can form

$$H^0(\mathcal{U}, C) = [F_*(\mathcal{U}), C]. \tag{12.5.4}$$

Example 12.5.6. A free basis element $[g_1,\dots,g_n]$ of $F_n(\mathcal{U})$ is equivalent to a sequence $[x,\lambda_0,\lambda_1,\dots,\lambda_n]$ such that $x \in U_{\lambda_0}\cap\cdots\cap U_{\lambda_n}$. Then we have the boundary formulae in $F_*(\mathcal{U})$:[196]

$$\begin{aligned}
\delta_2[x,\lambda,\mu,\nu] &= [x,\lambda,\nu]^{-1}[x,\lambda,\mu][x,\mu,\nu],\\
\delta_3[x,\lambda,\mu,\nu,\xi] &= [x,\lambda,\mu,\nu,\xi]^{[x,\nu,\xi]}\ [x,\mu,\nu,\xi]^{-1}[x,\lambda,\mu,\xi]^{-1}[x,\lambda,\nu,\xi].
\end{aligned}$$

So we can analyse this definition in the particular case when C is a crossed module of groups $\partial\colon M \to P$, and say that a *cocycle* $f = (f_1, f_2)$ of $\mathcal{U}$ with values in this crossed module consists of functions with values $f_1[x,\lambda,\mu,] \in P$, $f_2[x,\lambda,\mu,\nu] \in M$ and satisfying

$$\begin{aligned}
\partial f_2[x,\lambda,\mu,\nu] &= f_1([x,\lambda,\nu]^{-1}[x,\lambda,\mu][x,\mu,\nu]),\\
f_2\delta_3[x,\lambda,\mu,\nu,\xi] &= 1.
\end{aligned}$$

Now let the cover $\mathcal{V} = \{V_\alpha\}_{\alpha\in A}$ of X be a refinement of the cover $\mathcal{U}$. This means there is a function $\phi\colon A \to \Lambda$ such that for each $\alpha \in A$ we have $V_\alpha \subseteq U_{\phi(\alpha)}$. Such a *refinement map* defines a groupoid morphism $\phi_*\colon\ \mathrm{Equ}(\mathcal{V}) \to \mathrm{Equ}(\mathcal{U})$ by $(x,\alpha,\beta) \mapsto (x,\phi(\alpha),\phi(\beta))$. One easily checks that if $\psi\colon A \to \Lambda$ is another refinement map, then the two groupoid morphisms $\phi_*, \psi_*\colon\ \mathrm{Equ}(\mathcal{V}) \to \mathrm{Equ}(\mathcal{U})$ are homotopic by the homotopy h which assigns to (x,α) the element $(x,\phi\alpha) \to (x,\psi\alpha)$ of $\mathrm{Equ}(\mathcal{U})$.

The weight of this is that ϕ, ψ then induce homotopic morphisms $\phi_*, \psi_*\colon F(\mathcal{V}) \to F(\mathcal{U})$ of free crossed resolutions, by Remark 12.5.4; hence ϕ_*, ψ_* induce the same function

$$[F(\mathcal{U}), C] \to [F(\mathcal{V}), C].$$

This is the start of defining the nonabelian Čech cohomology of the space X with coefficients in the crossed complex C using refinements of open covers and taking inverse limits of the corresponding sets of homotopy classes.[197] □

12.6 Dimension 2 cohomology of a group

This section gives an account of the theory of nonabelian extensions of a group M by a group G, that is the aim is to classify extensions $1 \to M \to E \to G \to 1$. An immediate difference between this and the abelian case is that we do not get an action of G on M from such an extension. We have already mentioned in Example 12.5.2 the 'factor sets' which arise. It turns out that it is convenient to be more specific about how the actions arise by using extensions of the type of a crossed module.[198]

Definition 12.6.1. Let $\mathcal{M}$ denote the crossed module $\mu\colon M \to P$. An *extension* (i, p, σ) *of type* $\mathcal{M}$ of the group M by the group G is

(i) an exact sequence of groups

$$1 \to M \xrightarrow{i} E \xrightarrow{p} G \to 1$$

so that E operates on M by conjugation, and $i : M \to E$ is hence a crossed module; and

(ii) a morphism of crossed modules

$$\begin{tikzcd} 1 \arrow[r] & M \arrow[r, "i"] \arrow[d, equal] & E \arrow[d, "\sigma"] \\ & M \arrow[r, "\mu"] & P \end{tikzcd}$$

i.e. $\sigma i = \mu$ and $m^e = m^{\sigma e}$ for all $m \in M$, $e \in E$; thus the action of E on M is also via σ.

We shall write such an extension as

$$1 \longrightarrow M \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1, \qquad E \xrightarrow{\sigma} P.$$

Two such extensions of type $\mathcal{M}$

$$1 \longrightarrow M \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1, \qquad E \xrightarrow{\sigma} P,$$

$$1 \longrightarrow M \xrightarrow{i'} E' \xrightarrow{p'} G \longrightarrow 1, \qquad E' \xrightarrow{\sigma'} P.$$

are said to be *equivalent* if there is a morphism of exact sequences

$$\begin{tikzcd} 1 \arrow[r] & M \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\phi"'] & G \arrow[r] \arrow[d, equal] & 1 & E \arrow[r, "\sigma"] \arrow[d, "\phi"'] & P \arrow[d, equal] \\ 1 \arrow[r] & M \arrow[r] & E' \arrow[r] & G \arrow[r] & 1, & E' \arrow[r, "\sigma'"] & P. \end{tikzcd}$$

such that the right-hand square also commutes. Of course in this case ϕ is an isomorphism, by the 5-lemma, and hence equivalence of extensions is an equivalence relation.

We denote by

$$\mathrm{OpExt}_{\mathcal{M}}(G, M)$$

the set of equivalence classes of all extensions of type $\mathcal{M}$ of M by G. □

The usual theory of extensions of a group M by a group G considers extensions of the type of the crossed module $\chi_M : M \to \mathrm{Aut}\, M$. The advantages of replacing this by a general crossed module are first that the group $\mathrm{Aut}\, M$ is not a functor of M, so that the relevant cohomology theory in terms of χ_M appears to have no morphisms of coefficients, and second, that the more general case occurs geometrically.[199]

Theorem 12.6.2. *Suppose given a crossed sequence*

$$0 \to \pi \xrightarrow{i} F_2 \xrightarrow{\delta} F_1 \xrightarrow{\phi} G \to 1$$

and a crossed module $\mathcal{M} = (\mu : M \to P)$. Let $\mathcal{F}$ denote the crossed module $\delta : F_2 \to F_1$. Let $[\mathcal{F}, \mathcal{M}]^0$ denote the set of homotopy classes of morphisms $\mathbf{k} = (k^2, k^1) : \mathcal{F} \to \mathcal{M}$ of crossed modules, such that $k^2(i\pi) = 1$. Then there is a natural injection

$$\mathbf{E} : [\mathcal{F}, \mathcal{M}]^0 \to \mathrm{OpExt}_{\mathcal{M}}(G, M)$$

sending the class of a morphism $\mathbf{k}$ to the extension

$$1 \to M \to E(\mathbf{k}) \to G \to 1$$

where $E(\mathbf{k})$ is the quotient of the semidirect product group $F_1 \ltimes M$, in which F_1 acts on M via P. The function $\mathbf{E}$ is surjective if F_1 is a free group.

Proof. The heart of the proof is in the following proposition, which gives a formulation as a kind of pushout of the construction of nonabelian extensions of groups. This formulation is convenient for the development of the theory and the proof of theorems.

Proposition 12.6.3. *Suppose given a crossed sequence*

$$0 \to \pi \xrightarrow{i} F_2 \xrightarrow{\delta} F_1 \xrightarrow{\phi} G \to 1$$

and a morphism of groups $k^2 : F_2 \to M$, together with an action of F_1 on M such that

AP1) *if $m \in M$, $r \in F_2$ then $(k^2 r)^{-1} m (k^2 r) = m^{\delta r}$;*
AP2) *if $r \in F_2$, $x \in F_1$ then $k^2(r^x) = (k^2 r)^x$.*

Then there is a commutative square

$$\begin{tikzcd} F_2 \arrow[r, "\delta"] \arrow[d, "k^2"'] & F_1 \arrow[d, "k^1"] \\ M \arrow[r, "i_k"'] & E_k \end{tikzcd}$$

such that

CP1) $i_k \colon M \to E_k$ *is a crossed module;*
CP2) *if* $m \in M,\ x \in F_1$ *then* $m^x = m^{k^1 x}$*;*
CP3) *the square is universal for properties* CP1), CP2)*;*
CP4) *there is an exact sequence*

$$M \xrightarrow{i_k} E_k \xrightarrow{\psi} G \to 1;$$

CP5) *the morphism* i_k *is injective if and only if* $k^2(i\pi) = 1$.

Proof. Let C_k be the semidirect product group $F_1 \ltimes M$ formed with the given action of F_1 on M. Let C_k act on F_2 via the projection to F_1 and the action of F_1 on F_2.

We first prove that the function $\xi \colon F_2 \to C_k$, $r \mapsto (\delta r, (k^2 r)^{-1})$ is a morphism. Let $r, s \in F_2$. Then by the semidirect product rule

$$\begin{aligned} \xi(r)\xi(s) &= (\delta r\, \delta s, k^2((r^{-1})^{\delta s}\, s^{-1}) \\ &= (\delta r\, \delta s, k^2(s^{-1} r^{-1})) \\ &= \xi(rs). \end{aligned}$$

Next we prove ξ preserves the action. Let $r \in F_2$, $(x, m) \in C_k$. Then

$$\begin{aligned} (x,m)^{-1}\xi(r)(x,m) &= (x^{-1}, (m^{-1})^{x^{-1}})(\delta r, (k^2 r)^{-1})(x, m) \\ &= (x^{-1}(\delta r)x, (m^{-1})^{x^{-1}(\delta r)x}(k^2 r^{-1})^x m) \\ &= (x^{-1}(\delta r)x, (k^2 r^x)^{-1} m^{-1}(k^2 r^x)(k^2(r^{-1}))^x m) \\ &= \xi(r^x). \end{aligned}$$

Finally we prove easily the second crossed module rule:

$$\begin{aligned} r^{-1} s r &= s^{\delta r} && \text{by the crossed module rule for } \delta \\ &= s^{(\delta r, k^2 r^{-1})} && \text{by definition of the action of } C_k \\ &= s^{\xi r} && \text{as required.} \end{aligned}$$

Let $E_k = \operatorname{Cok}\xi$, and let $[x, m]$ denote the image in E_k of $(x, m) \in C_k$. Let $i_k : M \to E_k$ be given by $m \mapsto [1, m]$. Then the formula $m^{[x,n]} = n^{-1}m^x n$ gives, by AP1), a well-defined action of E_k on M which is easily shown to make i_k a crossed module.

Let $\psi : E_k \to G$ be given by $[x, m] \mapsto \phi(x)$. Then ψ is well defined and is the cokernel of i_k.

Suppose given a morphism of crossed modules

$$\begin{tikzcd} F_2 \arrow[r, "\delta"] \arrow[d, "k^2"'] & F_1 \arrow[d, "l"] \\ M \arrow[r, "\alpha"'] & P \end{tikzcd}$$

such that $m^x = m^{lx}$, $x \in F_1, m \in M$. Suppose $\omega : E_k \to P$ determines a morphism of crossed modules such that $\omega l = k^1$, $\omega i_k = \alpha$. Since $[x, m] = [x, 1][1, m]$, we easily check that $\omega[x, m] = (lx)(\alpha m)$. So such an ω is unique. On the other hand, we easily check this does define a morphism as required.

The morphism ψ of CP4) is defined by $\psi[x, m] = \phi x$. This gives the exact sequence.

Finally $i_k m = 1$ for all $m \in M$ is equivalent to $(1, m) = (\delta r, k^2 r^{-1})$ for some $r \in F_2$; this easily proves CP5). $\square$

Exercise 12.6.4. Complete the proof of Theorem 12.6.2, including the analysis of the equivalence of extensions.

Example 12.6.5. In the applications of Theorem 12.6.2 we would take a free crossed resolution $F(G)$ of a group G and let $\delta : F_2 \to F_1$ be the crossed module $\delta_2 : F_2(G) \to F_1(G)$ with $\phi : F_1(G) \to G$ given by the resolution. Then π is the module of identities among relations for the presentation $\langle X \mid R\rangle$ of G determined by the free bases X, R of $F_1(G)$, $F_2(G)$ respectively. $\square$

Example 12.6.6 (Extensions by a cyclic group). Let C_n denote the cyclic group of order n, written multiplicatively, and generated by an element c, and let the infinite cyclic group C_∞ be generated by x. The presentation $\langle x \mid x^n\rangle$ for C_n gives rise to the free crossed module $\delta : \mathbb{Z}[\mathsf{C}_n] \to \mathsf{C}_\infty$ where $\mathbb{Z}[\mathsf{C}_n]$ is also the free C_n module on a generator x_1 and $\delta(x_1) = x^n$. A morphism

$$\begin{tikzcd} \mathbb{Z}[\mathsf{C}_n] \arrow[r, "\delta"] \arrow[d, "k^2"'] & \mathsf{C}_\infty \arrow[d, "k^1"] \\ M \arrow[r, "\alpha"'] & P \end{tikzcd}$$

of crossed modules is thus specified by elements $q = k^1(x) \in P$, $a = k^2(x_1) \in M$ such that $\alpha a = q^n$. Further, $\operatorname{Ker}\delta$, the module of identities for the presentation, is the submodule generated by the element $x_1(1-c)$. Hence the condition $k^2(\operatorname{Ker}\delta) = 0$ is equivalent to $k^2(x_1(1-c)) = a(a^q)^{-1} = 1$, that is $a = a^q$. An equivalence $(a' : q') \simeq (a : q)$ of such data is given by a derivation $h : \mathsf{C}_\infty \to M$, and so by an element $b = h(x) \in M$, such that $q' = q(\alpha b)$ and

$$a' = ah(x^n) = ab^{q^{n-1}} b^{q^{n-2}} \dots b^2 b.$$

The extension group E determined by the data $(a : q)$ is the quotient of $\mathsf{C}_\infty \ltimes M$ by the element (x^n, a^{-1}).[200] □

Example 12.6.7 (The trefoil group). Let G be the trefoil group with presentation $\langle x, y \mid x^2 = y^3\rangle$. This is a one relator presentation whose relator is not a proper power, and so there are no identities among the relations.[201] Therefore the extension data of M by G of type $\alpha : M \to P$ is given by elements $q_x, q_y \in P$, $a_r \in M$, such that $\alpha a_r = (q_x)^2(q_y)^{-3}$. An equivalence $(a'_r : q'_x, q'_y) \simeq (a_r : q_x, q_y)$ of such data is given by elements $b, c \in M$ such that $q'_x = q_x(\alpha b)$, $q'_y = (q_y)(\alpha c)$ and $a'_r = a_r h(x^2 y^{-3})$ where h is the derivation $F\{x, y\} \to M$ given by $hx = b, hy = c$. Thus

$$\begin{aligned} h(x^2y^{-3}) &= h(x^2)^{q_y^{-3}} h(y^{-3}) \\ &= (b^{q_x} b)^{q_y^{-3}} (c^{-1})^{q_y^{-3}} (c^{-1})^{q_y^{-2}} (c^{-1})^{q_y^{-1}}. \end{aligned}$$

The group E determined by the extension data $(a_r : q_y, q_y)$ is the quotient of the semidirect product $F\{x, y\} \ltimes M$ by the element (x^2y^{-3}, a_r^{-1}). Here $F\{x, y\}$ acts on M by $a^x = a^{q_x}$, $a^y = a^{q_y}$, $a \in M$. □

Example 12.6.8 (Extensions by a product). The tensor product of crossed complexes as defined in Section 9.3.iii may be used to describe extensions by a product $G \times H$ of groups. Let $F_*(G)$, $F_*(H)$ be free crossed resolutions of groups G, H respectively. The tensor product $F_*(G) \otimes F_*(H)$ is then a free crossed resolution of $G \times H$. A proof of asphericity will be given in Corollary 15.8.5. It is proved in Theorem 9.6.1 that the tensor product of free crossed complexes is free on the tensor product of the free generators, so that in particular $F_*(G) \otimes F_*(H)$ is freely generated as a crossed complex by $a_i \otimes b_j$, where the a_i, b_j run over sets of free generators of $F_*(G)$, $F_*(H)$ respectively. Thus it is easy to specify morphisms from $F_*(G) \otimes F_*(H)$ to a crossed module or crossed complex. Further, generators for the module of identities for a presentation of the product $G \times H$ are the images under δ_3 of free generators of $(F_*(G) \otimes F_*(H))_3$, by asphericity. Such free generators are of the form $a_3 \otimes *$, $* \otimes b_3$, $a_2 \otimes b_1$, $a_1 \otimes b_2$ where a_i, b_j run over free generators of $F_i(G)$, $F_j(H)$ respectively.

This implies the following. Let $\langle X \mid R\rangle$, $\langle Y \mid S\rangle$ be presentations of G, H respectively, and let I, J be generating sets for the modules of identities for these presentations. Then a free crossed resolution $F_*(G)$ corresponding to X, R, I is in dimensions $\leqslant 3$ of the form

$$C_3(I) \xrightarrow{\delta_3} F_C(R) \xrightarrow{\delta_2} F(X)$$

where $C_3(I)$ is the free G-module on I, and similarly for $F_*(H)$. Thus in dimensions $\leqslant 3$, $F_*(G) \otimes F_*(H)$ has generators as follows, where for Z any set, $\bar{Z}$ denotes a set of formal generators $\bar{z}$, $z \in Z$:

- *dimension* 1: X, Y,
- *dimension* 2: $\bar{R}, \bar{S}, \{x \otimes y : x \in X,\ y \in Y\}$,
- *dimension* 3: $\bar{I}, \bar{J}, \{x \otimes \bar{s}, \bar{r} \otimes y : x \in X,\ y \in Y,\ r \in R,\ s \in S\}$.

The boundaries are given by

$$\begin{gathered}\delta_2\bar{r} = r, \quad \delta_2\bar{s} = s, \quad \delta_2(x \otimes y) = y^{-1}x^{-1}yx,\\ \delta_3\bar{i} = i, \quad \delta_3\bar{j} = j, \quad \delta_3(x \otimes \bar{s}) = \bar{s}^{-1}\bar{s}^{x}(x \otimes s)^{-1},\\ \delta_3(\bar{r} \otimes y) = (r \otimes y)\bar{r}^{-1}\bar{r}^{y}.\end{gathered}$$

Now the elements $x \otimes s$, $r \otimes y$ have to be expressed in terms of the free generators in dimension 2. This is done by using the biderivation rules

$$x \otimes uv = (x \otimes u)^{v}\,(x \otimes v),$$

$$\omega z \otimes y = (z \otimes y)(\omega \otimes y)^{z},$$

which are part of the crossed complex structure of the tensor product.

Note that in this example, we obtain nice generators of the module of identities for the product, by applying the boundary operator to free generators in dimension 3 of a crossed resolution.[202] □

Exercise 12.6.9. Examine free crossed resolutions and homotopical syzygies of the free abelian group $\mathbb{Z}^n$ using the tensor product of free crossed resolutions.[203] □

12.7 Crossed n-fold extensions and cohomology

The description of the second cohomology of a group in terms of extensions of groups led to a desire to find analogous interpretations of the third and higher cohomology groups. It turned out they could be described in terms of crossed n-fold extensions of a G-module M by the group G, as we explain in this section.[204]

Definition 12.7.1. A *crossed n-fold extension of M by G* is a crossed resolution E of G such that $E_{n+1} = M$ as a G-module, and $E_i = 0$ for all $i > n+1$. We can write E as an exact sequence:

$$E\colon = \quad 0 \longrightarrow M \xrightarrow{\partial_{n+1}} E_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_3} E_2 \xrightarrow{\partial_2} E_1 \xrightarrow{\phi} G \to 1.$$

This means of course that: the above sequence is exact; E_1 and the part above is a (truncated) crossed complex; and ϕ maps $\pi_1 E$ isomorphically to G. Thus for $n = 1$

we have exactly an *abelian extension of M by G*, since by the crossed module rule CM2), M abelian is equivalent to E_1 acts on M via G. A crossed 2-fold extension is also called a *crossed sequence*. $\square$

Example 12.7.2. Let G be a group and E a crossed resolution of G. Then $\mathrm{Cosk}^n E$ together with $\phi\colon E_1 \to G$ is the crossed n-fold extension

$$E^{n+1} \colon = \quad 0 \longrightarrow \mathrm{Ker}\,\partial_n \xrightarrow{i} E_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_3} E_2 \xrightarrow{\partial_2} E_1 \xrightarrow{\phi} G. \qquad \square$$

Definition 12.7.3. A *morphism* $E \to E'$ of crossed n-fold extensions of M by G is a morphism of crossed resolutions which induces the identity on M and on G as shown in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & E_n \arrow[r] \arrow[d] & E_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & E_1 \arrow[r, "\phi"] \arrow[d] & G \arrow[r] \arrow[d, "="] & 1 \\
0 \arrow[r] & M \arrow[r] & E'_n \arrow[r] & E'_{n-1} \arrow[r] & \cdots \arrow[r] & E'_1 \arrow[r, "\phi'"] & G \arrow[r] & 1
\end{tikzcd}$$

Two crossed n-fold extensions resolutions E, E' of M by G are *similar* if there is a ziz-zag (see page 579) of morphisms $E \to E'$. This relation is an equivalence relation, and we denote the quotient set by $\mathrm{OpExt}^n(G, M)$. $\square$

This quotient set may be given an abelian group structure called the 'Baer sum'.[205] Here we merely note that for $n > 1$ there is a class which we call 0 namely the class of the crossed n-fold extension

$$0 \longrightarrow M \xrightarrow{=} \underset{n}{M} \xrightarrow{0} \cdots \qquad \longrightarrow \underset{1}{G} \xrightarrow{\phi} G \longrightarrow 1$$

Further we shall give below a bijection

$$\mathrm{OpExt}^n(G, M) \cong H^{n+1}(G, M)$$

which then defines an abelian group structure on the set $\mathrm{OpExt}^n(G, M)$. The class in $H^{n+1}(G, M)$ of the crossed n-fold extension will be called its *Postnikov invariant*, or *k-invariant*.[206]

Remark 12.7.4. A crossed 2-fold extension is also called a *crossed sequence*. Such a crossed sequence

$$0 \to M \to E \to P \to G \to 1$$

determines and is determined up to equivalence by the crossed module $E \to P$. Thus we find as a particular case that crossed modules with kernel M and cokernel G can be classified by the k-invariant of the associated crossed sequence, i.e. by an element of the cohomology group $H^3(G, M)$. $\square$

Example 12.7.5. We recall the *dihedral crossed module* $\mu\colon \widetilde{D}_{2n} \to D_{2n}$ from Example 5.6.12. Here D_{2n}, $\widetilde{D}_{2n}$ have presentations

$$\langle x, y \mid x^n, y^2, xyxy\rangle, \quad \langle u, v \mid u^n, v^2, uvuv\rangle,$$

respectively, and $\mu(u) = x^2$, $\mu(v) = y$. We show the dihedral crossed module represents the trivial cohomology class in $H^3(\operatorname{Cok}\mu, \operatorname{Ker}\mu)$.

For n odd, we know that μ is an isomorphism, so the result is trivial.

For n even, we have $\operatorname{Ker}\mu \cong \operatorname{Cok}\mu \cong \mathsf{C}_2$ and we simply construct a morphism of crossed 2-fold extensions as in the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{C}_2 \arrow[r, "1"] \arrow[d, "="] & \mathsf{C}_2 \arrow[r, "0"] \arrow[d, "f_2"] & \mathsf{C}_2 \arrow[r, "1"] \arrow[d, "f_1"] & \mathsf{C}_2 \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathsf{C}_2 \arrow[r] & \widetilde{D}_{2n} \arrow[r, "\mu"'] & D_{2n} \arrow[r] & \mathsf{C}_2 \arrow[r] & 0
\end{tikzcd}$$

where if c denotes the nontrivial element of C_2 then $f_1(c) = x$, $f_2(c) = u^{n/2}$. □

Example 12.7.6. In this example, we give another crossed 2-fold extension α of A by G which represents 0 in its class. However to prove this triviality we use an intermediate crossed 2-fold extension β to give maps $0 \leftarrow \beta \to \alpha$, and it is not clear how to construct a direct map between 0 and α.

Let C_n denote the cyclic group of order n (including the case $n = \infty$), written multiplicatively, with generator t. Let $\gamma_n\colon \mathsf{C}_{n^2} \to \mathsf{C}_{n^2}$ be given by $t \mapsto t^n$. This defines a crossed module, with trivial operations. This crossed module represents the trivial cohomology class in $H^3(\mathsf{C}_n, \mathsf{C}_n)$, in view of the morphisms of crossed 2-fold extensions

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{C}_n \arrow[r, "1"] & \mathsf{C}_n \arrow[r, "0"] & \mathsf{C}_n \arrow[r, "1"] & \mathsf{C}_n \arrow[r] & 0 \\
0 \arrow[r] & \mathsf{C}_n \arrow[r, "{(1,0)}"] \arrow[u, "1"] \arrow[d, "1"'] & \mathsf{C}_n \times \mathsf{C}_\infty \arrow[r, "h"] \arrow[u, "p_1"] \arrow[d, "g"'] & \mathsf{C}_\infty \arrow[r, "\lambda"] \arrow[u, "\lambda"'] \arrow[d, "\mu"] & \mathsf{C}_n \arrow[r] \arrow[u, "1"'] \arrow[d, "1"] & 0 \\
0 \arrow[r] & \mathsf{C}_n \arrow[r, "i"'] & \mathsf{C}_{n^2} \arrow[r, "\gamma_n"'] & \mathsf{C}_{n^2} \arrow[r] & \mathsf{C}_n \arrow[r] & 0
\end{tikzcd}$$

where $g(t,1) = t^n$, $g(1,t) = t$, $h(t,1) = 1$, $h(1,t) = t^n$, $i(t) = t^n$ and λ, μ are given by $t \mapsto t$. You should check that each square of this diagram is commutative, and each row is exact. □

We now sketch the relation between crossed n-fold extensions and cohomology.

Proposition 12.7.7. *For a group G and G-module M, a crossed n-fold extension E of M by G determines a cohomology class $k_E \in H^{n+1}(G, M)$. Conversely, any such class determines a crossed n-fold extension of M by G.*

Proof. Suppose given the crossed n-fold extension as in Definition 12.7.1. Let F be a free crossed resolution of G. Since F is free and E is aspherical, there is a morphism $f : F \to E$ over the identity on G. Then f_{n+1} determines the required cohomology class.

Conversely, suppose given a morphism of G-modules $f_{n+1} \colon F_{n+1} \to M$ such that $f_{n+1}\delta_{n+2} = 0$. Suppose first $n > 1$. We form a crossed n-fold extension E of M by G on setting

$$E_i = \begin{cases} F_i & \text{if } i < n, \\ (F_n \times M)/D & \text{if } i = n \end{cases}$$

where D is the submodule of the product module which is generated by the elements $(\delta_{n+1}c, f_{n+1}c)$ for all $c \in F_{n+1}$. The morphism $i \colon M \to E_n$ is induced by the inclusion $m \mapsto (0, m)$ into the product, and the morphism $E_n \to F_{n-1}$ is induced by $(x, m) \mapsto \delta_n x$. To prove that i is injective, suppose that $im = (\delta_{n+1}c, f_{n+1}c)$ for some $c \in F_{n+1}$. Then $\delta_{n+1}c = 0$ and so $c = \delta_{n+2}c'$ for some $c' \in F_{n+2}$. By the condition $f_{n+1}\delta_{n+2} = 0$, we have $m = 0$.

If $n = 1$, then F_1 operates nontrivially on M via $\phi \colon F_1 \to G$, and instead of the product in the above formula we take the semidirect product $(F_1 \ltimes M)$.

We also want to show that similar crossed n-fold extensions E, E' give rise to cohomologous invariants. For this is enough to assume there is a morphism $g \colon E \to E'$. Let $f : F \to E$, $f' \colon F \to E'$ be morphisms. Then $gf : F \to E'$ and so gf, f' are homotopic. Hence their corresponding k-invariants are the same.

We omit further details. □

Exercise 12.7.8. Complete the details of the above proof. In particular, prove that the construction does give a crossed n-fold extension, i.e. verify exactness. Also show that homotopic k-invariants give rise to equivalent crossed n-fold extensions. For some extra points with regard to the case $n = 1$ we refer forward to Proposition 12.6.3. □

Now we give examples of crossed Q-modules with nontrivial Postnikov invariants. These examples arise from induced crossed modules, as in Example 5.6.11, and so actually arise as the 2-type of certain mapping cones of maps of classifying spaces of groups. Our examples also have Q with finite cyclic quotient and so we use the free crossed resolution of finite cyclic groups given in Example 10.2.4. Thus the point we want to stress here is that the use of crossed techniques is amenable to calculation. These examples show that success in computing a Postnikov invariant of a crossed n-fold extension of A by G is increased by having a convenient small free crossed resolution of G. Methods for the computation of such small free crossed resolution from a presentation for a finite group are given in Section 10.3.ii.[207]

Let us extract some more applications of the small free crossed resolution of C_a. We study the crossed module induced from $\mu \colon M \to P$ by the inclusion of a normal subgroup $\iota \colon P \lhd Q$ when the crossed module is also the inclusion $M \lhd P$ of a normal subgroup such that M is also normal in Q. Then Theorem 5.8.12 shows that the induced crossed module may be described as

$$\zeta\colon M \times (M^{\mathrm{ab}} \otimes I(Q/P)) \to Q$$

where $I(Q/P)$ denotes the augmentation ideal of the quotient group Q/P and, for $m, n \in M$, $x \in I(Q/P)$, the map ζ is defined by $\zeta(m, [n] \otimes x) = m \in Q$ and the action of $q \in Q$ is given by

$$(m, [n] \otimes x)^q = (m^q, [m^q] \otimes (\bar{q} - 1) + [n^q] \otimes x\bar{q}) \tag{12.7.1}$$

where $\bar{q}$ denotes the image of q in Q/P.

Remark 12.7.9. It might be imagined from this that the Postnikov invariant of this crossed module is trivial when $M = P$, since one could argue that the projection

$$\mathrm{pr}_2\colon P \times P^{\mathrm{ab}} \otimes I(Q/P) \to P^{\mathrm{ab}} \otimes I(Q/P)$$

should give a morphism κ from $\iota_* P$ to the crossed module $0\colon P^{\mathrm{ab}} \otimes I(Q/P) \to Q/P$ such that κ represents 0 in the cohomology group $H^3(Q/P, P^{\mathrm{ab}} \otimes I(Q/P))$. However, the projection pr_2 is a P-morphism, but is not in general a Q-morphism, as the description of the action in Equation (12.7.1) shows. In the next theorem we give a precise description of the Postnikov invariant of $\iota_* P$ when Q/P is cyclic of order n. □

Theorem 12.7.10. *Let P be a normal subgroup of Q such that Q/P is isomorphic to C_a, the cyclic group of order a. Let u be an element of Q which maps to the generator t of C_a under the quotient map. Then the first Postnikov invariant k^3 of the crossed module induced by the inclusion $P \to Q$ lies in a third cohomology group*

$$H^3(\mathsf{C}_a, P^{\mathrm{ab}} \otimes I(\mathsf{C}_a)).$$

This group is isomorphic to

$$P^{\mathrm{ab}} \otimes \frac{I(\mathsf{C}_a)}{I(\mathsf{C}_a)(1-t)}$$

and under this isomorphism the element k^3 is taken to the class of the element

$$[u^a] \otimes (1-t).$$

This class is in general not zero.

Proof. We first recall that a free crossed resolution of C_a is given by

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & \mathbb{Z}[\mathsf{C}_a] & \xrightarrow{\delta_4} & \mathbb{Z}[\mathsf{C}_a] & \xrightarrow{\delta_3} & \mathbb{Z}[\mathsf{C}_a] & \xrightarrow{\delta_2} & \mathsf{C}_\infty & \xRightarrow{\phi} & \mathsf{C}_a \\ & x_4 & & x_3 & & x_2 & & x_1 & & \end{array} \tag{12.7.2}$$

where the row gives free modules on x_4, x_3, x_2 respectively and the free group on x_1 with boundaries $\delta_2 x_2 = x_1^a$, $\delta_3 x_3 = x_2(1-t)$, $\delta_4 x_4 = x_3 N(a)$ and $N(a) = 1 + t + \cdots + t^{a-1}$, and t is a generator of C_a, so that $\phi(x_1) = t$.

So if A is a C_a-module, then by forming $\mathsf{Hom}_{\mathsf{C}_a}(-, A)$ with the above free crossed resolution, each term of which has a single free generator, we see that the cohomology group $H^3(\mathsf{C}_a, A)$ is the homology of the sequence of modules over C_a

$$A \xrightarrow{M} A \xrightarrow{N} A \tag{12.7.3}$$

where M, N are action by $1-t$ and $1+t+t^2+\cdots+t^{a-1}$ respectively.

In the case under discussion, $A = P^{\mathrm{ab}} \otimes I(\mathsf{C}_a)$. Now $I(\mathsf{C}_a)$ is generated by elements $1-t^r$, for $0 < r < a$ and $1 - t^r = (1 + t + \cdots + t^{r-1})(1-t)$ and hence $N(I(\mathsf{C}_a)) = 0$. So

$$H^3(\mathsf{C}_a, A) \cong P^{\mathrm{ab}} \otimes \frac{I(\mathsf{C}_a)}{I(\mathsf{C}_a)(1-t)}.$$

We have to determine the cohomology class represented by the crossed module

$$\xi \colon P \times A \to Q.$$

We consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{Z}[\mathsf{C}_a] \arrow[r, "\delta_4"] & \mathbb{Z}[\mathsf{C}_a] \arrow[r, "\delta_3"] \arrow[d, dashed, "f_3"] & \mathbb{Z}[\mathsf{C}_a] \arrow[r, "\delta_2"] \arrow[d, dashed, "f_2"] & \mathsf{C}_\infty \arrow[r, Rightarrow, "\phi"] \arrow[d, dashed, "f_1"] & \mathsf{C}_a \arrow[r] \arrow[d, "1"] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & A \arrow[r, "i"'] & P \times A \arrow[r, "\xi"'] & Q \arrow[r, Rightarrow, "\psi"'] & \mathsf{C}_a \arrow[r] & 0.
\end{tikzcd} \tag{12.7.4}$$

We are given $u \in Q$ such that $\phi u = t$. Then $u^a \in P$, since $Q/P \cong \mathsf{C}_a$. Let $w = (u^a, 0) \in P \times A$. Then by the description of the action in Equation (12.7.1), $w^u = (u^a, u^a \otimes (t-1))$.

Now we chose our morphism of crossed complexes. We define $f_1(x_1) = u$, and then $f_2(x_2) = w$. This ensures that $\xi f_2 = f_1 \delta_2$.

Then

$$\begin{aligned}
f_2\delta_3(x_3) &= w(w^{-1})^u \\
&= (u^a, 0)(u^a, [u^a] \otimes (t-1))^{-1} \\
&= (0, [u^a] \otimes (1-t))
\end{aligned}$$

which lies in $A = P^{\mathrm{ab}} \otimes I(\mathsf{C}_a)$. So we can define f_3 to have this value on x_3, and this gives the Postnikov invariant k^3 as required. □

Remark 12.7.11. The aim of giving this result is to show that such calculations can be obtained using a free crossed resolution. A Postnikov invariant is conveniently calculated by this route since a free crossed resolution has a structure analogous to that of a crossed sequence so that we can use freeness and the notion of morphism from one to the other. The traditional chain complex approach seems more *ad hoc*. □

Example 12.7.12. Here is a small example of the last theorem. Let $Q = \mathsf{C}_4$ with generator u and let P be the subgroup generated by u^2. The crossed module induced by the inclusion is $\xi\colon \mathsf{C}_2 \times \mathsf{C}_2 \to \mathsf{C}_4$ where each factor is included into C_4 which operates by twist on the product. The above theorem gives $H^3(\mathsf{C}_2, \mathsf{C}_2) \cong \mathsf{C}_2$ and this crossed module represents the nontrivial element of the group.[208] Note that it is not so easy to be more explicit on $I(\mathsf{C}_a)/(I(\mathsf{C}_a)(1-t))$ for $a > 2$. But if $a = 2$ then $(1-t)^2 = 1 - 2t + t^2 = 0 \bmod 2$. □

Remark 12.7.13. The examples in this section show how a representation of a cohomology class in terms of a crossed n-fold extension can be quite explicit algebraically, whereas the 'meaning' of a cohomology class is less clear, though it is interesting if we can know its order, or if it is not zero. Also the cohomology group relates of course many cohomology classes, and that is its main interest. By contrast, one can make elaborate algebraic manipulations with crossed n-fold extensions, using for example limits or colimits. Thus it was long believed that no algebraic description of the homotopy 2-type of a union of spaces was possible, yet the description of the 2-type of a union in terms of a pushout of crossed modules is clear and elegant. On the other hand the calculation of the corresponding Postnikov invariant of a pushout of crossed modules, or even of a second homotopy group, in terms of the given invariants of the parts of the union is fraught with difficulties, though we have made some such calculations in this section. Again the description of the free loop space in Example 9.3.8 is clear and complete, but has not yielded a calculation of the Postnikov invariants of the components of the free loop space. □

12.8 Concluding remarks to Part II

We have now ended this account of the theory and applications of crossed complexes, apart from the justification given in Part III of some fundamental results and methods. There are many ways in which this work might be pursued, and we try to indicate some of them and speculate on others in Chapter 16.

Notes

175 p. 396 The applications of crossed complexes to handlebody decompositions are discussed in [Sha93], Chapter VI, with different terminology.

176 p. 396 The notion of cohomology with a chain complex as coefficients was put forward in [Bro62] and applied to problems on the homotopy type of function spaces, particularly the calculation of k^Y where k is a cohomology operation. That thesis also noted and used the existence of many occurrences of exponential

laws involving a tensor product and internal hom. The main features of this work were published in [Bro64a], [Bro64c], [Bro66]. The first paper explains how to choose an isomorphism

$$\kappa\colon H^*(X \times Y, A) \cong H^*(X, H^*(Y, A))$$

for an abelian group A which is natural with respect to maps of X. The second paper applies this to find a convenient homotopy equivalence from $K(A,n)^Y$ to a product of Eilenberg–Mac Lane spaces, and so to do some calculations of k^Y for k a cohomology operation, with another example in the third paper. The existence of the isomorphism κ depends on working with chain complexes over the integers, and so does not work for chain complexes with operators. Nonetheless, the use in essence of monoidal closed categories were a model for the crossed complex development in the 1980s and described in this book.

177 p. 397 The notion of fibration of groupoids was defined in exercises in [Bro68], seen there as part of the 'homotopy theory of groupoids', and given a full account in [Bro70]. It was also seen as part of general homotopy theories in [And78]. Fibrations of groupoids are particularly useful in the theory of 'orbit groupoids', as shown by Higgins and Taylor, see the references in [Bro06], Chapter 11.

178 p. 397 This definition and the exact sequence given later in the book are due to Howie in [How79].

179 p. 397 The term *trivial fibration* has been used, but the term *acyclic* fibration is now more standard, see for example [DS95].

180 p. 401 That Crs with these notions is a model category was proved in [BG89b]; it implies, in virtue of the result of [BH81b], a model structure on the category of globular ∞-groupoids. That model structure is related to a model structure for higher categories in [AM11].

181 p. 403 This long exact sequence of a fibration of crossed complexes was first stated by Howie in [How79].

182 p. 403 The definition of fibration of groupoids was an exercise in [Bro68] and an account was published in detail in [Bro70], though with different conventions to those given here. For further applications of that part, see [HK81], [HK82].

183 p. 404 We refer to Exercises 1–5 in Section 10.7 of [Bro06], and previously cited papers, for applications of the lower part of the exact sequence, e.g. to problems in group theory. Notice also that there should be a useful Mayer–Vietoris type sequence for a pullback of a fibration of crossed complexes generalising that of [Bro06], 10.7.6. See also [BHK83].

184 p. 404 Homotopy classification theorems of the kind given here were proved by different methods by Olum in [Olu50], [Olu53], and were in essence found also by Whitehead in [Whi49b].

185 p. 408 For the classical approach to abstract kernels, see for example [ML63], Section IV.8. The relation with factor systems is shown in [BH82], [BP96].

186 p. 408 The discussion of the existence of a topological group structure on the universal cover was started by R. L. Taylor in [Tay54], with some interesting examples, and continued in [BM94]; in fact the methods of fibrations of crossed complexes explored above were developed in the latter paper, and further in [Bro08a]. The relation with the classifying space of the topological group was discussed in [BS76b]. The general theory of group extensions was developed by Taylor in [Tay53] and continued in [Tay55]. A more recent work relating to crossed modules is [CLV02].

187 p. 408 These applications of the Homotopy Classification Theorem appeared for the simplicial classifying space in [BH91]. An account of the main results on homotopy classification in [Whi49b] is given by Ellis in [Ell88b], for both the pointed and free case, and related to other results on homotopy classification such as those by Olum, [Olu53].

Ellis writes in [Ell88b]: "In view of the ease with which Whitehead's methods handle the classifications of Olum and Jajodia, it is surprising that the papers [Olu53] and [Jaj80] (both of which were written after the publication of [Whi49b]) make respectively no use, and so little use, of [Whi49b]."

"We note here that B. Schellenberg, who was a student of Olum, has rediscovered in [Sch73] the main classification theorems of [Whi49b]. The paper [Sch73] relies heavily on earlier work of Olum."

188 p. 409 For $n = 1$, this is well known (the Eilenberg–Mac Lane spaces do this), and for $n = 2$ it is essentially due to Mac Lane and Whitehead [MLW50]. Indeed, they prove that the 2-type (for which they use the term 3-type) of a reduced CW-complex X is described by the crossed module $\pi_2(X, X^1) \to \pi_1 X^1$, which is the same crossed module as arises for $n = 2$ in the proof of Theorem 12.3.1.

189 p. 410 This is due to Loday [Lod82].

190 p. 411 This corollary may also be obtained as a concatenation of results proved by J. H. C. Whitehead in [Whi49b]. It is also proved in general circumstances by Baues in his book [Bau89].

191 p. 412 This result on spaces of maps to an Eilenberg–Mac Lane space yields a result of Gottlieb [Got69] on the fundamental group of spaces of maps into an Eilenberg–Mac Lane space $K(H, 1)$.

192 p. 413 This type of result was first found in [EML50], and developed by Ando in [And57], and Huebschmann, [Hue80b]. The approach considered here appeared in a special case in [BM94], and in full in [Bro08a]. Related results are in [Whi50a], [CGCO02].

193 p. 415 The notion of cohomology with local coefficients was introduced by Steenrod in [Ste43]. An application to obstruction theory is given in [McC71], and there are many other applications, and relations for example with sheaf theory, as a web search shows. A recent work is [MP02]: the equivariant nature of those results suggests a possible relationship with [BGPT97], [BGPT01]. Cohomology with chains as coefficients was introduced in [Bro64a] and applied to function spaces in [Bro64c]. It is possible that some subtle invariants could be obtained from cohomology with coefficients in chains with a group(oid) of operators.

194 p. 418 The formulae for the differential given on this page are different in detail from those given in [Hue80a], [BH82], [Ton94]. This reflects the different conventions we have used. The formulae are forced on us by choices made in Chapter 13 for the equivalence of categories which is central to the work of this book, and which determine the tensor product formulae. The standard formula is obtained from this one by applying it to the 'transpose' of the standard simplex, so that in the above formulae the given ∂_i is replaced in dimension n by ∂_{n-i}.

195 p. 420 The details of the simplicial nerve as a right adjoint of Π are in [BH91]. This result is used in [BJ99].

196 p. 421 Formulae of this type go back to Dedecker, [Ded60] Compare also (2.6.5) of [Bre94]. These formulae are related to the notion of the simplicial nerve of a cover. For a more cubical approach, see [FMP11], Section 3.1; that paper strongly utilises the cubical methods of this book for certain local-to-global problems in differential geometry.

197 p. 421 See also [BS09] for an account of Čech cohomology of a space with coefficients in a topological '2-group', but without the machinery given here

198 p. 422 This idea was introduced by Dedecker in [Ded64]; see also Taylor [Tay53].

199 p. 423 Such examples are in [Tay54], [BM94]; the first gives examples of bundles and the second gives examples of covering maps of nonconnected topological groups. The following exposition is an adaptation of the work of [BP96].

200 p. 426 This classification of extensions by a cyclic group is given in [Zas49], Chapter III, Section 7, for the case of the crossed module $M \to \operatorname{Aut} M$.

201 p. 426 This was proved by Lyndon, [Lyn50]. See also [DV73] and [BH82] for more information.

202 p. 427 The above description explains the determination of extensions by a product of cyclic groups given in [Zas49]. Different conventions for the tensor product of crossed complexes have been adopted by Baues in [Bau91]. The use of identities among relations for discussing nonabelian extensions was given in [Tur38] and the exposition here of the Schreier theory is based on that in [BP96].

203 p. 427 This example is made a feature in [Lod00], [Ell04] but without using the tensor product of crossed complexes.

204 p. 427 The description of higher cohomology of a group in terms of crossed n-fold extensions was given in [Hue77], [Hue80a], and also in [Hol79]. See the Historical Note [ML79]. However these results are also part of a general theory of cohomology of algebras as in [Lue71], as pointed out in [Lue81]. They are also related to work of Glenn in [Gle82], and what is called the 'triple' views of cohomology, also used in [BFGM05].

205 p. 428 The definition may be found for the case $n = 1$ in [ML63], and in general in [Hue80a]. See also a general discussion in [Dan91].

206 p. 428 The theory of Postnikov decompositions of crossed complexes is discussed in [BFGM05]. It should be asked: what is the value, the usage, of the Postnikov invariant, considered as an element of an abelian group? It is of course interesting to know if it is not zero, or what is its order. How else can one 'get hold' of it, or do useful things with it?

207 p. 430 These methods of calculating a small free crossed resolution have been implemented in GAP4, see [HW03]. Related methods, using universal covering cell complexes, also implemented in GAP4, but without the crossed information, are in [Ell04].

208 p. 433 This example and others of nontriviality of the k-invariant of a crossed module have been discussed also in [Hue81b], [Hue81b].

Part III

Cubical ω-groupoids

Introduction to Part III

In Part II we have explored the techniques of crossed complexes, and hope we have shown convincingly that they are a powerful tool in algebraic topology. In this part, we give the proofs of the main theorems on which those tools depend.

To this end, we introduce the algebra of ω-groupoids, or in full, *cubical ω-groupoids with connections*. It was the way in which this algebra could be developed to model the geometry of cubes which suggested the possibility of the theory and calculations described in this book.

As intimated in Chapter 6 of Part I, the crucial advantages of cubical methods are the capacity to encode conveniently:

A) subdivision;
B) multiple composition as an algebraic inverse to subdivision;
C) commutative cubes, and their composition.

These properties allow us to prove a HHSvKT by verifying the required universal property: here A) and B) are used to give a candidate for a morphism, and C) is used to verify that this morphism is well defined.

A further advantage of the cubical methods is:

D) the formula $I^m \otimes I^n \cong I^{m+n}$ allows for a convenient modelling of homotopies and higher homotopies.

The techniques which enable analogous arguments for A)–C) in all dimensions are more elaborate than those of Part I. The main achievements are as follows:

- In order to *define* the notion of commutative shell, we have to relate the cubical theory of ω-groupoids to that of crossed complexes. This purely algebraic equivalence is established in Chapter 13, and is a central feature of this book.
- The proof in Chapter 14 that the natural definition of the fundamental ω-groupoid ρX_* of a filtered space actually is an ω-groupoid requires the techniques of *collapsing* for subcomplexes of a cube which were given in Chapter 11. These techniques are also used to prove the equivalence of the two functors ρ and Π under the equivalence of algebraic categories proved in Chapter 13.
- The proof of the HHSvKT for the functor ρ is also given in Chapter 14.
- The penultimate Chapter 15 constructs the monoidal closed structure on the category of ω-groupoids, and deduces the precise formulae for the equivalent structure on crossed complexes used in Part II. Also proved is the Eilenberg–Zilber type natural transformation $\rho(X_*) \otimes \rho(Y_*) \to \rho(X_* \otimes Y_*)$, for filtered spaces X_*, Y_*.
- The final Chapter 16 points to a number of areas which require further study and research to develop further this new foundation for algebraic topology.

Chapter 13

The algebra of crossed complexes and cubical ω-groupoids

As stated in the Introduction to Part III, this chapter contains the generalisation to all dimensions of the algebraic part of Chapter 6. There we proved the equivalence between the category XMod of crossed modules over groupoids and the category DGpds of double groupoids with connections. To obtain this equivalence we defined in Section 6.2 a functor

$$\gamma\colon \mathsf{DCatG} \to \mathsf{XMod}$$

and in Section 6.6 another functor

$$\lambda\colon \mathsf{XMod} \to \mathsf{DGpds}.$$

We proved also in Section 6.6 that these functors give an equivalence of categories. The composition $\gamma\lambda$ is clearly naturally isomorphic to the identity. Nevertheless, we had to work hard to prove that $\lambda\gamma$ is also isomorphic to the identity. We shall come back to this point later.

In this chapter we will follow analogous steps:[209] the first of these is to give the appropriate generalisation of both categories.

The generalisation of XMod has already been studied: it is the central algebraic category for most of Part II, namely the category Crs of crossed complexes. In the reduced case, this category had been studied in the literature, because of its connections with relative homotopy groups and with group cohomology.[210]

It was not so hard to write down a definition of ω-Gpds, the category of 'multiple groupoids with connections' or 'ω-groupoids', as a reasonable generalisation to all dimensions of DGpds, the category of double groupoids with connections. The definition and general properties of this category are given in Section 13.2 while earlier, in Section 13.1, we extend the notion of cubical sets given in Section 11.1 to include the structures of connections and compositions.

Once these two categories of ω-groupoids and of crossed complexes are fixed, it is easy to define the functor

$$\gamma\colon \omega\text{-}\mathsf{Gpds} \to \mathsf{Crs}.$$

As in Section 6.2, to associate a crossed complex to an ω-groupoid we take the elements of γG_n to be cubes with all faces but one (∂_1^- in our convention) concentrated at a point and the boundary maps are given by the restriction to the nontrivial face. All this is developed in Section 13.3.

As in the 2-dimensional case it is considerably more difficult to define the appropriate functor in the other direction

$$\lambda\colon \mathsf{Crs} \to \omega\text{-}\mathsf{Gpds}.$$

The still harder part is to give the natural equivalence $\lambda\gamma \simeq 1$, by showing that an ω-groupoid G may be rebuilt from the crossed complex γG it contains. The idea is essentially the same as that in Chapter 6 but requires considerably more careful organisation to carry it through.

This equivalence, which is completed in Section 13.6, is a purely algebraic equivalence between two algebraically defined categories. So we have to use only the algebraic definition, however much we rely on geometry for formulating the definitions and for structuring the proof. Each axiom for the two categories is used at least once, proving that all of them are needed.

Let us recall that in Chapter 6 to define the functor λ on a crossed module $\mathcal{M} = (\mu\colon M \to P)$ we used as 2-dimensional elements of $\lambda\mathcal{M}$ the 'squares of arrows in P commuting up to an element of M' as explained at the beginning of Section 6.6.

The clear generalisation of squares are the 'n-shells'. They are families of n-cubes that fit together as do the faces of an $(n+1)$-cube, that is they satisfy the appropriate face relations. These n-shells are studied in Section 13.5 where they are used to give the construction of right and left adjoints for the truncation functor.

It is more difficult to define a 'commutative n-shell'. But even in dimension 2 we found the 'commutative cube' rather an inconvenient idea and in Section 6.6 we worked instead with the 'folding map'. We explore this avenue in Section 13.4. We define first the 'folding' Φ_i in direction i and then the 'folding map'

$$\Phi = \Phi_1\Phi_2\dots\Phi_{n-1}\colon G_n \to \gamma G_n$$

as the composite of the foldings in decreasing order. The effect of Φ is to 'fold' all faces of a cube into one face, which in our convention is taken to be the $(-,1)$ face. This folding map allows us to say that an n-shell is commutative if and only if it folds to the trivial n-shell. We define the foldings in Section 13.4 and explore their behaviour with respect to all operators: i.e. faces, degeneracies, connections, compositions.

A main result is that every element $x \in G_n$ is determined by its total boundary $\boldsymbol{\partial} x$ and the folding Φx; this is a consequence of Proposition 13.5.10. In essence, this says that the folding process can be inverted and suggests how to construct λC_n inductively using pairs $(\mathbf{x}, \xi)$ where $\mathbf{x}$ is a 'shell' (generalisation of the total boundary) and $\xi \in C_n$ 'fills' the folding of the shell ($\delta\xi = \delta\Phi\mathbf{x}$). We work inductively using the coskeleton functor of Section 13.5; the construction of λ is done in Section 13.6.

In Chapter 6 we saw that connections and the folding map give a characterisation of commutative cubes. In the general case this may be taken as the definition of commutative n-cube, i.e. of the thin cubes. The *basic* thin n-cubes are images of degeneracies and connections: the general thin n-cubes are formed from the basic ones using negatives and compositions (see Definition 13.4.17). In Proposition 13.4.18 we prove that the thin n-cubes are exactly the elements that fold to the trivial one: i.e. they represent the 'commutative n-cubes', or, more precisely, the cubes with commutative boundary, or shell. Hence we obtain that *any composite of commutative cubes is commutative*. This is a key result for the proof of the Higher Homotopy Seifert–van Kampen Theorem in Chapter 14.

The last section (13.7) contains the algebraic Homotopy Addition Lemma (HAL) 13.7.1 and some of its consequences, which will be used in Chapter 14. The HAL gives an expression for the only nontrivial face of the folding of an n-shell ($\Sigma \mathbf{x} = \delta \Phi \mathbf{x}$). Thus a commutative shell is one having $\Sigma \mathbf{x} = 0$ and by Proposition 13.5.10 any commutative n-shell has a unique thin filler. The main consequence is that the thin cubes satisfy Dakin's axioms for T-complexes ([Dak77]):

- degenerate cubes are thin;
- any box has a unique thin filler (so ω-groupoids are fibrant cubical sets in a strong way);
- if a thin cube has all faces but one thin, then this last face is also thin.

This chapter involves a substantial amount of work, and checking of detail. The advantage of this is that we can often apply the main result, the equivalence of categories, without using the details, and even if the application seems simple, this simplicity may be deceptive, since powering it is a well crafted machine. Sufficient detail is given that all proofs should be checkable by a graduate student.

13.1 Connections and compositions in cubical sets

To generalise the category of double groupoids, it is important to notice that every double groupoid has an underlying 2-truncated cubical set. Moreover they have some extra 'degeneracies' that we have called connections. In this section we explore some definitions generalising these concepts to every dimension, adding extra structure to the cubical sets studied in Section 11.1.

A key example of a cubical set is the singular cubical set of a space $S^{\square}X$ (see Definition 11.1.10), which in this part we write also as KX. But we are interested in the filtered spaces whose definition and main properties were studied in Section 7.1.i, partly as these are a tool for studying spaces. There is a natural generalisation of the singular cubical set of a space to the filtered case, which we call the *filtered singular cubical set*.

Definition 13.1.1. For any filtered space X_* we denote by $R_n X_*$ the set of filtered maps $I^n_* \to X_*$ where I^n_* represents the standard n-cube with its standard cell structure as a product of n copies of $I = [0, 1]$.

The sets $R_n X_*$ for $n \geqslant 0$, together with the face and degeneracy maps defined for the singular cubical set of a space, form a cubical set called the *filtered singular cubical complex* of the filtered space X_*, which we write RX_*. $\square$

There is every reason to have a pictorial image for n-cubes very similar to the one we used for squares in Chapter 6 since it also useful here to state the laws of connections and compositions and to prove some results.

Remark 13.1.2. We now have to extend the conventions for multiple compositions which we used in Chapter 6. Since now we cannot picture all n dimensions, we have got to state which directions we are representing in any 2-dimensional picture, e.g. sometimes it is useful to show just one direction condensing all the orthogonal directions, as in

$$\partial_i^- u \ \boxed{\quad u \quad} \ \partial_i^+ u \qquad \begin{tikzcd} \cdot \arrow[r, "i"] \arrow[d, "\neq i"'] & {} \\ {} \end{tikzcd} .$$

The degeneracies can be represented by

$$\varepsilon_i(a) \;=\; a \ \boxed{\qquad} \ a \;=\; \boxed{\ = \ } \;=\; = \qquad \begin{tikzcd} \cdot \arrow[r, "i"] \arrow[d, "\neq i"'] & {} \\ {} \end{tikzcd} .$$
$\square$

Singular cubical sets have a lot of extra structure arising from geometric maps on cubes, and which we used in Part I for squares. We are going to give generalisations to all dimensions of connections and compositions.

Let us first generalise the connections studied in Section 6.5; these connections should be thought of as giving more forms in which an n-cube can be 'degenerate'.[211]

Definition 13.1.3. We say that a cubical set K has *connections* if it has additional structure maps

$$\Gamma_i \colon K_{n-1} \to K_n, \quad i = 1, 2, \dots, n-1$$

(called *connections*) satisfying the relations

$$\partial_i^\alpha \Gamma_j = \begin{cases} \Gamma_{j-1}\partial_i^\alpha & (i < j), \\ \Gamma_j \partial_{i-1}^\alpha & (i > j+1), \end{cases} \tag{1}$$

$$\partial_j^- \Gamma_j = \partial_{j+1}^- \Gamma_j = \mathrm{id},$$

$$\partial_j^+ \Gamma_j = \partial_{j+1}^+ \Gamma_j = \varepsilon_j \partial_j^+,$$

$$\Gamma_i \varepsilon_j = \begin{cases} \varepsilon_{j-1}\Gamma_i & (i < j), \\ \varepsilon_j \Gamma_{i-1} & (i > j), \end{cases} \tag{2}$$

$$\Gamma_j \varepsilon_j = \varepsilon_j^2 = \varepsilon_{j+1}\varepsilon_j,$$

$$\Gamma_i \Gamma_j = \Gamma_{j+1}\Gamma_i \quad (i \leqslant j). \tag{3}$$

$\square$

Remark 13.1.4. This definition generalises axioms CON 1, CON 2 of Definition 6.5.1 □

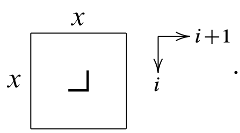
Example 13.1.5. 1. The singular cubical set KX of a space X is a cubical set with connections. The connection $\Gamma_i : K_{n-1} \to K_n$ is induced by the map $\gamma_i : I^n \to I^{n-1}$ defined by

$$\gamma_i(t_1, t_2, \dots, t_n) = (t_1, t_2, \dots, t_{i-1}, \max(t_i, t_{i+1}), t_{i+2}, \dots, t_n).$$

2. The connections of the previous example also give a structure of cubical set with connections to the filtered singular cubical set RX_* of a filtered space X_*. □

Remark 13.1.6. The connections are to be thought of as extra 'degeneracies'. (A degenerate cube of type $\varepsilon_j x$ has a pair of opposite faces equal and all other faces degenerate. A cube of type $\Gamma_i x$ has a pair of adjacent faces equal and all other faces of type $\Gamma_j y$ or $\varepsilon_j y$).

We can get a 2-dimensional picture of the connection Γ_i representing only the two dimensions i and $i+1$:

$\Gamma_i(x) =$ 

.

□

The singular cubical set KX of a space has another extra piece of structure which we will exploit in a substantial way: the possibility of 'adding together' cubes in a direction if the appropriate faces in this direction coincide. The multiple forms of this composition give a method of 'algebraic inverse to subdivision', as discussed for dimension 2 in the section on groupoids in our Introduction to the book (see p. xxii). The precise definition of the basic compositions is as follows.

Definition 13.1.7. A *cubical set with connections and compositions* is a cubical set K with connections in which each K_n has n partial compositions $+_i$ and n unary operations $-_i$ $i = 1, 2, \dots, n)$ satisfying the following axioms.

If $a, b \in K_n$, then $a +_i b$ is defined if and only if $\partial_i^+ a = \partial_i^- b$, and then for $\alpha = \pm$:

$$\begin{cases} \partial_i^-(a +_i b) = \partial_i^- a, \\ \partial_i^+(a +_i b) = \partial_i^+ b, \end{cases} \qquad \partial_i^\alpha(a +_j b) = \begin{cases} \partial_i^\alpha a +_{j-1} \partial_i^\alpha b & (i < j), \\ \partial_i^\alpha a +_j \partial_i^\alpha b & (i > j). \end{cases} \tag{1.i}$$

If $a \in K_n$, then $-_i a$ is defined and

$$\begin{cases} \partial_i^-(-_i a) = \partial_i^+ a, \\ \partial_i^+(-_i a) = \partial_i^- a, \end{cases} \qquad \partial_i^\alpha(-_j a) = \begin{cases} -_{j-1} \partial_i^\alpha a & (i < j), \\ -_j \partial_i^\alpha a & (i > j), \end{cases} \tag{1.ii}$$

$$\varepsilon_i(a +_j b) = \begin{cases} \varepsilon_i a +_{j+1} \varepsilon_i b & (i \leqslant j), \\ \varepsilon_i a +_j \varepsilon_i b & (i > j), \end{cases} \tag{2.i}$$

$$\varepsilon_i(-_j b) = \begin{cases} -_{j+1}\varepsilon_i a & (i \leqslant j), \\ -_j \varepsilon_i a & (i > j), \end{cases} \tag{2.ii}$$

$$\Gamma_i(a +_j b) = \begin{cases} \Gamma_i a +_{j+1} \Gamma_i b & (i < j), \\ \Gamma_i a +_j \Gamma_i b & (i > j), \end{cases} \tag{3.i}$$

$$\Gamma_j(a +_j b) = (\Gamma_j a +_{j+1} \varepsilon_j b) +_j (\varepsilon_{j+1} b +_{j+1} \Gamma_j b).$$

(This last equation is called the *transport law*.)

$$\Gamma_i(-_j a) = \begin{cases} -_{j+1}\Gamma_i a & (i < j), \\ -_j \Gamma_i a & (i > j). \end{cases} \tag{3.ii}$$

We have for $i \neq j$ and whenever both sides are defined,

$$(a +_i b) +_j (c +_i d) = (a +_j c) +_i (b +_j d). \tag{4.i}$$

These relations are called the *interchange laws*. Further:

$$\begin{aligned} -_i(a +_j b) &= (-_i a) +_j (-_i b) \quad \text{and} \quad -_i(-_j a) = -_j(-_i a) \quad \text{if } i \neq j, \\ -_j(a +_j b) &= (-_j b) +_j (-_j a) \quad \text{and} \quad -_j(-_j a) = a. \end{aligned} \tag{4.ii}$$

□

Example 13.1.8. 1. It is easily verified that the singular cubical set KX of a space X satisfies these axioms if $+_j$, $-_j$ are defined by

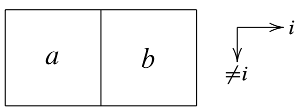
$$(a +_j b)(t_1, t_2, \dots, t_n) = \begin{cases} a(t_1, \dots, t_{j-1}, 2t_j, t_{j+1}, \dots, t_n) & (t_j \leqslant \frac{1}{2}), \\ b(t_1, \dots, t_{j-1}, 2t_j - 1, t_{j+1}, \dots, t_n) & (t_j \geqslant \frac{1}{2}) \end{cases}$$

whenever $\partial_j^+ a = \partial_j^- b$; and

$$(-_j a)(t_1, t_2, \dots, t_n) = a(t_1, \dots, t_{j-1}, 1 - t_j, t_{j+1}, \dots, t_n).$$

2. The faces and degeneracies of the previous example also give a structure of cubical set with connections and compositions to the filtered singular cubical set RX_* of a filtered space X_*. □

Remark 13.1.9. We have a 2-dimensional pictorial image of the composition $+_i$ given by



.

Also the interchange law can be stated in a matrix form. The diagram

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \qquad \begin{tikzcd}[sep=small] {} \arrow[r, "j"] \arrow[d, "i"'] & {} \\ {} & \end{tikzcd}$$

will be used to indicate that both sides of the equation (4.i) are defined and also to denote the unique composite of the four elements. With this notation, the transport law can be stated

$$\Gamma_j(a +_j b) = \begin{bmatrix} \Gamma_j a & \varepsilon_j b \\ \varepsilon_{j+1} b & \Gamma_j b \end{bmatrix} \qquad \begin{tikzcd}[sep=small] {} \arrow[r, "j+1"] \arrow[d, "j"'] & {} \\ {} & \end{tikzcd} .$$

□

Remark 13.1.10. The interchange law and forms of multiple composition were discussed in dimension 2 in Chapter 6 and you should refer back to that. The interchange laws in Definition 13.1.7 and the associativity laws (when they hold) have as a consequence that we can define the composition of some complicated arrays of elements in any cubical set G with associative compositions satisfying the interchange laws.[212]

A rectangular *array* of n-cubes is a family of n-cubes $x_{pq} \in G_n$ $(1 \leqslant p \leqslant P$, $1 \leqslant q \leqslant Q)$ satisfying for some $i \neq j$ the relations

$$\begin{aligned} \partial_i^+ x_{pq} &= \partial_i^- x_{p+1,q} \quad (1 \leqslant p < P, 1 \leqslant q \leqslant Q), \\ \partial_j^+ x_{pq} &= \partial_j^- x_{p,q+1} \quad (1 \leqslant p \leqslant P, 1 \leqslant q < Q). \end{aligned}$$

It is written $(x_{pq})_{\{1 \leqslant p \leqslant P, 1 \leqslant q \leqslant Q\}}$ or

$$(x_{pq}) = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1Q} \\ x_{21} & x_{22} & \dots & x_{2Q} \\ \dots & \dots & \dots & \dots \\ x_{P1} & x_{P2} & \dots & x_{PQ} \end{pmatrix} \qquad \begin{tikzcd}[sep=small] {} \arrow[r, "j"] \arrow[d, "i"'] & {} \\ {} & \end{tikzcd} .$$

An array (x_{pq}) has a unique *composite* $x = [x_{pq}] \in G_n$ obtained by applying the operations $+_i, +_j$ in any well-formed fashion; for example

$$x = (x_{11} +_i x_{21} +_i \dots +_i x_{P1}) +_j \dots +_j (x_{1Q} +_i x_{2Q} +_i \dots +_i x_{PQ}).$$

We write

$$[x_{pq}] = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1Q} \\ x_{21} & x_{22} & \dots & x_{2Q} \\ \dots & \dots & \dots & \dots \\ x_{P1} & x_{P2} & \dots & x_{PQ} \end{bmatrix} \qquad \begin{tikzcd}[sep=small] {} \arrow[r, "j"] \arrow[d, "i"'] & {} \\ {} & \end{tikzcd} .$$

The same is true for multi-dimensional arrays, and the most general situation can be described as follows. Let $(m) = (m_1, m_2, \dots, m_n)$ be a sequence of positive integers. A *composable array* in G_n of type (m) is a family of cubes $x_{(p)} \in G_n$, where $(p) = (p_1, p_2, \dots, p_n)$, $1 \leqslant p_i \leqslant m_i$, satisfying the relations

$$\partial_i^+ x_{(p)} = \partial_i^- x_{(p)'_i} \quad \text{for all } i$$

where $(p)'_i = (p_1, p_2, \ldots, p_{i-1}, p_i + 1, p_{i+1}, \ldots, p_n)$. We denote the unique composite in G_n of such an array by $[x_{(p)}]$. The previous case is obtained by taking $m_k = 1$ for $k \neq i, j$. We shall also sometimes write $[x_1, x_2, \ldots, x_r]_j$ for the linear composite $x_1 +_j x_2 +_j \cdots +_j x_r$, and an unlabeled $-x$ in such a composite will always mean $-_j x$. □

We introduce some notation for multiple compositions in the singular cubical sets KX and $R_n X_*$.

Remark 13.1.11. Let $(m) = (m_1, \ldots, m_n)$ be an n-tuple of positive integers and let

$$\phi_{(m)} \colon I^n \to [0, m_1] \times \cdots \times [0, m_n]$$

be the map $(x_1, \ldots, x_n) \mapsto (m_1 x_1, \ldots, m_n x_n)$. Then a *subdivision of type* (m) of a map $\alpha \colon I^n \to X$ is a factorisation $\alpha = \alpha' \circ \phi_{(m)}$; its *parts* are the cubes $\alpha_{(r)}$ where $(r) = (r_1, \ldots, r_n)$ is an n-tuple of integers with $1 \leqslant r_i \leqslant m_i$, $i = 1, \ldots, n$, and where $\alpha_{(r)} \colon I^n \to X$ is given by

$$(x_1, \ldots, x_n) \mapsto \alpha'(x_1 + r_1 - 1, \ldots, x_n + r_n - 1).$$

We then say that α is the *composite* of the cubes $\alpha_{(r)}$ and write $\alpha = [\alpha_{(r)}]$. The *domain* of $\alpha_{(r)}$ is then the set $\{(x_1, \ldots, x_n) \in I^n \mid r_i - 1 \leqslant x_i \leqslant r_i,\ 1 \leqslant i \leqslant n\}$.

The composite is *in direction* j if m_j is the only $m_i > 1$, and we then write $\alpha = [\alpha_1, \ldots, \alpha_{m_j}]_j$; the composite is *in the directions* j, k $(j \neq k)$ if m_j, m_k are the only $m_i > 1$, and we then write

$$\alpha = [\alpha_{rs}]_{j,k}$$

for $r = 1, \ldots, m_j$ and $s = 1, \ldots, m_k$.

These definitions and notations are some of the keys to our use of cubical methods in the proof of the Higher Homotopy Seifert–van Kampen Theorem, since they allow for

'algebraic inverses to subdivision'.[213] □

13.2 ω-groupoids

In this section we restrict to cubical sets with connections and compositions such that each composition gives a structure of groupoid. These objects give the category ω-Gpds of ω-groupoids which generalises the category DGpds of double groupoids studied in Chapter 6.

Definition 13.2.1. An *ω-groupoid* (or *cubical ω-groupoid*) $G = \{G_n\}_{n \geqslant 0}$ is a cubical set with connections and compositions in which each $+_j$ gives a groupoid structure on G_n such that for $x \in G_n$ the identity elements are

$$\eta^\alpha_j x = \varepsilon_j \partial^\alpha_j x$$

(the left identity when $\alpha = -$ and the right identity when $\alpha = +$) and the inverse is $-_j x$.

A *morphism* of ω-groupoids is a morphism of cubical sets preserving all the connections and all the groupoid operations. We denote the resulting category of ω-groupoids by ω-Gpds. □

Remark 13.2.2. Of course the compositions of the cubical singular set KX of a space X are not groupoid compositions, for the same reason as the usual composition of paths in a space do not form a category.[214] In dimension 1 it is easy to define the fundamental groupoid $\pi_1 X$ by taking homotopy classes rel end points.

For higher dimensions, there is a solution in the filtered case.[215] A major result in Chapter 14 is the definition of the fundamental ω-groupoid ρX_* of the filtered space X_*. The applications of this construction are a major theme of this book. □

Let us point out that in defining ω-groupoids some of the laws in Definition 13.1.7 are redundant.

Proposition 13.2.3. *If one assumes that each $+_j$ is a groupoid structure on G_n with identities $\eta_j^\alpha x$ for all $x \in G_n$ and inverse $-_j$, then one may omit parts* (1.ii), (2.ii), (3.ii) *and* (4.ii) *of all the laws in Definition* 13.1.7 *since they follow from the first parts and the groupoid laws. One may also rewrite the transport law* (3.i) *of the same definition in the form*

$$\Gamma_j(a +_j b) = (\Gamma_j a +_{j+1} \varepsilon_j b) +_j \Gamma_j b = (\Gamma_j a +_j \varepsilon_{j+1} b) +_{j+1} \Gamma_j b \qquad (3.\mathrm{i}^*)$$

and deduce that

$$\Gamma_j(-_j a) = (-_j \Gamma_j a) -_{j+1} \varepsilon_j a = (-_{j+1} \Gamma_j a) -_j \varepsilon_{j+1} a. \qquad (3.\mathrm{ii}^*)$$

Definition 13.2.4. An *ω-subgroupoid* of G is a sub cubical set closed under all the connections and all the operations $+_j, -_j$. Any set S of elements of G generates an ω-subgroupoid, namely, the intersection of all ω-subgroupoids containing S. Repeated applications of all the structure maps and operations allow one to build this ω-subgroupoid from S: first, it can be verified that the elements of the form $\varepsilon \dots \varepsilon \Gamma \dots \Gamma \partial \dots \partial x$ $(x \in S)$ make up the subcomplex-with-connections K generated by S; (here ∂ stands for various ∂_i^α, etc.) the ω-subgroupoid generated by S then consists, as again can be verified, of all composites of arrays of cubes of the form $-_i -_j \cdots -_l\, y$ $(y \in K)$. □

We also use finite-dimensional versions of the above structures and categories.

Definition 13.2.5. A *cubical n-groupoid* is an n-truncated cubical set

$$G = (G_n, G_{n-1}, \dots, G_0)$$

with connections, having m groupoid structures in dimension m $(m \leqslant n)$, and satisfying all the laws for an ω-groupoid in so far as they make sense.[216] We denote by ω-Gpds_n the category of cubical n-groupoids. The category ω-Gpds_2 is another name for the category DGpds of double groupoids, which we studied in Chapter 6, and was the prototype for ω-Gpds. $\square$

13.3 The crossed complex associated to an ω-groupoid

Analogously to Chapter 6, we consider for an ω-groupoid G the elements of G having all faces trivial but one. A main result is that these elements may be given the structure which was the main subject of Part II, namely that of crossed complex:

$$\gamma G\colon \cdots \to \gamma G_n \xrightarrow{\delta_n} \gamma G_{n-1} \xrightarrow{\delta_{n-1}} \dots \xrightarrow{\delta_3} \gamma G_2 \xrightarrow{\delta_2} G_1,$$

where $\delta_n = \partial_1^-$. We shall prove in the next few sections that crossed complexes are equivalent to ω-groupoids. Moreover, this associated crossed complex is obtained in such a way that the crossed complex $\gamma\rho X_*$ associated to the fundamental ω-groupoid ρX_* of the filtered space X_* is naturally isomorphic to ΠX_*, the fundamental crossed complex of a filtered space described in Section 7.1.v: the proof of this result is again delayed to the next chapter (see Theorem 14.4.1).

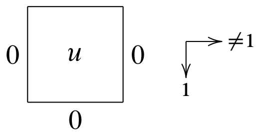
Let us start by defining γG_n as a set. The definition is motivated by the standard definition of relative homotopy groups, see p. 35.

Definition 13.3.1. For any ω-groupoid G and for $n \geqslant 2$ and $p \in G_0$, we define the set of n-cubes x all of whose faces except $\partial_1^- x$ are *concentrated* at p to be

$$\gamma G_n(p) = \{x \in G_n \mid \partial_i^\alpha x = (\varepsilon_1)^{n-1} p \text{ for all } (\alpha, i) \neq (-, 1)\}.$$

We observe that for any $p \in G_0$, such a concentrated r-cube $(\varepsilon_1)^r p$ is an identity for all compositions $+_k$ of G since $(\varepsilon_1)^r p = \varepsilon_k(\varepsilon_1)^{r-1} p$ for $1 \leqslant k \leqslant r$; accordingly, we will write 0 (sometimes 0_p) for such a cube $(\varepsilon_1)^r p$ $(p \in G_0)$. With this convention, we have the rules $\partial_i^\alpha 0 = 0$, $\varepsilon_i 0 = 0$, $\Gamma_i 0 = 0$. $\square$

Remark 13.3.2. An element of γG_n can be represented as



.

$\square$

Now we define the operations on the $\gamma G_n(p)$ which make them a family of groups, abelian for $n \geqslant 3$.

Proposition 13.3.3. *Let $n \geqslant 2$ and $p \in G_0$. Then for $2 \leqslant j \leqslant n$ each composition $+_j$ of G_n induces a group structure on $\gamma G_n(p)$. For $n \geqslant 3$ this group structure is independent of j and is abelian.*

Proof. The first part is easy to verify, while the last part is proved by applying the interchange law to the composites

$$\begin{bmatrix} x & 0_p \\ 0_p & y \end{bmatrix} \qquad \begin{bmatrix} 0_p & x \\ y & 0_p \end{bmatrix} \qquad \begin{array}{l} \longrightarrow k \\ \downarrow \\ j \end{array}$$

for $x, y \in \gamma G_n(p)$ and $2 \leqslant j < k \leqslant n$. □

Definition 13.3.4. We write $x + y$ for $x +_j y$ if $x, y \in \gamma G_n(p)$ and $2 \leqslant j \leqslant n$, and the zero element for this addition is 0_p. If $n = 1$ we also write $+$ for the groupoid operation $+_1$ on $\gamma G_1 = G_1$.

The face map $\partial_1^- \colon G_n \to G_{n-1}$ restricts to

$$\delta_n \colon \gamma G_n(p) \to \gamma G_{n-1}(p).$$

Let $n \geqslant 2$, $p, q \in G_0$. We define *the action* of $a \in G_1(p, q)$ on $x \in \gamma G_n(p)$ by

$$x^a = [-\varepsilon_1^{n-1} a,\ x,\ \varepsilon_1^{n-1} a]_n = \begin{array}{ccc} -a & \partial_1^- x & a \\ \hline \multicolumn{1}{|c|}{\mid\mid} & \multicolumn{1}{c|}{x} & \multicolumn{1}{c|}{\mid\mid} \\ \hline -a & 0 & a \end{array} \qquad \begin{array}{l} \longrightarrow n \\ \downarrow \\ \neq n \end{array} .$$

Also, if $x \in G_1(p)$, we define $x^a = -a + x + a$. □

We now check that these definitions imply γG is a crossed complex. We have seen that $\gamma G_n(p)$ is a group (abelian for $n \geqslant 3$) where $G_1(p) = G_1(p, p)$. It is also immediate that:

Proposition 13.3.5. *The maps δ_n are group homomorphisms and satisfy $\delta^2 = 0$.*

We now verify the main properties of the action.

Proposition 13.3.6. *Let $n \geqslant 2$, $p, q \in G_0$. For any $x \in \gamma G_n(p)$ and $a \in G_1(p, q)$ the element x^a defined above lies in $\gamma G_n(q)$, and the rule $(x, a) \mapsto x^a$ defines an action of the groupoid G_1 on the groupoid γG_n. This action is preserved by the map $\delta \colon \gamma G_n(p) \to \gamma G_{n-1}(p)$ for $n \geqslant 2$.*

Proof. We follow in essence the proof of Proposition 6.2.3, but use arrays rather than pictures.

First, note that, for $1 \leqslant i < n$,

$$\partial_i^\alpha (x^a) = [-\varepsilon_1^{n-2} a,\ \partial_i^\alpha x,\ \varepsilon_1^{n-2} a]_{n-1},$$

while $\partial_n^\alpha (x^a) = \varepsilon^{n-1} \partial_1^+ a = 0_q$. From this it follows that $x^a \in \gamma G_n(q)$ and $\delta(x^a) = (\delta x)^a$.

The equation

$$\varepsilon_1^{n-1}(a + b) = [\varepsilon_1^{n-1} a,\ \varepsilon_1^{n-1} b]_n$$

implies that $x^{a+b} = (x^a)^b$. □

We next show that the action by elements of $\delta_2(\gamma G_2)$ satisfies the crossed complex conditions.

Proposition 13.3.7. *Let $y \in \gamma G_2(p)$ and $a = \delta y$. If $x \in \gamma G_n(p)$, then $x^a = x$ for $n \geqslant 3$ and $x^a = -y + x + y$ for $n = 2$.*

Proof. If $a = \delta y$ and $n \geqslant 2$, the two ways of composing

$$\begin{bmatrix} -_n\varepsilon_1^{n-1}a & x & \varepsilon_1^{n-1}a \\ -_n\varepsilon_1^{n-2}y & 0_p & \varepsilon_1^{n-2}y \end{bmatrix} \qquad \begin{array}{l} \ulcorner\!\longrightarrow n \\ \downarrow \\ n-1 \end{array}$$

give $x^a = [-_n\varepsilon_1^{n-2}y,\ x,\ \varepsilon_1^{n-2}y]_n$, which is the result we require when $n = 2$. For $n \geqslant 3$ we may also compose

$$\begin{bmatrix} -_n -_{n-1}\varepsilon_1^{n-2}y & 0_p & -_{n-1}\varepsilon_1^{n-2}y \\ -_n\varepsilon_1^{n-2}y & x & \varepsilon_1^{n-2}y \end{bmatrix} \qquad \begin{array}{l} \ulcorner\!\longrightarrow n \\ \downarrow \\ n-1 \end{array}$$

in two ways to obtain, by what we have just proved, $x^a = x$. □

Putting together the above properties we obtain:

Theorem 13.3.8. *If G is an ω-groupoid then γG is a crossed complex, and this defines a functor*

$$\gamma\colon \omega\text{-}\mathsf{Gpds} \to \mathsf{Crs}.$$

By restriction, we also have a functor $\gamma\colon \omega\text{-}\mathsf{Gpds}_m \to \mathsf{Crs}_m$.

We shall show in Section 13.6 that the ω-groupoid G can be reconstructed from its associated crossed complex γG and hence that $\gamma\colon \omega\text{-}\mathsf{Gpds} \to \mathsf{Crs}$ is an equivalence of categories.

On our way to prove this result we will use (see p. 461) the alternative description of the action of G_1 on γG_n given in the next proposition, whose proof gives the first time in this section that we use the connections.

Proposition 13.3.9. *The action of G_1 on γG_n defined in Proposition 13.3.6 is also given by*

$$x^a = [-\varepsilon_1^{j-1}\varepsilon_2^{n-j}a,\ x,\ \varepsilon_1^{j-1}\varepsilon_2^{n-j}a]_j$$

for $x \in \gamma G_n(p)$, $a \in G_1(p,q)$ and any j with $2 \leqslant j \leqslant n$.

Proof. Let $2 \leqslant j \leqslant n$, and write $b_j = \varepsilon_1^{j-1}\varepsilon_2^{n-j}a = \varepsilon_n\varepsilon_{n-1}\dots\hat{j}\dots\varepsilon_1 a \in G_n$. Then b_j is an identity for all the compositions of G_n except $+_j$. Also $\partial_j^+(-_j b_j) = \partial_j^-(b_j) = 0$ and

$$\partial_{j+1}^\alpha(b_j) = \partial_j^\alpha(b_{j+1}) = \varepsilon_{n-1}\varepsilon_{n-2}\dots\hat{j}\dots\varepsilon_1 a = c,$$

say. Thus, if $j \geqslant 2$, we may form the composite

$$y = \begin{bmatrix} -_j -_{j+1}\Gamma_j c & -_j b_j & -_j\Gamma_j c \\ -_{j+1}b_{j+1} & x & b_{j+1} \\ -_{j+1}\Gamma_j c & b_j & \Gamma_j c \end{bmatrix} \qquad \begin{array}{l} \ulcorner\!\longrightarrow j+1 \\ \downarrow \\ j \end{array}.$$

Since b_{j+1} is an identity for $+_j$, the composite of the last column is $\varepsilon_j \partial_j^+ \Gamma_j c = 0_p$, and similarly the composites of the first column and of the first and last rows are 0_p. Hence, computing y by rows and by columns, we have

$$[-b_{j+1},\ x,\ b_{j+1}]_{j+1} = [-b_j,\ x,\ b_j]_j \quad (j \geqslant 2).$$

It follows that, for $j \geqslant 2$, $[-b_j,\ x,\ b_j]_j = [-b_n,\ x,\ b_n]_n$, which is the definition of x^a. □

13.4 Folding operations

As explained in the introduction to this chapter we have to take a detour to define the notion of a 'commutative n-shell'.[217] Instead of trying to make sense of all possible compositions of the $(n-1)$-faces, we just fold all faces into one face.

First we introduce a 'folding in the i-th direction' which for $i = 2$ is analogous to the 2-dimensional case. The composition of the foldings in all directions gives an operation Φ on cubes in an ω-groupoid G (or in a cubical n-groupoid) which has the effect of folding all faces of $x \in G_n$ onto the face $\partial_1^- \Phi x$. The resulting face can be seen as the 'ordered sum of the faces of x'. This operation Φ transforms x into an element of the associated crossed complex γG.

Later in this section we study the behaviour of the foldings with respect to the operators of an ω-groupoid, namely faces, degeneracies, connections and composition.

We end the section by proving that the thin elements (i.e. the composites of an array of degeneracies and connections) are just those folding to the trivial cube, i.e. those having 'commuting boundary'. This relation between thin elements and those having commuting boundary is crucial in the next chapter.

We emphasise again that these results and techniques, though with a geometric motivation, are purely algebraic, that is we use only the operations and laws that we have given. This is essential for the theory and the geometric applications.

Definition 13.4.1. In any cubical n-groupoid G, we define operations

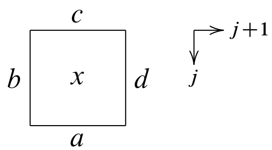
$$\Phi_j : G_m \to G_m,$$

for any $1 \leqslant j < m \leqslant n$, by the formula

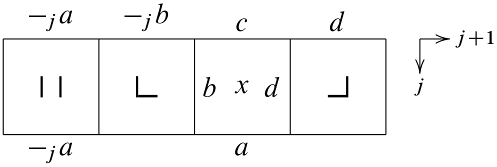
$$\Phi_j x = [-\varepsilon_j \partial_j^+ x,\ -\Gamma_j \partial_{j+1}^- x,\ x,\ \Gamma_j \partial_{j+1}^+ x]_{j+1}.$$

The map Φ_j is called the *folding in the j-th direction.* □

It is easy to check that the composite $\Phi_j x$ is defined. Writing a, b, c, d for the relevant faces of x,



the effect of Φ_j can be seen from the diagram

$\Phi_j x =$


in which unlabeled faces are appropriate degenerate cubes.

Next we study the various relations for the compositions of operations Φ_j with the operators of a cubical n-groupoid (i.e. faces, degeneracies, connections, compositions and inverses). Recall that to simplify the notation we have written $\eta_j^\alpha x$ for $\varepsilon_j \partial_j^\alpha x$, the left ($\alpha = 0$) or right ($\alpha = 1$) identity for x with respect to $+_j$.

We begin by the compositions of foldings and faces.

Proposition 13.4.2. *The faces of the folding in the j-th direction are given by:*

$$\partial_i^\alpha \Phi_j = \begin{cases} \Phi_{j-1}\partial_i^\alpha & (i < j), \\ \Phi_j \partial_i^\alpha & (i > j+1); \end{cases} \tag{i}$$

$$\partial_j^- \Phi_j x = [-\partial_j^+ x,\ -\partial_{j+1}^- x,\ \partial_j^- x,\ \partial_{j+1}^+ x]_j; \tag{ii}$$

$$\partial_{j+1}^\alpha \Phi_j = \partial_j^+ \Phi_j = \eta_j^+ \partial_j^+ = \eta_j^+ \partial_{j+1}^+. \tag{iii}$$

Proof. These are proved by using the laws for faces of degeneracies, connections and compositions contained in the Remark 11.1.5 and Definitions 13.1.3 and 13.1.7. We shall prove them using the array form.

(i) If $i < j$ then

$$\begin{aligned} \partial_i^\alpha \Phi_j x &= [-\partial_i^\alpha \eta_j^+ x,\ -\partial_i^\alpha \Gamma_j \partial_{j+1}^- x,\ \partial_i^\alpha x,\ \partial_i^\alpha \Gamma_j \partial_{j+1}^+ x]_j \\ &= [-\eta_{j-1}^+ \partial_i^\alpha x,\ -\Gamma_{j-1}\partial_j^- \partial_i^\alpha x,\ \partial_i^\alpha x,\ \Gamma_{j-1}\partial_j^+ \partial_i^\alpha x]_j \\ &= \Phi_{j-1}\partial_i^\alpha x. \end{aligned}$$

The case $i > j+1$ is similar.

(ii) This is proved by a routine argument of the same kind and we will omit all such routine proofs from now on.

(iii) As before,

$$\begin{aligned} \partial_j^+ \Phi_j x &= [-\partial_j^+ \eta_j^+ x,\ -\partial_j^+ \Gamma_j \partial_{j+1}^- x,\ \partial_j^+ x,\ \partial_j^+ \Gamma_j \partial_{j+1}^+ x]_j \\ &= [-\partial_j^+ x,\ \eta_j^+ \partial_{j+1}^- x,\ \partial_j^+ x,\ \eta_j^+ \partial_{j+1}^+ x]_j. \end{aligned}$$

But $\eta_j^+ \partial_{j+1}^- x$ and $\eta_j^+ \partial_{j+1}^+ x$ are identities for $+_j$, so

$$\partial_j^+ \Phi_j x = [-\partial_j^+ x,\ \partial_j^+ x]_j = \eta_j^+ \partial_j^+ x.$$

The other cases are easily verified. □

From this proposition we deduce immediately a formula which we will use later in this section.

Corollary 13.4.3. *With the notation of the above proposition*

$$\partial_{j+1}^\alpha \Phi_j \Phi_{j+1} \dots \Phi_{n-1} = \partial_j^+ \Phi_j \Phi_{j+1} \dots \Phi_{n-1} = \eta_j^+ \eta_{j+1}^+ \dots \eta_{n-1}^+ \partial_n^+.$$

Proof. This follows from (iii). □

Now we give the relation with degeneracies.

Proposition 13.4.4. *The foldings in the j-th direction behave on degeneracy operators as follows:*

$$\begin{cases} \Phi_j \varepsilon_i = \varepsilon_i \Phi_{j-1}, \ \Phi_j \eta_i^\alpha = \eta_i^\alpha \Phi_j & \text{if } i < j, \\ \Phi_j \varepsilon_i = \varepsilon_i \Phi_j, \quad \Phi_j \eta_i^\alpha = \eta_i^\alpha \Phi_j & \text{if } i > j+1; \end{cases} \tag{i}$$

$$\Phi_j \varepsilon_j = \eta_{j+1}^+ \varepsilon_j, \qquad \Phi_j \eta_j^\alpha = \eta_{j+1}^+ \eta_j^\alpha; \tag{ii}$$

$$\Phi_j \varepsilon_{j+1} = \eta_{j+1}^+ \varepsilon_j, \qquad \Phi_j \eta_{j+1}^\alpha = \eta_j^+ \eta_{j+1}^\alpha. \tag{iii}$$

and of course $\eta_{j+1}^+ \varepsilon_j = \eta_j^+ \varepsilon_{j+1}$.

Proof. (i) and (ii) are routine; the parts about $\Phi_j \eta_j^\alpha$ involve also the use of the previous proposition.

(iii)

$$\begin{aligned} \Phi_j \varepsilon_{j+1} x &= [-\eta_j^+ \varepsilon_{j+1} x,\ -\Gamma_j x, \varepsilon_{j+1} x,\ \Gamma_j x]_{j+1} \\ &= [-\eta_j^+ \varepsilon_{j+1} x]_{j+1} = [-\eta_{j+1}^+ \varepsilon_j x]_{j+1} \\ &= \eta_{j+1}^+ \varepsilon_j x. \end{aligned}$$

The other equations follow easily. □

From this proposition we deduce immediately another formula that we use later in this section.

Corollary 13.4.5. *With the notation of the above proposition*

$$\Phi_1 \Phi_2 \dots \Phi_{j-2} \eta_{j-1}^+ = \eta_1^+ \eta_2^+ \dots \eta_{j-1}^+, \quad \Phi_1 \Phi_2 \dots \Phi_{j-1} \varepsilon_j = \eta_1^+ \eta_2^+ \dots \eta_{j-1}^+ \varepsilon_j.$$

Proof. This follows from (iii) in the preceding proposition. □

Now we give the relations with connections:

Proposition 13.4.6. *The foldings in the j-th direction behave on connection operators as follows:*

$$\Phi_j \Gamma_i = \begin{cases} \Gamma_i \Phi_{j-1} & (i < j), \\ \Gamma_i \Phi_j & (i > j+1); \end{cases} \tag{i}$$

$$\Phi_j \Gamma_j = \varepsilon_j \eta_j^+ = \varepsilon_{j+1} \eta_j^+; \tag{ii}$$

$$\Phi_j \Gamma_{j+1} x = [-\Gamma_{j+1} \eta_j^+ x,\ -\Gamma_j x,\ \Gamma_{j+1} x,\ \Gamma_j \eta_{j+1}^+ x]_{j+1}. \tag{iii}$$

Proof. (i) and (iii) are routine. For (ii),

$$\begin{aligned} \Phi_j \Gamma_j x &= [-\eta_j^+ \Gamma_j x,\ -\Gamma_j \partial_{j+1}^- \Gamma_j x,\ \Gamma_j x, v\Gamma_j \partial_{j+1}^+ \Gamma_j x]_{j+1} \\ &= [-\varepsilon_j \eta_j^+ x,\ -\Gamma_j x,\ \Gamma_j x,\ \Gamma_j \eta_j^+ x]_{j+1} && \text{by 13.1.3} \\ &= [-\varepsilon_{j+1} \eta_j^+ x,\ \varepsilon_{j+1} \eta_j^+ x]_{j+1} && \text{by 11.1.5 and 13.1.3} \\ &= \varepsilon_{j+1} \eta_j^+ x = \varepsilon_j \eta_j^+ x. \end{aligned}$$

□

We now define for $n \geqslant 2$ the *folding operation*

$$\Phi \colon G_n \to \gamma G_n$$

by folding in each direction in decreasing order.

Definition 13.4.7. On G_0 and G_1 we define Φ as the identity map. We now define for $n \geqslant 2$

$$\Phi x = \Phi_1 \Phi_2 \dots \Phi_{n-1} x$$

for any $x \in G_n$. □

Let us see that the folding has the image we want, i.e. that Φx has all faces but one trivial. To do this, we introduce some notation.

Definition 13.4.8. For $x \in G_n$, we call $(\partial_1^+)^n x$ the *base-point* of x and denote it by βx. □

Proposition 13.4.9. *If $(\alpha, j) \neq (0, 1)$ then $\partial_j^\alpha \Phi = \varepsilon_1^{n-1} \beta$. Hence, for any $x \in G$, Φx lies in the associated crossed complex γG_n.*

Proof. If $2 \leqslant j \leqslant n$ then

$$\begin{aligned} \partial_j^\alpha \Phi &= \Phi_1 \Phi_2 \dots \Phi_{j-2} \partial_j^\alpha \Phi_{j-1} \dots \Phi_{n-1} && \text{by 13.4.2 (i)} \\ &= \Phi_1 \Phi_2 \dots \Phi_{j-2} \eta_{j-1}^+ \dots \eta_{n-1}^+ \partial_n^+ && \text{by 13.4.3} \\ &= \eta_1^+ \eta_2^+ \dots \eta_{n-1}^+ \partial_n^+ && \text{by 13.4.5} \\ &= \varepsilon_1^{n-1} (\partial_1^+)^n && \text{by 11.1.5.} \end{aligned}$$

If $j = 1$ and $n \geqslant 2$, then $\alpha = 1$ and the equation follows from Proposition 13.4.2 (iv) and Remark 11.1.5. The case $n = 1$ is trivial. Thus, for $x \in G_n$, we have $\partial_j^\alpha \Phi x = 0_p$ for $(\alpha, j) \neq (0, 1)$, where $p = \beta x$. This shows that $\Phi x \in \gamma G_n(p)$. $\square$

This gives the following important characterisation of the elements in γG as those invariant under the folding.

Corollary 13.4.10. *If $x \in G$, then x is in γG if and only if $\Phi x = x$. In particular $\Phi^2 y = \Phi y$ for all y in G.*

Proof. It is clear that if $x \in C_n(p) = (\gamma G_n)(p)$, then Definition 13.4.1 implies $\Phi_j x = x$. This implies $\Phi x = x$. $\square$

To end the study of the behaviour of the folding map with respect to the operators of a cubical set with connections, let us record the effect the folding map has on degeneracies and connections.

Proposition 13.4.11. *If $n \geqslant 2$, then on G_{n-1},*

$$\Phi\varepsilon_j = \varepsilon_1^n \beta \quad \textit{and} \quad \Phi\Gamma_j = \varepsilon_1^n \beta.$$

Proof. Making computations:

$$\begin{aligned}
\Phi_1\Phi_2 \ldots \Phi_{n-1}\varepsilon_j &= \Phi_1\Phi_2 \ldots \Phi_j\varepsilon_j\Phi_j\Phi_{j+1} \ldots \Phi_{n-2} && \text{by 13.4.4 (i)} \\
&= \Phi_1\Phi_2 \ldots \Phi_{j-1}\eta_{j+1}^+\varepsilon_j\Phi_j \ldots \Phi_{n-2} && \text{by 13.4.4 (ii)} \\
&= \Phi_1\Phi_2 \ldots \Phi_{j-1}\varepsilon_j\varepsilon_j\partial_j^+\Phi_j \ldots \Phi_{n-2} && \text{by 11.1.5} \\
&= \eta_1^+\eta_2^+ \ldots \eta_{j-1}^+\varepsilon_j\varepsilon_j\eta_j^+ \ldots \eta_{n-2}^+\partial_{n-1}^+ && \text{by 13.4.3 and 13.4.5} \\
&= \varepsilon_1^n\beta && \text{by 11.1.5.}
\end{aligned}$$

and

$$\begin{aligned}
\Phi_1\Phi_2 \ldots \Phi_{n-1}\Gamma_j &= \Phi_1\Phi_2 \ldots \Phi_j\Gamma_j\Phi_j\Phi_{j+1} \ldots \Phi_{n-2} && \text{by 13.4.6 (i)} \\
&= \Phi_1\Phi_2 \ldots \Phi_{j-1}\varepsilon_j\eta_j^+\Phi_j \ldots \Phi_{n-2} && \text{by 13.4.6 (ii)} \\
&= \varepsilon_1^n\beta && \text{as above.}
\end{aligned}$$ $\square$

We now study the behaviour of the folding map Φ with respect to composition and inverses. The rules are easy to state (see Proposition 13.4.14) but their proof involves more complicated rules for the partial foldings Φ_j.

Proposition 13.4.12. *We have the following relations of Φ_j with the compositions and inverses:*

$$\left.\begin{aligned}\Phi_j(x +_i y) &= \Phi_j x +_i \Phi_j y \\ \Phi_j(-_i x) &= -_i \Phi_j x\end{aligned}\right\} \quad \text{if } i \neq j, j+1; \tag{i}$$

$$\Phi_j(x +_j y) = [\Phi_j y, -\varepsilon_j\partial_{j+1}^+ y, \Phi_j x, \varepsilon_j\partial_{j+1}^+ y]_{j+1}; \tag{ii}$$

$$\Phi_j(x +_{j+1} y) = [-\eta_j^+ y, \Phi_j x, \eta_j^+ y, \Phi_j y]_{j+1}. \tag{iii}$$

Proof. (i) This is routine, using the interchange law for the directions i and $j+1$.

(ii) Let the relevant faces of x and y be given by

$$\begin{array}{ccc} & u & \\ a & \boxed{\;x\;} & b \\ & v & \end{array} \qquad \begin{array}{ccc} & v & \\ c & \boxed{\;y\;} & d \\ & w & \end{array} \qquad \begin{array}{l} \longrightarrow j+1 \\ \big\downarrow j \end{array} \quad .$$

Then

$$\Phi_j(x +_j y) = [-\varepsilon_j w, -\Gamma_j(a +_j c), (x +_j y), \Gamma_j(b +_j d)]_{j+1}.$$

Using the transport law, this can be written as the composite

$$A = \begin{bmatrix} -\varepsilon_j w & -\varepsilon_j c & -\Gamma_j a & x & \Gamma_j b & \varepsilon_j d \\ -\varepsilon_j w & -\Gamma_j c & -\varepsilon_{j+1} c & y & \varepsilon_{j+1} d & \Gamma_j d \end{bmatrix} \quad \begin{array}{l} \longrightarrow j+1 \\ \big\downarrow j \end{array}$$

where $-$ stands for $-_{j+1}$. Consider the composite

$$B = \begin{bmatrix} -\varepsilon_j w & -\varepsilon_j c & \varepsilon_j v & \varepsilon_j d & -\varepsilon_j d & -\varepsilon_j v & -\Gamma_j a & x & \Gamma_j b & \varepsilon_j d \\ -\varepsilon_j w & -\Gamma_j c & y & \Gamma_j d & -\varepsilon_j d & -\varepsilon_j v & -\varepsilon_j \eta_j^+ a & \varepsilon_j v & \varepsilon_j \eta_j^+ b & \varepsilon_j d \end{bmatrix} \quad \begin{array}{l} \longrightarrow j+1 \\ \big\downarrow j \end{array} .$$

By composing the columns first, we see that B is equal to the right-hand side of (ii). However, the composites of the rows of B are the same as the composites of the rows of A, since $\varepsilon_j \eta_j^+ b = \varepsilon_{j+1} \eta_j^+ b$ is an identity of the horizontal composition as well as the vertical one. Hence $A = B$.

(iii) This is routine. □

To state the behaviour of the folding map Φ with respect to compositions and inverses, we need some extra notation.

Definition 13.4.13. For $x \in G_n$, the *edges of x terminating at the base point*, $\beta x = (\partial_1^+)^n x$ will have special importance and we denote them by

$$u_i x = \partial_1^+ \partial_2^+ \dots \hat{\imath} \dots \partial_n^+ x$$

for all $1 \leqslant i \leqslant n$. □

Proposition 13.4.14. *Let $n \geqslant 2$ and $x, y, z \in G_n$ with $\partial_i^+ x = \partial_i^- y$. Then, in γG_n:*

$$\Phi(x +_i y) = \begin{cases} \Phi y + (\Phi x)^{u_1 y} & \text{if } n = 2 \text{ and } i = 1, \\ (\Phi x)^{u_i y} + \Phi y & \text{otherwise;} \end{cases} \tag{i}$$

$$\Phi(-_i z) = -(\Phi z)^{-u_i z}. \tag{ii}$$

Proof. (i) First consider the case $i = n \geqslant 2$. We have, by Proposition 13.4.12,

$$\begin{aligned}\Phi(x +_n y) &= \Phi_1\Phi_2 \dots \Phi_{n-2}[-\eta^+_{n-1}y,\ \Phi_{n-1}x,\ \eta^+_{n-1}y,\ \Phi_{n-1}y]_n \\ &= [-u,\ \Phi x,\ u,\ \Phi y]_n\end{aligned}$$

where

$$\begin{aligned}u &= \Phi_1\Phi_2 \dots \Phi_{n-2}\eta^+_{n-1}y \\ &= \eta^+_1\eta^+_2 \dots \eta^+_{n-1}y && \text{by 13.4.5} \\ &= \varepsilon_1^{n-1}u_n y && \text{by 11.1.5.}\end{aligned}$$

Hence $\Phi(x +_n y) = (\Phi x)^{u_n y} + \Phi y$ in this case.

In the remaining cases we have $1 \leqslant i < n$, so we may put

$$\begin{aligned}X &= \Phi_{i+1}\Phi_{i+2} \dots \Phi_{n-1}x, \\ Y &= \Phi_{i+1}\Phi_{i+2} \dots \Phi_{n-1}y,\end{aligned}$$

and then

$$\begin{aligned}\Phi(x +_i y) &= \Phi_1\Phi_2 \dots \Phi_i(X +_i Y) && \text{by 13.4.12 (i)} \\ &= \Phi_1 \dots \Phi_{i-1}[\Phi_i Y,\ -\varepsilon_i\partial^+_{i+1}Y,\ \Phi_i X,\ \varepsilon_i\partial^+_{i+1}Y]_{i+1} && \text{by 13.4.12 (ii)} \\ &= [\Phi y,\ -V,\ \Phi x,\ V]_{i+1} && \text{by 13.4.12 (i),}\end{aligned}$$

where

$$\begin{aligned}V &= \Phi_1 \dots \Phi_{i-1}\varepsilon_i\partial^+_{i+1}\Phi_{i+1} \dots \Phi_{n-1}y \\ &= \eta^+_1\eta^+_2 \dots \eta^+_{i-1}\varepsilon_i\eta^+_{i+1} \dots \eta^+_{n-1}\partial^+_n y && \text{by 13.4.3 and 13.4.5} \\ &= (\varepsilon_1)^i(\varepsilon_2)^{n-i-1}u_i y && \text{by 11.1.5.}\end{aligned}$$

Hence, by Proposition 13.3.9, $\Phi(x +_i y) = \Phi y + (\Phi x)^{u_i y}$ in this case. (Note that $i + 1 \geqslant 2$, so addition in direction $i + 1$ is addition in γG_n). If $n = 2$ and $i = 1$, this is the required formula. Otherwise, we have $n \geqslant 3$, so γG_n is commutative and the formula can be rewritten in the required form.

(ii) Put $x = -_i x, y = z$ in (i) and note that, by 13.4.11, $\Phi((-_i z) +_i z) = \Phi\varepsilon_i\partial^+_i z = \varepsilon_1^n\beta z = 0$ in γG_n. □

The folding map is idempotent. More precisely

Proposition 13.4.15. *For any* $1 \leqslant j \leqslant n - 1$, *we have*

$$\Phi\Phi_j = \Phi\colon G_n \to G_n.$$

Proof. By definition, for $x \in G_n$,

$$\begin{aligned}\Phi_j x &= [-\varepsilon_j \partial_j^+ x,\ -\Gamma_j \partial_{j+1}^- x,\ x,\ \Gamma_j \partial_{j+1}^+ x]_{j+1} \\ &= [a,\ b,\ x,\ c]_{j+1}, \quad \text{say.}\end{aligned}$$

By Proposition 13.4.11 and 13.4.14 (ii), Φa, Φb and Φc are all zero in γG_n, so Proposition 13.4.14 gives

$$\Phi\Phi_j x = (\Phi x)^u,$$

where

$$\begin{aligned}u &= u_{j+1} c \\ &= \partial_1^+ \ldots \partial_j^+ \partial_{j+2}^+ \ldots \partial_n^+ \Gamma_j \partial_{j+1}^+ x && \text{by definition of } u_{j+1} \\ &= \varepsilon_1 \partial_1^+ \partial_2^+ \ldots \partial_n^+ x && \text{by 11.1.5 and 13.1.3.}\end{aligned}$$

Thus $\Phi\Phi_j x = (\Phi x)^{\varepsilon_1 \beta x} = \Phi x$. □

Corollary 13.4.16. *The folding operation Φ is idempotent, i.e. for any n, we have*

$$\Phi\Phi = \Phi \colon G_n \to G_n.$$

We end this section with the definition of the thin n-cubes and their characterisation as those n-cubes that fold to the trivial cube; thus, in particular, a thin cube has commutative boundary. [218]

Definition 13.4.17. An element $x \in G_n$, for $n \geqslant 1$, is *thin* if it can be written as a composite of an array $x = [x_{(r)}]$, where each entry is either of the form $\varepsilon_i y$ or of the form $-_i -_j \cdots -_l \Gamma_m y$. □

The collection of all thin elements of G is clearly closed under all the ω-groupoid operations except possibly the face operations. It is useful to think of the thin elements as the most general kind of 'degenerate' cubes. They are important in the topological applications, see for example Theorem 14.2.9, and we establish their main properties in Section 13.7. For the present we prove only the following characterisation.

Proposition 13.4.18. *For $n \geqslant 1$ an element $x \in G_n$ is thin if and only if $\Phi x = 0$.*

Proof. We have shown that $\Phi\varepsilon_j y = 0, \Phi\Gamma_j y = 0$ for all $y \in G_{n-1}$ (see Proposition 13.4.11). It follows from Proposition 13.4.14 that $\Phi x = 0$ whenever x is thin. To see the converse, we recall the definition

$$\Phi_j x = [-\varepsilon_j \partial_j^+ x,\ -\Gamma_j \partial_{j+1}^- x,\ x,\ \Gamma_j \partial_{j+1}^+ x]_{j+1}$$

which can be rewritten as

$$x = [\Gamma_j \partial_{j+1}^- x,\ \varepsilon_j \partial_j^+ x,\ \Phi_j x,\ -\Gamma_j \partial_{j+1}^+ x]_{j+1}.$$

These two equations show that $\Phi_j x$ is thin if and only if x is thin. Hence Φx is thin if and only if x is thin. In particular, if $\Phi x = 0$ (i.e. $\Phi x = \varepsilon_1^n \beta x$) then Φx is thin, so x is also thin. □

13.5 n-shells: coskeleton and skeleton

To work inductively on an ω-groupoid, we have at each step n to restrict our attention to dimensions $\leqslant n$ and the minimal part accompanying it. To this end, it is useful to introduce the n-skeleton of an ω-groupoid as the ω-subgroupoid generated by the part of dimensions $\leqslant n$, analogously to the constructions for crossed complexes in Section 7.1.vi. Again, it is useful to make the construction a bit more categorical.[219]

Definition 13.5.1. If we ignore the elements of dimension higher than n in an ω-groupoid we obtain a cubical n-groupoid. This gives the *n-truncation* functor

$$\mathrm{tr}_n : \omega\text{-}\mathsf{Gpds} \to \omega\text{-}\mathsf{Gpds}_n. \qquad \square$$

We shall show that tr_n has both a right adjoint $\mathrm{cosk}^n : \omega\text{-}\mathsf{Gpds}_n \to \omega\text{-}\mathsf{Gpds}$, the *$n$-coskeleton functor* (Definition 13.5.5) and a left adjoint $\mathrm{sk}^n : \omega\text{-}\mathsf{Gpds}_n \to \omega\text{-}\mathsf{Gpds}$, the *$n$-skeleton functor* (Definition 13.5.14).

We will see that both can be described in terms of 'shells', i.e. families of r-cubes that fit together as the faces of an $(r+1)$-cube do. A trivial example is the total boundary of an n-cube.

For any cubical n-groupoid $G = (G_n, G_{n-1}, \dots, G_0)$ we will construct an ω-groupoid $\mathrm{cosk}^n\, G$ by adding 'shells' in all dimensions $\geqslant n$. To check that $\mathrm{cosk}^n\, G$ is an ω-groupoid we need to explain how to apply faces, degeneracies and connections to these shells. As a consequence, we describe the result of applying the folding operations Φ_i and Φ to these shells. In particular, we prove that Φ commutes with the total boundary.

All these results may be used to prove the existence and uniqueness of fillers for n-shells. Associated to any n-cube $x \in G_n$ we have its total boundary $\boldsymbol{\partial} x$ and its folding Φx satisfying $\boldsymbol{\partial}\Phi x = \Phi \boldsymbol{x}$. Conversely, for any $\mathbf{x} \in \square' G_{n-1}$ and $\xi \in \gamma G_n(\beta \mathbf{x})$ and $n \geqslant 2$ such that $\delta\xi = \delta\Phi\mathbf{x}$ exists $x \in G_n$ with $\boldsymbol{\partial} x = \mathbf{x}$ and $\Phi x = \xi$ is. This x is unique and it is denoted $x = \langle \mathbf{x}, \xi\rangle$. This property and notation allows the reconstruction of G from γG.

We finish the section by constructing sk^n the n-skeleton functor as an ω-subgroupoid of cosk^n, and proving that it is the left adjoint of tr_n.

Definition 13.5.2. In any cubical set K, an *n-shell* is a family $\mathbf{x} = (x_i^\alpha)$ of n-cubes $(i = 1, 2, \dots, n+1; \alpha = \pm)$ satisfying

$$\partial_j^\beta x_i^\alpha = \partial_{i-1}^\alpha x_j^\beta \quad \text{for } 1 \leqslant j < i \leqslant n+1 \text{ and } \alpha, \beta = \pm.$$

We denote by $\square' K_n$ the set of all n-shells of K. We usually write shells in boldface. $\square$

Example 13.5.3. Notice that the faces $\{\partial_j^\alpha y\}$ for any $(n+1)$-cube y form an n-shell $\boldsymbol{\partial} y$ that we call its *total boundary*. It could be said that an n-shell is just a collection of n-cubes which forms a candidate to be the total boundary of an $(n+1)$-cube. If this $(n+1)$-cube exists it is called a *filler* of the n-shell. $\square$

Now, to any n-truncated cubical set we associate an $(n+1)$-truncated cubical set by adding the n-shells.

Definition 13.5.4. Let $K = (K_n, K_{n-1}, \dots, K_0)$ be an n-truncated cubical set.

To give to $K' = (\square' K_n, K_n, K_{n-1}, \dots, K_0)$ the structure of $(n+1)$-truncated cubical set we need only to define faces and degeneracies involving the top dimension.

Thus the faces

$$\partial_i^\alpha : \square' K_n \to K_n$$

are given by $\partial_i^\alpha \mathbf{x} = x_i^\alpha$ for any $\mathbf{x} \in \square' K_n$, and, the degeneracies

$$\boldsymbol{\varepsilon}_j : K_n \to \square' K_n$$

are given by $\boldsymbol{\varepsilon}_j y = \mathbf{z}$, for any $y \in K_n$, where

$$z_i^\alpha = \begin{cases} \varepsilon_{j-1} \partial_i^\alpha y & (i < j), \\ \varepsilon_j \partial_{i-1}^\alpha y & (i > j), \\ y & (i = j). \end{cases} \tag{i}$$

Clearly the cubical rules of 11.1.5 are satisfied.

If K has also connections, we can define connections on K' by:

$$\boldsymbol{\Gamma}_j : K_n \to \square' K_n$$

given by $\boldsymbol{\Gamma}_j y = \mathbf{w}$, where

$$w_i^\alpha = \begin{cases} \Gamma_{j-1} \partial_i^\alpha y & (i < j), \\ \Gamma_j \partial_{i-1}^\alpha y & (i > j+1), \end{cases} \quad \begin{matrix} w_j^- = w_{j+1}^- = y, \\ w_j^+ = w_{j+1}^+ = \eta_j^+ y. \end{matrix} \tag{ii}$$

Again this is the definition needed for the connections to satisfy the relations in Definition 13.1.3. In this way K' becomes an $(n+1)$-truncated cubical set with connections.

If K has compositions, we can also define compositions in $\square' K_n$ as follows. Let $\mathbf{x}, \mathbf{y} \in \square' K_n$ with $y_j^- = x_j^+$. Define $\mathbf{x} +_j \mathbf{y} = \mathbf{t}$ and $-_j \mathbf{x} = \mathbf{s}$, where (cf. 13.1.7)

$$\begin{gathered} \begin{cases} t_j^- = x_j^-, \\ t_j^+ = y_j^+, \end{cases} \qquad t_i^\alpha = \begin{cases} x_i^\alpha +_{j-1} y_i^\alpha & (i < j), \\ x_i^\alpha +_j y_i^\alpha & (i > j), \end{cases} \\ \begin{cases} s_j^- = x_j^+, \\ s_j^+ = x_j^-, \end{cases} \qquad s_i^\alpha = \begin{cases} -_{j-1} x_i^\alpha & (i < j), \\ -_j x_i^\alpha & (i > j). \end{cases} \end{gathered} \tag{iii}$$

Then K' becomes an $(n+1)$-truncated cubical set with connections and compositions.

Moreover, if K is a cubical n-groupoid, then K' is a cubical $(n+1)$-groupoid. The verification of these facts is a tedious but entirely routine computation. $\square$

The coskeleton functor can now be obtained by iteration of this construction.

Definition 13.5.5. For any cubical n-groupoid $G = (G_n, G_{n-1}, \dots, G_0)$ we define its *n-coskeleton* by

$$(\operatorname{cosk}^n G)_m = \begin{cases} G_m & \text{for } m \leqslant n, \\ \square'^{\,m-n} G_n & \text{for } m > n \end{cases}$$

with operations defined as above. $\square$

Proposition 13.5.6. *If* $G = (G_n, G_{n-1}, \dots, G_0)$ *is a cubical n-groupoid, then* $\operatorname{cosk}^n G$ *is an ω-groupoid. This construction gives a functor*

$$\operatorname{cosk}^n \colon \omega\text{-}\mathsf{Gpds}_n \to \omega\text{-}\mathsf{Gpds}$$

which is right adjoint to tr_n.

Proof. By definition, it is clear that $\operatorname{cosk}^n G$ is an ω-groupoid.

If H is any ω-groupoid and $\theta_k \colon H_k \to G_k$ are defined for $k \leqslant n$ so as to form a morphism of cubical n-groupoids from $\operatorname{tr}_n H$ to G, then there is a unique extension to a morphism of ω-groupoids $\theta \colon H \to \operatorname{cosk}^n G$ defined inductively by

$$\theta_m y = \mathbf{z}, \quad \text{where } z_i^\alpha = \theta_{m-1} \partial_i^\alpha y,\ m > n.$$

This shows that cosk^n is right adjoint to tr_n. $\square$

Proposition 13.5.7. *If* $G = (G_n, G_{n-1}, \dots, G_0)$ *is a cubical n-groupoid, then all elements of* $\operatorname{cosk}^n G$ *in dimension $n+2$ and higher are thin.*

Proof. To prove the result it is enough to show that, for any ω-groupoid G, elements of $\square'^2 G_r$ are always thin, or equivalently, by Proposition 13.4.18, that their foldings are trivial.

Let $\mathbf{z} \in \square'^{\,2} G_r$ and $\mathbf{w} = \Phi \mathbf{z}$. Then $\mathbf{w} \in \square'^{\,2} G_r$ and all its $(r+1)$-dimensional faces $\partial_i^\alpha \mathbf{w}$ are 0_p, where $p = \beta \mathbf{z}$, except possibly $\partial_1^- \mathbf{w}$. Let us check that this one is also 0_p.

The condition that all $(r+1)$-faces but one are 0_p implies that *all* the r-dimensional faces of $\mathbf{w}$ are 0_p. Hence $\partial_1^- \mathbf{w}$ is an r-shell all of whose faces are 0_p. By definition, therefore $\partial_1^- \mathbf{w} = 0_p$.

Hence $\mathbf{w}$ itself is an $(r+1)$-shell all of whose faces are 0_p and therefore $\mathbf{w} = 0_p$. By Proposition 13.4.18, $\mathbf{z}$ is thin. $\square$

We next see that the total boundary commutes with the folding.

Proposition 13.5.8. *For any element x of dimension at least two in any cubical m-groupoid*

$$\Phi \boldsymbol{\partial} x = \boldsymbol{\partial} \Phi x.$$

Proof. Given an n-shell $\mathbf{y} = (y_i^\alpha) \in \square' G_n$, we obtain n-shells $\Phi_i \mathbf{y}$ and

$$\Phi\mathbf{y} = \Phi_1\Phi_2 \ldots \Phi_{n-1}\mathbf{y}.$$

By Proposition 13.4.9, all faces of $\Phi\mathbf{y}$ except $\partial_1^-\Phi\mathbf{y}$ are 0_p, where $p = \beta\mathbf{y} = (\partial_1^+)^n y_1^+$.

If H is a given ω-groupoid, then adjointness gives a canonical morphism

$$\theta\colon H \to \mathrm{Cosk}^n H = \mathrm{cosk}^n(\mathrm{tr}_n H),$$

with $\theta_{n+1}x = \boldsymbol{\partial} x$ for $x \in H_{n+1}$. Since θ preserves the folding operations we have the result. □

Remark 13.5.9. Note that by Proposition 13.4.2 the faces of $\Phi_j x$ depend only on the faces of x, and this gives a recipe for $\Phi_j \boldsymbol{\partial} x$. □

We can now prove that an n-shell $\mathbf{x} \in \square' G_{n-1}$ has a unique filler $x \in G_n$ for each element $\xi \in \gamma G_n(p)$ having the same boundary as the folding $\Phi\mathbf{x}$. This is the key to the inductive reconstruction of an ω-groupoid G from its associated crossed complex γG (Theorem 13.6.2), a construction which in essence arises from the fact that the folding operations are invertible, given complete information on the needed boundary.

Proposition 13.5.10. *Let G be an ω-groupoid, and let γG be its associated crossed complex. Let $\mathbf{x} \in \square' G_{n-1}$ and $\xi \in \gamma G_n(p)$, where $p = \beta\mathbf{x}$ and $n \geqslant 2$. Then a necessary and sufficient condition for the existence of $x \in G_n$ such that $\boldsymbol{\partial} x = \mathbf{x}$ and $\Phi x = \xi$ is that $\delta\xi = \delta\Phi\mathbf{x}$. Furthermore, if x exists, it is unique and it is denoted $x = \langle \mathbf{x}, \xi\rangle$.*

Proof. Clearly the condition is necessary, since if $\boldsymbol{\partial} x = \mathbf{x}$ and $\Phi x = \xi$, then $\boldsymbol{\partial}\Phi x = \Phi\boldsymbol{\partial} x = \Phi\mathbf{x}$, by the previous proposition, so $\delta\Phi\mathbf{x} = (\Phi\mathbf{x})_1^- = \partial_1^-\Phi x = \delta\xi$.

Suppose, conversely, that we are given $\mathbf{x}$ and ξ with $\delta\xi = \delta\Phi\mathbf{x}$, i.e. $\partial_1^-\xi = (\Phi\mathbf{x})_1^-$. Since all other faces of ξ and $\Phi\mathbf{x}$ are concentrated at p, this condition is equivalent to $\boldsymbol{\partial}\xi = \Phi\mathbf{x}$, an equation in $\square' G_{n-1}$. We have to show that there is a unique $x \in G_n$ such that $\boldsymbol{\partial} x = \mathbf{x}$ and $\Phi x = \xi$.

Since $\Phi\mathbf{x} = \Phi_1\Phi_2 \ldots \Phi_{n-1}\mathbf{x}$, by induction, it is enough to show that if $y \in G_n$ and $\boldsymbol{\partial} y = \Phi_i\mathbf{z}$ for some $1 \leqslant i \leqslant n-1$ and $\mathbf{z} \in \square' G_{n-1}$, then there is a unique $z \in G_n$ with $\boldsymbol{\partial} z = \mathbf{z}$ and $\Phi_i z = y$. But this is clear since the equation

$$[-\varepsilon_i\partial_i^+ z,\ -\Gamma_i\partial_{i+1}^- z,\ z,\ \Gamma_i\partial_{i+1}^+ z]_{i+1} = y$$

becomes

$$[-\varepsilon_i z_i^+,\ -\Gamma_i z_{i+1}^-,\ z,\ \Gamma_i z_{i+1}^+]_{i+1} = y$$

under the stated conditions, and therefore has a unique solution for z in terms of y and $\mathbf{z}$. It is easy to check that this z has boundary $\mathbf{z}$. □

An easy consequence of this and Proposition 13.4.18 is a characterisation of when an n-shell has a thin filler, plus the fact that this filler is unique.

Corollary 13.5.11. *A thin element of an ω-groupoid is determined by its faces. Given a shell $\mathbf{x}$, there is a thin element t with $\partial t = \mathbf{x}$ if and only if $\delta\Phi\mathbf{x} = 0$.*

Proof. Put $\xi = 0$ in Proposition 13.5.10 and use that t is thin if and only if $\Phi t = 0$ (Proposition 13.4.18). $\square$

Definition 13.5.12. A shell $\mathbf{x}$ will be called a *commuting shell* if its folding is trivial, i.e. if $\delta\Phi\mathbf{x} = 0$. This can be interpreted as 'the sum of its folded faces is 0'. By the previous corollary, a commuting shell has a thin filler and that filler is unique.

Another consequence of Corollary 13.5.11 is that any ω-groupoid G can be recovered from its associated crossed complex γG.

Proposition 13.5.13. *Let G be an ω-groupoid. Then the substructure γG generates G as ω-groupoid.*

Proof. Let H be any ω-subgroupoid of G containing γG. Then $\gamma H = \gamma G$ by definition. We show inductively that $H_n = G_n$.

This is true for $n = 0, 1$ since $\gamma G_0 = G_0, \gamma G_1 = G_1$.

Suppose $x \in G_n (n \geqslant 2)$. Then $\Phi x \in \gamma G_n$ and, by induction hypothesis, $\partial x \in \square' H_{n-1}$. By Proposition 13.5.10, there is a unique $y \in H_n$ with $\partial y = \partial x$ and $\Phi y = \Phi x$. But x is the unique element of G_n with this property, so $H_n = G_n$. $\square$

We shall finish the section by constructing the n-skeleton functor sk^n as a substructure of cosk^n and proving that it is the left adjoint of tr_n.

Definition 13.5.14. Given a cubical n-groupoid $G = (G_n, G_{n-1}, \dots, G_0)$, the *$n$-skeleton* $\mathrm{sk}^n\, G$ of G is the ω-subgroupoid of $\mathrm{cosk}^n\, G$ generated by G. $\square$

There is a characterisation of sk^n in terms of commuting shells.

Proposition 13.5.15. *Given a cubical n-groupoid $G = (G_n, G_{n-1}, \dots, G_0)$, the n-skeleton*

$$\mathrm{sk}^n\, G = S$$

where S is defined by

$$S_m = \begin{cases} G_m & \text{if } m \leqslant n, \\ \{\mathbf{x} \in \square' S_{m-1} \mid \delta\Phi\mathbf{x} = 0\} & \text{if } m > n. \end{cases}$$

i.e. for $m > n$, $\mathrm{sk}^n\, G_m$ consists entirely of thin elements, namely, the commuting shells. Moreover, for $m \geqslant n+2$, $\mathrm{cosk}^n\, G_m = \mathrm{sk}^n\, G_m$, i.e. all shells in $\square' S_{m-1}$ are commuting shells.

Proof. It is clear that $S \subseteq \mathrm{cosk}^n\, G$. By Proposition 13.5.7 all elements of S_m are thin for $m > n$.

Clearly, S is closed under face maps, degeneracy maps and connections (since $\varepsilon_j y$ and $\Gamma_j y$ are always thin).

Also, by induction on m, S_m is closed under $+_i, -_i$ $(1 \leqslant i \leqslant m)$; for if $\mathbf{x}, \mathbf{y} \in S_m$ $(m > n)$ and $\mathbf{x} +_i \mathbf{y}$ is defined, then $\mathbf{x} +_i \mathbf{y}$ has faces in S_{m-1} (by induction hypothesis) and $\delta\Phi(\mathbf{x} +_i \mathbf{y}) = 0$ because composites of thin elements in $\operatorname{cosk}^n G$ are thin. Thus $\mathbf{x} +_i \mathbf{y} \in S_m$, and similarly $-_i\mathbf{x} \in S_m$. Hence S is an ω-subgroupoid of $\operatorname{cosk}^n G$.

By Corollary 13.5.11, any ω-subgroupoid of $\operatorname{cosk}^n G$ containing S_{m-1} (for $m > n$) must contain S_m, so S is generated by G and $S = \operatorname{sk}^n G$.

To prove the last statement, if $m \geqslant n + 2$, all shells in $\operatorname{cosk}^n G_m = \Box'^{m-n} G_k$ are thin by Proposition 13.5.7 and therefore satisfy $\delta\Phi\mathbf{x} = 0$ by Corollary 13.5.11. □

Proposition 13.5.16. *The functor* $\operatorname{sk}^n : \omega\text{-}\mathsf{Gpds}_n \to \omega\text{-}\mathsf{Gpds}$ *is left adjoint to* tr_n.

Proof. If H is any ω-groupoid and $\psi : G \to \operatorname{tr}_n H$ is a morphism of cubical n-groupoids, then ψ extends uniquely to a morphism of ω-groupoids $\psi : \operatorname{sk}^n G \to H$ inductively.

For $m > n$, consider a commuting shell $\mathbf{x} \in \Box' \operatorname{sk}^n G_{m-1}$. Since the elements $\psi_{m-1} x_i^\alpha$ form a commuting shell in H, by Corollary 13.5.11 exists $t \in H_m$ thin such that $\partial_i^\alpha t = \psi_{m-1} x_i^\alpha$ for $1 \leqslant i \leqslant m$ and $\alpha = 0, 1$. Then, we define $\psi_m \mathbf{x} = t$. □

Given an ω-groupoid G, we define $\operatorname{Sk}^n G = \operatorname{sk}^n(\operatorname{tr}_n G)$ and call this, by abuse of language, the *n-Skeleton* of G. There is a unique morphism $\sigma : \operatorname{Sk}^n G \to G$ of ω-groupoids (the adjunction) which is the identity in dimensions 0, 1, 2, …, n. Let us prove that the image is what we would call intuitively the n-skeleton of G, i.e. the ω-groupoid of G generated by G_n.

Proposition 13.5.17. *The adjunction* $\sigma : \operatorname{Sk}^n G \to G$ *is an injection and identifies* $\operatorname{Sk}^n G$ *with the ω-subgroupoid of G generated by G_n.*

Proof. For $m = 0, 1, 2, \dots, n$, $\sigma_m : G_m \to G_m$ is the identity map.

Then, for $m > n$, $(\operatorname{Sk}^n G)_m$ is the set of commuting shells in $\Box'_{m-1}(\operatorname{Sk}^n G)$, by Proposition 13.5.15. Suppose that, for some $m > n$, $\sigma_{m-1} : (\operatorname{Sk}^n G)_{m-1} \to G_{m-1}$ is an injection. For any $\mathbf{x} \in (\operatorname{Sk}^n G)_m$, the elements $\sigma_{m-1} x_i^\alpha$ form a commuting shell $\mathbf{y}$ in $\Box' G_{m-1}$ and $\sigma_m \mathbf{x}$ is the unique thin element t of G_m with $\boldsymbol{\partial} t = \mathbf{y}$. Thus $x_i^\alpha = \sigma_{m-1}^{-1} y_i^\alpha = \sigma_{m-1}^{-1} \partial_i^\alpha t$ is uniquely determined by t for all (i, α) and therefore σ_m is an injection. This shows, inductively, that σ is an injection.

Now G_n generates $\operatorname{tr}_n G$ as cubical n-groupoid (even as n-truncated cubical set) and therefore generates $\operatorname{Sk}^n G$ as ω-groupoid, by Proposition 13.5.15. It follows that G_n generates the image of $\operatorname{Sk}^n G$ in G. □

Corollary 13.5.18. *If G is an ω-groupoid, and $n \geqslant 0$ then the crossed complexes* $\gamma \operatorname{Sk}^n G$ *and* $\operatorname{Sk}^n \gamma G$ *coincide.*

13.6 The equivalence of ω-groupoids and crossed complexes

In this section we construct a functor

$$\lambda\colon \mathsf{Crs} \to \omega\text{-}\mathsf{Gpds}$$

which together with γ gives an equivalence of categories.

The key idea for constructing λ in such a way that there is an equivalence $\lambda\gamma \simeq 1_{\omega\text{-}\mathsf{Gpds}}$ comes from Proposition 13.5.10, which shows that any element of G_n is determined by its total boundary and its folding.

We have proved that given $\mathbf{x} \in \Box' G_{n-1}$, $\xi \in \gamma G_n$ with $\delta\xi = \delta\Phi\mathbf{x}$ there is a unique element $x \in G_n$ such that $\boldsymbol{\partial} x = \mathbf{x}$ and $\Phi x = \xi$. We write $\langle \mathbf{x}, \xi\rangle = x$.

To define λG we use these elements $\langle \mathbf{x}, \xi\rangle$. It is clear how to express its faces, degeneracies and connections of G following Definition 13.5.4. Our next proposition shows how to define the compositions.

Proposition 13.6.1. *If $x = \langle \mathbf{x}, \xi\rangle$, $y = \langle \mathbf{y}, \eta\rangle$ in G_n, and $x_i^+ = y_i^-$, then*

$$x +_i y = \begin{cases} \langle \mathbf{x} +_1 \mathbf{y}, \eta + \xi^{u_1\mathbf{y}}\rangle & \text{if } n = 2 \text{ and } i = 1,\\ \langle \mathbf{x} +_i \mathbf{y}, \xi^{u_1\mathbf{y}} + \eta\rangle & \text{otherwise,}\end{cases}$$

and

$$-_i x = \langle -_i\mathbf{x}, -\xi^{-u_i\mathbf{x}}\rangle.$$

Proof. This follows immediately from Proposition 13.4.14 and the rule

$$\boldsymbol{\partial}(x +_i y) = \boldsymbol{\partial} x +_i \boldsymbol{\partial} y. \qquad \Box$$

These results show how to construct from any crossed complex C an ω-groupoid $G = \lambda C$ with $\gamma G \cong C$.

Theorem 13.6.2. *There is a functor λ from the category Crs of crossed complexes to the category ω-Gpds of ω-groupoids such that $\lambda\colon \mathsf{Crs} \to \omega\text{-}\mathsf{Gpds}$ and $\gamma\colon \omega\text{-}\mathsf{Gpds} \to \mathsf{Crs}$ are inverse equivalences.*

Proof. Let C be any crossed complex. We construct an ω-groupoid $G = \lambda C$ and an isomorphism of crossed complexes $\sigma\colon C \to \gamma G$ by induction on dimension.

We start with $G_0 = C_0$, $G_1 = C_1$, so that (G_1, G_0) is a groupoid. We write γG_n (in any cubical complex) for the set of n-cubes x with all faces except $\partial_1^- x$ concentrated at a point. Then $\gamma G_0 = C_0$, $\gamma G_1 = C_1$, and we take $\sigma_0\colon C_0 \to \gamma G_0$ and $\sigma_1\colon C_1 \to \gamma C_1$ to be the identity maps.

Suppose, inductively, that we have defined G_r and $\sigma_r\colon C_r \to \gamma G_r$ for $0 \leqslant r < n$ (where $n \geqslant 2$) so that $(G_{n-1}, G_{n-2}, \dots, G_0)$ is a cubical $(n-1)$-groupoid and $(\sigma_{n-1}, \sigma_{n-2}, \dots, \sigma_0)$ is an isomorphism of $(n-1)$-truncated crossed complexes. Then $(\Box' G_{n-1}, G_{n-1}, \dots, G_0)$ is a cubical n-groupoid and we define

$$G_n = \{(\mathbf{x}, \xi) \mid \mathbf{x} \in \Box' G_{n-1},\ \xi \in C_n,\ \delta\Phi\mathbf{x} = \sigma_{n-1}\delta\xi\}.$$

For $y \in G_{n-1}$, let $\varepsilon_j y = (\boldsymbol{\varepsilon}_j y, 0)$, where $\boldsymbol{\varepsilon}_j$ is defined in Definition 13.5.4 (i). Then $\varepsilon_j y \in G_n$, since $\Phi \boldsymbol{\varepsilon}_j y = 0$ by Proposition 13.4.11. The maps $\varepsilon_j : G_{n-1} \to G_n$, together with the obvious face maps $\partial_i^\alpha : G_n \to G_{n-1}$ defined by $\partial_i^\alpha(\mathbf{x}, \xi) = x_i^\alpha$, give $(G_n, G_{n-1}, \dots, G_0)$ the structure of an n-truncated cubical set.

Similarly one can define connections $\Gamma_j : G_{n-1} \to G_n$ by $\Gamma_j y = (\boldsymbol{\Gamma}_j y, 0)$, where $\boldsymbol{\Gamma}_j$ is defined in Definition 13.5.4 (ii), and the laws in Definition 13.1.3 are clearly satisfied, since they are satisfied by $\boldsymbol{\Gamma}_j$.

Recalling Proposition 13.6.1, we define operations $+_i$, $-_i$. For $(\mathbf{x}, \xi), (\mathbf{y}, \eta) \in G_n$ with $x_i^+ = y_i^-$, let

$$(\mathbf{x}, \xi) +_i (\mathbf{y}, \eta) = \begin{cases} (\mathbf{x} +_1 \mathbf{y}, \eta + \xi^{u_1 \mathbf{y}}) & \text{if } n = 2 \text{ and } i = 1, \\ (\mathbf{x} +_i \mathbf{y}, \xi^{u_1 \mathbf{y}} + \eta) & \text{otherwise,} \end{cases}$$

and

$$-_i(\mathbf{x}, \xi) = (-_i \mathbf{x}, -\xi^{u_i \mathbf{x}}).$$

By Proposition 13.4.14, for all $(n, i) \neq (2, 1)$,

$$\begin{aligned} \delta\Phi(\mathbf{x} +_i \mathbf{y}) &= \delta((\Phi\mathbf{x})^{u_i \mathbf{y}} + \Phi\mathbf{y}) \\ &= (\sigma_{n-1}\delta\xi)^{u_i \mathbf{y}} + \sigma_{n-1}\delta\eta \\ &= \sigma_{n-1}\delta(\xi^{u_i \mathbf{y}} + \eta), \end{aligned}$$

so G_n is closed under $+_i$. The case $n = 2$, $i = 1$ is similar. Also

$$\delta\Phi(-_i \mathbf{x}) = \delta(-\Phi\mathbf{x})^{-u_i \mathbf{x}} = \sigma_{n-1}\delta(-\xi^{u_i \mathbf{x}}),$$

and therefore $-_i \mathbf{x} \in G_n$.

We claim that $(G_n, G_{n-1}, \dots, G_0)$ is now a cubical n-groupoid. Firstly, it is clear that, for $t \in G_{n-1}$, $\varepsilon_i t$ acts as an identity for $+_i$, and that $-_i$ is an inverse operation for $+_i$. The associative law is verified as for semi-direct products of groups. Secondly, the laws (1),(2) and (3) of Definition 13.1.7 are true for $\square' G_{n-1}$. It remains, therefore, to prove the interchange law (4i) (from which (4ii) follows, using the groupoid laws).

Let $1 \leqslant i < j \leqslant n$ and let $x = (\mathbf{x}, \xi)$, $y = (\mathbf{y}, \eta)$, $z = (\mathbf{z}, \zeta)$, $t = (\mathbf{t}, \tau)$ be elements of G_n such that the composite shell

$$\mathbf{w} = \begin{bmatrix} \mathbf{x} & \mathbf{y} \\ \mathbf{z} & \mathbf{t} \end{bmatrix} \qquad \begin{tikzcd} \cdot \arrow[r, "j"] \arrow[d, "i"'] & {} \\ {} & \end{tikzcd}$$

is defined. Let $g = \partial_1^+ \partial_2^+ \dots \hat{\imath}^+ \dots \hat{\jmath}^+ \dots \partial_n^+ \mathbf{t} \in G_2$ have boundary

$$\begin{tikzcd} \cdot \arrow[r, "c"] \arrow[d, "b"'] \arrow[dr, phantom, "g"] & \cdot \arrow[d, "d"] \\ \cdot \arrow[r, "a"'] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "2"] \arrow[d, "1"'] & {} \\ {} & \end{tikzcd} \quad .$$

Then

$$(x +_i z) +_j (y +_i t) = (\mathbf{w}, \omega), \quad (x +_j y) +_i (z +_j t) = (\mathbf{w}, \omega'),$$

say, and we have to show that $\omega = \omega'$ in C_n.

If $n = 2$ then $i = 1$ and $j = 2$ and we find that

$$\omega = (\zeta + \xi^b)^a + (\tau + \eta^d), \quad \omega' = (\zeta^a + \tau) + (\xi^c + \eta)^d.$$

To show that these are equal, it is enough to show that $\xi^{b+a} + \tau = \tau + \xi^{c+d}$. But this follows from the crossed module laws since

$$\delta\tau = \sigma_1\delta\tau = \delta\Phi\mathbf{t} = \delta\Phi g = -a - b + c + d$$

and therefore

$$-\tau + \xi^{b+a} + \tau = (\xi^{b+a})^{\delta\tau} = \xi^{c+d}.$$

If $n > 2$, we find that

$$\omega = (\xi^b + \zeta)^a + \eta^d + \tau, \quad \omega' = (\xi^c + \eta^d) + \zeta^a + \tau,$$

and since addition is now commutative, the equation $\omega = \omega'$ reduces to $\xi^{a+b} = \xi^{c+d}$, that is, $\xi^{\delta\Phi g} = \xi$. But, by induction hypothesis, we have an isomorphism $\sigma_2 \colon C_2 \to \gamma G_2$ preserving the crossed module structure, and if $\theta \in C_2$ is the element with $\sigma_2(\theta) = \Phi g$, then $\xi^{\delta\Phi g} = \xi^{\delta\theta} = \xi$ by the crossed complex laws. This completes the proof of the interchange law.

We now have a cubical n-groupoid $(G_n, G_{n-1}, \ldots, G_0)$, and we must identify γG_n. For any $\xi \in C_n(p)$, let $\mathbf{d}\xi$ denote the shell $\mathbf{x} \in \Box' G_{n-1}$ with $x_1^- = \sigma_{n-1}\delta\xi$ and all other x_i^α concentrated at p. Define

$$\sigma_n\xi = (\mathbf{d}\xi, \xi).$$

Clearly $\sigma_n\xi \in \gamma G_n$ and every element of γG_n is of this form. The bijection $\sigma_n \colon C_n \to \gamma G_n$ is compatible with the boundary maps since $\delta\sigma_n\xi = \partial_1^-\sigma_n\xi = \sigma_{n-1}\delta\xi$. It preserves addition because, for $\xi, \eta \in C_n(p)$,

$$\begin{aligned}(\mathbf{d}\xi, \xi) + (\mathbf{d}\eta, \eta) &= (\mathbf{d}\xi +_n \mathbf{d}\eta, \xi^{u_n\mathbf{d}\eta} + \eta)\\ &= (\mathbf{d}(\xi + \eta), \xi + \eta).\end{aligned}$$

Furthermore, if $\xi \in C_n(p)$ and $a \in C_1(p, q) = G_1(p, q)$, then

$$\begin{aligned}(\sigma_n\xi)^a &= -_n\varepsilon_1^{n-1}a +_n \sigma_n\xi +_n \varepsilon_1^{n-1}a\\ &= (-_n\boldsymbol{\varepsilon}_1\varepsilon_1^{n-2}a, 0) +_n (\mathbf{d}\xi, \xi) +_n (\boldsymbol{\varepsilon}_1\varepsilon_1^{n-2}a, 0)\\ &= (\mathbf{y}, \xi^a),\end{aligned}$$

in all cases. Since $(\sigma_n\xi)^a \in \gamma G_n$, it follows that $\mathbf{y} = \mathbf{d}(\xi^a)$, making σ_n an isomorphism of crossed complexes up to dimension n.

This completes the inductive step in our construction, and we therefore obtain an ω-groupoid $G = \lambda C$ and an isomorphism $\sigma\colon C \to \gamma G$ of crossed complexes. This ω-groupoid has the following universal property: If G' is any ω-groupoid and $\sigma'\colon C \to \gamma G'$ any morphism of crossed complexes then there is a unique morphism $\theta\colon G \to G'$ of ω-groupoids making the diagram

$$\begin{tikzcd} C \arrow[r, "\sigma"] \arrow[dr, "\sigma'"'] & \gamma G \arrow[r, hook] \arrow[d, "\gamma\theta"] & G = \lambda C \arrow[d, "\theta"] \\ & \gamma G' \arrow[r, hook] & G' \end{tikzcd}$$

commute.

We define θ inductively, starting with $\theta_0 = \sigma'_0$, $\theta_1 = \sigma'_1$. For $n \geqslant 2$, each $x' \in G'_n$ is uniquely of the form $\langle \mathbf{x}', \xi' \rangle$ where $\mathbf{x}' \in \square' G'_{n-1}, \xi' \in \gamma G_n$ and $\delta\Phi\mathbf{x}' = \delta\xi'$. We define $\theta_n\colon G_n \to G'_n$ by $(\mathbf{x}, \xi) \mapsto \langle \mathbf{x}', \xi' \rangle$, where $(x')^\alpha_i = \theta_{n-1} x^\alpha_i$ and $\xi' = \sigma'_n \xi$. This definition is forced, and it clearly gives a morphism of ω-groupoids.

From this universal property, it follows that the functor $\lambda\colon \mathsf{Crs} \to \omega\text{-}\mathsf{Gpds}$ is left adjoint to $\gamma\colon \omega\text{-}\mathsf{Gpds} \to \mathsf{Crs}$.

The adjunction $\sigma_c\colon C \to \gamma\lambda C$ is an isomorphism for all C, so $1_{\mathsf{Crs}} \simeq \gamma\lambda$. Also, the adjunction $\lambda\gamma G' \to G'$ is obtained by putting $G = \gamma G', \sigma' =$ identity, in which case θ is an isomorphism $\lambda\gamma G' \to G'$, as is clear from its definition. Hence $\lambda\gamma \simeq 1_{\omega\text{-}\mathsf{Gpds}}$ and we have inverse equivalences λ and γ between Crs and ω-Gpds. $\square$

13.7 The Homotopy Addition Lemma and properties of thin elements

Another very important property of ω-groupoids is that they are fibrant cubical sets (see Section 11.3.i), i.e. that any n-box has a filler.

Moreover the n-boxes have a set of canonical fillers, i.e. the thin elements giving an ω-groupoid the structure of T-complex.[220]

The proof of both these facts may be deduced from Proposition 13.5.10 via an algebraic Homotopy Addition Lemma that expresses the only nontrivial face of the folding of a shell in term of the elements of the shell.

Lemma 13.7.1 (Homotopy Addition Lemma). *Let G be an ω-groupoid (or a cubical m-groupoid with $m \geqslant n$). Let $\mathbf{x} \in \square' G_n$ and define $\Sigma\mathbf{x} \in C_n = (\gamma G)_n$ by*

$$\Sigma\mathbf{x} = \begin{cases} -x_1^+ - x_2^- + x_1^- + x_2^+ = -\Phi x_1^+ - \Phi x_2^- + \Phi x_1^- + \Phi x_2^+ & \text{if } n = 1, \\ -\Phi x_3^+ - (\Phi x_2^-)^{u_2\mathbf{x}} - \Phi x_1^+ + (\Phi x_3^-)^{u_3\mathbf{x}} + \Phi x_2^+ + (\Phi x_1^-)^{u_1\mathbf{x}} & \text{if } n = 2, \\ \sum_{i=1}^{n+1} (-1)^i \{\Phi x_i^+ - (\Phi x_i^-)^{u_i\mathbf{x}}\} & \text{if } n \geqslant 3 \end{cases}$$

where $u_i = \partial_1^+ \partial_2^+ \dots \hat{\imath} \dots \partial_{n+1}^+$ as in Definition 13.4.13. Then $\delta\Phi\mathbf{x} = \Sigma\mathbf{x}$ in all cases. Hence, if t is a thin element of G, then $\Sigma\boldsymbol{\partial} t = 0$.

Proof. The case $n = 1$ is trivial, so we assume $n \geqslant 2$. First, notice

$$\begin{aligned} \delta\Phi\mathbf{x} &= \Phi\delta\Phi\mathbf{x} && (\text{because } \delta\Phi\mathbf{x} \in C_n) \\ &= (\Phi\partial_1^-\Phi\mathbf{x})^{u_1\Phi\mathbf{x}} && (\text{because } u_1\Phi\mathbf{x} = \varepsilon_1\beta\mathbf{x}) \\ &= \Sigma\Phi\mathbf{x}. \end{aligned}$$

So, we have to prove $\Sigma\Phi\mathbf{x} = \Sigma\mathbf{x}$. It is enough to show that $\Sigma\Phi_j\mathbf{x} = \Sigma\mathbf{x}$ for $j = 1, 2, \ldots, n$.

To prove that $\Sigma\Phi_j\mathbf{x} = \Sigma\mathbf{x}$, put $\mathbf{y} = \Phi_j\mathbf{x}$ (for fixed j). By Proposition 13.4.2, we have

$$\begin{aligned} y_i^\alpha &= \begin{cases} \Phi_{j-1}x_i^\alpha & (i < j), \\ \Phi_j x_i^\alpha & (i > j+1); \end{cases} \\ y_{j+1}^\alpha &= y_j^+ = \eta_j^+ x_j^+; \\ y_j^- &= [-x_j^+, -x_{j+1}^-, x_j^-, x_{j+1}^+]_j. \end{aligned}$$

Hence, by Proposition 13.4.15 and Proposition 13.4.18,

$$\begin{aligned} \Phi y_i^\alpha &= \Phi x_i^\alpha (i \neq j, j+1), \\ \Phi y_{j+1}^\alpha &= \Phi y_j^+ = 0, \\ \Phi y_j^- &= \Phi[-x_j^+, -x_{j+1}^-, x_j^-, x_{j+1}^+]_j. \end{aligned} \tag{$*$}$$

We write $a_j = [-x_j^+, -x_{j+1}^-, x_j^-, x_{j+1}^+]_j$ and use Proposition 13.4.14 to compute Φa_j.

First we study the case $(n, j) \neq (2, 1)$. Then

$$\Phi a_j = -(\Phi x_j^+)^{p_j} - (\Phi x_{j+1}^-)^{q_j} + (\Phi x_j^-)^{r_j} + \Phi x_{j+1}^+,$$

where $p_j = u_j a_j$, $q_j = u_j[x_j^+, a_j]_j$, $r_j = u_j x_{j+1}^+$. By the relations in Definition 13.1.7, u_j is a morphism of groupoids from $(G_n, +_j)$ to $(G_1, +)$ so

$$p_j = -u_j x_j^+ - u_j x_{j+1}^- + u_j x_j^- + u_j x_{j+1}^+$$

in G_1, and $q_j = u_j x_j^+ + p_j$. The four terms of p_j are the edges of the square

$$s_j = \partial_1^+\partial_2^+ \ldots \hat{j}\widehat{j+1} \ldots \partial_n^+\mathbf{x};$$

hence $p_j = \Sigma\boldsymbol{\partial} s_j = \delta\Phi s_j$. Also $u_j x_j^+ = u_{j+1}\mathbf{x}$ and $u_j x_{j+1}^+ = u_j\mathbf{x}$, so

$$\Phi y_j^- = \Phi a_j = -(\Phi x_j^+)^{\delta\Phi s_j} - (\Phi x_{j+1}^-)^{u_{j+1}\mathbf{x}+\delta\Phi s_j} + (\Phi x_j^-)^{u_j\mathbf{x}} + \Phi x_{j+1}^+. \tag{$**$}$$

We have to differentiate two subcases.

If $n \geqslant 3$ then $\delta\Phi s_j$ acts trivially on C_n, since $C = \gamma G$ is a crossed complex, and addition is commutative. Hence by $(*)$,

$$\begin{aligned}\Sigma\mathbf{y} &= \sum_{i=1}^{n}(-1)^i\{\Phi y_i^+ - (\Phi y_i^-)^{u_i\mathbf{y}}\}\\ &= \sum_{i\neq j,j+1}(-1)^i\{\Phi x_i^+ - (\Phi x_i^-)^{u_i\Phi_j\mathbf{x}}\} + (-1)^{j+1}(\Phi y_j^-)^{u_j\Phi_j\mathbf{x}}.\end{aligned}$$

But $u_i\Phi_j\mathbf{x} = u_i\mathbf{x}$ if $i \neq j, i \neq j+1$; and $u_j\Phi_j\mathbf{x} = 0$; so substituting from $(**)$ we find $\Sigma\mathbf{y} = \Sigma\mathbf{x}$.

If $n = 2$ and $j = 2$ then $s_2 = \partial_1^+\mathbf{x} = x_1^+$, and $\delta\Phi s_2 = \delta\Phi x_1^+$ acts on C_2 by $a^{\delta\Phi s_2} = -\Phi x_1^+ + a + \Phi x_1^+$. Hence $(**)$ becomes

$$\Phi y_2^- = -\Phi x_1^+ - \Phi x_2^+ - (\Phi x_3^-)^{u_3\mathbf{x}} + \Phi x_1^+ + (\Phi x_2^-)^{u_2\mathbf{x}} + \Phi x_3^+$$

which, together with $(*)$, gives

$$\begin{aligned}\Sigma\mathbf{y} &= -\Phi y_3^+ - (\Phi y_2^-)^{u_2\Phi_2\mathbf{x}} - \Phi y_1^+ + (\Phi y_3^-)^{u_3\Phi_2\mathbf{x}} + \Phi y_2^+ + (\Phi y_1^-)^{u_1\Phi_2\mathbf{x}}\\ &= 0 - \Phi y_2^- - \Phi x_1^+ + 0 + 0 + (\Phi x_1^-)^{u_1\mathbf{x}}\\ &= \Sigma\mathbf{x}.\end{aligned}$$

Finally, in the case $n = 2$, $j = 1$, we have

$$\begin{aligned}\Phi y_1^- &= \Phi[-x_1^+, -x_2^-, x_1^-, x_2^+]_1\\ &= \Phi x_2^+ + (\Phi x_1^-)^{r_1} - (\Phi x_2^-)^{q_1} - (\Phi x_1^+)^{p_1}\end{aligned}$$

by Proposition 13.4.14, where p_1, q_1, r_1 are as defined above. As in the previous cases, this gives

$$\Phi y_1^- = \Phi x_2^+ + (\Phi x_1^-)^{u_1\mathbf{x}} - \Phi x_3^+ - (\Phi x_2^-)^{u_2\mathbf{x}} - \Phi x_1^+ + \Phi x_3^+$$

and hence

$$\Sigma\mathbf{y} = -\Phi x_3^+ + (\Phi x_3^-)^{u_3\mathbf{x}} + \Phi x_2^+ + (\Phi x_1^-)^{u_1\mathbf{x}} - \Phi x_3^+ - \Phi(x_2^-)^{u_2\mathbf{x}} - \Phi x_1^+ + \Phi x_3^+.$$

Writing $b = (\Phi x_3^-)^{u_3\mathbf{x}} + \Phi x_2^+ + (\Phi x_1^-)^{u_1\mathbf{x}} - \Phi x_3^+$ and $c = -(\Phi x_2^-)^{u_2\mathbf{x}} - \Phi x_1^+$, it can be verified that $\delta b = -\delta c$, and hence, by the crossed module laws, $b + c = c + b^{\delta c} = c + b^{-\delta b} = c + b$. It follows easily that $\Sigma\mathbf{y} = \Sigma\mathbf{x}$, as required.

To prove the last statement, if t is thin, then $\Sigma\boldsymbol{\partial} t = \delta\Phi\boldsymbol{\partial} t = \partial_1^-\Phi t = 0$ by Lemma 13.5.8 and Proposition 13.4.18. □

Remark 13.7.2. The element $\Sigma\mathbf{x}$ in the case $n = 2$ is in the centre of $C_2(\beta\mathbf{x})$, because conjugation by $\Sigma\mathbf{x} = \delta\Phi x$ is the same as action by $\delta\delta\Phi\mathbf{x} = 0$. Hence $\Sigma\mathbf{x}$ can be rewritten, for example, by permuting its terms cyclically. □

Proposition 13.7.3. *Let G be an ω-groupoid. Then each box in G has a unique thin filler. In particular, G is a fibrant cubical set.*

Proof. Let $\mathbf{y}$ be an n-box with missing (γ, k)-face. The result is trivial if $n = 0$, so we assume $n \geqslant 1$. By Corollary 13.5.11, it is enough to prove that there is a unique n-cube y_k^γ which closes the box $\mathbf{y}$ to form an n-shell $\bar{\mathbf{y}}$ with $\delta\Phi\bar{\mathbf{y}} = \Sigma\bar{\mathbf{y}} = 0$.

If $n \geqslant 2$, the edges of the given box $\mathbf{y}$ form the complete 1-skeleton of an $(n+1)$-cube; in particular, $\mathbf{y}$ determines the $n+1$ edges $w_i = u_i\mathbf{y}$ terminating at $\beta\mathbf{y}$. We write $F(s_i^\alpha)$ for the word in the indeterminates $s_i^\alpha (i = 1, 2, \dots, n+1; \alpha = 0, 1)$ obtained from the formula for $\Sigma\mathbf{x}$ in Lemma 13.7.1 by substituting s_i^α for Φx_i^α and the given edges $w_i = u_i\mathbf{y}$ for $u_i\mathbf{x}$. If $n = 1$, then $F(s_i^\alpha) = -s_1^+ - s_2^- - s_1^- + s_2^+$ does not involve the w_i.

If we put $\mathbf{z}_i^\alpha = \boldsymbol{\partial} y_i^\alpha$ for $(\alpha, i) \neq (\gamma, k)$, then the $\mathbf{z}_i^\alpha$ form a box of $(n-1)$-shells, and there is a unique $(n-1)$-shell $\mathbf{z}_k^\gamma$ which closes this box to form an n-shell $\bar{\mathbf{z}} \in \square^2 G_{n-1}$. Since δ preserves addition and the action of the edges w_i, we find

$$F(\delta\Phi\mathbf{z}_i^\alpha) = \delta F(\Phi\mathbf{z}_i^\alpha) = \delta\Sigma\bar{\mathbf{z}} = \delta^2\Phi\bar{\mathbf{z}} = 0. \qquad (*)$$

Next, put $\zeta_i^\alpha = \Phi y_i^\alpha$ for $(\alpha, i) \neq (\gamma, k)$ and let $\zeta_k^\gamma \in C_n$ be the unique element determined by the equation $F(\zeta_i^\alpha) = 0$. Then

$$\delta\zeta_i^\alpha = \delta\Phi y_i^\alpha = \delta\Phi\mathbf{z}_i^\alpha \text{ for } (\alpha, i) \neq (\gamma, k),$$

while

$$F(\delta\zeta_i^\alpha) = 0.$$

From these equations and $(*)$ we deduce that $\delta\zeta_k^\gamma = \delta\Phi\mathbf{z}_k^\gamma$ also. Hence, by Proposition 13.5.10, there is a unique $y_k^\gamma \in G_n$ such that $\boldsymbol{\partial} y_k^\gamma = \mathbf{z}_k^\gamma$ and $\Phi y_k^\gamma = \zeta_k^\gamma$; this y_k^γ completes the box $\mathbf{y}$ to form a shell $\bar{\mathbf{y}}$ with $\Sigma\bar{\mathbf{y}} = F(\zeta_i^\alpha) = 0$, as required. □

Proposition 13.7.4. *Let t be a thin element in an ω-groupoid. If all faces except one of t are thin, then the remaining face is also thin.*

Proof. Let the faces of t be $t_i^\alpha (i = 1, 2, \dots, n; \alpha = 0, 1)$. By Proposition 13.4.18, $\Phi t_i^\alpha = 0$ for $(\alpha, i) \neq (\gamma, k)$ say, so $\Sigma\boldsymbol{\partial} t = \pm(\Phi t_k^\gamma)^w$ for some edge w of t. But t is thin so, by the Homotopy Addition Lemma 13.7.1, $\Sigma\boldsymbol{\partial} t = 0$. Hence $\Phi t_k^\gamma = 0$ and t_k^γ is thin. □

The thin elements of an ω-groupoid have another property which is crucial in the proof of the HHSvKT in the next chapter; it is used in proving Lemma 14.3.5 on page 496 to show that a constructed element of an ω-groupoid is independent of the choices in the construction. It is also used to relate the fundamental ω-groupoid ρX_* and fundamental crossed complex ΠX_* of a filtered space (Proposition 14.5.1). For the following proposition, we need Definition 11.1.6 of cubical operators not involving a given face.

Proposition 13.7.5. *Let G be an ω-groupoid and x a thin element of G_{n+1}. Suppose that for $m = 1, \dots, n$ and each face operator $d\colon G_{n+1} \to G_m$ not involving ∂^-_{n+1} or ∂^+_{n+1}, the element dx is thin. Then $x = \varepsilon_{n+1}\partial^-_{n+1}x$ and hence*

$$\partial^-_{n+1}x = \partial^+_{n+1}x.$$

Proof. The proof is by induction on n, the case $n = 0$ being trivial since a thin element in G_1 is degenerate.

The inductive assumption thus implies that every face $\partial^\tau_i x$ with $i \neq n+1$ is of the form $\varepsilon_n\partial^-_n\partial^\tau_i x$. So the box consisting of all faces of x except $\partial^+_{n+1}x$ is filled not only by x but also by $\varepsilon_{n+1}\partial^-_{n+1}x$. Since a box in G has a unique thin filler (Proposition 13.7.3), it follows that $x = \varepsilon_{n+1}\partial^-_{n+1}x$. □

Remark 13.7.6. The properties of thin elements in Propositions 13.7.3 and 13.7.4, together with the fact that degenerate cubes are thin, can be taken as axioms for 'cubical T-complexes' or 'cubical sets with thin elements'.[221] Precisely, a (*cubical*) *T-complex* is a cubical set with a distinguished set of elements called 'thin', satisfying:

(i) all degenerate cubes are thin;

(ii) every box has a unique thin filler;

(iii) if a thin cube has all faces except one thin then the last face is also thin.[222]

□

Remark 13.7.7. If G is any ω-groupoid, we may define the fundamental groupoid $\pi_1 G$ and the homotopy groups $\pi_n(G, p)$ ($p \in G_0$, $n \geqslant 2$) as follows. For $a, b \in G_1(p, q)$, define $a \sim b$ if there exists $c \in G_2$ such that

$$\partial^-_1 c = a, \quad \partial^+_1 c = b, \quad \partial^-_2 c = \varepsilon_1 p, \quad \partial^+_2 c = \varepsilon_1 q.$$

Then $\sim$ is a congruence relation on G_1 and we define $\pi_1 G = G_1/\sim$. For $n \geqslant 2$ and $p \in G_0$, let $Z_n(G, p) = \{x \in G_n \mid \partial^\alpha_1 x = \varepsilon^{n-1}_1 p$ for all $(\alpha, i)\}$. Then the $+_i$ ($i = 1, 2, \dots, n$) induce on $Z_n(G, p)$ the same abelian group structure. Two elements x, y of $Z_n(G, p)$ are *homotopic*, $x \sim y$, if there exists $h \in G_{n+1}$ such that $\partial^-_{n+1}h = x$, $\partial^+_{n+1}h = y$ and $\partial^\alpha_i h = \varepsilon^n_1 p$ for $i \neq n+1$. This is a congruence relation on $Z_n(G, p)$ and we define $\pi_n(G, p)$ to be the quotient group $Z_n(G, p)/\sim$.

Now G is a fibrant cubical set, by Proposition 13.7.3, so, there is a standard procedure suggested in Proposition 11.3.27 and in that Section 11.3.iii, for defining $\pi_1 G$ and $\pi_n(G, p)$, without using the compositions $+_i$. As sets they coincide with the definitions above, but their groupoid and group structures are defined by a procedure using only the properties of Kan fillers. □

It is not hard to see that the special properties of thin fillers in G ensure that the groupoid and group structures obtained in this way coincide with those induced by the compositions $+_i$.

It is also clear that the groupoid $\pi_1 G$ and the groups $\pi_n(G, p)$ coincide with the *fundamental groupoid* $\pi_1\gamma G$ and the *homology groups* $H_n(\gamma G, p)$ of the crossed complex γG (see the definitions in Section 7.1.iv). □

We will later (in the proof of Theorem 15.6.1) need the following result.

Proposition 13.7.8. *Let G, H be ω-groupoids and let $f : G \to H$ be a morphism of the underlying cubical sets with connections which also preserves the thin structures. Then f is a morphism of ω-groupoids.*

Proof. This involves the fact that the compositions can be recovered from the thin structures, which is the main result of [BH81c], showing the equivalence of cubical T-complexes and ω-groupoids. In our terms, this can be shown as follows.

In Proposition 11.3.27 we showed how the fundamental groupoid of a fibrant cubical set can be defined. In the case of a cubical T-complex K, with unique Kan fillers, this method actually determines a groupoid structure $+_1$ on K_1.

By using the functor $P^n K$ we can similarly get a composition, and in fact a groupoid structure, $+_n$ derived from the T-structure.

However we showed in Chapter 6, that double groupoids with connection admit rotations, which exchange the two groupoid structures in dimension 2 (Theorems 6.4.10 and 6.4.11). In higher dimensions, this argument gives an operation of the symmetry group S_n in dimension n of an ω-groupoid G, interchanging the operations $+_i$. Hence any addition $+_i$ is determined by the thin structure.

The same argument applies to H and hence f preserves the compositions $+_i$. □

Remark 13.7.9. It will be shown in Remark 14.6.4 that the crossed complex ΠI^n_* has one generator for each cell of I^n, with defining relations given by the cubical Homotopy Addition Lemma 9.9.4. The corresponding statement for the ω-groupoid ρI^n_* is that it is the free ω-groupoid on a single generator in dimension n; this is the subject of Section 14.6, as part of the description of the free ω-groupoid on a cubical set (Proposition 14.6.3). □

Notes

209 p. 443 Most of the results of this chapter are taken from [BH81].

210 p. 443 For the single vertex case, the term 'crossed complex' is due to Huebschmann in [Hue77], [Hue80a], but the concept has a longer history. The immediate interest for Huebschmann was the generalisation of a result on an interpretation in terms of classes of crossed sequences of $H^3(G, M)$, the third cohomology of a group G with coefficients in a G-module M. More background to this is given in [ML79]. There is also more to say. Lue in [Lue81] shows that his earlier work in [Lue71] can be specialised to the case of the cohomology of groups with respect to the variety of abelian groups to give the interpretation of $H^{n+1}(G, M)$ in terms of n-fold crossed sequences which we have derived in Chapter 12. The first formal

definition of crossed complex in the single vertex case seems to have been in Blakers, [Bla48], using the term 'group system'. Almost contemporaneously, Whitehead in [Whi49b] uses the term 'homotopy system' for what we call a free crossed complex. He remarks there that the concept is a translation using chain complexes and relative homotopy groups of the term 'natural system' on p. 1216 of [Whi41b]. That paper shows a deep study of the work of Reidemeister, [Rei34], on chain complexes with operators; see [Rei50] for a later account.

211 p. 446 Cubical sets with this, and other, structures have also been considered by Évrard [Evr76]. See also M. Grandis and L. Mauri, [GM03], which deals with various normal forms, including those for cubical sets with connections. Various papers use connections $\Gamma_i^{\pm}$, for example [AABS02], [Hig05]; these are not needed in this book because we are dealing with multiple groupoid structures. The work of [Mal09] shows that cubical sets with connections have good properties for models of homotopy, in that they form a 'strict test category'. Here 'strict' means that the geometric realisation of the cartesian product of cubical sets has the same homotopy type as the cartesian product of the realisations. The failure of this fact for the original cubical sets of [Kan55] was a main reason for abandoning cubical for simplicial sets. Another was that cubical groups are in general not fibrant cubical sets: but cubical groups with connections are fibrant, [Ton92].

212 p. 449 These results apply also to the more general n-fold categories defined in [BH81b]. The question of defining compositions of more complicated subdivisions in double categories is studied in [DP93b], see also [DP93a].

213 p. 450 Several authors refer to the singular simplicial complex of a space X as a kind of ∞-groupoid and write it ΠX, see for example [Lur09], [Ber02], and also discussions on the ncatlab. However there seems little discussion of the singular cubical complex in similar terms.

214 p. 451 See [Bro09b] for another construction which yields a strict multiple category.

215 p. 451 This raises the question of the significance of the fact that this construction can be made in the case of a filtered topological space but not in any obvious manner for just a space. There are results in dimension 2 for the case of a Hausdorff space, see [BHKP02], [BKP05], but it is not so clear how to obtain applications from this. There are intriguing analogous constructions in the smooth case, see for example in dimension 2 [FMP11].

216 p. 451 The term 'cubical n-groupoid' is analogous to that of 'globular n-groupoid' which is the more common form in the literature. The definitions and relation between the two are given in [BH81b], [Bro08b].

217 p. 455 For an extension of these methods to the case of cubical ω-categories, see [Hig05], [Ste06].

218 p. 462 Thin elements are also used in higher categorical rather than groupoid situations, see for example [Str87], [Hig05, Ste06], [Ver08a].

219 p. 463 Here we follow the notation and terminology of Duskin [Dus75].

220 p. 472 This notion was discovered in the simplicial context by Dakin in [Dak77]. The simplicial account was completed by Ashley in his 1978 PhD thesis published as [Ash88]. The work of these two students was a key input to the work of [BH81], [BH81a]. See also [Bro83].

221 p. 476 The definition was first given by Dakin [Dak77] in the simplicial case. These axioms are related to the axioms for a group composition given in [Lev57]: the last of these axioms is essentially the dimension 3 simplicial case of the third axiom for thin elements, and is rightly linked by Levi to associativity of the composition.

222 p. 476 We have shown that every ω-groupoid is a T-complex, and it is a remarkable fact (see [BH81c], [BH81b]) that the converse is also true: all the ω-groupoid structure can be recovered from the set of thin elements using these three assumptions. Thus the category of cubical T-complexes is equivalent (in fact isomorphic) to the category of ω-groupoids; it is therefore, by 13.6.2, equivalent to the category of crossed complexes. Ashley has shown [Ash88] that the category of simplicial T-complexes is also equivalent to the category of crossed complexes (see also [NT89a] for the relation to 'hypergroupoids') – this should be called the Ashley Theorem. He has also shown that this result generalises the theorem of Dold and Kan [Dol58], [Kan58b], [May67] which gives an equivalence between the category of simplicial abelian groups and the category of chain complexes; the T-complex structure on a simplicial abelian group is obtained by defining the thin elements to be sums of degenerate elements. For more information on the cubical case, see also [BH03]. Some writers use the term 'complicial set' rather than T-complex, for analogous concepts, see for example [Ver08a], [Ver08b].

Chapter 14
The cubical homotopy ω-groupoid of a filtered space

This chapter contains the construction and applications of the cubical higher homotopy groupoid ρX_* of a filtered space X_*. Without the idea for this construction, the major results of this book would not have been conjectured, let alone proved.[223]

The definition of ρX_* as a cubical set with connections is easy: it is a quotient of RX_*, the filtered cubical singular complex of X_*, by the relation of *thin homotopy rel vertices*. The difficult part is to prove that the compositions on RX_* are inherited by ρX_*, so that it becomes an ω-groupoid: the proof is a generalisation of that in dimension 2, but needs an organisation of the collapsing of cubes in order to fill some 'holes' starting in low dimensions. It is remarkable that there is exactly enough room, in the filtration sense, to fill these holes as required; this gives one confidence that the definitions are the right ones, and that filtered spaces do have a special role in algebraic topology.

These collapsings which were introduced in Section 11.3.i also enable a proof of a key result, the Fibration Theorem 14.2.7 stating that the projection $p\colon RX_* \to \rho X_*$ is a fibration of cubical sets. Some more precise properties of this fibration are a key to later results. For example, since ρX_* is an ω-groupoid, it has a notion of thin element, by Definition 13.4.17: these elements we call *algebraically thin*. There is also a notion of a *geometrically thin*, or deficient, element of $(\rho X_*)_n$ for any $n \geqslant 1$, namely those which have a representative $f : I^n_* \to X_*$ such that $f(I^n) \subseteq X_{n-1}$. The precise Fibration Theorem implies these two notions coincide (see Theorem 14.2.9).[224]

The main part of this chapter gives proofs of a Higher Homotopy Seifert–van Kampen Theorems (HHSvKT) both for ω-groupoids (Section 14.3) and for crossed complexes (Section 14.4). In Section 14.3.1 we prove the result for ω-groupoids. It shows, in succinct terms, that the functor ρ preserves certain colimits of connected filtered spaces.

The proof of the Higher Homotopy Seifert–van Kampen Theorem (HHSvKT) for ω-groupoids (Section 14.3) follows the same structure as the proofs of the Seifert–van Kampen Theorems in dimension 1 and 2, given in Part I (Section 1.6 and Theorem 6.8.2). It goes as follows.

Let $\mathcal{U} = \{U^\lambda\}_{\lambda\in\Lambda}$ be an open cover of a space X with a given filtration X_* and assume that for any finite intersection U^ν of elements of $\mathcal{U}$, the induced filtration U^ν_* is connected. The theorem says that in the induced diagram

$$\bigsqcup_{\nu\in\Lambda^2} \rho U^\nu_* \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_{\lambda\in\Lambda} \rho U^\lambda_* \xrightarrow{c} \rho X_*$$

c is the coequaliser of a and b in the category ω-Gpds of ω-groupoids.

To prove the universality condition of the coequaliser, we show that for any ω-groupoid G and morphism of ω-groupoids

$$f\colon \bigsqcup_{\lambda\in\Lambda} \rho U^{\lambda}_{*} \to G$$

which is compatible with the double intersections (i.e. $fa = fb$), there is a unique morphism

$$f'\colon \rho X_* \to G$$

such that $f'c = f$. This morphism f' is constructed by using choices to construct a map

$$F\colon RX_* \to G$$

following the same pattern as in the proof of Theorem 6.8.2.

For any element $\alpha \in R_n(X_*)$ we choose a subdivision $\alpha = [\alpha_{(r)}]$ such that each $\alpha_{(r)}$ lies in some element $U^{\lambda(r)} \in \mathcal{U}$. The connectivity conditions imply, analogously to the 2-dimensional case, that there are elements $\theta^{\lambda} \in R_n(X_*)$ and a thin homotopy $h\colon \alpha \equiv \theta$ such that in the subdivision given by α we have $h_{(r)}\colon \alpha_{(r)} \equiv \theta_{(r)}$, $\theta_{(r)} \in R_n X_*$ and $h_{(r)}$ lies in $U^{\lambda(r)}$. We define

$$F(\alpha) = [f^{\lambda(r)}\theta_{(r)}]$$

the composite of the array.

The central part of the proof is to show that F is well defined up to homotopy. Here we diverge from the proof of the theorem in dimension 2. There, the Homotopy Addition Lemma was used in dimension 2 to see that any composition of commuting 3-shells is also a commuting 3-shell. In higher dimensions, the 'commuting $(n+1)$-shells' are replaced by the thin elements defined in 13.4.17. The geometric characterisation of thin elements already stated is crucial in the proof.

The Higher Homotopy Seifert–van Kampen Theorem (HHSvKT) for crossed complexes (Section 14.3) follows from the HHSvKT for ω-groupoids using the equivalence of categories in Section 13.6

$$\gamma\colon \omega\text{-}\mathsf{Gpds} \to \mathsf{Crs}$$

from the category of ω-groupoids to the category of crossed complexes. In the case of the ω-groupoid ρX_*, we prove that $\gamma\rho X_*$ is naturally isomorphic to the fundamental crossed complex ΠX_* of the filtered space X_* (Theorem 14.4.1). This isomorphism gives the Higher Homotopy Seifert–van Kampen Theorem (HHSvKT) for fundamental crossed complexes (8.1.5) whose applications have been described in Chapter 8, assuming the Higher Homotopy Seifert–van Kampen Theorem for the ω-groupoid ρX_*.

Section 14.5 shows that every ω-groupoid and every crossed complex arise up to isomorphism from our functors from filtered spaces. This shows our axioms for these structures to be optimal. These results are surprisingly used in Section 14.6 to show that the functor ρ from cubical sets to ω-groupoids is left adjoint to the forgetful functor, and so gives the free ω-groupoid on a cubical set. This result is very useful for our next chapter on monoidal closed structures.

Section 14.7 gives a final link with classical results by showing how these cubical methods help in proving the classical Absolute Hurewicz Theorem, and also relate to an exact sequence of J. H. C. Whitehead which includes the Hurewicz morphism from homotopy to homology. This exact sequence is necessary for applications of our fundamental homotopy classification of maps to the classifying space of a crossed complex, since it gives useful conditions for a space Y to be of the homotopy type of BC for some crossed complex C.

Section 14.8 relates previous work to a classical theorem of Dold and Kan giving an equivalence between chain complexes and simplicial abelian groups. Here we give a cubical analogue.

Some results of this chapter in Section 14.5 link with results of Chapter 11 and indeed depend on results on realisations of cubical sets stated there. It is of course important that this chapter is otherwise independent of results from Part II, though we assume some of the definitions given there.

14.1 The cubical homotopy ω-groupoid of a filtered space

It is natural to associate to a filtered space X_* its filtered singular cubical set RX_* which in dimension n is the set of filtered maps $I^n_* \to X_*$. The aim is to define a homotopy relation on RX_* which gives an ω-groupoid ρX_* whose associated crossed complex is exactly ΠX_* as used in Part II. The fact that the 0-dimensional part of ΠX_* is just X_0 suggests that the homotopy relation we require is *thin homotopy rel vertices*. It turns out that this works: the 'rel vertices' condition is enough to start the inductive constructions required to prove that the compositions on RX_* are inherited by ρX_*.[225]

Definition 14.1.1. Let X_*, Y_* be filtered spaces. A *thin homotopy* $f : Y_* \otimes I_* \to X_*$ is a map $h : Y \times I \to X$ such that $f(Y_s \times I) \subseteq X_s$, $s \geqslant 0$, and it is *rel vertices* if $f|Y_0 \times I$ is the constant homotopy. Such a homotopy is as usual also written $f_t : Y_* \to X_*$, $0 \leqslant t \leqslant 1$, and called a homotopy from f_0 to f_1. Two elements $\alpha, \beta \in R_n X_*$ are *thinly homotopic rel vertices* if there is a thin homotopy $f : I^n \times I \to Y$ from α to β rel vertices. We write $\alpha \equiv \beta$.[226]

The set of equivalence classes of elements of $R_n X_*$ under thin homotopy rel vertices is written $\rho_n X_*$, and the class of $\alpha \in R_n X$ is written $\langle\!\langle \alpha \rangle\!\rangle$. So we have a quotient map $p : RX_* \to \rho_n X_*$. □

It is easy to check that the connections and the face and degeneracy maps of RX_* are inherited by ρX_*, giving it the structure of cubical complex with connections. In order to generalise to all dimensions the arguments given in dimension 2 in Chapter 6 for constructing the compositions, we need some preliminary definitions and results.

Definition 14.1.2. Let B be a subcomplex of I^n, let $m \geqslant 2$, and let $B \times I^m$ be given the product cell structure, so that the skeletal filtration gives a filtered space $B_* \otimes I^m_*$. Let

$$h \colon B \times I^m \to X$$

be a map. Fixing the i-th coordinate of I^m at the value t, where $0 \leqslant t \leqslant 1$, we obtain a map

$$\partial^t_i h \colon B \times I^{m-1} \to X.$$

If X_* is a filtered space, and $\partial^t_i h \colon B_* \otimes I^{m-1}_* \to X_*$ is a filtered map for each $0 \leqslant t \leqslant 1$, we say h is a *thin homotopy in the i-th direction of* I^m. In such case we write $h \colon \alpha \equiv_i \beta$ where $\alpha = \partial^0_i h, \beta = \partial^1_i h$. It is easy to see that the relation $\equiv_i$ defined on filtered maps $B_* \otimes I^{m-1}_* \to X_*$ by the existence of such an h is an equivalence relation independent of i, $1 \leqslant i \leqslant m$. □

Definition 14.1.3. A map $h \colon B_* \otimes I^2_* \to X_*$ is called a *thin double-homotopy* if it is a thin homotopy in each of the two directions of I^2; this is equivalent to

$$h(B_s \times I^2) \subseteq X_{s+1}, \quad h(B_s \times \partial I^2) \subseteq X_s, \quad s = 0, 1, 2, \dots .$$

(If K is a *proper* subcomplex of I^2, and $k \colon B \times K \to X$ satisfies $k(B_s \times K) \subseteq X_s$, $s = 0, 1, 2, \dots$, then by an abuse of language we also call k a thin double-homotopy.) □

Consider now a filtered space X_*.

Proposition 14.1.4. *Let B, C be subcomplexes of I^n such that $B \searrow C$. Let*

$$f \colon B_* \otimes \partial I^2_* \to X_*, \quad g \colon C_* \otimes I^2_* \to X_*$$

be two thin double homotopies which agree on $C \times \partial I^2$. Then their union $f \cup g$ extends to a thin double-homotopy $h \colon B_* \otimes I^2_* \to X_*$.

Proof. It is sufficient to consider the case of an elementary collapse $B \overset{e}{\searrow} C$. Suppose then $B = C \cup a$, $C \cap a = \partial a \setminus b$, where a is an s-cell and b is an $(s-1)$-face of a.

Let $r \colon a \times I^2 \to (a \times \partial I^2) \cup ((\partial a \setminus b) \times I^2)$ be a retraction. Then r defines an extension $h \colon B \times I^2 \to X$ of $f \cup g$. Since f is a thin double-homotopy,

$$h(a \times \partial I^2) = f(a \times \partial I^2) \subseteq X_s,$$

and since g is a thin double-homotopy

$$h((\partial a \setminus b) \times I^2) = g((\partial a \setminus b) \times I^2) \subseteq X_s.$$

Hence $h(a \times I^2) \subseteq X_s$, and in particular $h(b \times I^2) \subseteq X_s$. These conditions, with those on f and g, imply that h is a thin double-homotopy. □

Corollary 14.1.5. *Let X_* be a filtered space and let B be a subcomplex of I^n such that B collapses to one of its vertices. Then any thin double-homotopy rel vertices*

$$f : B_* \otimes \partial I^2_* \to X_*$$

extends to a thin double-homotopy rel vertices $h: B_* \otimes I^2_* \to X_*$.

Proof. Let v be a vertex of B such that $B \searrow \{v\}$. Now $f(\{v\} \times \partial I^2) \subseteq X_0$. Since the homotopies are rel vertices, $f|_{\{v\}\times\partial I^2}$ extends to a constant map $g: \{v\} \times I^2 \to X$ with image in X_0. Thus g is a thin double-homotopy. By Proposition 14.1.4, $f \cup g$ extends to a thin double-homotopy $h: B_* \otimes I^2_* \to X_*$. □

We now show that the compositions in RX_* are inherited by the quotient to give ρX_* the structure of ω-groupoid. This gives us the definition of the fundamental homotopy groupoid of a filtered space.

Definition 14.1.6. Let X_* be a filtered space. A *composition* $+_i$ on $\rho_n X_*$ is defined for $i = 1, \dots, n$ as follows:

Let $\langle\!\langle \alpha \rangle\!\rangle, \langle\!\langle \beta \rangle\!\rangle \in \rho_n X_*$ satisfy $\partial_i^+ \langle\!\langle \alpha \rangle\!\rangle = \partial_i^- \langle\!\langle \beta \rangle\!\rangle$. Then $\partial_i^+ \alpha \equiv \partial_i^- \beta$, so we may choose $h: I^n \to X$, a thin homotopy rel vertices in the i-th direction, so that $[\alpha, h, \beta]_i$ is defined in $R_n X_*$. We let

$$\langle\!\langle \alpha \rangle\!\rangle +_i \langle\!\langle \beta \rangle\!\rangle = \langle\!\langle [\alpha, h, \beta]_i \rangle\!\rangle$$

and prove next in Theorem 14.1.7 that this composition is well defined. □

Theorem 14.1.7. *If X_* is a filtered space, then the compositions on RX_* induce compositions on ρX_* which, together with the induced face and degeneracy maps and connections, give ρX_* the structure of ω-groupoid.*

Proof. We have to prove that the definition of the composition $+_i$ given in Definition 14.1.6 is independent of the representatives. For this it is sufficient, by symmetry, to suppose $i = n$.

To prove independence of choices, let $\alpha' \in \langle\!\langle \alpha \rangle\!\rangle$ and $\beta' \in \langle\!\langle \beta \rangle\!\rangle$ be alternative choices. Let $h': \partial_i^+ \alpha' \equiv \partial_i^- \beta'$ be any thin homotopy rel vertices, so that $[\alpha', h', \beta']_n$ is defined. We must prove that

$$\langle\!\langle [\alpha, h, \beta]_n \rangle\!\rangle = \langle\!\langle [\alpha', h', \beta']_n \rangle\!\rangle.$$

By construction there exist thin homotopies

$$k : \alpha \equiv \alpha', \quad l : \beta \equiv \beta'$$

in the $(n + 1)$-st direction.

We view I^{n+1} as a product $I^{n-1} \times I^2$ and define a thin double-homotopy rel vertices

$$f : I^{n-1} \times \partial I^2 \to X$$

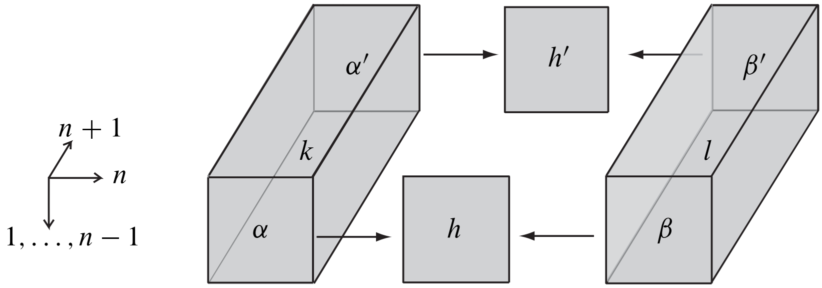


Figure 14.1. Compositions are well defined.

by $f(x,t,0) = h(x,t)$, $f(x,t,1) = h'(x,t)$, $f(x,0,t) = k(x,1,t)$, $f(x,1,t) = l(x,0,t)$, where $x \in I^{n-1}$ and $t \in I$. Figure 14.1 illustrates the situation. (Compare Figure 6.3 on p. 159.)

By Corollary 14.1.5 with $B = I^{n-1}$, f extends to a thin double-homotopy

$$H\colon I^{n-1} \times I^2 \to X.$$

Then $[k, H, l]_n$ is well defined and is a thin homotopy $[\alpha, h, \beta]_n \equiv [\alpha', h', \beta']_n$. This completes the proof that $+_n$, and by symmetry $+_i$, is well defined.

Suppose now that $\alpha +_i \beta$ is defined in $R_n X_*$. Let $h\colon \partial_i^+ \alpha \equiv_i \partial_i^- \beta$ be the constant thin homotopy in the i-th direction. Then $\alpha +_i \beta$ is thinly homotopic to $[\alpha, h, \beta]_i$ and so $\langle\!\langle \alpha +_i \beta \rangle\!\rangle = \langle\!\langle \alpha \rangle\!\rangle +_i \langle\!\langle \beta \rangle\!\rangle$. Thus the operations $+_i$ on $\rho_n X_*$ are induced by those on $R_n X_*$ in the usual algebraic sense.

Further, if $\langle\!\langle \alpha \rangle\!\rangle +_i \langle\!\langle \beta \rangle\!\rangle$ is defined in $\rho_n X_*$, then we may choose representatives α' , β' of $\langle\!\langle \alpha \rangle\!\rangle$, $\langle\!\langle \beta \rangle\!\rangle$ such that $\alpha' +_i \beta'$ is defined and represents $\langle\!\langle \alpha \rangle\!\rangle +_i \langle\!\langle \beta \rangle\!\rangle$ (for example we may take $\alpha' = \alpha$, $\beta' = h +_i \beta'$ where $h\colon \partial_i^+ \alpha \equiv_i \partial_i^- \beta$).

Defining $-_i \langle\!\langle \alpha \rangle\!\rangle = \langle\!\langle -_i \alpha \rangle\!\rangle$, one easily checks that $+_i$ and $-_i$ make $\rho_n X_*$ a groupoid with initial, final and identity maps ∂_i^-, ∂_i^+ and ε_i.

The laws for ε_j, ∂_j^τ, Γ_j of a composite $\langle\!\langle \alpha \rangle\!\rangle +_i \langle\!\langle \beta \rangle\!\rangle$ follow from the laws in $R_n X_*$ by choosing the representatives α, β so that $\alpha +_i \beta$ is defined.

Finally, we must verify the interchange law for $+_i$, $+_j$ $(i \neq j)$. By symmetry, it is sufficient to assume $i = n-1$, $j = n$.

Suppose that

$$\langle\!\langle \alpha \rangle\!\rangle +_{n-1} \langle\!\langle \beta \rangle\!\rangle, \quad \langle\!\langle \gamma \rangle\!\rangle +_{n-1} \langle\!\langle \delta \rangle\!\rangle, \quad \langle\!\langle \alpha \rangle\!\rangle +_n \langle\!\langle \gamma \rangle\!\rangle, \quad \langle\!\langle \beta \rangle\!\rangle +_n \langle\!\langle \delta \rangle\!\rangle$$

are defined in $\rho_n X_*$. We choose representatives $\alpha, \beta, \gamma, \delta$ and construct in $R_n X_*$ most of the composite

$$\begin{bmatrix} \alpha & k & \gamma \\ h & H & h' \\ \beta & k' & \delta \end{bmatrix}_{n-1,n} . \qquad (*)$$

Here the thin homotopies h, h' in the $(n-1)$-st direction and the thin homotopies k, k' in the n-th direction already exist, because the appropriate composites are defined, but H has to be defined (by 'filling the hole').

To construct H, we define a thin double-homotopy

$$f\colon I_*^{n-2} \otimes \partial I_*^2 \to X_*$$

by $f(x,0,t) = k(x,1,t)$, $f(x,1,t) = k'(x,0,t)$, $f(x,t,0) = h(x,t,1)$, $f(x,t,1) = h'(x,t,0)$ where $x \in I^{n-2}$, and $t \in I$. By Corollary 14.1.5, f extends to a thin double-homotopy

$$H\colon I_*^{n-2} \otimes I_*^2 \to X_*.$$

Then the composite $(*)$ is defined in $R_n X$ and the interchange law that

$$\begin{bmatrix} \langle\!\langle\alpha\rangle\!\rangle & \langle\!\langle\gamma\rangle\!\rangle \\ \langle\!\langle\beta\rangle\!\rangle & \langle\!\langle\delta\rangle\!\rangle \end{bmatrix}_{n-1,n}$$

is well defined is readily deduced by evaluating $(*)$ in two ways.

This completes the proof that ρX_* is an ω-groupoid. □

Remark 14.1.8. An examination of the proof shows that there is exactly enough 'filtered room' to accommodate the proof. This is evidence for the construction giving the 'right' concept. □

Definition 14.1.9. We call ρX_* the *homotopy ω-groupoid* (or the *fundamental ω-groupoid*, or *cubical homotopy groupoid*) of the filtered space X_*.

A filtered map $f\colon X_* \to Y_*$ clearly defines a map $Rf\colon RX_* \to RY_*$ of cubical sets with connections and compositions, and a map $\rho f\colon \rho X_* \to \rho X_*$ of ω-groupoids. So we have a functor

$$\rho\colon \mathsf{FTop} \to \omega\text{-}\mathsf{Gpds}.$$ □

The behaviour of ρ with regard to homotopies of filtered maps will be studied in the next chapter. At this stage we can use standard results in homotopy theory to prove:

Proposition 14.1.10. *Let $f\colon X_* \to Y_*$ be a filtered map of filtered spaces such that each $f_n\colon X_n \to Y_n$ is a homotopy equivalence. Then $\rho f\colon \rho X_* \to \rho Y_*$ is an isomorphism of ω-groupoids.*

Proof. This is immediate from [tDKP70], 10.11. Here is some detail of a different proof in the form of a set of remarks and exercises.

1) First recall that a set of conditions on functions can often be represented as describing a limit of sets of functions. In particular if A_*, X_* are n-truncated filtered spaces, i.e. $A_i = A_n$, $X_i = X_n$ for $i > n$, then $\mathsf{FTop}(A_*, X_*)$ can be represented as an iterated pullback.

2) We now want to prove that if A_* is n-truncated and $f\colon X_* \to Y_*$ is such that each $f_n\colon X_n \to Y_n$ is a homotopy equivalence, then so also is the induced map on

the topological spaces $f_*\colon \mathsf{FTop}(A_*, X_*) \to \mathsf{FTop}(A_*, Y_*)$. Start with the fact that if $f\colon X \to Y$ is a homotopy equivalence, then so also is the induced map of function spaces $\mathsf{Top}(A, X) \to \mathsf{Top}(A, Y)$, which is easy to prove, and that if $i\colon A \to B$ is a cofibration, then

$$i^*\colon \mathsf{Top}(B, X) \to \mathsf{Top}(A, X)$$

is a fibration.

3) Now use 1) to apply inductively the cogluing theorem from [BH70][227] to show by induction that if A_* is an n-truncated cofibred filtered space and $f\colon X_* \to Y_*$ is such that each $f_n\colon X_n \to Y_n$, $n \geqslant 0$ is a homotopy equivalence, then so also is the induced map $\mathsf{FTop}(A_*, X_*) \to \mathsf{FTop}(A_*, Y_*)$. □

14.2 The fibration and deformation theorems

In this section we provide all the technical results on extensions of thin homotopies needed for the further development of the theory.

The main result is the Deformation Theorem 14.2.5 which explains how to extend a thin homotopy from a special kind of subcomplex $B \subseteq I^n$ to the full I^n.

To get this Deformation Theorem we use some consequences of the construction of thin double homotopies in Corollary 14.1.5. Particularly useful is the filter homotopy extension property of Proposition 14.2.4. The proofs use the methods of collapsing from Section 11.3.i already seen at work in Proposition 14.1.4.

We finish the section with some consequences of the Deformation Theorem; perhaps the most intuitively striking is Proposition 14.2.8 which gives the possibility of lifting composable arrays of homotopy classes of filtered maps to composable arrays of maps.

Let us begin with consequences of Corollary 14.1.5.

Proposition 14.2.1. *Let $B \subseteq A$ be subcomplexes of I^n such that B collapses to one of its vertices. Let X_* be a filtered space. Let $\alpha, \beta\colon A_* \to X_*$ be filtered maps and let*

$$\psi\colon \alpha \equiv \beta, \quad \phi\colon \alpha|_B \equiv \beta|_B$$

be thin homotopies rel vertices. Then there is a thin double-homotopy

$$H\colon A_* \otimes I^2_* \to X_*$$

such that H is a homotopy rel end maps of ψ to a thin homotopy

$$H_1\colon \alpha \equiv \beta$$

extending ϕ.

Proof. Let $L = (I \times \{0\}) \cup (\partial I \times I)$. Define

$$l\colon (A \times L) \cup (B \times I \times \{1\}) \to X$$

by $l(x,t,0) = \psi(x,t)$, $l(x,0,t) = \alpha(x)$, $l(x,1,t) = \beta(x)$, $l(y,t,1) = \phi(y,t)$, $x \in A$, $y \in B, t \in I$. Then $f = l\,|_{B\times\partial I^2}$ and $k = l\,|_{A\times L}$ are thin double-homotopies.

By Corollary 14.1.5, f extends to a thin double-homotopy $h\colon B \times I^2 \to X$.

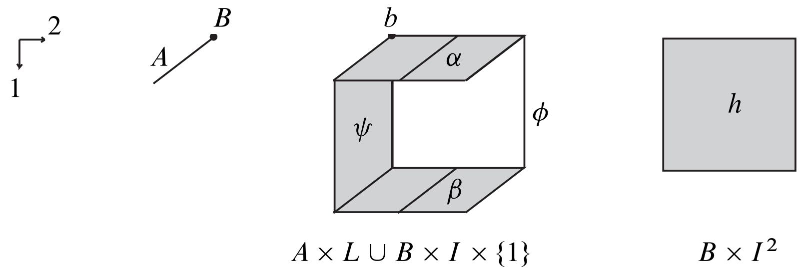


Figure 14.2. Starting the construction.

We are going to extend the map

$$k \cup h\colon (A \times L) \cup (B \times I^2) \to X$$

to a thin double-homotopy $H: A \times I^2 \to X$ by induction on the dimension of $A \setminus B$.

Suppose that H^s is a thin double-homotopy defined on $(A \times L) \cup ((A^s \times B) \times I^2)$, extending $H^{-1} = k \cup h$. For each $(s+1)$-cell a of $A \setminus B$, choose a retraction

$$r_a\colon a \times I^2 \to (a \times L) \cup (\partial a \times I^2).$$

These retractions extend H^s to H^{s+1} defined also on $A^{s+1} \times I^2$. Since $r_a(a \times I^2) \subseteq X_{s+1}$, it follows that H^{s+1} is also a thin double-homotopy.

Clearly $H = H^n$ is a thin double-homotopy as required. □

Corollary 14.2.2. *Let B, A, X_* be as in Proposition 14.2.1. If $\alpha, \beta\colon A_* \to X_*$ are maps which are thin homotopic rel vertices, then any thin homotopy rel vertices $\alpha|_B \equiv \beta|_B$ extends to a thin homotopy $\alpha \equiv \beta$.* □

We need to pay attention to the filtered maps which 'drop filtration' by at least one; we call these deficient.

Definition 14.2.3. If $f: Y_* \to X_*$ is a filtered map, where Y_* is a CW-complex with its skeletal filtration, we say that f is *deficient on a cell a of Y* if $\dim a = s$ but $f(a) \subseteq X_{s-1}$. In particular, a filtered map $I^n_* \to X_*$ is *deficient* if it is deficient on the top dimensional cell of I^n. □

Proposition 14.2.4 (Thin homotopy extension property). *Let B, A be subcomplexes of I^n such that $B \subseteq A$. Let*

$$f\colon A \times \{0\} \cup B \times I \to X$$

be a map such that $f|_{A\times\{0\}}$ *is a filtered map and* $f|_{B\times I}$ *is a thin homotopy rel vertices. Then* f *extends to a thin homotopy*

$$h\colon A\times I \to X.$$

Further, h *can be chosen so that if* f *is deficient on a cell* $a\times\{0\}$ *of* $(A\setminus B)\times\{0\}$*, then* h *is deficient on* $a\times\{1\}$*.*

Proof. The proof of this proposition is an easy induction on the dimension of the cells of $A\setminus B$, using retractions $a\times I\to a\times\{0\}\cup\partial a\times I$ for each cell a of $A\setminus B$. □

Now we can proceed to the proof of the Deformation Theorem which is needed as a technical tool for the results of the next section. The proof uses the results on partial boxes from Section 11.3.i.

Theorem 14.2.5 (Deformation Theorem). *Let* X_* *be a filtered space, and let* $\alpha\in R_nX_*$*. Any filtered map*

$$\gamma\colon B_*\to X_*$$

defined on a partial box $B\subseteq I^n$ *such that for each* $(n-1)$*-face* a *of* B*, the maps* $\alpha|_a$*,* $\gamma|_a$ *are thin homotopic rel vertices has an extension to a filtered map*

$$\beta\colon I^n_*\to X_*$$

that is thin homotopic to α*.*

Further, if α *is deficient (i.e.* $\alpha(I^n)\subseteq X_{n-1}$*), then* β *may be chosen to be deficient.*

Proof. Let B_1 be any $(n-1)$-cell contained in B. We choose a chain

$$B=B_s\searrow B_{s-1}\searrow\cdots\searrow B_1$$

of partial boxes and $(n-1)$-cells $a_1,a_2,\dots,a_{s-1}$ as in Theorem 11.3.5.

We construct thin homotopies $\phi_i:\alpha|_{B_i}\equiv\gamma|_{B_i}$ by induction on i, starting with ϕ_1 any thin homotopy $\alpha|_{B_1}\equiv\gamma|_{B_1}$. Suppose ϕ_i has been constructed and extends ϕ_{i-1}. Then $\phi_i|_{(a_i\cap B_i)}$ is defined. Since $a_i\cap B_i$ is a partial box, it collapses to any of its vertices. Since $\alpha|_{a_i}\equiv\gamma|_{a_i}$, the homotopy $\phi_i|_{(a_i\cap B_i)}$ extends, by Corollary 14.2.2, to a thin homotopy $\alpha|_{a_i}\equiv\gamma|_{a_i}$; this, with ϕ_i, defines ϕ_{i+1}.

Finally, we apply the thin homotopy extension property (Proposition 14.2.4) to extend $\phi_s:\alpha|_B\equiv\gamma$ to a thin homotopy $\alpha\equiv\beta$, for some β extending γ. The last part of Proposition 14.2.4 gives the final part of this theorem. □

For some applications of the Deformation Theorem, it is convenient to work in the category of cubical sets. Recall that we write $\mathbb{I}^n$ for the free cubical set on one generator c^n of dimension n (See Definition 11.1.7). Then an element γ of dimension n of a cubical set C determines a unique cubical map $\hat\gamma\colon\mathbb{I}^n\to C$ such that $\hat\gamma(c^n)=\gamma$ (Proposition 11.1.8). As a useful abuse of notation we are going to ‘drop the hat’.

In particular, a filtered map $\gamma\colon I^n_* \to X_*$ determines a unique cubical map $\gamma\colon \mathbb{I}^n \to RX_*$ such that $\gamma(c^n) = \gamma$. Also, if B is a subcomplex of the geometric n-cube I^n then B determines a cubical subset, also written B, of the cubical set $\mathbb{I}^n$, and a filtered map $\gamma\colon B_* \to X_*$ determines uniquely a cubical map $\hat{\gamma}\colon B \to RX_*$. The same may be said about homotopy classes of maps $[\gamma]$.

We can now rewrite the Deformation Theorem in the category of cubical sets as follows:

Corollary 14.2.6. *Let B be a box in $\mathbb{I}^n$ and let $i\colon B \to \mathbb{I}^n$ be the inclusion. Let X_* be a filtered space, and suppose given a commutative diagram of cubical maps*

$$\begin{tikzcd} B \arrow[r, "\gamma"] \arrow[d, "i"'] & RX_* \arrow[d, "p"] \\ \mathbb{I}^n \arrow[r, "{[\alpha]}"'] & \rho X_* . \end{tikzcd}$$

Then there is a cubical map

$$\beta\colon \mathbb{I}^n \to RX_*$$

such that $\beta i = \gamma$, $p\beta = [\alpha]$, i.e. extends γ and induces $[\alpha]$.

Further, if $[\alpha](c^n)$ has a deficient representative, then β may be chosen so that $\beta(c^n)$ is deficient. □

The following result is an easy and memorable consequence of the first part of Corollary 14.2.6. This result is used later in this chapter in Theorem 14.7.9, which has been used in applications to the homotopy classification of maps in Section 12.3.

Theorem 14.2.7 (Fibration Theorem). *Let X_* be a filtered space. Then the quotient map*

$$p\colon RX_* \to \rho X_*$$

is a fibration of cubical sets. □

Another application of Corollary 14.2.6 is to the lifting of subdivisions from $\rho_n X_*$ to $R_n X_*$. For the proof of this, and of the Higher Homotopy Seifert–van Kampen Theorem 14.3.1, we require the following construction.

Let $(m) = (m_1, \dots, m_n)$ be an n-tuple of positive integers. The subdivision of I^n with small n-cubes $c_{(r)}$, $(r) = (r_1, \dots, r_n)$, $1 \leqslant r_i \leqslant m_i$, where $c_{(r)}$ lies between the hyperplanes $x_i = (r_i - 1)/m_i$ and $x_i = r_i/m_i$ for $i = 1, \dots, n$, is called the subdivision of I^n of *type* (m).

Proposition 14.2.8 (Lifting arrays of homotopy classes). *Let X_* be a filtered space and*

$$\langle\!\langle \alpha \rangle\!\rangle = [\langle\!\langle \alpha_r \rangle\!\rangle]$$

a subdivision of an element $\langle\langle\alpha\rangle\rangle \in \rho_n X_*$. *Then there is an element* $\beta \in R_n X_*$ *and a subdivision*

$$\beta = [\beta_{(r)}]$$

of β, *where all* $\beta_{(r)}$ *lie in* $R_n X_*$ *such that* $\langle\langle\beta\rangle\rangle = \langle\langle\alpha\rangle\rangle$ *and* $\langle\langle\beta_{(r)}\rangle\rangle = \langle\langle\alpha_{(r)}\rangle\rangle$ *for all* (r).

Further, if each $\langle\langle\alpha_{(r)}\rangle\rangle$ *has a deficient representative, then the* $\beta_{(r)}$, *and hence also* β, *may be chosen to be deficient.*

Proof. Let K be the cell complex of the subdivision of I^n of the same type as the given subdivision of $\langle\langle\alpha\rangle\rangle$. Then K collapses to a vertex, see diagram (10.4.vi), page 363, so that there is a chain

$$K = A_s \searrow A_{s-1} \searrow \cdots \searrow A_1 = \{v\}$$

of elementary collapses, where $A_{i+1} = A_i \cup a_i$ for some cell a_i of K, and $A_i \cap a_i$ is a box in a_i.

We now work in terms of the corresponding cubical sets $K = A_s, A_{s-1}, \dots, A_1$, where K has unique nondegenerate elements $c_{(r)}$ of dimension n. The subdivision of $\langle\langle\alpha\rangle\rangle$ determines a unique cubical map

$$g\colon K \to \rho X_*$$

such that $g(c_{(r)}) = \langle\langle\alpha_{(r)}\rangle\rangle$. We construct inductively maps

$$f_i\colon A_i \to RX_*,$$

for $i = 1, \dots, s$, such that f_i extends f_{i-1}, produces $g|_{A_i}$, and $f_{i+1}(a_i)$ is deficient if $g(a_i)$ has a deficient representative. The induction is started by choosing $f_1(v)$ to be any element such that $pf_1(v) = g(v)$. The inductive step is given by Corollary 14.2.6.

Let

$$f = f_s\colon K \to RX_*,$$

and let $\beta_{(r)} = f(c_{(r)})$ for all (r). Then the $\beta_{(r)}$ compose in $R_n X_*$ to give an element $\beta = [\beta_{(r)}]$ as required. □

Recall from Definition 13.4.17 that in any ω-groupoid G, an element $x \in G_n$ is thin if it can be written as a composite $x = [x_{(r)}]$ with each entry of the form $\varepsilon_j y$ or of the form a repeated negative of $\Gamma_j y$. The following characterisation of thin elements of $\rho_n X_*$ is essential for later work.

Theorem 14.2.9 (Geometric characterisation of thin elements). *Let* $n \geqslant 2$ *and let* X_* *be a filtered space. Then an element of* $\rho_n X_*$ *is thin if and only if it has a deficient representative.*

Proof. We suppose $n \geqslant 2$ and that α in $R_n X_*$ is deficient. Define $\Psi_i \alpha \in R_n X_*$ by

$$\Psi_i \alpha = [-\varepsilon_i \partial_i^+ \alpha, -\Gamma_i \partial_{i+1}^- \alpha, \alpha, \Gamma_i \partial_{i+1}^+ \alpha]_{i+1}$$

where $-$ denotes $-_{i+1}$. Let $\Psi\alpha = \Psi_1 \dots \Psi_{n-1}\alpha$; then $\Psi\alpha$ also is deficient.

In Section 13.4 we defined for any ω-groupoid, and hence also for $\rho_n X_*$, a 'folding operation' Φ. The above formula for Ψ is the same as that for Φ. It follows that $p\Psi = \Phi p$, where $p\colon RX_* \to \rho X_*$ is the quotient map. So by Proposition 13.4.9, if $(\tau, j) \neq (-, 1)$ then $\partial_1^\tau \Phi p(\alpha) = \varepsilon_1^{n-1}[x]$ for some $[x] \in \rho_0 X = \pi_0 X_0$.

Thus if B is the box in I^n with base $\partial_1^+ I^n$, then for each $(n-1)$-cell a of B, $\Psi\alpha\,|_a$ is thin homotopic to the constant map at x. By the Deformation Theorem 14.2.5, $\Psi\alpha$ is thin homotopic to an element β such that $\beta(B) = \{x\}$, and such that β is deficient. Therefore, the homotopy of β to the constant map at x, defined by a strong deformation retraction of I^n onto B, is a thin homotopy giving $p\Psi\alpha = p\beta = 0$. So $\Phi p\alpha = 0$. By Proposition 13.4.18, $\langle\!\langle \alpha \rangle\!\rangle = p\alpha$ is thin.

For the other implication, suppose that $\langle\!\langle \alpha \rangle\!\rangle$ is thin. Then $\langle\!\langle \alpha \rangle\!\rangle$ has a subdivision $\langle\!\langle \alpha \rangle\!\rangle = [\langle\!\langle \alpha_{(r)} \rangle\!\rangle]$ in which each $\alpha_{(r)}$ is deficient. By Proposition 14.2.8, $\langle\!\langle \alpha \rangle\!\rangle$ has a deficient representative. □

14.3 The HHSvKT Theorem for ω-groupoids

Suppose for the rest of this section that X_* is a filtered space. We suppose given a cover $\mathcal{U} = \{U^\lambda\}_{\lambda\in\Lambda}$ of X such that the interiors of the sets of $\mathcal{U}$ cover X. For each $\nu \in \Lambda^n$ we set $U^\nu = U^{\nu_1} \cap \cdots \cap U^{\nu_n}$, $U_i^\nu = U^\nu \cap X_i$. Then $U_0^\nu \subseteq U_1^\nu \subseteq \cdots$ is called the *induced filtration* U_*^ν of U^ν. So the fundamental ω-groupoids in the following *ρ-diagram* of the cover are well defined:

$$\bigsqcup\nolimits_{\nu\in\Lambda^2} \rho U_*^\nu \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup\nolimits_{\lambda\in\Lambda} \rho U_*^\lambda \xrightarrow{c} \rho X_*.$$

Here $\bigsqcup$ denotes disjoint union, which is the same as coproduct in the category of ω-groupoids; a, b are determined for each index $\nu = (\lambda, \mu) \in \Lambda^2$ by the inclusions

$$a_\nu\colon U^\lambda \cap U^\mu \to U^\lambda, \quad b_\nu\colon U^\lambda \cap U^\mu \to U^\mu;$$

and c is determined by the inclusions $c_\lambda\colon U^\lambda \to X$.[228]

Theorem 14.3.1 (HHSvKT theorem for ω-groupoids). *Suppose that for every finite intersection U^ν of elements of $\mathcal{U}$, the induced filtration U_*^ν is connected. Then*

(Con) *X_* is connected;*

(Iso) *in the above ρ-diagram c is the coequaliser of a, b in the category of ω-groupoids.*

Proof. The proof of (Con) will be made on the way to verifying the universal property which proves (Iso).

Suppose we are given a morphism

$$f' \colon \bigsqcup_{\lambda \in \Lambda} \rho U_*^\lambda \to G \tag{*}$$

of ω-groupoids such that $f'a = f'b$. We have to show there is a unique morphism $f \colon \rho X_* \to G$ of ω-groupoids such that $fc = f'$. It will be clear that if a morphism f satisfying $fc = f'$ exists, then it must be given by the following recipe. The problem is to show that this recipe gives a well-defined morphism.

Let i_λ be the inclusion of $\rho U_*^\lambda \to \bigsqcup_{\lambda\in\Lambda} \rho U_*^\lambda$. Let $p_\lambda \colon RU_*^\lambda \to \rho U_*^\lambda$ be the quotient map, and let $F_\lambda = f' i_\lambda p_\lambda \colon RU_*^\lambda \to G$. We can use these F_λ to construct $F\theta$ in G_n for certain kinds of elements θ in $R_n X_*$.

1. Suppose that θ in $R_n X_*$ is such that θ lies in some set U^λ of $\mathcal{U}$. Then θ determines uniquely an element θ^λ of $R_n U_*^\lambda$, and the rule $f'a = f'b$ implies that an element of G_n,

$$F\theta = F_\lambda \theta^\lambda,$$

is determined by θ.

2. Suppose given a subdivision $[\theta_{(r)}]$ of an element θ of $R_n X_*$ such that each $\theta_{(r)}$ is in $R_n X_*$ and also lies in some $U^{\lambda(r)}$ of $\mathcal{U}$. Since the composite $\theta = [\theta_{(r)}]$ is defined, it is easy to check, again using $f'a = f'b$, that the elements $F\theta_{(r)}$ form a composable array in G_n. We write the composition as $F\theta$,

$$F\theta = [F\theta_{(r)}]$$

although *a priori* it could depend on the subdivision chosen.

3. Suppose now that α is an arbitrary element of $R_n X_*$. The construction from α of an element g in G_n and the proof that g depends only on the class of α in $\rho_n X_*$ are based on the following lemma which generalises Lemma 6.8.3.

Lemma 14.3.2. *Let $\alpha \colon I^n \to X$ and let $\alpha = [\alpha_{(r)}]$ be a subdivision of α such that each $\alpha_{(r)}$ lies in some set $U^{\lambda(r)}$ of $\mathcal{U}$. Then there is a homotopy $h \colon \alpha \simeq \theta$ with $\theta \in R_n X_*$ such that in the subdivision $h = [h_{(r)}]$ determined by that of α, each homotopy $h_{(r)} \colon \alpha_{(r)} \simeq \theta_{(r)}$ satisfies:*

- (i) *$h_{(r)}$ lies in $U^{\lambda(r)}$;*
- (ii) *$\theta_{(r)}$ belongs to $R_n X_*$;*
- (iii) *if some m-dimensional face of $\alpha_{(r)}$ lies in X_j, so also do the corresponding faces of $h_{(r)}$ and $\theta_{(r)}$;*
- (iv) *if v is a vertex of I^n and $\alpha(v) \in X_0$ then h is the constant homotopy on v.*

Proof. Let K be the cell-structure on I^n determined by the subdivision $\alpha = [\alpha_{(r)}]$. Let $L_m = K^m \times I \cup K \times \{0\}$. We construct maps

$$h_m \colon L_m \to X$$

for $m = 0, \dots, n$ such that h_m extends h_{m-1}, starting with $h_{-1} = \alpha$. Further we construct h_m to satisfy the following conditions, for each m-cell e of K:

(i)$_m$ if e is contained in the domain of $\alpha_{(r)}$, then $h_m(e \times I) \subseteq U^{\lambda(r)}$;

(ii)$_m$ $h_m\,|_{e\times\{1\}}$ is an element of $R_m(X_*)$;

(iii)$_m$ if α maps e into X_j, then $h_m(e \times I) \subseteq X_j$;

(iv)$_m$ if $\alpha\,|_e\colon e \to X$ is a filtered map, then h is constant on e.

For an m-cell e of K, let j be the smallest integer such that α maps e into X_j. Let U^e be the intersection of all the sets $U^{\lambda(s)}$ such that e is contained in the domain of $\alpha_{(s)}$.

Let $h_m\,|_{K\times 0}$ be given by α, and for those cells e of K such that $\alpha|_e$ is filtered, let h_m be the constant homotopy on $e \times I$.

Let e be a 0-cell of K. If $\alpha(e)$ does not lie in X_0, then, since U^e_* is connected, there is be a path in U^e joining e to a point of X_0. We define h_0 on $e \times I$ by using this path.

Let $m \geqslant 1$. The construction of h_m from h_{m-1} is as follows on those m-cells e such that the restriction of α to e is not a filtered map. If $j \leqslant m$, then h_{m-1} can be extended to h_m on $e \times I$ by means of a retraction $\alpha \times I \to e \times \{0\} \cup \partial e \times I$. If $j > m$ the restriction of h_{m-1} to the pair $(e \times \{0\} \cup \partial e \times I, \partial e \times I)$ determines an element of $\pi_m(U^e_j, U^e_{m-1})$. By $\phi(X_*, m)$, h_{m-1} extends to h_m on $e \times I$ mapping into U^e_j and such that $e \times \{1\}$ is mapped into U^e_m. □

Corollary 14.3.3. *Let $\alpha \in R_n X_*$. Then there is a thin homotopy rel vertices $h\colon \alpha \equiv \theta$ such that $F\theta$ is defined in G_n.*

Proof. Choose a subdivision $\alpha = [\alpha_{(r)}]$ such that $\alpha_{(r)}$ lies in some set $U^{\lambda(r)}$ of $\mathcal{U}$. Lemma 14.3.2 gives a thin homotopy $h\colon \alpha \equiv \theta$ and subdivision $\theta = [\theta_{(r)}]$ as required. □

We will show in Lemma 14.3.5 below that this element $F\theta$ depends only on the class of α in $\rho_n X_*$. But first we can now prove that X_* is connected.

Proof of (Con). The condition $\phi(X_*, 0)$ is clear since each point of X_j belongs to some U^λ and so may be joined in U^λ to a point of X_0.

Let $J^{m-1} = I \times \partial I^{m-1} \cup \{1\} \times I^{m-1}$. Let $j > m > 0$, $x \in X_0$ and let $[\alpha] \in \pi_m(X_j, X_{m-1}, x)$, so that $\alpha\colon (I^m, \{0\} \times I^{m-1}, J^{m-1}) \to (X_j, X_{m-1}, x)$. By Lemma 14.3.2, α is deformable as a map of triples into X_m.

This proves X_* is connected. □

Remark 14.3.4. Up to this stage, our proof of the HHSvKT is very like the proof for the 2-dimensional case given in 6.8. We now diverge from that proof for two reasons. First, the form of the Homotopy Commutativity Lemma given in 6.7.6 is not so easily stated in higher dimensions. So we employ thin elements, since these are elements with 'commuting boundary'. Second, we can now arrange that the proof is nearer in structure to the 1-dimensional case, for example the proof of the classical Seifert–van Kampen Theorem given in 1.6. □

Two facts about ω-groupoids which make the proof work are that composites of thin elements are thin (as is obvious from Definition 13.4.17), and Proposition 13.7.5.

Suppose now that $h' : \alpha \equiv \alpha'$ is a thin homotopy between elements of $R_n X_*$, and $h : \alpha \equiv \theta$, $h'' : \alpha' \equiv \theta'$ are thin homotopies constructed as in Corollary 14.3.3, so that $F\theta$, $F\theta'$ are defined. From the given thin homotopies we can obtain a thin homotopy $H : \theta \equiv \theta'$. So to prove $F\theta = F\theta'$ it is sufficient to prove the following key lemma. In fact, it could be said that the previous machinery has been developed in order to give expression to this proof.

Lemma 14.3.5. *Let $\theta, \theta' \in R_n X_*$ and let $H : \theta \equiv \theta'$ be a thin homotopy. Suppose*

$$\theta = [\theta_{(r)}], \quad \theta' = [\theta'_{(s)}]$$

are subdivisions into elements of $R_n X_*$ each of which lies in some set of $\mathcal{U}$. Then in G_n,

$$[F\theta_{(r)}] = [F\theta'_{(s)}].$$

Proof. Suppose $\theta_{(r)}$ lies in $U^{\lambda(r)} \in \mathcal{U}$, $\theta'_{(s)}$ lies in $U^{\lambda'(s)} \in \mathcal{U}$, for all (r), (s). Now $\theta = \partial^-_{n+1} H$, $\theta' = \partial^+_{n+1} H$. We choose a subdivision $H = [H_{(t)}]$ such that each $H_{(t)}$ lies in some set $V^{(t)}$ of $\mathcal{U}$ and so that on $\partial^-_{n+1} H$ and $\partial^+_{n+1} H$ it induces refinements of the given subdivisions of θ and θ' respectively. Further, this subdivision can be chosen fine enough so that $\partial^-_{n+1} H_{(t)}$, if it is a part of $\theta_{(r)}$, lies in $U^{\lambda(s)}$, and $\partial^+_{n+1} H_{(t)}$, if it is part of $\theta'_{(s)}$, lies in $U^{\lambda'(s)}$. So we can and do choose $V^{(t)} = U^{\lambda(r)}$ in the first instance, $V^{(t)} = U^{\lambda'(s)}$ in the second instance (and avoid both cases holding together by choosing, if necessary, a finer subdivision).

We now apply Lemma 14.3.2 with the substitution of $n+1$ for n, H for α, K for θ, and (t) for (r), to obtain in $R_{n+2} X_*$ a thin homotopy $h : H \equiv K$ such that in the subdivision $h = [h_{(t)}]$ determined by that of H, each homotopy $h_{(t)} : H_{(t)} \simeq K_{(t)}$ satisfies

(i) $h_{(t)}$ lies in $V^{(t)}$;
(ii) $K_{(t)}$ belongs to $R_{n+1} X_*$;
(iii) if some m-dimensional face of $H_{(t)}$ lies in X_j, so also do the corresponding faces of $h_{(t)}$ and $K_{(t)}$.

Now $k = \partial^-_{n+1} h$, $k' = \partial^+_{n+1} h$ are thin homotopies $k : \theta \equiv \phi$, $k' : \theta' \equiv \phi'$, say. Further, the previous choices ensure that in the subdivision $k = [k_{(r)}]$ induced by that of θ, $k_{(r)}$ is a thin homotopy $\theta_{(r)} \equiv \phi_{(r)}$ (by (iii)) and lies in $U^{\lambda(r)}$ (by (i)). It follows that $F\theta_{(r)} = F\phi_{(r)}$ in G_n and hence $F\theta = F\phi$. Similarly $F\theta' = F\phi'$, so it is sufficient to prove $F\phi = F\phi'$.

We have a thin homotopy $K : \phi \equiv \phi'$ and a subdivision $K = [K_{(t)}]$ such that each $K_{(t)}$ belongs to $R_{n+1} X_*$ and lies in some $V^{(t)}$ of $\mathcal{U}$. Thus $FK = [FK_{(t)}]$ is defined in G_{n+1}. Further, the induced subdivisions of $\partial^-_{n+1} FK$, $\partial^+_{n+1} FK$ refine the subdivisions $[F\phi_{(r)}]$, $[F\phi'_{(s)}]$ respectively. Hence $\partial^-_{n+1} FK = F\phi$, $\partial^+_{n+1} FK = F\phi'$, and it is sufficient to prove $\partial^-_{n+1} FK = \partial^+_{n+1} FK$. For this we apply Proposition 13.7.5.

Let d be a face operator from dimension $n+1$ to dimension m, and not involving ∂^-_{n+1} or ∂^+_{n+1}. Let $\sigma = d(H)$, $\tau = d(K)$. Then σ is deficient (since H is a filter homotopy) and so by the choice of h in accordance with (iii), τ is deficient. In the subdivision $\tau = [\tau_{(u)}]$ induced by the subdivision $K = [K_{(t)}]$, $\tau_{(u)} \in R_m X_*$ and is deficient. By Theorem 14.2.9, the $F\tau_{(u)} \in G_m$ are thin, and hence their composite $F\tau \in G_m$ is thin. But $FK = [FK_{(t)}]$ has, by its construction, the property that $dFK = F\tau$. So dFK is thin. By Proposition 13.7.5, $\partial^-_{n+1} FK = \partial^+_{n+1} FK$. □

Proof of (Iso). We have completed the proof that there is a well-defined function $f\colon \rho_n X_* \to G_n$ given by $f(\langle\!\langle \alpha \rangle\!\rangle) = F(\theta)$, where θ is constructed as in Corollary 14.3.3. These maps $f\colon \rho_n X_* \to G_n, n \geqslant 0$, determine a morphism $f\colon \rho X_* \to G$ of ω-groupoids. By its construction, f satisfies $fc = f'$ and is the only such morphism. Thus the proof of Theorem 14.3.1 is complete. □

14.4 The HHSvKT for crossed complexes

In order to interpret the HHvK Theorem 14.3.1, we relate the ω-groupoid ρX_* to the fundamental crossed complex ΠX_* of Part II.

In Section 13.3 we have defined a functor

$$\gamma\colon \omega\text{-}\mathsf{Gpds} \to \mathsf{Crs}$$

associating a crossed complex γG to any ω-groupoid G.

Now we prove that for any filtered space the crossed complex $\gamma\rho X_*$ is canonically isomorphic to ΠX_* the fundamental crossed complex used throughout Part II.[229] Hence we can translate Theorem 14.3.1 to obtain the HHvK Theorem for crossed complexes (Theorem 8.1.5) whose consequences we have studied in Part II.

Theorem 14.4.1. *If X_* is a filtered space then $\gamma\rho X_*$ is naturally isomorphic to ΠX_*.*

Proof. It is clear that the dimension 1 groupoids in both structures are the same.

Let $n \geqslant 2$, and $x \in X_0$. We construct an isomorphism

$$\theta_n\colon \pi_n(X_n, X_{n-1}, x) \to (\gamma\rho X_*)_n.$$

The elements of $\pi_n(X_n, X_{n-1}, x)$ are homotopy classes of maps of triples

$$\alpha\colon (I^n, \partial^-_1 I^n, B) \to (X_n, X_{n-1}, x),$$

where B is the box in I^n with base $\partial^+_1 I^n$. Such a map α defines a filtered map

$$\theta'\alpha\colon I^n_* \to X_*$$

with the same values as α, and $\theta'\alpha$ is constant on B.

If α is homotopic to β (as maps of triples), then $\theta'\alpha$ is thin homotopic to $\theta'\beta$, and so θ' induces a map $\theta_n : \pi_n(X_n, X_{n-1}, x) \to (\gamma\rho X_*)_n$. But addition in the relative homotopy group $\pi_n(X_n, X_{n-1}, x)$ is defined using any $+_i$, $i \geqslant 2$. So θ_n is a morphism of groups.

Suppose α represents in $\pi_n(X_n, X_{n-1}, x)$ an element mapped to 0 by θ_n. Then there is a filter homotopy rel vertices

$$H : \theta'\alpha \equiv x_*,$$

where x_* is the constant map at x. Now we want a map of triples

$$F : (I^n \times I, \partial_1^- I^n \times I, B \times I) \to (X_n, X_{n-1}, x)$$

with $F_0 = \alpha$ and $F_1 = x_*$. We know that $\alpha|_B$ is constant. By Corollary 14.2.2 and since B collapses to a vertex (by Corollary 11.3.7), the constant thin homotopy $\theta'\alpha|_B \equiv x_*|_B$ extends to a thin homotopy $\theta'\alpha \equiv x_*$. This thin homotopy defines a homotopy $F : \alpha \simeq x_*$. So θ_n is injective.

We now prove θ_n surjective. Let $\langle\!\langle \gamma \rangle\!\rangle \in (\gamma\rho X_*)_n$. Then for each $(n-1)$-face a of B, $\gamma|_a$ is thin homotopic to $\tilde{x}|_a$ (where $\tilde{x}$ is the constant map $B \to X_*$ at x). By the deformation Theorem 14.2.5, γ is thin homotopic to a map $\gamma' : I^n \to X_*$ extending $\tilde{x}$. Hence θ_n is surjective.

The isomorphism θ also preserves the boundary maps δ. To complete the proof, we only have to show that θ preserves the action of C_1 on C.

Let α represent an element of $\pi_n(X_n, X_{n-1}, x)$, and let ξ represent an element of $\pi_1 X_1(x, y)$. A standard method of constructing $\beta = \alpha^\xi$ representing an element of $\pi_n(X_n, X_{n-1}, y)$ (as seen in Section 2.1) is to use the homotopy extension property as follows. Let $\xi' : B \times I \to X_*$ be $(x, t) \mapsto \xi(t)$. Then ξ' is a homotopy of $\alpha|_B$ which extends to a homotopy $h : \alpha \simeq \beta$, and we set $\alpha^\xi = \beta$. We want to prove that $\theta_n[\alpha^\xi] = \theta_n[\beta] = (\theta_n[\alpha])^{[\xi]}$. So, if we recall that h is constructed by extending ξ' over $\partial_1^- I^n \times I$ using a retraction of $\partial_1^- I^n \times I$ to its box with base $\partial_1^- I^n \times \{0\}$, and then extending again using a retraction of $I^n \times I$ to its box with base $I^n \times \{0\}$. Thus h is a filtered map $I_*^{n+1} \to X_*$ with h and $\partial_i^\tau h$ $(i \neq n+1)$ deficient; hence $[h]$ and $\partial_i^\tau [h]$ $(i \neq n+1)$ are thin (Theorem 14.2.9). Therefore the folding map $\Phi : \rho_n X_* \to \rho_n X_*$ defined in Section 13.4 vanishes on these elements by Proposition 13.4.18 and so the Homotopy Addition Lemma 13.7.1 reduces to

$$\Phi\partial_{n+1}^+[h] = (\Phi\partial_{n+1}^-[h])^{u_{n+1}[h]}.$$

By Corollary 13.4.10, Φ is the identity on $(\gamma\rho X_*)_n$, to which belong both

$$\partial_{n+1}^+[h] = \theta_n[\beta] \quad \text{and} \quad \partial_{n+1}^-[h] = \theta_n[\alpha].$$

Further $u_{n+1}[h] = [\xi]$. So

$$\theta_n[\beta] = (\theta_n[\alpha])^{[\xi]}.$$

Thus θ preserves the operations.

Finally, the naturality of θ is clear. □

Proof of Theorem 8.1.5. Since the functor γ is an equivalence of categories, we obtain immediately from the previous theorem and the HHSvK Theorem 14.3.1 for ω-groupoids, the HHSvK Theorem 8.1.5 for crossed complexes. □

Proposition 14.4.2. *Let $n \geqslant 2$ and let $c^n \in \rho_n I^n_*$ be the class of the identity map $I^n_* \to I^n_*$. Then $\pi_n(I^n, \partial I^n, 1)$ is isomorphic to $\mathbb{Z}$ and is generated by $\theta^{-1}\Phi c^n$.*

Proof. There exists an alternative definition of relative homotopy groups, namely $\pi'_n(X, Y, x)$ is the set of homotopy classes of maps $(I^n, \partial I^n, 1) \to (X, Y, x)$, with addition induced by a map $I^n \to I^n \bigvee I^n$. An isomorphism $\xi \colon \pi_n(X, Y, x) \to \pi'_n(X, Y, x)$ is induced by $\alpha \mapsto \alpha'$ where (in the notation of the proof of Theorem 14.4.1) if $\alpha \colon (I^n, \partial_1^- I^n, B) \to (X, Y, x)$, then $\alpha' \colon (I^n, \partial I^n, 1) \to (X, Y, x)$ has the same values as α. (Here $1 = (1, \dots, 1)$ is the base point of I^n.)

Let $\rho_n(I^n_*, 1)$ be the set of y in $\rho_n I^n_*$ such that $(\partial_1^+)^n y = 1$. Then a map

$$\eta \colon \rho_n(I^n_*, 1) \to \pi'_n(I^n, \partial I^n, 1)$$

is induced by $\beta \mapsto \beta'$ where $\beta \colon I^n_* \to I^n_*$ satisfies $\beta(1) = 1$, and β' has the same values as β. Clearly $\eta\theta = \xi$.

A standard deduction from Corollary 8.3.12 is that $\pi'_n(I^n, \partial I^n, 1)$ is isomorphic to $\mathbb{Z}$ and is generated by a^n, the class of the identity map. Now clearly $\eta c^n = a^n$. Also, it is easily checked that for any $y \in \rho_n(I^n_*, 1)$ and $j = 1, \dots, n-1$, we have $\eta\Phi_j y = \eta y$. Hence $\eta\Phi c^n = \eta c^n = a^n$. The result now follows. □

From now on, we identify ΠX_* with $\theta \Pi X_* = \gamma\rho X_*$ for any filtered space X_*.

14.5 Realisation properties of ω-groupoids and crossed complexes

In this section, we show that each of the functors ρ and Π from FTop to respectively ω-groupoids and crossed complexes are *representative functors*, i.e. all ω-groupoids and all crossed complexes are, up to isomorphism, values of these functors. An implication of this is that the axioms for these structures well reflect the properties of these functors.

Let G be any ω-groupoid and define G^m to be the ω-subgroupoid of G generated by all elements of dimension $\leqslant m$. Then G^m has only thin elements in dimension greater than m and is the largest such ω-groupoid. In fact,

$$G^m \cong \mathrm{Sk}^m G = \mathrm{sk}^m(\mathrm{tr}^m G)$$

as described in Section 13.5, and by abuse of language we call it the *m-skeleton of G* (not to be confused with the m-skeleton of G considered as a cubical set). We define the *skeletal filtration* of G to be

$$G^*: \quad G^0 \subseteq G^1 \subseteq \cdots .$$

The elements of G^m_n are the same as those of G_n for $n \leqslant m$; and for $n > m$, G^m_n can be described inductively as the set of thin elements of G_n whose faces are in G^m_{n-1}.

Since G^m is an ω-groupoid, it is a fibrant cubical set. Therefore if $x \in G_0$, and $0 < l < m$, the r-th relative homotopy group $\pi_r(G^m, G^l, x)$ is defined for $r \geqslant 2$. So there is a crossed complex ΠG^* which in dimension $n \geqslant 2$ is the family of groups $\pi_n(G^n, G^{n-1}, x)$, $x \in G_0$, and in dimension 1 is the groupoid $\pi_1 G^1$.

Proposition 14.5.1. *If G^* is the skeletal filtration of an ω-groupoid G then the crossed complex ΠG^* is naturally isomorphic to γG. Further, G^* is connected.*

Proof. The elements of $\pi_n(G^n, G^{n-1}, p)$, $p \in G_0$, $n \geqslant 2$, are classes of elements x of G_n such that $\partial^\tau_i x = \varepsilon^{n-1}_1 p$ for $(\tau, i) \neq (0, 1)$, two such elements x, y being equivalent if there is an $h \in G^n_{n+1}$ such that $\partial^-_{n+1} h = x$, $\partial^+_{n+1} h = y$, $\partial^\tau_i h = \varepsilon^n_1 p$ for $(\tau, i) \neq (0, 1)$ and $i \neq n + 1$, and $\partial^-_1 h \in G^{n-1}_n$. Then h is thin, as is dh for any face operator d not involving ∂^-_{n+1} or ∂^+_{n+1}. It follows from Proposition 13.7.5 that $x = y$. Thus $\pi_n(G^n, G^{n-1}, p)$ can be identified with $C_n(p) = (\gamma_n G^*)(p)$.

The identification of the groupoid $\pi_1 G^1$ with G_1 is simple, as is the identification of the boundary maps. The identification of the operations may be carried out in a similar manner to the proof of Theorem 14.4.1.

Finally, that G^* is connected follows from the fact that $G^r_n = G_n$ for $r \geqslant n$. □

Remark 14.5.2. By Corollary 13.5.18, the crossed complex filtration associated to this filtration is precisely also the filtration by skeleta. That is,

$$\gamma \operatorname{sk}^n G = \operatorname{sk}^n(\gamma G). \qquad \square$$

We now use the geometric realisation $|K|$ of a cubical set K as described in Section 11.1.iii of Chapter 11. If G is an ω-groupoid, then $|UG|$ denotes the geometric realisation of the underlying cubical set UG of G.[230] Note that UG has a filtration by the usual cubical skeleta, and this is written $U_* G$. Note that for all n, $U_n G$ is a subcubical set of UG^n.

Proposition 14.5.3. *Let G be an ω-groupoid, G^* its skeletal filtration, and let $|UG^*|$ be the filtration of $|G|$ induced by the filtration $|G^n|$. Then there is a natural isomorphism of ω-groupoids*

$$G \cong \rho |UG^*|.$$

Proof. By the previous remarks and Proposition 14.5.1 we have natural isomorphisms

$$\gamma G \cong \Pi G^* \cong \Pi |UG^*|.$$

The result follows since $\Pi|UG^*| \cong \gamma\rho|UG^*|$ and γ is an equivalence of categories. $\Box$

Corollary 14.5.4. *If C is a crossed complex, there is a filtered space X_* such that C is isomorphic to ΠX_*.*

Proof. Let G be the ω-groupoid λC (cf. 13.6) and let $X_* = |UG^*|$. By Proposition 14.5.3, $C \cong \Pi X_*$. $\Box$

Remark 14.5.5. This result contrasts with Whitehead's example of a crossed complex C which is of dimension 5, has $\pi_1 C = \mathbb{Z}_2$, is free in each dimension but is not isomorphic to ΠX_* for the skeletal filtration X_* of any CW-complex X (see [Whi49b]).[231] $\Box$

Remark 14.5.6. Note also that when $X = |\lambda C|$, the absolute homotopy groups $\pi_n(X, x)$ are isomorphic to $\pi_1(C, x)$ for $n = 1$, $H_n(C, x)$ for $n \geqslant 2$ by Remark 13.7.7 of Section 13.7. Thus Corollary 14.5.4 generalises a cubical version of the construction of Eilenberg–Mac Lane spaces. We will see in the next section that there is a natural isomorphism $\lambda C \cong NC$ where NC is the nerve of C defined in Section 11.4.ii of Chapter 11. Hence $|\lambda C|$ is essentially the classifying space BC of the crossed complex C. $\Box$

14.6 Free properties

In this section, we give the important notion of the free ω-groupoid $\rho(K)$ on a cubical set K. We show that the methods of this Part III recover some results obtained in Chapter 11 by different methods.

Definition 14.6.1. Let K be a cubical set. We write $\rho(K)$ for $\rho(|K_*|)$. An element $k \in K_n$ defines a cubical map $\hat{k} : \mathbb{I}^n \to K$, and so a map of spaces $|\hat{k}| : |\mathbb{I}^n| \to |K|$. But $|\mathbb{I}^n| = I^n$ and so we have a filtered map $\bar{k} : I^n_* \to |K_*|$. This gives a cubical map $i : K \to \rho K$, which we call the *canonical map*. $\Box$

Proposition 14.6.2. *For any cubical set K, the canonical cubical map $i_K : K \to \rho K$ makes ρK the free ω-groupoid on K.*

Proof. Let G be an ω-groupoid, and let $f : K \to UG$ be a cubical map. Then f induces a filtered map $|K_*| \to |U_*G|$, which composes with the inclusion $|U_*G| \to |UG^*|$ to give $|f| : |K_*| \to |UG_*|$. The natural isomorphism $i_G : G \to \rho|G^*|$ and the natural map $i' : K \to U\rho|K_*|$ give a commutative diagram

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "i'"'] & UG \arrow[d, "Ui_G"] \\ U\rho|K_*| \arrow[r, "U\rho|f|"'] \arrow[ur, dashed, "U\tilde{f}"] & U\rho|UG^*|. \end{tikzcd}$$

Thus $\tilde{f} = (i_G)(\rho|f|)\colon \rho|K_*| \to \rho|UG^*|$ is a morphism of ω-groupoids extending f. Its uniqueness follows if we can show that $\rho|K_*|$ is generated, as an ω-groupoid, by $i'(K)$. But $\rho|K_*|$ is generated by the crossed complex $\gamma\rho|K_*| = \Pi|K_*|$ which it contains, by Corollary 13.5.13. Also $\Pi|K_*|$ is generated, as crossed complex, by the cells of $|K_*|$, i.e. by nondegenerate elements of K, by Corollary 9.6.5. So uniqueness is proved. □

Corollary 14.6.3. *The homotopy ω-groupoid ρI^n_* is the free ω-groupoid on the class $c^n \in \rho_n I^n_*$ of the identity map.*

Remark 14.6.4. The above corollary will be used in Section 15.6. We can now recover by these methods what was deduced in Chapter 11 from the HHSvKT and the tensor product results, namely the description of the crossed complex ΠI^n_*, compare Proposition 9.9.6. The cell complex I^n has one cell for each cubical face operator d from dimension n to r, $0 \leqslant r \leqslant n$, and d determines a characteristic map $\tilde{d}\colon I^r_* \to I^n_*$ for this cell. Then $\tilde{d}$ induces $\rho(\tilde{d})\colon \rho I^r_* \to \rho I^n_*$ and $\rho(\tilde{d})(c^r) = dc^n$. Since $\rho(\tilde{d})$ is a morphism of ω-groupoids, it follows that $\rho(\tilde{d})(\Phi c^r) = \Phi dc^n$. Hence ΠI^n_* has generators Φdc^n for each face operator d from dimension n to r, $0 \leqslant r \leqslant n$. The boundary $\delta\Phi dc^n$ is given by the HAL 13.7.1. □

Corollary 14.6.5. *If G is an ω-groupoid, then there is a natural bijection*

$$G_n \cong \mathsf{Crs}(\Pi I^n_*, \gamma G).$$

Proof. $G_n \cong \mathsf{Gpds}(\rho I^n_*, G) \cong \mathsf{Crs}(\Pi I^n_*, \gamma G)$. □

Remark 14.6.6. This corollary gives another description of the functor $\lambda\colon \mathsf{Crs} \to \omega\text{-}\mathsf{Gpds}$, the inverse equivalence of γ, namely that λ is naturally equivalent to $C \mapsto \mathsf{Crs}(\Pi I^n_*, C)$. In view of the explicit description of ΠI^n_* given above, a morphism $f\colon \Pi I^n_* \to C$ of crossed complexes is describable as a family $\{f(d)\}$ where d runs through all the cubical face operators from dimension n to dimension r $(0 \leqslant r \leqslant n)$, $f(d) \in C_r$, and the elements $f(d)$ are required to satisfy the relations (compare with Proposition 9.9.6 and Lemma 13.7.1)

$$\delta f(d) = \begin{cases} \sum_{i=1}^{r}(-1)^i\{f(\partial_i^+ d) - f(\partial_i^- d)^{f(u_i d)}\} & (r \geqslant 4), \\ -f(\partial_3^+ d) - f(\partial_2^- d)^{f(u_2 d)} - f(\partial_1^+ d) + f(\partial_3^- d)^{f(u_3 d)} & \\ \qquad\qquad + f(\partial_2^+ d) + f(\partial_1^- d)^{f(u_1 d)} & (r = 3), \\ -f(\partial_1^+ d) - f(\partial_2^- d) + f(\partial_1^- d) + f(\partial_2^+ d) & (r = 2), \end{cases}$$

and $\delta^\tau f(d) = f(\partial_1^\tau d)$ $(r = 1)$. (These relations imply that $f(d) \in C_r(p)$ where $p = f(\beta d)$). □

Corollary 14.6.7. *For any cubical set K, there is a natural isomorphism $\gamma\rho(K) \cong \Pi|K_*|$.*

By virtue of this corollary we identify these two crossed complexes and write either as $\Pi(K)$. So we have a functor $\Pi\colon \mathsf{Cub} \to \mathsf{Crs}$. Again we obtain a result of Chapter 11, namely Proposition 11.4.9.

Corollary 14.6.8. *The functor* $\Pi\colon \mathsf{Cub} \to \mathsf{Crs}$ *is left adjoint to the nerve functor* $N\colon \mathsf{Crs} \to \mathsf{Cub}$.

Proof. This follows from the fact that the functor $\rho\colon \mathsf{Cub} \to \omega\text{-}\mathsf{Gpds}$ is left adjoint to $U\colon \omega\text{-}\mathsf{Gpds} \to \mathsf{Cub}$, that $\Pi = \gamma\rho$, that $N = \lambda U$, and that γ and λ give the equivalence of the categories of ω-groupoids and crossed complexes. □

Remark 14.6.9. The fact that the functor $\rho\colon \mathsf{Cub} \to \omega\text{-}\mathsf{Gpds}$ has a right adjoint implies that ρ preserves all colimits. However, the Higher Homotopy Seifert–van Kampen Theorem 14.3.1 is *not* an immediate consequence of this fact since that theorem is about the functor $\rho\colon \mathsf{FTop} \to \omega\text{-}\mathsf{Gpds}$ from *filtered spaces* to ω-Gpds, and one of the conditions for $\rho \operatorname{colim} U_* \cong \operatorname{colim} \rho U_*$ is that each filtered space U_* should be connected, in the sense of 14.3. It would be interesting to know whether this Higher Homotopy Seifert–van Kampen Theorem can be deduced from the fact that $\rho\colon \mathsf{Cub} \to \omega\text{-}\mathsf{Gpds}$ preserves all colimits. □

14.7 Homology and homotopy

The homology groups of a cubical set K are defined as follows. First we form the chain complex $C'(K)$ where $C'_n(K)$ is the free abelian group on K_n, and with boundary

$$\partial k = \sum_{i=1}^{n} (-1)^i (\partial_i^- k - \partial_i^+ k). \tag{14.7.1}$$

It is easily verified that this gives a chain complex, i.e. $\partial\partial = 0$. However if K is a point, i.e. K_n is a singleton for all n, then the homology groups of $C'(K)$ are $\mathbb{Z}$ in even dimensions, whereas we want the homology of a point to be zero in dimensions > 0. We therefore normalise, i.e. factor $C'(K)$ by the subchain complex generated by the degenerate cubes. This gives the chain complex $C_*(K)$ of K, and the homology groups of this chain complex are defined to be the homology groups of K. In particular the homology groups of $S^{\square}X$ are the (cubical) singular homology groups of the space X.[232]

Let X_* be a filtered space. Then RX_* is a fibrant cubical set and ρX_* is an ω-groupoid, and hence also a fibrant cubical set by Proposition 13.7.3. A direct proof that ρX_* is fibrant can be given using Theorem 14.2.5.

The following proposition is one step towards the Hurewicz theorem.[233] In the proof, a useful lemma is that if (Y, Z) is a cofibred pair, and $f\colon (Y, Z) \to (X, A)$ is a map of pairs which is deformable (as a map of pairs) into A, then f is deformable into A rel Z ([Bro06], 7.4.4).

Proposition 14.7.1. *Let X_* be a filtered space such that the following conditions $\psi(X_*, m)$ hold for all $m \geqslant 0$:*

- $\psi(X_*, 0)$: *The map $\pi_0 X_0 \to \pi_0 X$ induced by inclusion is surjective.*
- $\psi(X_*, 1)$: *Any path in X joining points of X_0 is deformable in X rel end points to a path in X_1.*
- $\psi(X_*, m)(m \geqslant 2)X$: *For all $\nu \in X_0$, the map*

$$\pi_m(X_m, X_{m-1}, \nu) \to \pi_m(X, X_{m-1}, \nu)$$

induced by inclusion is surjective.

Then the inclusion $i\colon RX_* \to KX = S^{\square}X$ is a homotopy equivalence of cubical sets.

Proof. There exist maps $h_m\colon K_m X \to K_{m+1}X$, $r_m\colon K_m X \to K_m X$ for $m \geqslant 0$ such that

(i) $\partial^-_{m+1} h_m = 1$, $\partial^+_{m+1} h_m = r_m$;
(ii) $r_m(KX) \subseteq R_m X_*$ and $h_m | R_m X_* = \varepsilon_{m+1}$;
(iii) $\partial^\tau_i h_m = h_{m-1}\partial^\tau_i$ for $1 \leqslant i \leqslant m$ and $\tau = -, +$;
(iv) $h_m \varepsilon_j = \varepsilon_j h_{m-1}$ for $1 \leqslant j \leqslant m$.

Such r_m, h_m are easily constructed by induction, starting with $h_{-1} = \emptyset$, and using $\psi(X_*, m)$ to define $h_m \alpha$ for elements α of $K_m X$ which are not degenerate and do not lie in $R_m X_*$. Here is a picture for h_1:

$$\begin{tikzcd}
r\partial_1^- k \arrow[r, dash, "h_0\partial_1^- k"] \arrow[d, dashed, dash, "rk"'] \arrow[dr, phantom, "h_1 k"] & \partial_1^- k \arrow[d, dash, "k"] \\
r\partial_1^+ k \arrow[r, dash, "h_0\partial_1^+ k"'] & \partial_1^+ k
\end{tikzcd}$$

These maps define a retraction $r\colon KX \to RX_*$ and a homotopy $h \simeq ir$ rel RX_*. □

Corollary 14.7.2. *If the conditions $\psi(X_*, m)$ of the proposition hold for all $m \geqslant 0$, then the inclusion $i\colon RX_* \to KX$ induces a homotopy equivalence of chain complexes and hence an isomorphism of all homology and homotopy groups.*

Proof. The result on homotopy is standard, and that on homology follows from the development in [Mas80]. □

Corollary 14.7.3. *If X_* is the skeletal filtration of a CW-complex, then the inclusion $RX_* \to S^{\square}X$ is a homotopy equivalence of fibrant cubical sets.* □

14.7.i Relative Hurewicz Theorem: dimension 1

In this section we identify the total abelianisation of the groupoid $\pi_1(X, A)$ in certain cases.[234] One reason for including this result is that it gives a natural generalisation of a classical result in algebraic topology, which is the case when A is a singleton, see p. 10. The other reason is that we use the result in Proposition 8.4.2 which when X_* is a CW-filtration identifies $\nabla \Pi X_*$ with the cellular chains of the universal covers of X. The latter is a commonly used construction in algebraic topology, and indeed in essence goes back to Reidemeister.

Definition 14.7.4. Let $C_*(X)$ denote the chain complex of normalised cubical singular chains of the space X. We now coin a term: let

$$C_*(X \operatorname{rel}_0 A)$$

for a subspace A of X denote the sub chain complex of $C_*(X)$ generated for $n \geqslant 1$ by singular cubes $f : I^n \to X$ which map the vertices of I^n into A, and in which $C_0(X \operatorname{rel}_0 A) = 0$, so that all elements of $C_1(X \operatorname{rel}_0 A)$ are cycles. We write $H_*(X \operatorname{rel}_0 A)$ for the homology of this chain complex. □

For the notion of *total abelianisation* G^{totab} of a groupoid G, see Section A.8: this functor is the left adjoint of the inclusion of categories from abelian groups to groupoids.

Theorem 14.7.5. *Let A be a subspace of the space X. Then a Hurewicz morphism*

$$\omega : \pi_1(X, A) \to H_1(X \operatorname{rel}_0 A)$$

is defined and induces an isomorphism

$$\omega' : \pi_1(X, A)^{\mathrm{totab}} \to H_1(X \operatorname{rel}_0 A).$$

Proof. For each path class $[f] \in \pi_1(X, A)$ the representative f determines a generator of $C_1(X \operatorname{rel}_0 A)$. Differing choices of f yield homologous elements of $C_1(X \operatorname{rel}_0 A)$, so this defines ω as a function. If $f \circ g$ is a composite of paths with vertices in A then the diagram

$$\begin{tikzcd} \cdot \arrow[r, "f \circ g"] \arrow[d, "f"'] & \cdot \arrow[d, "1"] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd} \tag{14.7.2}$$

extends to a map of $I^2 \to X$ with vertices mapped to A whose boundary shows that ω is a morphism to $H_1(X \operatorname{rel}_0 A)$. It hence defines $\omega' : \pi_1(X, A)^{\mathrm{totab}} \to H_1(X \operatorname{rel}_0 A)$.

Now $C_1(X \operatorname{rel}_0 A)$ is free abelian on the nondegenerate paths $f : I \to X$ with vertices in A. So a morphism $\eta : C_1(X \operatorname{rel}_0 A) \to \pi_1(X, A)^{\mathrm{ab}}$ is defined by sending f to its class in $\pi_1(X, A)^{\mathrm{ab}}$. It is easy to check that $\eta\partial_2 = 0$, so that η defines a morphism $H_1(X \operatorname{rel}_0 A) \to \pi_1(X, A)^{\mathrm{totab}}$, and that η is inverse to ω'. □

Next we relate $H_*(X \operatorname{rel}_0 A)$ to the standard relative homology.

For a subspace A of X, we define the filtered space X_A to be A in dimension 0 and X in dimensions > 0. Our next result generalises a classical case when X is path connected and A consists of a single point.

Proposition 14.7.6. *If A meets each path component of X, then the inclusion*

$$C_*(X_A) \to C_*(X)$$

is a chain equivalence.

Proof. This is an immediate consequence of Corollary 14.7.2. □

We say $C_*(A)$ is *concentrated in dimension* 0 if $C_i(A) = 0$ for $i > 0$. This occurs for example if A is totally path disconnected, and so if A is discrete.

Theorem 14.7.7 (Relative Hurewicz Theorem: dimension 1). *If A is totally path disconnected and meets each path component of X then the natural morphism*

$$\pi_1(X, A)^{\text{totab}} \to H_1(X, A)$$

is an isomorphism.

Proof. We define A_* to be the constant filtered space with value A. So we regard A_* as a sub-filtered space of X_A.

We consider the morphism of exact sequences of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & C_*(A) \arrow[r] \arrow[d, "="] & C_*(X) \arrow[r] \arrow[d, "i"] & C_*(X, A) \arrow[r] \arrow[d, "j"] & 0 \\
0 \arrow[r] & C_*(A_*) \arrow[r] & C_*(X_A) \arrow[r] & C_*(X_A, A_*) \arrow[r] & 0
\end{tikzcd} \tag{14.7.3}$$

where classically the first sequence defines relative homology $H_*(X, A)$, and the second sequence defines $H_*(X_A, A_*)$. Under our assumptions, the morphism i is a homotopy equivalence and hence so also is j (since all the chain complexes are free in each dimension).

Our assumption that A is totally path disconnected implies that $C_i(A) = 0$ for $i > 0$. This implies that $C_*(X_A, A_*) \cong C_*(X \operatorname{rel}_0 A)$. So the theorem follows from Theorem 14.7.5 and Proposition 14.7.6. □

14.7.ii Absolute Hurewicz Theorem and Whitehead's exact sequence

We now outline a proof of the Absolute Hurewicz Theorem (Theorem 14.7.8) using Corollary 14.7.2 and the Homotopy Addition Lemma in the following form. Let $n \geqslant 2$, and let $\beta\colon (I^{n+1}, I^{n+1}_{n-1}) \to (X, \nu)$ be a map. Then each $\partial^\tau_i \beta$ represents an element β^τ_i of $\pi_n(X, \nu)$, and we have

$$\sum_{i=1}^{n+1} (-1)^i (\beta^-_i - \beta^+_i) = 0. \tag{14.7.4}$$

This follows from the form of the Homotopy Addition Lemma given in (13.7.1) applied to the ω-groupoid ρX_* where X_* is the filtered space with $X_i = \{\nu\}$, $i < n$, $X_i = X$, $i \geqslant n$.

Theorem 14.7.8 (Absolute Hurewicz Theorem). *If X is an $(n-1)$-connected pointed space for $n \geqslant 2$, then $H_i X = 0$ for $0 < i < n$ and the Hurewicz map*

$$\omega_n\colon \pi_n X \to H_n X$$

is an isomorphism.

Proof. Again, let X_* be the filtered space with

$$X_i = \{\nu\}, \ \ i < n, \qquad X_i = X, \ \ i \geqslant n.$$

Then X_* satisfies the condition $\psi(X_*, m)$ of Proposition 14.7.1 for all $m \geqslant 0$ and so $i\colon RX_* \to KX$ is a homotopy equivalence. But $H_i RX_* = 0$ for $0 < i < n$; hence $H_i X = H_i KX = 0$ for $0 < i < n$.

For $m \geqslant 0$ let $C_m X_*$ denote the group of (normalised) m-chains of RX_*. Then every element of $C_n X_*$ is a cycle, and the basis elements $\alpha \in R_n X_*$ of $C_n X_*$ are maps $I^n \to X$ with $\alpha(\partial I^n) = \{\nu\}$. So they determine elements $[\alpha]$ of $\pi_n(X, \nu)$, and $\alpha \mapsto [\alpha]$ determines a morphism $C_n X_* \to \pi_n(X, \nu)$. But by Equation (14.7.4), this morphism annihilates the group of boundaries. So it induces a map $H_n X \to \pi_n(X, \nu)$ which is easily seen to be inverse to the Hurewicz map. □

We know from Theorem 14.2.7 that if X_* is a filtered space, then $p\colon RX_* \to \rho X_*$ is a fibration of cubical sets. Notice that if $\nu \in X_0$, then ν also belongs to RX_* and to the fibre of p over ν.

Theorem 14.7.9 (Whitehead's exact sequence[235]). *Let X_* be a filtered space, and let $\nu \in X_0$. Let F be the fibre of $p\colon RX_* \to \rho X_*$ over ν, so that also $\nu \in F_0$. Then:*

(i) *There is an exact sequence*

$$\begin{aligned}\cdots \to &\pi_n(F, \nu) \to \pi_n(RX_*, \nu) \to \pi_n(\rho X_*, \nu) \\ &\to \cdots \to \pi_1(F, \nu) \to \pi_1(RX_*, \nu) \to \pi_1(\rho X_*, \nu) \to 1.\end{aligned}$$

(ii) *The group $\pi_n(F, \nu)$ is isomorphic to the image of the morphism*

$$i_n\colon \pi_n(X_{n-1}, \nu) \to \pi_n(X_n, \nu)$$

induced by inclusion.

(iii) *If all the X_n are Hausdorff, have universal covers $\widetilde{X}_n(\nu)$ for all $\nu \in X_0$, and the filtration X_* is connected and satisfies the conditions of Proposition 14.7.1, then the above exact sequence is equivalent to one of the form*

$$\cdots \to \boldsymbol{\Gamma}_n(X,\nu) \to \pi_n(X,\nu) \xrightarrow{\omega} H_n(\widetilde{X}(\nu)) \to \cdots$$

where ω is called the Hurewicz morphism. In particular, these conditions hold if X_* is a CW-filtration.

Proof. (i) This is just the exact sequence of the fibration of cubical sets

$$F \to RX_* \xrightarrow{p} \rho X_*.$$

(We leave to you the proof of the exact homotopy sequences of a fibration of fibrant cubical sets.)

(ii) We define a map $\theta\colon \pi_n(F,\nu) \to \pi_n(X_n,\nu)$.

An element of $\pi_n(F,\nu)$ is represented by an element α in dimension n of the cubical set F such that α has all its faces at the base point ν. Since F is a subcubical set of RX_*, α determines $\alpha'\colon (I^n,\partial I^n) \to (X_n,\nu)$ with the same values as α; and θ is determined by $\alpha \mapsto \alpha'$.

We now prove $\operatorname{Im}\theta \subseteq \operatorname{Im} i_n$. If $\alpha \in F_n$, then $p(\alpha) = \langle\langle \alpha \rangle\rangle = \varepsilon_1^n \nu$ in $\rho_n X$, and so α is thin homotopic to $\bar{\nu}$, the constant map at ν. Suppose further that α has all its faces at the base point. Let B be the box in I^n with base $\partial_n^- I^n$. By Corollary 14.2.2, the constant thin homotopy $\bar{\nu}|B \equiv \alpha|B$ extends to a thin homotopy $h\colon \bar{\nu} \equiv \alpha$. Let $\beta = \partial_n^+ h, k = \Gamma_n \beta$. Then $h +_n k$ is a thin homotopy $\bar{\nu} +_n \beta \simeq \alpha +_n \bar{\nu}$, rel ∂I^n. Let $\beta'\colon (I^n,\partial I^n) \to (X_{n-1},\nu)$ be the map with the same values as β. Then $\alpha' \simeq i\beta'$.

We next prove $\operatorname{Im} i_n \subseteq \operatorname{Im}\theta$. Let $\alpha'\colon (I^n,\partial I^n) \to (X_{n-1},\nu)$ represent an element of $\pi_n(X_{n-1},\nu)$. Let $\alpha\colon I_*^n \to X_*$ have the same values as α'. Then $\Gamma_n\alpha$ is a thin homotopy $\alpha \equiv \bar{\nu}$, so that $\alpha \in F_n$. Clearly $\theta[\alpha] = i_n[\alpha']$.

For the final part of (ii), we prove θ injective. Suppose $\theta[\alpha] = 0$. Then there is a homotopy $h\colon \alpha' \simeq \bar{\nu}$ of maps $(I^n,\partial I^n) \to (X_n,\nu)$. Clearly $h \in R_{n+1}X_*$. However, $\Gamma_{n+1}h$ is a thin homotopy $h \equiv \bar{\nu}$. Therefore $h \in F_{n+1}$, and so $[\alpha] = 0$.

(iii) We have proved in Proposition 8.4.2 that with these conditions and for $n \geqslant 2$, $H_n(\Pi X_*,\nu) \cong H_n(\widetilde{X}(\nu))$, and in Corollary 14.7.3, that the inclusion $RX_* \to S^{\square}X$ is a homotopy equivalence. So this exact sequence follows from that in (i). $\square$

Definition 14.7.10. We say X_* is a *J_n-filtered space* if for $0 \leqslant i < n$ and $\nu \in X_0$, the map

$$\pi_{i+1}(X_i,\nu) \to \pi_{i+1}(X_{i+1},\nu)$$

induced by inclusion is trivial.[236] $\square$

Corollary 14.7.11. *If X_* is a J_n-filtered space, then each fibre of $p\colon RX_* \to \rho X_*$ is n-connected, and the induced maps $\pi_i RX_* \to \pi_i \rho X_*$, $H_i RX_* \to H_i \rho X_*$, of homotopy and homology, are isomorphisms for $i \leqslant n$ and epimorphisms for $i = n+1$.* $\square$

14.8 The cubical Dold–Kan Theorem

We have shown in Chapter 13 the equivalence of the categories of crossed complexes and of cubical ω-groupoids with connections. In this section we relate this result to a famous theorem of Dold and Kan stating that the categories of chain complexes and of simplicial abelian groups are equivalent, and which we have already mentioned in Remark 9.10.6[237]. We use the notion of *structure internal to a category*, more information on which can be found in many texts and expositions of category theory.

The basic elements of what we say next are well known, but are given for completeness.

Suppose we are given an action of a group P on the right of a group M such that the action $\phi\colon M \times P \to M$ is a morphism of groups. Then, as is well known, the action is trivial. The proof is easy: let $m \in M$, $p \in P$. Then $m^p = \phi(m, p) = \phi(m, 1)\phi(1, p) = m^1 1^p = m$. It follows that a crossed module internal to the category of groups is just a morphism of abelian groups.

We need to consider below the more general case of crossed modules over groupoids. Internally to the category of groups, these are more complicated; but internally to the category of abelian groups they are again equivalent to morphisms of abelian groups.

Theorem 14.8.1 (Cubical Dold–Kan Theorem). *Let $\mathcal{A}$ be an additive category with kernels. The following categories, defined internally to $\mathcal{A}$, are equivalent.*

- $\mathbb{B}_1$: *The category of chain complexes.*
- $\mathbb{B}_2$: *The category of crossed complexes*
- $\mathbb{B}_3$: *The category of cubical sets with connection.*
- $\mathbb{B}_4$: *The category of cubical ω-groupoids with connections.*

Proof. By working on the morphism sets, we can as usual assume that we are working in the category of abelian groups. Note that the theorem corresponding to the title of this section is the equivalence $\mathbb{B}_1 \simeq \mathbb{B}_3$.

$\mathbb{B}_1 \simeq \mathbb{B}_2$: By a chain complex we shall always mean a sequence of objects and morphisms $\partial\colon A_n \to A_{n-1}$, $n \geqslant 1$, such that $\partial\partial = 0$. Let C be a crossed complex internal to $\mathcal{A}$. The associated chain complex αC will be defined by

$$\begin{aligned}
(\alpha C)_0 &= C_0,\\
(\alpha C)_1 &= \operatorname{Ker}\,(\delta_0 : C_1 \to C_0),\\
(\alpha C)_n &= C_n(0), \quad n \geqslant 2.
\end{aligned}$$

The crossed complex ΘA associated to a chain complex A will be defined by

$$\begin{aligned}
(\Theta A)_0 &= A_0,\\
(\Theta A)_1 &= A_1 \times A_0,\\
(\Theta A)_n &= A_n \times A_0, \quad n \geqslant 2.
\end{aligned}$$

The groupoid structure on ΘA in dimension 1 is defined as usual by $s(a, y) = y + \partial a$, $t(aa, y) = y$ and with composition $(b, x)(a, y) = (b+a, y)$. The structure on $(\Theta A)_n$ for $n \geqslant 2$ is that the only addition is $(b, x)+(c, x) = (b+c, x)$. The operation of $(\Theta A)_1$ on $(\Theta A)_n$, $n \geqslant 2$, is $(b, x)^{(a,y)} = (b, y)$. This gives our first equivalence, between chain complexes and crossed complexes internal to $\mathcal{A}$. Notice that Θ is essentially the special case of the functor Θ in Section 7.4.v in which the acting groupoid H is the trivial group.

$\mathbb{B}_2 \simeq \mathbb{B}_3$: An equivalence between crossed complexes and cubical ω-groupoids with connections internally to the category of sets is established in previous sections. Although choices are involved in this, the end result is a natural equivalence. It follows that this can be applied internally to a category $\mathcal{A}$, simply by applying it to the morphism sets $\mathcal{A}(X, A)$ for all objects X of $\mathcal{A}$. This yields our equivalence between crossed complexes and cubical ω-groupoids with connections internal to $\mathcal{A}$.

In Remark 14.6.6 we have shown that the equivalence $\lambda : \mathsf{Crs} \to \omega$-groupoids may be given by

$$\lambda(C)_n = \mathsf{Crs}(\Pi(I_*^n), C).$$

If we apply this to $C = \Theta A$ as above we find that

$$\begin{aligned} \lambda(\Theta A)_n &= \mathsf{Crs}(\Pi(I_*^n), \Theta A) \\ &= \mathsf{Chn}(\nabla\Pi(I_*^n), A) \qquad \text{(by adjointness)} \\ &= \mathsf{Chn}(C_*(I_*^n), A), \end{aligned}$$

which is the cubical analogue of the classical formulation of the Dold–Kan theorem.

$\mathbb{B}_3 \simeq \mathbb{B}_4$: Let K be a cubical abelian group with connections, in the sense of previous sections. The following is an easy result, related to work of Section 2.5 [238].

Lemma 14.8.2. *If G is an abelian group, and if $s, t : G \to G$ are endomorphism of G such that $st = s$, $ts = t$, then we can define a groupoid structure on G with source and target maps s, t by*

$$g \circ h = g - tg + h,$$

for $g, h \in G$ with $tg = sh$, and this defines on G the structure of groupoid internal to abelian groups.

This result can be applied to K_n, $n \geqslant 1$, and for each $i = 1, \dots, n$, with

$$s_i = \varepsilon_i \partial_i^-, \quad t_i = \varepsilon_i \partial_i^+,$$

giving n compositions and so a cubical complex with compositions and connections in the sense of Definition 13.1.3. The interchange law is easily verified, and there remains essentially only the transport law for the connections, which is again simple, showing that K is now a cubical ω-groupoid with connections. It is easy to see that the functor thus defined is adjoint to the forgetful functor $\mathbb{B}_4 \to \mathbb{B}_3$. □

Remark 14.8.3. The lack of an equivalence between chain complexes and cubical abelian groups (without connections) was a further reason for abandoning the use of cubical sets for simplicial sets. The pros and cons of simplicial sets versus cubical sets with connections need further development and argument.[239] □

Notes

223 p. 480 Most of the results of this chapter come from [BH81a]. In that paper a condition J_0 was imposed on the filtered spaces, namely that each loop in X_0 is contractible in X_1: this condition was sufficient to define $\rho(X_*)$ without requiring the homotopy relation $\equiv$ defined in Section 14.1 to be rel vertices. However this implied that $\rho(X_*)$, and similarly ΠX_*, in dimension 0 was $\pi_0(X_0)$, which does not agree with usual conventions; the condition J_0 is also inconvenient in practice since it is rarely satisfied by the filtered space $\mathsf{FTOP}(X_*, Y_*)$. Thus the arguments of this book differ from those in [BH81a] in the first stage of various inductive processes.

224 p. 480 There is a current tendency to write ΠX for the singular simplicial complex of a space X and to term ΠX an ∞-groupoid of some kind, see for example [Ber02], [Lur09]. Now Ashley in [Ash88] gives in the simplicial case a fibration theorem analogous to Theorem 14.2.7, so various questions arise. One is whether the singular cubical complex might be a better candidate for such a lax ∞-groupoid, and indeed it is possible to give a definition using 'rectangles' which makes the compositions $+_i$ into category structures, see [Bro09b]. The other is whether in the filtered case, the Fibration Theorem 14.2.7 gives some interesting control over the laxity of the totality of structures on RX_*.

225 p. 482 Brown wishes to thank C. T. C. Wall who in conversation after a splinter group talk by Brown at a British Mathematical Colloquium in 1975 pointed out that the 'free face' argument was likely to work. Collapsing arguments go back to the paper of J. H. C. Whitehead, [Whi41b]. The term 'nuclei' in that paper refers to what was later called 'simple homotopy type', [Whi50b].

226 p. 482 This concept is called 'filter homotopy ' in [BH81a].

227 p. 487 The cogluing theorem from [BH70] is the dual of a gluing theorem which was first published in [Bro68], is now available in [Bro06], and has been set up in abstract homotopy in for example [Bau89], and Theorem 7.1 in [KP97], with different proofs. See also [tD08], (5.3.3) Proposition. One advantage of the original proof in [Bro68], [Bro06] is that it gives more control of the homotopies. Using function spaces, the cogluing theorem implies the gluing theorem.

228 p. 492 The indexing by Λ^2 implies there is repetition in the first disjoint union, and also that pairs (λ, λ) occur. This could be avoided by totally ordering Λ and then considering only pairs (λ, μ) such that $\lambda < \mu$, but that seems not worth the effort.

229 p. 496 An analogous argument to the following is used in [Bro08b] to show that a globular higher homotopy groupoid is well defined.

230 p. 499 It is proved in [Ant00] that the realisation with connections collapsed has the same homotopy type as the usual realisation. The paper [Mal09] shows that the former realisation has better properties with respect to the cartesian product than the usual realisation.

231 p. 500 There is an exposition on obstructions to the realisability of chain complexes in [Bau91], Proposition A.2, p. 136.

232 p. 502 A full exposition of this cubical homology theory is in [Mas80]. It is proved in [EML53a] using acyclic models that the cubical singular homology groups are isomorphic to the simplicial singular homology groups. Recent works using cubical methods are [GNAPGP88], [BJT10], [Isa11].

233 p. 502 This retraction proposition should be compared with the special case discussed in [Mas80], Section III.7. The history of classical papers on singular homology and the Hurewicz Theorem shows the use of deformation theorems of the type of Proposition 14.7.1, as for example in Blakers [Bla48]. However our use of cubical methods rather than the traditional chain complexes and simplicial methods, simplifies the proofs; one reason is that cubical methods are easier than simplicial methods for constructing homotopies.

234 p. 504 The results of this section are taken from [Bro11].

235 p. 506 Theorem 14.7.9 gives a certain exact sequence considered by Whitehead in the paper [Whi50c]. The term 'Whitehead exact sequence' may be found in many papers. The reason for our choosing the notation $\mathbf{\Gamma}$ in this theorem is that Whitehead used the notation Γ, which is used in this book for the cubical connections. The methods of Whitehead in [Whi50c] for his exact sequence are more direct and he also proves a remarkable determination of $\mathbf{\Gamma}_3 X$ as the value of a 'universal quadratic functor' Γ on $\pi_2(X)$. This is related to results in [BL87]. For further work in this area see [MW10] and the references there.

236 p. 507 The condition that X_* be a J_n-filtered space is in the CW-complex case precisely the condition that X is a J_n-complex in the sense of [Whi49b], and is also by Theorem 14.7.9 equivalent to $p\colon RX_* \to \rho X_*$ being an n-equivalence. Thus these results are related to the results of [Ada56] which give necessary and sufficient conditions for X to be a J_n-complex.

237 p. 508 This theorem was stated and proved by Dold in [Dol58]. The reason for the addition of the name of Kan to the theorem is that an earlier preprint by Dold was purely combinatorial, and then Kan shed a huge light on this by showing that the functor K from chain complexes to simplicial abelian groups could be defined by the formula $K(A)_n = \mathsf{Chn}(C_*(\Delta^n), A)$. Dold also states that his results were found earlier by Kan. Our formulation of the cubical analogue is taken from [BH03], which also includes the case of globular ω-groupoids, using the equivalence of these with crossed complexes proved in [BH81b]. Part of the result, namely relating a groupoid to a morphism of abelian groups is essentially in [Gro68].

Thus we see that various results related to, or generalisations of, the Dold–Kan Theorem are shown in the diagram:

$$\begin{tikzcd}
\text{polyhedral } T\text{-complexes} \arrow[r, leftrightarrow, "{[\text{Jon88}]}"] & \text{cubical } T\text{-complexes} & \omega\text{-groupoids} \arrow[l, "{[\text{BH81c}]}"'] \arrow[dd, "{[\text{BH81b}]}"] \arrow[ddl, leftrightarrow, "{[\text{BH81}]}"'] \\
\text{simplicial } J\text{-groupoids} \arrow[dr, leftrightarrow, "{[\text{NT89b}]}"] & & \\
\text{simplicial } T\text{-complexes} \arrow[r, leftrightarrow, "{[\text{Ash88}]}"'] & \text{crossed complexes} \arrow[r, leftrightarrow, "{[\text{BH81b}]}"'] & \infty\text{-groupoids}
\end{tikzcd}$$

where each arrow denotes an explicit equivalence of categories, and the citations give full details. There could be a case for using notations such as Π^{cell}, Π^{cub}, Π^{glob}, Π^{simp} for the fundamental object of a filtered space lying in each of the categories Crs, ω-Gpds, ∞-groupoids (also called strict globular ω-groupoids, [Bro08b]), simplicial T-complexes, respectively.

Generalisations of the Dold–Kan Theorem to more general categories than additive categories with kernels are given in [Bou07], while the paper [CC91] gives general structure on a nonabelian chain complex C of groups to be able to reconstruct a simplicial group G of which C is the Moore complex. This generalises a result in [Con84] on 2-crossed modules. Further references to and applications of 2-crossed modules are in [FM11]; they also occur in [Bİ03a].

238 p. 509 This result comes from [Gro68], and is also a special case of a nonabelian result on cat[1]-groups, [Lod82], where the condition $[\mathrm{Ker}\, s,\ \mathrm{Ker}\, t] = 1$ is required, and is here trivially satisfied.

239 p. 510 As an example, there has been little exploitation of the fact that cubical groups with connections are fibrant cubical sets, [Ton92], whereas the corresponding result for simplicial groups is well used.

Chapter 15
Tensor products and homotopies

We now explain the final and vital piece of algebraic structure which gives power to the machinery of crossed complexes, particularly to the Homotopy Classification Theorem 11.4.19: this is the monoidal closed structure on crossed complexes, and its properties, which were stated in Chapter 9.[240] Our justification of these properties is in terms of the category ω-Gpds of cubical ω-groupoids, where the corresponding monoidal closed structure has a simple and convenient expression. In this category it is also easy to construct a natural transformation of Eilenberg–Zilber type

$$\eta\colon \rho X_* \otimes \rho Y_* \to \rho(X_* \otimes Y_*)$$

and so this may be transferred to the category of crossed complexes and the functor Π via the equivalence of categories γ and Theorem 14.4.1.

The design of this chapter is as follows. In Section 15.1 we extend to ω-groupoids the structure of monoidal closed category constructed for cubical sets in Chapter 11. This part is straightforward.

Then, in Section 15.2 we study the translation of the closed structure from ω-groupoids to crossed complexes using the details of the inverse equivalences

$$\gamma\colon \omega\text{--}\mathsf{Gpds} \rightleftarrows \mathsf{Crs}\colon \lambda$$

getting the fairly complicated description of the closed category structure for closed complexes given in Part II. In some sense this difficulty is an advantage, since the results of the story are fairly easy to use, see Chapter 9, and when we do use these results, we know we have a powerful machine in the background, so that the applications have the potential of being highly nontrivial.

In Section 15.3, we define the natural transformation η mentioned above. Again this result can be transferred to crossed complex giving the important Theorem 9.8.1, that there is a natural transformation

$$\zeta\colon \Pi X_* \otimes \Pi Y_* \to \Pi(X_* \otimes Y_*)$$

for all filtered spaces X_*, Y_* and which is an isomorphism if X_*, Y_* are CW-filtrations. In Section 15.4 we establish the symmetry of the tensor product which, by contrast with the other results, is easier to prove for crossed complexes than for ω-groupoids. It is interesting to note that the tensor product of cubical sets is not symmetric, as pointed out in Proposition 11.2.20: the extra structure of ω-groupoids is needed to define the symmetry map $G \otimes H \to H \otimes G$. In Section 15.5 we give a brief account of the case of ω-groupoids with base-point.

In the last two sections we give a dense subcategory of the category of ω-groupoids, and use this to show certain covering crossed complexes of tensor product of crossed complexes are also a tensor product of coverings. We use this to prove that the tensor product of two free aspherical crossed complexes is also aspherical. This is a useful result for our earlier Chapters 10 and 12 on resolutions and on homotopy classification of maps and cohomology.

For further background on monoidal closed categories, see Section C.7 of Appendix C.

15.1 Monoidal closed structure on ω-groupoids

The category ω-Gpds of ω-groupoids is a convenient algebraic model for certain geometric constructions. In particular it is well-suited for the discussion of homotopies and higher homotopies and their composition.

The precise definition of ω-groupoid is in a previous chapter, Section 13.2; recall that an ω-groupoid is a cubical set with extra structures of connections and compositions, the latter giving groupoid structures. The internal hom functor for cubical sets developed in Section 11.2.iii generalises immediately to ω-groupoids as follows.[241]

Definition 15.1.1. Any ω-groupoid G has an underlying cubical set and we have given in Definition 11.2.10 the n-fold left path cubical set P^nG. It is

$$(P^nG)_r = G_{n+r},$$

with cubical operators

$$\partial^\alpha_{n+1}, \partial^\alpha_{n+2}, \dots, \partial^\alpha_{n+r} \colon (P^nG)_r \to (P^nG)_{r-1}, \quad \alpha = +, -,$$

and

$$\varepsilon_{n+1}, \varepsilon_{n+2}, \dots, \varepsilon_{n+r} \colon (P^nG)_{r-1} \to (P^nG)_r.$$

Now, we can define connections

$$\Gamma_{n+1}, \Gamma_{n+2}, \dots, \Gamma_{n+r-1} \colon (P^nG)_{r-1} \to (P^nG)_r$$

and compositions

$$+_{n+1},\ +_{n+2},\ \dots,\ +_{n+r} \quad \text{on } (P^nG)_r.$$

They make P^nG an ω-groupoid since the laws to be checked are just a subset of the ω-groupoid laws of G. We call P^nG the *n-fold* (*left*) *path ω-groupoid* of G. □

The operators of G not used in P^mG give maps

$$\partial^\alpha_1, \dots, \partial^\alpha_m \colon P^mG \to P^{m-1}G,$$
$$\varepsilon_1, \dots, \varepsilon_m \colon P^{m-1}G \to P^mG,$$
$$\Gamma_1, \dots, \Gamma_{m-1} \colon P^{m-1}G \to P^mG$$

which are morphisms of ω-groupoids and obey the cubical laws. The unused additions of G define partial compositions $+_1, +_2, \ldots, +_m$ on $P^m G$ which, by the ω-groupoid laws for G, are compatible with the ω-groupoid structure of $P^m G$.

Definition 15.1.2. The *'internal hom' ω-groupoid* $\omega\text{-}\mathsf{GPDS}(G, H)$ is defined for any ω-groupoids G, H by

$$\omega\text{-}\mathsf{GPDS}_m(G, H) = \omega\text{-}\mathsf{Gpds}(G, P^m H),$$

with cubical operators

$$\partial_1^\alpha, \ldots, \partial_m^\alpha \colon \omega\text{-}\mathsf{GPDS}(G, H)_m \to \omega\text{-}\mathsf{GPDS}(G, H)_{m-1},$$
$$\varepsilon_1, \ldots, \varepsilon_m \colon \omega\text{-}\mathsf{GPDS}(G, H)_{m-1} \to \omega\text{-}\mathsf{GPDS}(G, H)_m,$$

connections

$$\Gamma_1, \ldots, \Gamma_{m-1} \colon \omega\text{-}\mathsf{GPDS}(G, H)_{m-1} \to \omega\text{-}\mathsf{GPDS}(G, H)_m$$

and compositions

$$+_1, \ \ldots, \ +_m \quad \text{on } \omega\text{-}\mathsf{GPDS}(G, H)_m$$

all induced by the similarly numbered operations on H.

We make $\omega\text{-}\mathsf{GPDS}(G, H)$ a functor in G and H (contravariant in G) in the obvious way: if $g \colon G \to G'$ and $h \colon H \to H'$ are morphisms, the corresponding morphism

$$\omega\text{-}\mathsf{GPDS}(g, h) \colon \omega\text{-}\mathsf{GPDS}(G, H) \to \omega\text{-}\mathsf{GPDS}(G', H')$$

is given, in dimension r, by

$$\omega\text{-}\mathsf{GPDS}(g, h)_r(f) = (P^r h) \circ f \circ g,$$

for each $f \colon G \to P^r H$. □

Remark 15.1.3. Thus in dimension 0, $\omega\text{-}\mathsf{GPDS}(G, H)$ consists of all morphisms $G \to H$, while in dimension n it consists of *n-fold* (*left*) *homotopies* $G \to H$.

The definition of tensor products of ω-groupoids is harder. We require that $- \otimes G$ be left adjoint to $\omega\text{-}\mathsf{GPDS}(G, -)$ as a functor from $\omega\text{-}\mathsf{GPDS}$ to $\omega\text{-}\mathsf{GPDS}$, and this determines $\otimes$ up to natural isomorphism.

One way of getting the tensor product is using the power of generalities, because the representability of the functor $\omega\text{-}\mathsf{GPDS}(F, \omega\text{-}\mathsf{GPDS}(G, -))$ can be asserted on general grounds. The point is that $\omega\text{-}\mathsf{GPDS}$ is an equationally defined category of many sorted algebras in which the domains of the operations are defined by finite limit diagrams. General theorems on such algebraic categories imply that $\omega\text{-}\mathsf{GPDS}$ is complete and cocomplete and that it is monadic over the category Cub of cubical sets.[242]

We are going to follow an alternative path strengthening the bicubical maps of Section 11.2.i to bimorphisms. The definition requires for any ω-groupoid H the *transposition* TH (see Definition 11.2.19): here we just say that TH has the same elements as H but has its cubical operations, connections and compositions numbered in reverse order.

Definition 15.1.4. For any ω-groupoids F, G, H a *bimorphism* $f: (F, G) \to H$ is a family of maps

$$f_{pq}: F_p \times G_q \to H_{p+q} \quad (p, q \geqslant 0)$$

such that

(i) for each $x \in F_p$, the map

$$f_x = f(x, -): G \to P^p H$$

given by $y \mapsto f(x, y)$ is a morphism of ω-groupoids;

(ii) for each $y \in G_q$ the map

$$f_y = f(-, y): F \to T(P^q)TH$$

given by $x \mapsto f(x, y)$ is a morphism of ω-groupoids. □

These bimorphisms may be reinterpreted in terms of morphisms.

Proposition 15.1.5. *There is a natural one-one correspondence between*

1. *bimorphisms* $(F, G) \to H$*, and*
2. *morphisms* $f: F \to \omega\text{-}\mathsf{GPDS}(G, H)$.

Proof. The conditions in the definition of a bimorphism from (F, G) to H, may be interpreted as saying that condition (i) gives maps $F_p \to \omega\text{-}\mathsf{GPDS}_p(G, H)$ for each p, and condition (ii) states that these combine to give a morphism of ω-groupoids $F \to \omega\text{-}\mathsf{GPDS}(G, H)$. □

Definition 15.1.6. We define the *ω-groupoid tensor product* $F \otimes G$ as given by the bimorphism

$$\chi: (F, G) \to F \otimes G$$

universal with respect to bimorphisms $(F, G) \to H$. We shall denote $\chi(x, y)$ by $x \otimes y$.

The universality condition says of course that every bimorphism $f: (F, G) \to H$ factors uniquely as $(x, y) \mapsto \hat{f}(x \otimes y)$ where $\hat{f}: F \otimes G \to H$ is a morphism of ω-groupoids. □

Proposition 15.1.7. *The tensor product is associative: i.e. for all ω-groupoids E, F, G there is a natural isomorphism*

$$(E \otimes F) \otimes G \cong E \otimes (F \otimes G).$$

Proof. Both sides of the above equation are determined by a universal property with respect to 'trimorphisms' from E, F, G. □

Remark 15.1.8. We do not delve into the question of coherence for the monoidal closed structures considered in this book. One reason is lack of space. The other reason is that because tensor products are universal for bimorphisms, the coherence properties for the tensor product derive ultimately from the coherence properties of various cartesian products, and coherence there derives from the universal properties of that product. This relationship seems not to have been studied abstractly. The subject of coherence is standard in many references on category theory. □

Proposition 15.1.9 (Exponential law for ω-groupoids). *For any ω-groupoid G, the functor ω-$\mathsf{GPDS}(G, -)$ is right adjoint to the functor $- \otimes G$; so there are bijections*

$$\omega\text{-}\mathsf{Gpds}(F \otimes G, H) \cong \omega\text{-}\mathsf{Gpds}(F, \omega\text{-}\mathsf{GPDS}(G, H))$$

natural with respect to ω-groupoids F, G, H.

Proof. We get the bijection just by putting together the previous definitions and the universality condition. □

This proposition can be strengthened in a standard way:

Proposition 15.1.10. *For ω-groupoids F, G, H there is a natural equivalence*

$$\omega\text{-}\mathsf{GPDS}(F \otimes G, H) \cong \omega\text{-}\mathsf{GPDS}(F, \omega\text{-}\mathsf{GPDS}(G, H)).$$

Proof. We can use Proposition 15.1.9 repeatedly and the associativity of the tensor product to give for any ω-groupoid E a natural isomorphism

$$\omega\text{-}\mathsf{Gpds}(E, \omega\text{-}\mathsf{GPDS}(F \otimes G, H)) \cong \omega\text{-}\mathsf{Gpds}(E, \omega\text{-}\mathsf{GPDS}(F, \omega\text{-}\mathsf{GPDS}(G, H))).$$

The result follows. □

We will show in Section 15.4 that the tensor product of ω-groupoids is symmetric, although the isomorphism $G \otimes H \cong H \otimes G$ is not an obvious one.

We now show that, as in the tensor product of R-modules, the tensor product for ω-groupoids may also be given by a presentation.

We may specify an ω-groupoid by a *presentation*, that is, by giving a set of generators in each dimension and a set of defining relations of the form $u = v$, where u, v are well-formed formulae of the same dimension made from generators and the operators $\partial_i^\alpha, \varepsilon_i, \Gamma_i, +_i, -_i$.

Given ω-groupoids F, G, we now give an alternative, but equivalent, definition of $F \otimes G$ by giving a presentation of it as an ω-groupoid. The universal property of the presentation will then give the required adjointness.

Definition 15.1.11. Let F, G be ω-groupoids. We define $F \otimes G$ to be the ω-groupoid generated by elements in dimension $n \geqslant 0$ of the form $x \otimes y$ where $x \in F_p,\ y \in G_q$ and $p + q = n$, subject to the following defining relations (plus, of course, the laws for ω-groupoids)

(i) $\partial_i^\alpha(x \otimes y) = \begin{cases} (\partial_i^\alpha x) \otimes y & \text{if } 1 \leqslant i \leqslant p, \\ x \otimes (\partial_{i-p}^\alpha y) & \text{if } p+1 \leqslant i \leqslant n; \end{cases}$

(ii) $\varepsilon_i(x \otimes y) = \begin{cases} (\varepsilon_i x) \otimes y & \text{if } 1 \leqslant i \leqslant p+1, \\ x \otimes (\varepsilon_{i-p} y) & \text{if } p+1 \leqslant i \leqslant n+1; \end{cases}$

(iii) $\Gamma_i(x \otimes y) = \begin{cases} (\Gamma_i x) \otimes y & \text{if } 1 \leqslant i \leqslant p, \\ x \otimes (\Gamma_{i-p} y) & \text{if } p+1 \leqslant i \leqslant n; \end{cases}$

(iv) $(x +_i x') \otimes y = (x \otimes y) +_i (x' \otimes y)$ if $1 \leqslant i \leqslant p$ and $x +_i x'$ is defined in F;

(v) $x \otimes (y +_j y') = (x \otimes y) +_{p+j} (x \otimes y')$ if $1 \leqslant j \leqslant q$ and $y +_j y'$ is defined in G. □

Remark 15.1.12. There are quite a few relations that can be deduced from this definition, for example:

(vi) $-_i(x \otimes y) = \begin{cases} (-_i x) \otimes y & \text{if } 1 \leqslant i \leqslant p, \\ x \otimes (-_{i-p} y) & \text{if } p+1 \leqslant i \leqslant n \end{cases}$

and

(vii) $(\varepsilon_{p+1} x) \otimes y = x \otimes (\varepsilon_1 y)$. □

15.1.i Relations between the internal homs for cubes and for ω-groupoids

We now use the free ω-groupoid ρK on a cubical set K, which gives the left adjoint

$$\rho \colon \mathsf{Cub} \to \omega\text{-}\mathsf{Gpds}$$

to the forgetful functor

$$U \colon \omega\text{-}\mathsf{Gpds} \to \mathsf{Cub},$$

to relate the monoidal closed structures of Cub and ω-Gpds. This will enable us to tie in the theory with results in Sections 11.2, 11.4 on the nerve of a crossed complex.

It is easy to see that $\rho(K)$ is the ω-groupoid generated by elements $[k]$ for all $k \in K$ with defining relations given by $\partial_i^\alpha[k] = [\partial_i^\alpha k]$ and $\varepsilon_i[k] = [\varepsilon_i k]$ for all $n \geqslant 1$ and face and degeneracy maps $\partial_i^\alpha \colon K_n \to K_{n-1}$ and $\varepsilon_i \colon K_{n-1} \to K_n$.

This notation is consistent with our previous use of $\rho(K)$ as the fundamental ω-groupoid of the filtered space $|K_*|$ because, for any cubical set K, $\rho(K) \cong \rho(|K_*|)$, by the HHSvKT, as a deduction from Theorem 14.3.1. In particular we will write $\mathbf{I}^n$ for the ω-groupoid $\rho(\mathbb{I}^n)$, which is also the free ω-groupoid on one generator of dimension n.

Proposition 15.1.13. *For a cubical set L and an ω-groupoid G, there is a natural isomorphism of cubical sets*

$$U(\omega\text{-}\mathsf{GPDS}(\rho(L), G)) \cong \mathsf{CUB}(L, UG).$$

Proof. Let us get first the bijections at every dimension, i.e.

$$\omega\text{-}\mathsf{GPDS}_r(\rho(L), G) \cong \mathsf{CUB}_r(L, UG)$$

for all $r \geqslant 0$.

They follow from the adjointness since the bijections

$$\omega\text{-}\mathsf{GPDS}_r(\rho(L), G) = \omega\text{-}\mathsf{Gpds}(\rho(L), P^r G) \cong \mathsf{Cub}(L, UP^r G) = \mathsf{CUB}_r(L, UG)$$

are compatible with the cubical operators. □

From this proposition we easily deduce that the free ω-groupoid functor preserves the tensor product.

Proposition 15.1.14. *If K, L are cubical sets, there is a natural isomorphism of ω-groupoids*

$$\rho K \otimes \rho L \cong \rho(K \otimes L).$$

Proof. From the previous Proposition 15.1.13 and the closed category structures of Cub and ω-Gpds, we get the bijection of cubical sets

$$\begin{aligned}
&U(\omega\text{-}\mathsf{GPDS}(\rho(K \otimes L), G)) \\
&\quad\cong \mathsf{CUB}(K \otimes L, UG) && \text{by 15.1.13} \\
&\quad\cong \mathsf{CUB}(K, \mathsf{CUB}(L, UG)) && \text{since Cub is monoidal closed} \\
&\quad\cong \mathsf{CUB}(K, U(\omega\text{-}\mathsf{GPDS}(\rho(L), G))) && \text{by 15.1.13} \\
&\quad\cong U(\omega\text{-}\mathsf{GPDS}(\rho(K), \omega\text{-}\mathsf{GPDS}(\rho(L), G))) && \text{by 15.1.13} \\
&\quad\cong U(\omega\text{-}\mathsf{GPDS}(\rho(K) \otimes \rho(L), G)) && \text{since } \omega\text{-Gpds is monoidal closed.}
\end{aligned}$$

This natural bijection gives in dimension 0

$$\omega\text{-}\mathsf{Gpds}(\rho(K \otimes L), G) \cong \omega\text{-}\mathsf{Gpds}(\rho(K) \otimes \rho(L), G)$$

from which the proposition follows. □

We get as a consequence the following relation among the ω-groupoids $\rho(\mathbb{I}^n)$ freely generated by one element in dimension n, $n \geqslant 0$.

Corollary 15.1.15. *There are natural isomorphisms of ω-groupoids*

$$\rho(\mathbb{I}^m) \otimes \rho(\mathbb{I}^n) \cong \rho(\mathbb{I}^{m+n}).$$

Proposition 15.1.16. (i) $\rho(\mathbb{I}^n) \otimes -$ *is left adjoint to* $P^n\colon\ \omega\text{-}\mathsf{GPDS} \to \omega\text{-}\mathsf{GPDS}$.
(ii) $- \otimes \rho(\mathbb{I}^n)$ *is left adjoint to* $\omega\text{-}\mathsf{GPDS}(\rho(\mathbb{I}^n), -)$.
(iii) $\omega\text{-}\mathsf{GPDS}(\rho(\mathbb{I}^n), -)$ *is naturally isomorphic to* TP^nT.

Proof. (i) There are natural bijections

$$\begin{aligned}\omega\text{-}\mathsf{Gpds}(\rho(\mathbb{I}^n) \otimes H, K) &\cong \omega\text{-}\mathsf{Gpds}(\rho(\mathbb{I}^n), \omega\text{-}\mathsf{GPDS}(H, K))\\ &\cong \omega\text{-}\mathsf{GPDS}_n(H, K) = \omega\text{-}\mathsf{Gpds}(H, P^n K).\end{aligned}$$

(ii) This is a special case of Proposition 15.1.9.

(iii) It follows from (i) that $TP^nT\colon \omega\text{-}\mathsf{GPDS} \to \omega\text{-}\mathsf{GPDS}$ has left adjoint $T(\rho(\mathbb{I}^n) \otimes T(-)) \cong - \otimes T\rho(\mathbb{I}^n)$. But the obvious isomorphism $T\mathbb{I} \to \mathbb{I}$ induces an isomorphism $T\rho(\mathbb{I}^n) \cong \rho(\mathbb{I}^n)$, so $- \otimes T\rho(\mathbb{I}^n)$ is naturally isomorphic to $- \otimes \rho(\mathbb{I}^n)$. The result now follows from (ii). □

Remark 15.1.17. It was proved in Section 14.6 that $\rho(\mathbb{I}^n)$ is the fundamental ω-groupoid $\rho(I^n_*)$ of the n-cube with its skeletal filtration. We will show, by similar methods, that for any cubical set K, there is a natural isomorphism $\rho(K) \cong \rho(|K_*|)$, where $|K_*|$ is the geometric realisation of K, with its skeletal filtration. Thus Proposition 15.1.14 gives an isomorphism

$$\rho(|K_*| \otimes |L_*|) \cong \rho(|K_*|) \otimes \rho(|L_*|)$$

which will be generalised in Corollary 15.3.3 to an isomorphism

$$\rho(X_*) \otimes \rho(Y_*) \cong \rho(X_* \otimes Y_*)$$

for arbitrary CW-complexes X, Y. □

15.2 The monoidal closed structure on crossed complexes revisited

It is an easy exercise to prove that given a monoidal closed category C and an equivalent category C', we can use the equivalence to transfer the closed category structure from C to C'. Thus the monoidal closed structure defined on $\omega\text{-}\mathsf{GPDS}$ in Section 15.1 can be transferred to the category Crs by defining for arbitrary crossed complexes C and D

$$C \otimes D = \gamma(\lambda C \otimes \lambda D) \quad \text{and} \quad \mathsf{CRS}(C, D) = \gamma(\omega\text{-}\mathsf{GPDS}(\lambda C, \lambda D)).$$

Remark 15.2.1. There is one aspect of the notion of monoidal categories which should in principle be given more coverage than we are giving, namely the various coherence laws which are part of the standard definition, see Section C.7 and Chapter VII in [ML71]. These laws will not be important for our purposes, and so we leave their investigation in our cases to the reader. Because the tensor product is defined in the various cases by a universal property of 'bi'-morphisms of various types, coherence properties may be deduced from those for the usual cartesian product of sets, where coherence follows from the universal property. An example on p. 160 of [ML71] shows that the cartesian product cannot always be taken to give a strict monoidal structure.[243] □

Our goal in this section is to derive our monoidal closed structure on the category Crs from that on ω-groupoids and so arrive at the definitions already given in Section 9.3.

We begin with the translation of the internal hom functor; this can be done explicitly because the internal hom is defined by families of functions satisfying certain conditions. Then we translate the concept of bimorphism since that is essentially a 'morphism of morphisms'. The definition of the tensor product is not so explicit, since it is given in terms of a presentation, which makes it difficult to identify the elements of a tensor product.

The difficulty in passing from presentations in ω-GPDS to presentations in Crs may be illustrated by the example $\rho(\mathbb{I}^n)$. In ω-GPDS, this is free on one generator in dimension n; however, the corresponding crossed complex $\gamma\rho(\mathbb{I}^n) \cong \Pi(I^n_*)$ requires, for each r-dimensional face d of I^n, a generator $x(d)$ in dimension r, with defining relations of the form

$$\delta(x(d)) = \sum_{(\alpha,i)} \{x(\partial_i^\alpha d)\},$$

where the formula for the 'sum of the faces' on the right is given by the Homotopy Addition Lemma 13.7.1.

15.2.i The internal hom on crossed complexes

As we have seen we could define

$$\mathsf{CRS}(C, D) = \gamma(\omega\text{-}\mathsf{GPDS}(\lambda C, \lambda D))$$

for any crossed complexes C, D and get a closed category structure on Crs. We want to describe the structure of $\mathsf{CRS}(C, D)$ in terms internal to the crossed complexes C, D and arrive at the definition of left (or right) m-fold homotopy for crossed complexes given in Definition 9.3.3, i.e. a pair (F, f) where f is a morphism of crossed complexes and F has degree m over f satisfying some conditions.

So, for two ω-groupoids G, H we have to study $\gamma(\omega\text{-}\mathsf{GPDS}(G, H))_m$, whose elements are m-fold homotopies of ω-groupoids which satisfy an extra degeneracy condition (almost all faces are degenerate). Thus we need to examine such homotopies.

The main technical tool for changing a cube to another one with extra degeneracies is the folding map Φ. Thus we are going to use the folding map to relate both kinds of m-fold homotopies.

Proposition 15.2.2. *Let G, H be ω-groupoids, let $\psi\colon G \to H$ be an m-fold left homotopy. We may define*

(i) *a morphism of crossed complexes*

$$f\colon \gamma G \to \gamma H$$

defined by $f = \partial_1^+ \partial_2^+ \dots \partial_m^+ \psi$;

(ii) *a homotopy over f,*

$$F\colon \gamma G \to \gamma H,$$

given by $F = \Phi\psi$.

This m-fold left homotopy of crossed complexes (F, f) is said to be associated to ψ.

Proof. The part (i) is clear since

$$\partial_1^+ \partial_2^+ \dots \partial_m^+ \psi\colon G \to H$$

is a morphism of ω-groupoids. Thus it maps γG to γH and restricts to a morphism

$$f\colon \gamma G \to \gamma H$$

of crossed complexes.

Part (ii) is much longer since we have to check all conditions for a homotopy in Definition 9.3.3.

Let us begin with the *base point*. Let us see that F is a map over f. For any $c \in (\gamma G)_n$, the base point is

$$\begin{aligned}\beta F(c) = \beta\Phi\psi(c) = \beta\psi(c) &= \partial_1^+ \partial_2^+ \dots \partial_{m+n}^+ \psi(c)\\ &= \partial_1^+ \dots \partial_m^+ \psi(\partial_1^+ \dots \partial_n^+ c) = f(\beta c).\end{aligned}$$

Thus $F(c) \in f_0\beta(c)$.

The other conditions for (F, f) to be a homotopy follow from the formulae for $\Phi(x +_i y)$ in Proposition 13.4.14.

First the *operations*. Recall that in previous notation, for a k-dimensional cube x,

$$u_i x = \partial_1^+ \dots \partial_{i-1}^+ \partial_{i+1}^+ \dots \partial_k^+ x.$$

• If $c + c'$ is defined in $(\gamma G)_1 = G_1$, then

$$\begin{aligned}F(c + c') = \Phi\psi(c +_1 c') &= \Phi(\psi(c) +_{m+1} \psi(c'))\\ &= (\Phi\psi(c))^u + \Phi\psi(c') = h(c)^u + h(c'),\end{aligned}$$

where $u = u_{m+1}\psi(c') = \partial_1^+ \dots \partial_m^+ \psi(c') = f(c')$.

• Similarly, if $n \geqslant 2$ and $c + c'$ is defined in $(\gamma G)_n$, then

$$F(c + c') = F(c)^u + F(c'),$$

where

$$u = u_{m+n}\psi(c') = \partial_1^+ \dots \partial_m^+ \partial_{m+1}^+ \dots \partial_{m+n-1}^+ \psi(c') = \partial_1^+ \dots \partial_m^+ \psi(\partial_1^+ \dots \partial_{n-1}^+ c').$$

But since $c' \in (\gamma G)_n$, the element $\partial_1^+ \dots \partial_{n-1}^+ c'$ of $(\gamma G)_1$ is the identity element $\varepsilon_1 \beta c'$; so $u = f(\varepsilon_1 \beta c')$ is also an identity element and $F(c + c') = F(c) + F(c')$.

Now the *action*. If c^t is defined, where $c \in (\gamma G)_n$ $(n \geqslant 2)$ and $t \in (\gamma G)_1$, then

$$\begin{aligned} F(c^t) = \Phi\psi(c^t) &= \Phi\psi(-_n \varepsilon_1^{n-1} t +_n c +_n \varepsilon_1^{n-1} t) \\ &= -_{m+n}\varepsilon_{m+1}^{n-1}\psi(t) +_{m+n} \psi(c) +_{m+n} \varepsilon_{m+1}^{n-1}\psi(t) \\ &= -(\Phi\varepsilon_{m+1}^{n-1}\psi(t))^u + (\Phi\psi(c))^v + \Phi\varepsilon_{m+1}^{n-1}\psi(t) \end{aligned}$$

for certain edges $u, v \in (\gamma H)_1$.

But $n \geqslant 2$, so $\varepsilon_{m+1}^{n-1}\psi(t)$ is degenerate and $\Phi\varepsilon_{m+1}^{n-1}\psi(t) = 0$ for Proposition 13.4.18. Hence

$$F(c^t) = F(c)^v,$$

where $v = u_{m+n}(\varepsilon_{m+1}^{n-1}\psi(t)) = \partial_1^+ \dots \partial_{m+n-1}^+ \varepsilon_{m+1}^{n-1}\psi(t) = \partial_1^+ \dots \partial_m^+ \psi(t) = f(t)$ giving the result. □

Remark 15.2.3. Notice that given an m-fold left homotopy $\psi : G \to H$ of ω-groupoids, the m-fold left homotopy of crossed complexes associated to this, (F, f), satisfies an extra condition with respect to the folding map, namely:

$$F(\Phi x) = \Phi\psi(\Phi x) = \Phi\psi(\Phi_1 \dots \Phi_{n-1} x) = \Phi\Phi_{m+1} \dots \Phi_{m+n-1}\psi(x) = \Phi\psi(x)$$

using Proposition 13.4.15. We call this extra condition

(Fold) $F(\Phi x) = \Phi\psi(x).$ □

So we have associated to any m-fold left homotopy between ω-groupoids an m-fold left homotopy between the associated crossed complexes satisfying the extra condition (Fold). Now we prove that the former homotopy between ω-groupoids may be reconstructed from the homotopy between the associated crossed complexes.

Proposition 15.2.4. *Let G, H be ω-groupoids, and F be any m-fold left homotopy from γG to γH beginning at f. Then there is a unique m-fold left homotopy $\psi : G \to H$ such that F is the associated homotopy and satisfies the extra condition about degeneration of the faces*

(Deg) $\partial_i^\alpha \psi(x) = \varepsilon_1^{m-1} \hat{f}(x)$

for $1 \leqslant i \leqslant m$, $\alpha = 0, 1$ and $(\alpha, i) \neq (0, 1)$ and all $x \in G$, where $\hat{f} : G \to H$ denotes the unique morphism of ω-groupoids extending the morphism $f : \gamma G \to \gamma H$ of crossed complexes.

Proof. We are looking for the existence and uniqueness of an m-fold left homotopy $\psi : G \to H$ having F as associated homotopy and satisfying the extra conditions

(Deg) $\partial_i^\alpha \psi(x) = \varepsilon_1^{m-1} \hat{f}(x)$ for $i \leqslant m$, $(\alpha, i) \neq (0,1)$, and

(Fold) $\Phi\psi(x) = F(\Phi x)$.

Using these conditions we construct ψ inductively.

When $n = 0$, all faces but one of $\psi(x)$ are specified by (Deg). The elements $z_i^\alpha = \varepsilon_1^{m-1} \hat{f}(x) = \varepsilon_1^{m-1} f(x)$ of H_{m-1} for $(\alpha, i) \neq (0,1)$ form a box and the Homotopy Addition Lemma (13.7.1) gives a unique last face z_1^- such that $\delta\Phi\mathbf{z} = \Sigma\mathbf{z}$ has the value $\delta F(\Phi x) \in (\gamma H)_{m-1}$. Proposition 13.5.10 then gives a unique filler $\psi(x)$ for the box such that $\Phi(\psi(x))$ has the value $F(\Phi x)$. (Of course, one must verify that $\delta F(\Phi x) = \delta F(x)$ has the same basepoint as the given box, but this is clear since $\beta F(x) = \beta f(x)$).

Now suppose that $n \geqslant 1$ and assume that $\psi(x)$ is already defined for all x of dimension $< n$ and that it satisfies (Deg) and (Fold) for all such x. Assume further that ψ satisfies all the conditions for an m-fold left homotopy in so far as they apply to elements of dimension $< n$.

Then, for $x \in G_n$ we need to find $\psi(x) \in H_{m+n}$ satisfying (amongst others) the conditions

$$\begin{cases} \partial_j^\alpha \psi(x) = \varepsilon_1^{m-1} \hat{f}(x) & \text{for } 1 \leqslant j \leqslant m,\ (\alpha, j) \neq (0,1), \\ \partial_{m+j}^\alpha \psi(x) = \psi(\partial_j^\alpha x) & \text{for } 1 \leqslant j \leqslant n, \\ \Phi\psi(x) = F(\Phi x). \end{cases} \tag{15.2.1}$$

It is straightforward to verify that the specified faces of $\psi(x)$ form a box whose basepoint is $t\hat{f}(x) = f(\Phi x) = F(\Phi x)$ and therefore, as in the case $n = 0$, there is a unique $\psi(x)$ satisfying these conditions.

To complete the induction we have only to verify that this $\psi(x)$ has all the defining properties of an m-fold homotopy.

For example, to prove that

$$\psi(x +_i y) = \psi(x) +_{m+i} \psi(y), \tag{15.2.2}$$

we first note that $\partial_{m+i}^+ \psi(x) = \psi(\partial_i^+ x) = \psi(\partial_i^- y) = \partial_{m+i}^- \psi(y)$ so that

$$z = \psi(x) +_{m+i} \psi(y) \tag{15.2.3}$$

is defined. We then verify easily, using the induction hypotheses, that the faces of z other than $\partial_1^- z$ are given by

$$\begin{cases} \partial_j^\alpha z = \varepsilon_1^{m-1} \hat{f}(x +_i y) & \text{for } 1 \leqslant j \leqslant m,\ (\alpha, j) \neq (0,1), \\ \partial_{m+j}^\alpha z = \psi(\partial_j^\alpha (x +_i y)) & \text{for } 1 \leqslant j \leqslant n. \end{cases}$$

Also

$$\Phi z = \Phi(\psi(x) +_{m+i} \psi(y)) = (\Phi\psi(x))^u + \Phi\psi(y),$$

by Proposition 13.4.14, where

$$u = u_{m+i}\psi(y) = \partial_1^+ \dots \partial_m^+ \psi(u_i y) = \hat{f}(u_i y) = f(u_i y).$$

But it may be verified that

$$F(\Phi(x +_i y)) = F(\Phi x)^{f(u_i y)} + F(\Phi y)$$

using the defining properties of F and formulae of Proposition 13.4.14. (In the case $n = 1, i = 1$ one needs to observe that addition in $(\gamma H)_{m+n}$ is commutative). Hence

$$\Phi z = F(\Phi(x +_i y))$$

in all cases.

The uniqueness of $\psi(x)$ satisfying conditions (15.2.1) implies that

$$z = \psi(x +_i y),$$

and this proves (15.2.2). The other properties of ψ are proved in a similar way. □

These propositions set up for $m \geqslant 1$ a bijection between m-fold left homotopies $\gamma G \to \gamma H$ and elements of $\gamma(\omega\text{-}\mathsf{GPDS}(G, H))_m$, namely m-fold left homotopies $\psi : G \to H$ which satisfy the extra degeneracy condition:

(Deg) $\partial_i^\alpha \psi(x) = \varepsilon_1^{m-1} \partial_1^+ \partial_2^+ \dots \partial_m^+ \psi(x)$ for $i \leqslant m$, $(\alpha, i) \neq (0, 1)$.

(Note that if $\partial_i^\alpha u = \varepsilon_1^{m-1} v$, then v must be $\partial_1^+ \dots \partial_m^+ u$).

We complete this correspondence by defining a 0-fold left (or right) homotopy of crossed complexes $C \to D$ to be a morphism $f : C \to D$. We then have:

Proposition 15.2.5. *The elements of* $\mathsf{CRS}(C, D)$ *in dimension* $m \geqslant 0$ *are in natural one-one correspondence with the* m*-fold left homotopies from* C *to* D.

In view of this result we will, from now on, identify $\mathsf{CRS}(C, D)$ with the collection of morphisms and left homotopies from C to D. The operations which give this collection the structure of a crossed complex can be deduced from the above correspondence. They will also be described later in internal terms.

15.2.ii Bimorphisms on crossed complexes

Next, we need to relate the concepts of bimorphism of ω-groupoids given in Definition 15.1.4 with that of bimorphism of crossed complexes introduced in Definition 9.3.10. This section is the technical heart of the work on establishing the monoidal closed structure on the category of crossed complexes.

We are going to use extensively the previous section since in both cases a bimorphism may be interpreted by fixing the first variable as a family of m-fold left homotopies one for each element of dimension m (see Definition 15.1.4 and 9.3.10) and we know from the previous section how both kinds of m-fold homotopies are related, essentially by the folding map.

Before entering in the proof of this correspondence let us state a result that will be used later.

Lemma 15.2.6. *If $\chi\colon (F,G) \to H$ is a bimorphism of ω-groupoids, then $\chi(x,y)$ is thin whenever x or y is thin.*

Proof. We have just remarked that $\chi_y\colon F \to P^n H$ is a morphism of ω-groupoids. If x is thin in F, it follows that $\chi(x,y)$ is a thin element of $P^n H$. But the thin elements of $P^n H$ are a subset of the thin elements of H. □

Proposition 15.2.7. *Let F, G, H be ω-groupoids with associated crossed complexes $\gamma F, \gamma G, \gamma H$. If*

$$\chi\colon (F,G) \to H$$

is any bimorphism of ω-groupoids, then we have an associated bimorphism of crossed complexes

$$\xi\colon (\gamma F, \gamma G) \to \gamma H$$

defined by $\xi(c,d) = \Phi\chi(c,d)$ for any $c \in \gamma F$ and $d \in \gamma G$.

Proof. To check that ξ is a bimorphism of crossed modules we have to see the behaviour with respect to source and target, actions and operations and boundary maps is according to Definition 9.3.10.

With respect to the *base point*,

$$\beta\xi(c,d) = \beta\Phi\chi(c,d) = \beta\chi(c,d) = \chi(\beta c, \beta d) = \xi(\beta c, \beta d).$$

With respect to *actions and operations*: For $c \in (\gamma F)_0$, the map $\chi_c\colon G \to H$ is a morphism of ω-groupoids. Thus

$$\Phi\chi_c\colon \gamma G \to \gamma H$$

is a morphism of crossed complexes.

Similarly, by Proposition 15.2.2, if $c \in (\gamma F)_m$ is fixed, then the map $\chi_c\colon G \to H$ is an m-fold left homotopy of ω-groupoids. Thus the map

$$\xi_c = \Phi\chi_c\colon \gamma G \to \gamma H$$

is an m-fold left homotopy $\gamma G \to \gamma H$ over the morphism $\xi_{\beta c} = \Phi\chi_{\beta c}$.

The morphism $\chi_{\beta c}$ maps γG into γH, so $\xi(\beta c, d) = \Phi\chi(\beta c, d) = \chi(\beta c, d)$ and ξ_c is an m-fold homotopy over $\xi_{\beta c}$.

Now we repeat the same process with respect to the second variable. Note that in this version for n-fold right homotopies $\gamma F \to \gamma H$ the formula $f(d) = \partial_1^+ \dots \partial_m^+ \psi(d)$

is replaced by $f(c) = \partial^+_{n+1} \dots \partial^+_{n+m} \psi(c)$. Hence, if $d \in (\gamma G)_n$, the right homotopy $c \mapsto \Phi\chi(c,d)\colon \gamma F \to \gamma H$ has base morphism

$$c \mapsto \partial^+_{n+1} \dots \partial^+_{n+m} \chi(c,d) = \chi(c, \beta d).$$

With respect to *boundary maps* we use the Homotopy Addition Lemma 13.7.1; in order to compute $\delta\xi(c,d) = \delta\Phi\chi(c,d)$ we need to compute $\Phi\partial^\alpha_i \chi(c,d)$ for each face of $\chi(c,d)$ and sum them according to the formulae in Lemma 13.7.1.

To compute $\delta\xi(c,d)$ in the case $m \geqslant 2, n \geqslant 2$ we note that the faces of c and d other than $\partial^-_1 c$, $\partial^-_1 d$ are all thin, so all but two faces of $\chi(c,d)$ are thin by Lemma 15.2.6, and we conclude that $\Phi\partial^\alpha_i \chi(c,d) = 0$ except when $\alpha = 0$ and $i = 1$ or $m+1$. The appropriate formula of the Homotopy Addition Lemma 13.7.1 now gives

$$\begin{aligned}\delta\Phi\chi(c,d)\delta\xi(c,d) &= \delta\Phi\chi(c,d) \\ &= (\Phi\chi(\partial^-_1 c, d))^v + (-1)^m (\Phi\chi(c, \partial^-_1 d))^w \\ &= \xi(\delta c, d)^v + (-1)^m \xi(c, \delta d)^w,\end{aligned}$$

where

$$v = u_1\chi(c,d) = \chi(u_1 c, \beta d) \quad \text{and} \quad w = u_{m+1}\chi(c,d) = \chi(\beta c, u_1 d).$$

Since $c \in \gamma F$, $d \in \gamma G$, both $u_1 c$ and $u_1 d$ are identities, so v, w act trivially and we obtain the formula

$$\delta\xi(c,d) = \xi(\delta c, d) + (-1)^m \xi(c, \delta d).$$

The other formulae of Definition 9.3.10 are proved in the same way using the different forms of the Homotopy Addition Lemma 13.7.1 in various cases. Thus ξ is a bimorphism of crossed complexes. □

Proposition 15.2.8. *Let F, G, H be ω-groupoids with corresponding crossed complexes γF, γG, γH. Given any bimorphism*

$$\xi\colon (\gamma F, \gamma G) \to \gamma H$$

of crossed complexes, there is a unique bimorphism

$$\chi\colon (F, G) \to H$$

of ω-groupoids satisfying $\xi(c,d) = \Phi\chi(c,d)$ for $c \in \gamma F$ and $d \in \gamma G$.

Proof. For each $c \in (\gamma F)_m$ we have an m-fold left homotopy

$$(\xi_c, f_c)\colon \gamma G \to \gamma H.$$

By Proposition 15.2.4, there is a unique m-fold left homotopy

$$\psi_c\colon G \to H$$

satisfying the conditions

$$\begin{cases} \Phi\psi_c(d) = \xi_c(d) = \xi(c, d) & \text{for } d \in D, \\ \psi_c \in \gamma(\omega\text{-}\mathsf{GPDS}(G, H)). \end{cases} \tag{$**$}$$

The required bimorphism χ must yield such an n-fold left homotopy $y \mapsto \chi(c, y)$, so the definition $\chi(c, y) = \psi_c(y)$ is forced. Furthermore, since γF generates F as ω-groupoid by Proposition 13.5.13 and $\chi(x, y)$ must preserve first variable x, for fixed y, the values $\chi(c, y)$ for $c \in \gamma F$, $y \in G$ determine χ completely. Thus χ is unique if it exists.

To prove that the required bimorphism χ exists we first note that we have a map $c \mapsto \psi_c$ from γF to $\gamma(\omega\text{-}\mathsf{GPDS}(G, H))$ of degree 0 and we will show that it is a morphism of crossed complexes where the crossed complex structure of $\gamma(\omega\text{-}\mathsf{GPDS}(G, H))$ has been given in Definition 9.3.5

We need to show that $\psi_{c+c'} = \psi_c + \psi_{c'}$, $\psi_{c^t} = \psi_c^{\psi_t}$, $\psi_{\delta c} = \delta\psi_c$ if $c \in (\gamma F)_m (m \geqslant 2)$, and $\psi_{\delta^\alpha c} = \delta^\alpha \psi_c$ if $c \in (\gamma F)_1$. Using the characterisation $(**)$ of ψ_c and the fact that $\psi_c + \psi_{c'}$, $\psi_c^{\psi_t}$, etc. are all elements of $\gamma(\omega\text{-}\mathsf{GPDS}(G, H))$, it is enough to prove that, for $d \in \gamma G$,

(i) $\Phi(\psi_c(d) +_m \psi_{c'}(d)) = \xi(c + c', d)$ if $c + c'$ is defined in $(\gamma F)_m$,

(ii) $\Phi(-_m \varepsilon_1^{m-1}\psi_t(d) +_m \psi_c(d) +_m \varepsilon_1^{m-1}\psi_t(d)) = \xi(c^t, d)$ if $t \in A_1$ and c^t is defined in $(\gamma F)_m (m \geqslant 2)$,

(iii) $\Phi(\partial_1^- \psi_c(d)) = \xi(\delta c, d)$ if $c \in (\gamma F)_m$, $m \geqslant 2$,

(iv) $\Phi(\partial_1^\alpha \psi_c(d)) = \xi(\delta^\alpha c, d)$ if $c \in (\gamma F)_1$, $\alpha = \pm$.

The calculations for (i) and (ii) are similar to calculations done in the proof of Proposition 15.2.4. For example, in (ii), if $c \in (\gamma F)_m$, $d \in (\gamma G)_n$, then $\Phi(\varepsilon_1^{m-1}\psi_t(d)) = 0$, so

$$\Phi(-_m \varepsilon_1^{m-1}\psi_t(d) +_m \psi_c(d) +_m \varepsilon_1^{m-1}\psi_t(d)) = (\Phi\psi_c(d))^v = \xi(c, d)^v$$

where

$$\begin{aligned} v = u_m \varepsilon_1^{m-1}\psi_t(d) &= \partial_1^+ \dots \partial_{m-1}^+ \partial_{m+1}^+ \dots \partial_{m+n}^+ \varepsilon_1^{m-1}\psi_t(d) \\ &= \partial_2^+ \dots \partial_{n+1}^+ \psi_t(d) = \psi_t(\partial_1^+ \dots \partial_n^+ b) \\ &= \psi_t(\beta d) = \xi(t, \beta d) \quad \text{(since } \Phi = \text{id in dimension 1).} \end{aligned}$$

Hence $\xi(c, d)^v = \xi(c, d)^{\xi(t, \beta d)} = \xi(c^t, d)$ since $c \mapsto \xi(c, d)$ is an n-fold right homotopy with base morphism $c \mapsto \xi(c, d)$.

The calculations for (iii) and (iv) use the Homotopy Addition Lemma 13.7.1 and the behaviour of ξ with respect to the boundary map. For example, to prove (iii) we observe that $\Phi\psi_c(d) = \xi(c, d)$ and $\delta\Phi\psi_c(d) = \Sigma\{\Phi\partial_i^\alpha \psi_c(d)\}$, the sum of the folded faces on the right being calculated by the appropriate formula of the Homotopy Addition Lemma 13.7.1, depending on the dimensions of c and d. Now $c \in \gamma F$ and $d \in \gamma G$ so most terms in this sum are 0. In the case $m \geqslant 2$, $n \geqslant 2$, two terms survive and one of these, $\Phi\partial^-_{m+1}\psi_c(d)$, we can calculate: because ψ_c is an m-fold left homotopy of ω-groupoids, $\Phi\partial^-_{m+1}\psi_c(d) = \Phi\psi_c(\partial^-_1 d) = \xi(c, \delta d)$. Hence the Homotopy Addition Lemma 13.7.1 says

$$\delta\xi(c,d) = \Phi\partial^-_1\psi_c(d) + (-1)^m\xi(c,\delta d).$$

Comparing this with the defining property

$$\delta\xi(c,d) = \xi(\delta c,d) + (-1)^m\xi(c,\delta d)$$

we obtain (iii). The other cases are similar. This proves that $c \mapsto \psi_c$ is a morphism of crossed complexes from γF to $\gamma(\omega\text{-}\mathsf{GPDS}(G,H))$.

It therefore extends uniquely to a morphism of ω-groupoids $x \mapsto \psi_x$, say, from F to $\omega\text{-}\mathsf{GPDS}(G,H)$. But now the definition $\chi(x,y) = \psi_x(y)$ gives a bimorphism of ω-groupoids $\chi\colon (F,G) \to H$ such that $\Phi\chi(c,d) = \Phi\psi_c(d) = \xi(c,d)$ for $c \in \gamma F$, $d \in \gamma G$, and this completes the proof. □

15.2.iii The tensor product of crossed complexes

Last, we want to describe tensor products of crossed complexes. Let C, D be crossed complexes. If we choose ω-groupoids F, G such that $C = \gamma F$, $D = \gamma G$, we should have

$$C \otimes D = \gamma(F \otimes G).$$

If we consider the universal bimorphism of ω-groupoids

$$\chi\colon (F,G) \to F \otimes G,$$

it is clear that the bimorphism of crossed complexes

$$\xi\colon (C,D) \to C \otimes D$$

given by the restriction of the composition $\Phi\chi$ is universal with respect to bimorphisms of crossed complexes from (C,D).

By the universality of the bimorphism of crossed complexes

$$\xi\colon (C,D) \to \gamma(F \otimes G),$$

it is clear that $-\otimes D$ is the left adjoint to $\mathsf{Crs}(D,-)$.

A warning about notation. For any $c \in C = \gamma F$ and $d \in D = \gamma G$, we have already defined their tensor product by

$$c \otimes d = \chi(c,d) \in F \otimes G.$$

Clearly we have good reason for writing also

$$c \otimes d = \xi(c, d) \in C \otimes D.$$

We shall keep $c \otimes d$ for this last definition, while writing $c \hat{\otimes} d$ for the tensor product element in $F \otimes G$.

The Definition 9.3.10 of a bimorphism now gives the presentation of $C \otimes D$ described in Definition 9.3.12.

This completes the derivation of the monoidal closed structure on the category Crs.

15.2.iv Another description of the internal hom in Crs

We now go back to $\mathsf{CRS}(C, D)$ and produce a description of its crossed complex structure in terms of the crossed complex structures of C and D.

Recall from Definition 7.1.12 that $\mathbb{F}(m)$ is the crossed complex freely generated by one generator a in dimension m. Any element of $\mathsf{CRS}_m(C, D)$ corresponds to a morphism $\mathbb{F}(m) \to \mathsf{CRS}(C, D)$, or, equivalently, to a bimorphism $\xi\colon (\mathbb{F}(m), C) \to D$. If $m = 0$ the given element is the morphism

$$\psi_a\colon C \to D$$

defined by $\psi_a(c) = \xi(a, c)$.

If $m \geqslant 1$ then $\psi_a(c) = \xi(a, c),\ f_a(c) = \xi(\beta a, c)$ defines the m-fold left homotopy $\psi_a = (\psi_a, f_a)$.

Similarly, if two elements of $\mathsf{CRS}(C, D)$ are given, we may choose A to be the free crossed complex on two generators of appropriate dimensions and represent both the given elements as induced by the same bimorphism $\xi\colon (A, C) \to D$ for suitable fixed values of the first variable. We have seen that the map $a \mapsto \psi_a$ from A to $\mathsf{CRS}(C, D)$ given in this way by ξ is a morphism of crossed complexes, so we can now read off the crossed complex operations on $\mathsf{CRS}(C, D)$ from the bimorphism laws of Definition 9.3.10 for ξ.

For example, given $(F, f) \in \mathsf{CRS}_m(C, D)(m \geqslant 2)$ we determine $\delta(F, f)$ as follows. Write $(F, f) = (F_a, f_a)$ for suitable $a \in A$ as above, where $F_a(c) = \xi(a, c)$, $f_a(c) = \xi(\beta a, c)$. Then $\delta(F, f) = (F_{\delta a}, f_{\delta a})$. We note that $f_{\delta a} = f$ since $\delta\beta a = \beta a$. We write δF for $F_{\delta a}$, so that $\delta(F, f) = (\delta F, f)$. Now $(\delta F)(c) = \xi(\delta a, c)$ is given by the formula in Definition 9.3.10 in terms of known elements, namely (assuming $m \geqslant 2$)

$$\xi(\delta a, c)$$

$$= \begin{cases} \delta(\xi(a, c)) + (-1)^{m+1}\xi(a, \delta c) & \text{if } c \in C_n \ (n \geqslant 2), \\ (-1)^{m+1}\xi(a, \delta^- c)^{\xi(\beta a, c)} + (-1)^m \xi(a, \delta^+ c) + \delta(\xi(a, c)) & \text{if } c \in C_1, \\ \delta(\xi(a, c)) & \text{if } c \in C_0. \end{cases}$$

In other words

$$(\delta F)(c)$$
$$= \begin{cases} \delta(F(c)) + (-1)^{m+1}F(\delta c) & \text{if } c \in C_n \ (n \geqslant 2), \\ (-1)^{m+1}F(\delta^- c)^{f(c)} + (-1)^m F(\delta^+ c) + \delta(F(c)) & \text{if } c \in C_1, \\ \delta(h(c)) & \text{if } c \in C_0. \end{cases} \qquad (***)$$

This automatic procedure gives the crossed complex structure of $\mathsf{CRS}(C, D)$ as stated in Definition 9.3.5.

15.2.v Crossed complexes and cubical sets

We now revisit Section 11.4 in the light of the current results. Recall that in Definition 11.4.3 we set

$$\Pi K = \int^{\square, n} K_n \times \Pi \mathbb{I}^n$$

for a cubical set K, so that ΠK is freely generated by the nondegenerate cubes of K with boundaries given by the cubical Homotopy Addition Lemma.

Proposition 15.2.9. *For any cubical set K, there is a natural isomorphism of crossed complexes*

$$\gamma\rho K \cong \Pi K.$$

Proof. We have since $K \cong \int^{\square,n} K_n \times \mathbb{I}^n$

$$\begin{aligned} \rho K &\cong \rho \int^{\square,n} K_n \times \mathbb{I}^n \\ &\cong \int^{\square,n} \rho(K_n \times \mathbb{I}^n) \\ &\cong \int^{\square,n} K_n \times \rho(\mathbb{I}^n) \end{aligned}$$

and since γ is an equivalence of categories

$$\begin{aligned} \gamma\rho K &\cong \int^{\square,n} K_n \times \gamma\rho(\mathbb{I}^n) \\ &\cong \Pi K. \end{aligned}$$

This proves the proposition. $\square$

For any cubical set K we define the *fundamental crossed complex of* K to be $\Pi(K) = \gamma\rho(K)$. Propositions 15.1.14 and 15.1.15 then give immediately

Theorem 15.2.10. *If K, L are cubical sets, there is a natural isomorphism of crossed complexes*
$$\Pi(K) \otimes \Pi(L) \cong \Pi(K \otimes L).$$
In particular
$$\Pi(\mathbb{I}^m) \otimes \Pi(\mathbb{I}^n) \cong \Pi(\mathbb{I}^{m+n}).$$

For any crossed complex C we define the *cubical nerve* of C to be $NC = U\lambda C$, which is a cubical set. Since ρ is left adjoint to U, $\Pi = \gamma\rho$ is left adjoint to $N = U\lambda$, but we now prove a stronger result. We observe that, for any ω-groupoid G and any cubical set L, $\mathsf{Cub}(L, UG)$ has a canonical ω-groupoid structure induced by the structure of G (see Proposition 15.1.13). In particular $\mathsf{Cub}(L, NC)$ is an ω-groupoid and Proposition 15.1.13 gives

Theorem 15.2.11. *For any cubical set L and any crossed complex C, there are natural isomorphisms of crossed complexes*
$$\mathsf{Crs}(\Pi L, C) \cong \gamma(\omega\text{-}\mathsf{GPDS}(\rho L, \lambda C)) \cong \gamma(\mathsf{Cub}(L, NC)).$$

By taking cubical nerves and connected components we obtain

Corollary 15.2.12. *Let L be a cubical set and C be a crossed complex.*

(i) *There is a natural isomorphism of cubical sets*
$$\mathsf{Cub}(L, NC) \cong N(\mathsf{Crs}(\Pi L, C)).$$

(ii) *There is a natural bijection*
$$[L, NC] \cong [\Pi L, C],$$
where $[-,-]$ denotes the set of homotopy classes of morphisms in Cub *or in* Crs*, as the case may be.*

15.3 The Eilenberg–Zilber natural transformation

We now prove the important Theorem 9.8.1 that if X_*, Y_* are filtered spaces, then there is a natural transformation
$$\zeta : \Pi X_* \otimes \Pi Y_* \to \Pi(X_* \otimes Y_*)$$
which is an isomorphism if X_*, Y_* are CW-filtrations.[244]

In view of the previous sections, it is sufficient to prove a similar result for ω-groupoids.

Theorem 15.3.1. *If X_* and Y_* are filtered spaces, then there is a natural morphism*

$$\eta\colon \rho X_* \otimes \rho Y_* \to \rho(X_* \otimes Y_*)$$

such that:

i) *η is associative;*

ii) *if $*$ denotes a singleton space or crossed complex, then the following diagrams are commutative:*

$$\begin{tikzcd} \rho X_* & (\rho X_*) \otimes * \arrow[l, "\cong"'] \arrow[d, "\eta"] \\ & \rho(X_* \otimes *), \arrow[from=1-1, to=2-2, "\cong"'] \end{tikzcd} \qquad \begin{tikzcd} * \otimes \rho X_* \arrow[r, "\cong"] \arrow[d, "\eta"'] & \rho X_* \arrow[dl, "\cong"] \\ \rho(* \otimes X_*); & \end{tikzcd}$$

iii) *η is commutative in the sense that if $T_c\colon G \otimes H \to H \otimes G$ is the transposition and $T_t\colon X_* \otimes Y_* \to Y_* \otimes X_*$ is the twisting map, then the following diagram is commutative*

$$\begin{tikzcd} \rho X_* \otimes \rho Y_* \arrow[r, "\eta"] \arrow[d, "T_c"'] & \rho(X_* \otimes Y_*) \arrow[d, "\rho(T_t)"] \\ \rho Y_* \otimes \rho X_* \arrow[r, "\eta"] & \rho(Y_* \otimes X_*). \end{tikzcd}$$

Proof. To construct a natural morphism

$$\eta\colon \rho X_* \otimes \rho Y_* \to \rho(X_* \otimes Y_*)$$

all we need is to construct a bimorphism of ω -groupoids

$$\eta'\colon (\rho X_*, \rho Y_*) \to \rho(X_* \otimes Y_*).$$

Let $f\colon I_*^p \to X_*$, $g\colon I_*^q \to Y_*$ be representatives of elements of $\rho_p X_*$, $\rho_q Y_*$ respectively. We define $\eta'([f],[g])$ to be the class of the composite

$$I_*^{p+q} \cong I_*^p \otimes I_*^q \xrightarrow{f \otimes g} X_* \otimes Y_*.$$

It is easy to check that $\eta'([f],[g])$ is independent of the choice of representatives. Also, the conditions that η' be a bimorphism are almost automatic. Thus, we have a natural morphism η.

The proofs of (i) (associativity) and (ii) (preserves base point) are clear.

The proof of (iii) (symmetry) follows from the description of the isomorphism $G \otimes H \to H \otimes G$ of ω-groupoids as given by $x \otimes y \mapsto (y^* \otimes x^*)^*$ where, in the geometric case $G = \rho X_*$, $x \mapsto x^*$ is induced by the map $(t_1, \dots, t_p) \mapsto (t_p, \dots, t_1)$ of the unit cube.

This gives conditions (i)–(iii) of Theorem 9.8.1 □

Remark 15.3.2. The above construction of η should be compared with the proof of Proposition 11.4.13. □

Corollary 15.3.3. *For any filtered spaces X_*, Y_* there is a natural transformation*

$$\zeta \colon \Pi X_* \otimes \Pi Y_* \to \Pi(X_* \otimes Y_*)$$

which is an isomorphism if X_*, Y_* are CW-filtrations.

Proof. The first part is an immediate consequence of Theorem 15.3.1 and the translation of the monoidal structure from ω-groupoids to crossed complexes.

To prove that ζ is an isomorphism for CW-filtrations, recall that $X_* \otimes Y_*$ is a CW-filtration, and so the crossed complex $\Pi(X_* \otimes Y_*)$ is free, with basis the characteristic maps of the product cells $e^p \times e^q$ of $X_* \otimes Y_*$. So the theorem follows from Theorem 9.6.1 that the tensor product of free crossed complexes is also free. □

15.4 The symmetry of tensor products

We have seen that in the category Cub, the map $x \otimes y \mapsto y \otimes x$ does not give an isomorphism $K \otimes L \to L \otimes K$; indeed it is easy to construct examples of cubical sets K, L such that $K \otimes L$ and $L \otimes K$ are not isomorphic. However, in ω-GPDS, and Crs the situation is different. Although the map $x \otimes y \mapsto y \otimes x$ still does not give an isomorphism $K \otimes L \to L \otimes K$, there is a less obvious map which does. This is easiest to see in Crs.

Theorem 15.4.1. *Let C, D be crossed complexes. Then there is a natural isomorphism $C \otimes D \to D \otimes C$ which, for $c \in C_m$, $d \in D_n$, sends the generator $c \otimes d$ to $(-1)^{mn} d \otimes c$. This isomorphism, combined with the structure studied until now, makes the category of crossed complexes a symmetric monoidal closed category.*

Proof. One merely checks that the relations defining the tensor product are preserved by the map $c \otimes d \mapsto (-1)^{mn} d \otimes c$. The necessary coherence and naturality conditions are obviously satisfied. □

Remark 15.4.2. This proof is unsatisfactory because, although it is clear that $c \otimes d \mapsto d \otimes c$ does not preserve the relations of the tensor product, the fact that $c \otimes d \mapsto (-1)^{mn} d \otimes c$ does preserve them seems like a happy accident. A better explanation is provided by the transposition functor T (see Sections 11.2 and 15.1). □

For a cubical set K, TK is not in general isomorphic to K. But for any ω-groupoid G and any crossed complex C we will construct isomorphisms $G \to TG$ and $C \to TC$. Since in all these categories we have obvious natural isomorphisms $T(X \otimes Y) \cong TY \otimes TX$, this implies the symmetry $X \otimes Y \cong Y \otimes X$.

For an ω-groupoid G, the *transpose* TG has the same elements as G but has all its operations ∂_i^α, ε_i, Γ_i, $+_i$, $-_i$ numbered in reverse order with respect to i (but not with respect to $\alpha = \pm$): compare Definition 11.2.19. For a crossed complex C, TC is defined, of course, as $\gamma(T\lambda C)$. The calculation expressing this crossed complex in terms of the crossed complex structure of C is straightforward (though it needs a clear head).

Proposition 15.4.3. *The crossed complex TC is defined, up to natural isomorphism, in the following way:*

(i) $(TC)_0 = C_0$ *as a set;*
(ii) $(TC)_2 = C_2^{\mathrm{op}}$ *as a groupoid;*
(iii) $(TC)_n = C_n$ *as a groupoid for $n = 1$ and $n \geqslant 3$;*
(iv) *the action of $(TC)_1$ on $(TC)_n (n \geqslant 2)$ is the same as the action of C_1 on C_n;*
(v) *the boundary map $T\delta\colon (TC)_{n+1} \to (TC)_n$ is given by*

$$T\delta = (-1)^n \delta\colon C_{n+1} \to C_n.$$

We note that $-\delta\colon C_2 \to C_1$ is an anti-homomorphism, that is a homomorphism $C_2^{\mathrm{op}} \to C_1$, as required; the map $+\delta\colon C_3 \to C_2^{\mathrm{op}}$ is also a homomorphism because the image is in the centre of C_2. In higher dimensions the groupoids C_n and C_n^{op} are the same.

Corollary 15.4.4. *Let $\lfloor x \rfloor$ denote the integer part of a real number x. For any crossed complex C there is a natural isomorphism $\tau\colon C \to TC$ given by*

$$\tau(c) = (-1)^{\lfloor n/2 \rfloor} c \quad \textit{for } c \in C_n.$$

Remark 15.4.5. The somewhat surprising sign $(-1)^{\lfloor n/2 \rfloor}$ is forced by the signs in Proposition 15.4.3; it is less surprising when one notices that it is the signature of the permutation which reverses the order of $(1, 2, \dots, n)$. The symmetry map of Theorem 15.4.1 now comes from the map

$$c \otimes d \to \tau^{-1}(\tau d \otimes \tau c) = (-1)^k d \otimes c,$$

where $k = \lfloor m/2 \rfloor + \lfloor n/2 \rfloor - \lfloor (m+n)/2 \rfloor$, which is 0 if m or n is even, and -1 if both are odd. □

Let G be an ω-groupoid and $C = \gamma G$. Then $G \cong \lambda C$ and the isomorphism $\tau\colon C \to TC$ extends uniquely to an isomorphism $\tau\colon G \to TG$. This isomorphism can be viewed as a 'reversing automorphism' $x \mapsto x^*$ of G, that is, a map of degree 0 from G to itself which preserves the operations while reversing their order (e.g. $(x +_i y)^* = x^* +_{n-i+1} y^*$ in dimension n). The isomorphism $G \otimes H \to H \otimes G$ for ω-groupoids is then given by

$$x \otimes y \mapsto (y^* \otimes x^*)^*.$$

The element x^* should be viewed as a transposeof the cube x.

Remark 15.4.6. In the geometric case $G = \rho(X_*)$, x^* is induced from x by the map $(t_1, \dots, t_n) \mapsto (t_n, \dots, t_1)$ of the unit n-cube.

The operation $*$ is preserved by morphisms of ω-groupoids, because of the naturalness of $\tau\colon 1 \to T$. It follows from the notion of density which we deal with in Section 15.6 that the operation $*$ can be written in terms of the ω-groupoid operations ∂_i^α, ε_i, Γ_i, $+_i$, $-_i$, but the formulae needed for this seem rather complicated. □

15.5 The pointed case

We consider briefly the notions of tensor product and homotopy in the categories ω-Gpds_* and Crs_* of pointed ω-groupoids and pointed crossed complexes. Here the objects have a distinguished element $*$ in dimension 0 and all morphisms are to preserve the base points.

Definition 15.5.1. For any ω-groupoid H with basepoint $*$, the ω-groupoid $P^m H$ has basepoint $0_* = \varepsilon_1^m(*)$, the constant cube at $*$. An *m-fold pointed (left) homotopy* $h\colon G \to H$ is a morphism $h\colon G \to P^m H$ preserving basepoints, that is, a homotopy h with $h(*) = 0_*$. Clearly, all such pointed homotopies form an ω-subgroupoid ω-$\mathsf{GPDS}_*(G, H)$ of ω-$\mathsf{GPDS}(G, H)$ since $0_* = \varepsilon_1^m(*)$ is an identity for all the compositions $+_i (1 \leqslant i \leqslant m)$. This ω-subgroupoid has as basepoint the trivial morphism $G \to H$ which sends each element of dimension n to $0_* = \varepsilon_1^n(*)$. Thus we have an internal hom functor ω-$\mathsf{GPDS}_*(G, H)$ in the pointed category ω-Gpds_*. The pointed morphisms from F to ω-$\mathsf{GPDS}_*(G, H)$ are in one-one correspondence with the *pointed bimorphisms* $\chi\colon (F, G) \to H$, that is, bimorphisms χ satisfying the extra conditions

$$\begin{cases} \chi(x, *) = 0_* & \text{for all } x \in F, \\ \chi(*, y) = 0_* & \text{for all } y \in G. \end{cases} \tag{i}$$

To retain the correspondence between bimorphisms $(F, G) \to H$ and morphisms $F \otimes G \to H$, we must therefore add corresponding relations to the definition of the tensor product. Thus, for pointed ω-groupoids F, G, we define $F \otimes_* G$ to be the ω-groupoid with generators $x \otimes_* y$, $(x \in F, y \in G)$, basepoint $* = * \otimes_* *$, and defining relations the same as in Definition 11.2.5 together with

$$\begin{cases} x \otimes_* * = 0_* & \text{for all } x \in F, \\ * \otimes_* y = 0_* & \text{for all } y \in G. \end{cases} \tag{ii}$$

These equations are to be interpreted as $x \otimes_* * = * \otimes_* y = *$ when x, y have dimension 0, so that $(F \otimes_* G)_0 = F_0 \wedge G_0$, the product $F_0 \times G_0$ with $F_0 \times * \cup * \times G_0$ identified to a point. □

Theorem 15.5.2. *The pointed tensor product and hom functor described above define a symmetric monoidal closed structure on the pointed category ω-Gpds_*.*

15.6 Dense subcategories

The notion of dense subcategory is very useful in many categories of algebraic objects; it allows properties of a category C to be deduced from properties of the dense subcategory. In our case, we will use this for the category of ω-groupoids, deducing some properties from those of the full subcategory on the objects $\mathbf{I}^n$, which denote the free ω-groupoids on a generator of dimension n. We know by results of Section 14.6 that $\mathbf{I}^n$ is isomorphic to $\rho\mathbb{I}^n$ and to ρI^n_*.[245]

Our aim in this section is to explain and prove the theorem:

Theorem 15.6.1. *The full subcategory* $\widehat{\mathbf{I}}$ *of* ω-Gpds *on the objects* $\mathbf{I}^n$ *is dense in* ω-Gpds.

We recall the definition of a dense subcategory. First, in any category C, a morphism $f: C \to D$ induces a natural transformation $f_*: \mathsf{C}(-, C) \to \mathsf{C}(-, D)$ of functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$. Conversely, any such natural transformation is induced in this way by a (unique) morphism $C \to D$ (see Appendix A.2).

If J is a subcategory of C, then each object C of C gives a functor

$$\mathsf{C}^{|\mathsf{J}}(-, C)\colon \mathsf{J}^{\mathrm{op}} \to \mathsf{Set}$$

and a morphism $f: C \to D$ of C induces a natural transformation of functors

$$f_*\colon \mathsf{C}^{|\mathsf{J}}(-, C) \Rightarrow \mathsf{C}^{|\mathsf{J}}(-, D).$$

The subcategory J is said to be *dense* in C if every such natural transformation arises from a morphism. More precisely, there is a functor $\eta\colon \mathsf{C} \to \mathsf{CAT}(\mathsf{J}^{\mathrm{op}}, \mathsf{Set})$ defined in the above way, and J is *dense* in C if η is full and faithful.

Example 15.6.2. Consider the Yoneda embedding

$$\Upsilon\colon \mathsf{C} \to \mathsf{C}^{\mathrm{op}}\text{-}\mathsf{Set} = \mathsf{CAT}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$$

where C is a small category. Then each object $K \in \mathsf{C}^{\mathrm{op}}$-$\mathsf{Set}$ is a colimit of objects in the image of Υ and this is conveniently expressed in terms of coends as that the natural morphism

$$\int^{c} (\mathsf{C}^{\mathrm{op}}\text{-}\mathsf{Set}(\Upsilon c, K) \times \Upsilon c) \;\to\; K$$

is an isomorphism. Thus the Yoneda image of C is dense in C^{op}-Set. □

Example 15.6.3. Let $\mathbb{Z}$ be the cyclic group of integers. Then $\{\mathbb{Z}\}$ is a generating set for the category Ab of abelian groups, but the full subcategory of Ab on this set is not dense in Ab. In order for a natural transformation to specify not just a function $f: A \to B$ but a morphism in Ab, we have to enlarge this subcategory to the full subcategory also including $\mathbb{Z} \oplus \mathbb{Z}$. As an exercise, you should try finding dense subcategories of other categories of general algebraic interest, such as groups, rings, groupoids, $\dots$. □

Example 15.6.4. Consider the Yoneda embedding

$$\Upsilon\colon \mathsf{C} \to \mathsf{C}^{\mathrm{op}}\text{-}\mathsf{Set} = \mathsf{Cat}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$$

where C is a small category. Then each object $K \in \mathsf{C}^{\mathrm{op}}\text{-}\mathsf{Set}$ is a colimit of objects in the image of Υ and this is conveniently expressed in terms of coends as that the natural morphism

$$\left(\int^{c} (\mathsf{C}^{\mathrm{op}}\text{-}\mathsf{Set}(\Upsilon c, K) \times \Upsilon c)\right) \to K$$

is an isomorphism. Thus the Yoneda image of C is dense in $\mathsf{C}^{\mathrm{op}}\text{-}\mathsf{Set}$. □

Proof of Theorem 15.6.1. Let G, H be ω-groupoids and let

$$\mathsf{f}\colon \omega\text{-}\mathsf{Gpds}_{\mathcal{I}}(-, G) \to \omega\text{-}\mathsf{Gpds}_{\mathcal{I}}(-, H)$$

be a natural transformation. We define $f : G \to H$ as follows.

Let $x \in G_n$. Then x defines $\hat{x} : \mathbf{I}^n \to G$. We set $f(x) = \mathsf{f}(\hat{x})(c^n) \in H_n$. We have to prove f preserves all the structure.

For example, we prove that $f(\partial_i^{\pm} x) = \partial_i^{\pm} f(x)$. Let $\bar{\partial}_i^{\pm} : \mathbf{I}^{n-1} \to \mathbf{I}^n$ be given by having value $\partial_i^{\pm} c^n$ on c^{n-1}. The natural transformation condition implies that $\mathsf{f}(\bar{\partial}_i^{\pm})^* = (\bar{\partial}_i^{\pm})^* \mathsf{f}$. On evaluating this on $\hat{x}$ we obtain $f(\partial_i^{\pm} x) = \partial_i^{\pm} f(x)$ as required. In a similar way, we prove that f preserves the operations ε_i, Γ_i.

Now suppose that $t \in G_n$ is thin in G. We prove that $f(t)$ is thin in H.

Consider the morphism of ω-groupoids $\hat{t} : \mathbf{I}^n \to G$. Let B be the box consisting of all faces but one of c_n. Then B has a unique thin filler b_t. Now $\hat{t}(B)$ consists of all faces but one of t, and so is filled by t. Since $\hat{t}$ preserves thin elements, we must have $\hat{t}(b_t) = t$. Let $\bar{b}: \mathbf{I}^n \to \mathbf{I}^n$ be the unique morphism such that $\bar{b}(c^n) = b_t$. Then the natural transformation condition implies $f(t) = \mathsf{f}(\hat{t})(c^n) = \mathsf{f}(\hat{t})(b_t)$. Since b_t is thin, it follows that $f(t)$is thin. Thus f preserves the thin structure.

Now Proposition 13.7.8 implies that the operations $+_i$ are preserved by f. This completes the proof of Theorem 15.6.1. □

We can now conveniently represent each ω-groupoid as a coend.

Corollary 15.6.5. *The subcategory* $\widehat{\mathbf{I}}$ *of* ω-Gpds *is dense and for each object* G *of* ω-Gpds *the natural morphism*

$$\int^{n} \omega\text{-}\mathsf{Gpds}(\mathbf{I}^n, G) \times \mathbf{I}^n \to G$$

is an isomorphism.

Proof. This is a standard consequence of the property of $\widehat{\mathbf{I}}$ being dense. □

Corollary 15.6.6. *The full subcategory of* Crs *generated by the objects* ΠI_*^n *is dense in* Crs.

Proof. This follows from Theorem 14.4.1, which gives the fact that the equivalence $\gamma\colon \omega\text{-}\mathsf{Gpds} \to \mathsf{Crs}$ takes $\mathbf{I}^n$ to ΠI^n_*. □

Remark 15.6.7. It is easy to find a generating set of objects for the category Crs, namely the free crossed complexes on single elements, given in fact by ΠE^n_*, where E^n_* is the usual cell decomposition of the unit ball, with one cell for $n = 0$ and otherwise three cells. It is not so obvious how to construct directly from this generating set a dense subcategory, and then a dense subcategory closed under tensor products, of Crs.[246] □

15.7 Fibrations and coverings of ω-groupoids

The definitions of covering morphism and of fibration of crossed complexes were given in Sections 10.1 and 12.1 respectively. We now give corresponding conditions for ω-groupoids.[247]

Theorem 15.7.1. *Let $p\colon G \to H$ be a morphism of ω-Gpdss. Then the morphism of crossed complexes $\gamma(p)\colon \gamma(G) \to \gamma(H)$ is a fibration (covering morphism) if and only if $p\colon G \to H$ is a Kan fibration (covering map) of cubical sets.*

Proof. As regards fibrations this is the result of Proposition 12.1.13. The restriction to covering morphisms follows in a similar way. □

Corollary 15.7.2. *Let $p\colon K \to L$ be a morphism of ω-groupoids such that the underlying map of cubical sets is a fibration. Then the pullback functor*

$$f_*\colon \omega\text{-}\mathsf{Gpds}/L \to \omega\text{-}\mathsf{Gpds}/K$$

has a right adjoint and so preserves colimits.

Proof. This is immediate from Theorem 15.7.1 and results of Howie stated as Theorem 10.1.12. □

Remark 15.7.3. It seems likely that a covering ω-groupoid of a free ω-groupoid is also free. □

15.8 Application to the tensor product of covering morphisms

Our aim is to prove the following:

Theorem 15.8.1. *The tensor product of two covering morphisms of crossed complexes is a covering morphism.*

Remark 15.8.2. The reason why we have to give an indirect proof of this result is that the definition of covering morphism involves *elements* of crossed complexes; but it is difficult to specify exactly the elements of a tensor product whose definition is perforce by generators and relations. □

It is sufficient to assume that all the crossed complexes involved are connected. We will also work in the category of ω-groupoids, and prove the following:

Theorem 15.8.3. *Let G, H be connected ω-groupoids with base points x, y respectively, and let $p\colon \widetilde{G} \to G$ be the covering morphism determined by the subgroup M of $\pi_1(G,x)$. Let $\phi\colon C \to G \otimes H$ be the covering morphism determined by the subgroup $M \times \pi_1(H,y)$ of*

$$\pi_1(G \otimes H,(x,y)) \cong \pi_1(G,x) \times \pi_1(H,y).$$

Then there is an isomorphism $\psi\colon C \to \widetilde{G} \otimes H$ such that $(p \otimes 1_H)\psi = \phi$, and, consequently,

$$p \otimes 1_H \colon \widetilde{G} \otimes H \to G \otimes H$$

is a covering morphism.

Proof. [248] First we know from Corollary 15.1.15 that the tensor product in ω-Gpds satisfies

$$\mathbf{I}^m \otimes \mathbf{I}^n \cong \mathbf{I}^{m+n}.$$

It follows that the tensor product $G \otimes H$ of ω-groupoids G, H satisfies

$$G \otimes H \cong \int^{m,n} \omega\text{-Gpds}(\mathbf{I}^m, G) \times \omega\text{-Gpds}(\mathbf{I}^n, H) \times (\mathbf{I}^m \otimes \mathbf{I}^n). \tag{15.8.1}$$

We suppose G, H are reduced, i.e. that G_0, H_0 are singletons. Let $(x,y) \in G_0 \times H_0$. Now let $\phi\colon C \to G \otimes H$ be a covering morphism determined by the subgroup $M \times \pi_1(H,y)$ of

$$\pi_1(G,x) \times \pi_1(H,y) \cong \pi_1(G \otimes H,(x,y)).$$

Let $p\colon \widetilde{G} \to G$ be the covering morphism determined by the subgroup M. By Corollary 15.7.2, pullback ϕ^* by ϕ preserves colimits. Hence

$$\begin{aligned} C &\cong \phi^*\Big(\int^{m,n} \omega\text{-Gpds}(\mathbf{I}^m, G) \times \omega\text{-Gpds}(\mathbf{I}^n, H) \times (\mathbf{I}^m \otimes \mathbf{I}^n)\Big) \\ &\cong \int^{m,n} \phi^*(\omega\text{-Gpds}(\mathbf{I}^m, G) \times \omega\text{-Gpds}(\mathbf{I}^n, H)) \times (\mathbf{I}^m \otimes \mathbf{I}^n) \end{aligned}$$

and so because of the construction of C by the specified subgroup:

$$\begin{aligned} &\cong \int^{m,n} \omega\text{-Gpds}(\mathbf{I}^m, \widetilde{G}) \times \omega\text{-Gpds}(\mathbf{I}^m, H) \times (\mathbf{I}^m \otimes \mathbf{I}^n) \\ &\cong \widetilde{G} \otimes H. \end{aligned}$$

□

Corollary 15.8.4. *The tensor product of covering morphisms of ω-groupoids is again a covering morphism.*

Proof. Because tensor product commutes with disjoint union, it is sufficient to restrict to the connected case. Since the composition of covering morphisms is again a covering morphisms, it is sufficient to restrict to the case of $p \otimes 1$ for a single covering morphism p. But this case is given by Theorem 15.8.3. □

The proof of Theorem 15.8.1 follows immediately.

This finally enables us to prove Theorem 10.2.16.[249]

Corollary 15.8.5. *The tensor product of free, aspherical crossed complexes is also free and aspherical.*

Proof. Let F, F' be free aspherical crossed complexes. It is sufficient to assume they are connected. Since F, F' are aspherical, their universal covers $\widetilde{F}$, $\widetilde{F}'$ are acyclic. Since they are also free, they are contractible, by a Whitehead type theorem, B.8.1. But the tensor product of free crossed complexes is free, by Theorem 9.6.1. Therefore $\widetilde{F} \otimes \widetilde{F}'$ is contractible, and hence acyclic. Therefore $F \otimes F'$ is aspherical. □

Notes

240 p. 513 A background to these results was the work on the homotopy type of function spaces in [Bro62], [Bro64a], [Bro64c], in which the Dold–Kan Theorem relating chain complexes and simplicial abelian groups was a central tool, as was the monoidal closed structure on chain complexes and on various other categories needed for that work, though the words 'monoidal closed' were hardly used at the time. This is a reason for emphasising in Section 14.8 the relations between that theorem and the work on the equivalence of various algebraic categories with that of crossed complexes.

The results of the first five sections of this chapter are taken from [BH87], except for Section 15.3 on the Eilenberg–Zilber transformation ζ for crossed complexes which comes from [BH91]. The results of the last three sections come from [BS10].

241 p. 514 These methods are used in [AABS02] to give a monoidal closed structure for cubical ω-categories with connections and hence, because of the equivalence proved in that paper, to obtain a monoidal closed structure for globular ω-categories. These ideas are related to what is also called the Gray tensor product of 1- and 2-categories, see for example [Cra99] and the references there.

242 p. 515 This follows from general theorems of Freyd [Fre72], Bastiani–Ehresmann [BE72] and Coates [Coa74]. G. Janelidze has pointed out that these types of

results should be seen in the general context of the equivalent terms of ‘locally finitely presentable’ and ‘essentially algebraic’ categories, and that there is a wide and recent literature on these topics, giving results on completeness and cocompleteness. Note that in the case of colimits of crossed complexes we have earlier given reference to [Law04], [Man76]. Further background to algebraic theories is in Volume 2 of [Joh02].

243 p. 521 The coherence laws should be taken into account when making constructions such as ‘free internal monoids with respect to tensor’, as discussed for crossed complexes in [BB93]. A discussion of free monoids is in [Lac10], but is concerned with the case where the monoidal category is not closed, and tensoring does not distribute over coproducts, so that the usual geometric series does not apply.

244 p. 532 The more general result is proved in [BB93] that ζ is an isomorphism if X_*, Y_* are connected and cofibred, where the latter means that all inclusions $X_n \to X_{n+1}, n \geqslant 0$, are closed cofibrations, and similarly for Y_*. A good example of such a filtration is $(BC)_*$, the classifying space of a crossed complex filtered by the skeleta of C. The application in *loc. cit.* is to what is called the *tensor algebra* of a pointed crossed complex, generalising previous work in [BC92] on the tensor algebra of a finite group.

245 p. 537 The results of these last two sections are taken from [BS10]. For more on dense subcategories see [ML71] and many other books on category theory; for a discussion of the relation between dense subcategories and the Yoneda Lemma see [Pra09]. Essentially, the notion of density is especially required for categories of algebraic structures, while the Yoneda Lemma seems adequate for categories of geometric structures, such as simplicial or cubical sets.

246 p. 539 The paper [BH81b] gives an equivalence between the category Crs of crossed complexes and the category there called ∞-groupoids and now commonly called globular ω-groupoids. Thus the above corollary yields also a dense subcategory, based on models of cubes, in the latter category. Compare this to the approach to the Gray tensor product of 2-categories in [Str88], and of globular ∞-categories in [Cra99]; compare also [AABS02], Section 10.

247 p. 539 The paper [AM11] relates the notion of fibration for crossed complexes also to that for globular ω-categories and groupoids.

248 p. 540 The PhD thesis [Day70] of Brian Day addressed the problem of extending a promonoidal structure on a category $\mathcal{A}$ along a dense functor $J : \mathcal{A} \to \mathcal{X}$ into a suitably complete category $\mathcal{X}$ to obtain a closed monoidal structure on $\mathcal{X}$. The two published papers [Day70a], [Day72] are only part of the thesis and represent components towards the density result. The formulae in, and the spirit of, Day’s work suggested the approach to the present results in the paper [BS10]. However, here the category $\mathcal{A}$ is actually small (consisting of cubes) and monoidal, and so

is an easy case of Day's general setting. The same simplification occurs in the approach to the Gray tensor product of 2-categories in [Str88], and of globular ∞-categories in [Cra99], Proposition 4.1.

249 p. 541 Tonks proved in [Ton94], Theorem 3.1.5, that the tensor product of free crossed resolutions of a group is a free crossed resolution: his proof used the crossed complex Eilenberg–Zilber–Tonks Theorem 10.4.14.

Chapter 16
Future directions?

Yet all experience is an arch wherethro'
Gleams that untravelled world, whose margin fades
Forever and forever when I move.

from 'Ulysses', by Alfred Lord Tennyson

We have now come to the end of our description of this intricate structure. We hope to have shown how it fits together and allows a new approach to algebraic topology, based on filtered spaces and homotopically defined functors on such structured spaces, and in which some nonabelian information in dimension 2 and the actions of the fundamental groupoid are successfully taken into account. We also wanted to convey how a key to the success of the theory has been the good modelling of the geometry by the algebra, and the way the algebra gives power and reality to some basic intuitions, revealing underlying processes.

We have presented the material in a way which we hope will convince you that the intricacy of the justification of the theory does not detract from the fact that crossed complexes theory are usable as a tool even without knowing exactly why they work. That is, we have given a pedagogical order rather than a logical and structural order. It should be emphasised that the order of discovery followed the logical order! The conjectures were made and verified in terms of ω-groupoids, and we were amazed that the theory of crossed complexes, which was in essence already available, fitted with this so nicely.

It is also surprising that this corpus of work followed from a simple aesthetic question posed in 1964–65, to find a determination of the fundamental group of the circle which avoided the detour of setting up covering space theory. This led to nonabelian cohomology, [Bro65a], and then to groupoids, [Bro67]. The latter suggested the programme of seeing how much could be done of a rewriting of homotopy theory replacing the word 'group' by 'groupoid', and if so whether the result was an improvement! This naive question raised some new prospects.

There is much more to do, and we explain some potential areas of work in the next section. It is not expected that these questions and problems are of equal interest or solvability!

Some of these matters discussed are speculative; it seems right to quote here from a letter of Alexander Grothendieck dated 14/06/83:

> Of course, no creative mathematician can afford not to "speculate", namely to do more or less daring guesswork as an indispensable source of inspiration. The trouble is that, in obedience to a stern tradition, almost nothing of

> this appears in writing, and preciously little even in oral communication. The point is that the disrepute of "speculation" or "dream" is such, that even as a strictly private (not to say secret!) activity, it has a tendency to vegetate – much like the desire and drive of love and sex, in too repressive an environment.

Any new idea has to be caught as it flashes across the mind, or it might vanish; talking about ideas can help to make them real, though it can also raise some funny looks from superior persons!

16.1 Problems and questions

There are a number of standard methods and results in algebraic topology to which the techniques of crossed complexes given here have not been applied, or applied only partially. So we leave these open for work to be done, and for you to decide how the uses in these areas of crossed complexes and related structures can advance the subjects of algebraic topology and homological algebra. We expect you to use texts and the internet for additional references and sources for further details, with the usual cautions about not relying totally on all that is there. Also you must do your own assessment of the possible value of these questions.

Problem 16.1.1. There has been surprisingly little general use in low-dimensional topology and geometric group theory of the HHSvKT for crossed modules, Theorem 2.3.1: this theorem is not even mentioned in [HAMS93], though some consequences are given. We mention again the important work of Papakyriakopoulos on relations between group theory and the Poincaré conjecture, [Pap63], which uses Whitehead's theorem on free crossed modules which, as shown in Theorem 5.4.8, is but one application of the 2-dimensional SvKT. Of course the Poincaré Conjecture has been resolved by different, and differential, rather than combinatorial or group theoretic, means. Recent uses of the 2-dimensional Seifert–van Kampen Theorem are by [KFM08], [FM09]. Perhaps even more surprising uses could be made of the triadic results in [BL87], [BL87a], relating to surgery problems, and borrowing methods from [Ell93]? See also [FM11]. □

Problem 16.1.2. Investigate applications of the enrichment of the category FTop over the monoidal closed category Crs in the spirit of the work on 2-groupoids in [KP02]. In fact, as an exercise, translate the work of the last paper into the language of crossed complexes and their internal homs. □

Problem 16.1.3. Investigate and apply Mayer–Vietoris type exact sequences for a pullback of a fibration of crossed complexes, analogous to that given for a pullback of a covering morphism of groupoids in [Bro06], Section 10.7. See also [HK81], [BHK83]. □

Problem 16.1.4. Can one use crossed complexes to give a finer form of Poincaré Duality? For an account of this duality, see for example Chapter 16 in [tD08]. This might require developing cup and cap products, which should be no problem, and also coefficients in an object with an analogue of a 'ring structure'. These could be the crossed differential algebras (i.e. monoid objects in the monoidal category Crs) considered in [BT97], and the braided regular crossed modules of [BG89a], further developed in [AU07]. See also the paper [Bro10b]. One would like to relate these ideas to older intuitions for Poincaré duality as explained in for example [ST80]. □

Problem 16.1.5. Another standard area in algebraic topology is *fixed point theory*, which includes the Lefschetz theory, involving homology, and also Nielsen theory, involving the fundamental group. Can these be combined? Perhaps one needs abstract notions for the Lefschetz number analogous to those found for the Euler characteristic, and with values in some ring generalising the integers? Relevant papers on this are perhaps [Hea05], [Pon09], [PS09]. Note that the last two papers use symmetric monoidal categories, and all use groupoid techniques. □

Problem 16.1.6. Are there possible results on the fundamental crossed complex of an orbit space of a filtered space analogous to those for the fundamental groupoid of an orbit space given in [Bro06], Chapter 11? Some related work is in [HT82]. But in Chapter 11 of [Bro06] a key result is on path lifting. Can one get some homotopy lifting using subdivisions of a square and the retraction arguments used in the proof of Proposition 14.2.8? □

Problem 16.1.7. Are there applications of crossed complexes to the nonabelian cohomology of fibre spaces? Could the well developed acyclic model theory and fibre spaces of [GM57] be suitably modified and used? The spectral sequence of filtered crossed complexes has been developed by Baues in [Bau89], but surely more work needs to be done. Note also that while the theory of simplicial fibre bundles is well developed, the cubical theory has problems because the categorical product of cubical sets has poor homotopical properties. This might be solved by using cubical sets with connections: the paper [Mal09] on the geometric realisation of such structures is surely relevant, as is [FMP11]. □

Problem 16.1.8. The category Gpds of groupoids does not satisfy some properties analogous to those of the category of groups, for example is not semi-abelian in the sense of [JMT02]. However it seems that each fibre of the functor $\mathrm{Ob}\colon \mathsf{Gpds} \to \mathsf{Set}$ is semi-abelian. Is it reasonable to investigate for purposes of homological algebra the general situation of fibrations of categories such that each fibre is semi-abelian, and can such a generalised theory be helpfully applied to crossed complexes? □

Problem 16.1.9. Can one apply to the cubical collapses of Section 11.3.i the methods of finite topological spaces as applied to collapses of simplicial complexes in [BM09]? □

Problem 16.1.10. Is there a nonabelian *homological perturbation theory* for constructing nonabelian twisted tensor products from fibrations? As a start in the literature, see

[BL91]. Or for constructing small free crossed resolutions of groups? References for the standard theory, and the important relation to twisted tensor products, may be found by a web search. □

Problem 16.1.11. The standard theory of chain complexes makes much use of double chain complexes. Double crossed complexes have been defined in [Ton94] but presumably there is much more to be done here. □

Problem 16.1.12. The theory of equivariant crossed complexes has already been developed in [BGPT97], [BGPT01]. However notions such as fibrations of crossed complexes have not been applied in that area. □

Problem 16.1.13. Can one make progress with nonabelian cohomology operations? The tensor product of crossed complexes is symmetric, as proved in Section 15.4. So if K is a simplicial set, then we can consider the noncommutativity of the diagonal map $\Delta\colon \pi|K| \to \pi|K| \otimes |K|$. If T is the twisting map $A \otimes B \to B \otimes A$, then there is a natural homotopy $T\Delta \simeq \Delta$, by the usual acyclic models argument. This look like the beginnings of a theory of nonabelian Steenrod cohomology operations. Does such a theory exist and does it hold any surprises? By contrast, [Bau89] gives an obstruction to the existence of a Pontrjagin square with local coefficients. □

Problem 16.1.14. One use of chain complexes is in defining Kolmogorov–Steenrod homology. One takes the usual net of polyhedra defined as the nerves of open covers of a space X, with maps between them induced by choices of refinements. The result is a homotopy coherent diagram of polyhedra. This is also related to Čech homology theory. It is shown in [Cor87] that a strong homology theory results by taking the chain complexes of this net, and forming the chain complex which is the homotopy inverse limit. What sort of strong homology theory results from using the fundamental crossed complexes of the nerves instead of the chain complexes? Is there a kind of ‘strong fundamental groupoid’, and could this be related to defining universal covers of spaces which are not locally ‘nice’? □

Problem 16.1.15. There are a number of areas of algebraic topology where chain complexes with a group of operators are used, for example [Coh73], [RW90]. Is it helpful to reformulate this work in terms of crossed complexes? Note that Section 17 of [Whi50b] is given in terms of crossed complexes, but the exposition there is sparse; we have earlier related this work to that of Baues in [Bau89], p. 357. A related work on simple homotopy theory is [Bro92], which is also related to generalisations of Tietze equivalences of presentations. Standard expositions of simple homotopy theory, for example [Coh73], are in terms of chain complexes with operators. It may be worth going back to the paper which introduced many of these ideas, namely [Whi41b]. Note that simple homotopy theory is applied to manifolds using filtrations defined by a Morse function in [Maz65]. □

Problem 16.1.16. Another example for the last problem of replacing chain complexes by crossed complexes is the work of Dyer and Vasquez in [DV73] on CW-models for one-relator groups. Can that work be helpfully reworked in terms of crossed complexes and the techniques of Chapter 10? The paper [Lod00] gives some problems on identities among relations. □

Problem 16.1.17. Can the use of crossed complexes in Morse theory explained by Sharko in [Sha93] be further developed? He writes at the beginning of Chapter VII:

> The need to make use of homotopy systems [i.e. free crossed complexes] in order to study Morse functions on non-simply connected closed manifolds or on manifolds with one boundary component arises from the failure of the chain complexes constructed from the Morse functions and gradient-like vector fields to capture completely the geometric aspects of the problem. This relates to application of the Whitney lemma to the reduction of the number of points of intersection of manifolds of complementary dimensions.

Problem 16.1.18. Baues and Tonks in [BT97] use crossed complexes to study the cobar construction. But the original work on the cobar construction in [AH56] was cubical. Can one do better by using many base points instead of just loop spaces, and also using ω-groupoids instead of crossed complexes? □

Problem 16.1.19. Find applications of these nonabelian constructions to configuration space theory and mapping space theory, particularly the theory of spaces of rational maps. More generally, one can look at areas where the standard tools are simplicial abelian groups, classifying spaces, and some notion of freeness. □

Problem 16.1.20. A further aim is to use these methods in the theory of stacks and gerbes, and more generally in differential topology and geometry. The ideas of Section 12.5.i are hopefully a start on this. The paper [FMP11] uses directly methods of our ω-groupoids, and for similar reasons to ours, but in the context of smooth manifolds rather than filtered spaces. □

Problem 16.1.21. Investigate the relation between the cocycle approach to Postnikov invariants and that given using triple cohomology and crossed complexes in [BFGM05]. □

Problem 16.1.22. One starting intuition for the proof of the HHSvKT was the wish to algebraicise the proof of the cellular approximation theorem due to Frank Adams, and given in [Bro68], [Bro06]. Now a subtle proof of an excision connectivity theorem of Blakers and Massey is given in [tD08], Section 6.9. Can one use methods of crossed squares or cat^n-groups to algebraicise this proof? □

Problem 16.1.23. It would be good to have another proof of the main result of [BB93], using cubical ω-groupoids. Perhaps one needs also some of the methods of [tD08], Section 6.9? □

Problem 16.1.24. There are many problems associated with generalisation of the HHSvKT to n-cubes of spaces as given in [BL87], [BL87a]. For a survey, and references to related literature, see [Bro92]. Recent works in this area are [EM10], [MW10]. It is not clear what should be the appropriate generalisation to a many base point approach of the work on the fundamental cat^n-group of an n-cube of spaces explained in [BL87], [Gil87]. Note the idea of a fundamental double groupoid of a map of spaces in [BJ04]. Can this be generalised to n-cubes of spaces? Grothendieck remarked in 1985 to Brown that the idea that (strict) n-fold groupoids model homotopy n-types was 'absolutely beautiful!'. Some relation of cat^n-groups to other models is developed in [Pao09]. □

Problem 16.1.25. The term ∞-groupoid has been used for the simplicial singular complex $S^{\Delta}X$ of a space X and this has also been written ΠX. See for example [Ber02], [Lur09], [JT07]. However the axiomatic properties of the *cubical* singular complex $S^{\square}X$, with its multiple compositions which we use greatly in this book, have not been much investigated. We mention [Ste06] as an approach to using Kan fillers in a categorical situation. □

Problem 16.1.26. The area of homological algebra has been invigorated by the notion of triangulated category and related areas, see for example [Nee01], [Kün07]. These are related to chain complexes, also called differential graded objects. However the work of Fröhlich and of Lue, for which see references in [Lue71], shows the relevance of general notions of crossed modules. Crossed modules and triangulated categories are also used in [MTW10]. Again work of Tabuada [Tab09], [Tab10] relates Postnikov invariants and monoidal closed categories. But this is done for dg-objects without the crossed module environment.

Problem 16.1.27. One intention of the work of Mosa, [Mos87], was to start on working out the homological algebra of algebroids (rings with several objects) by defining crossed resolutions of algebroids and obtaining a monoidal closed structure on crossed complexes of algebroids. However even the conjectured equivalence between crossed complexes of algebroids and higher dimensional cubical algebroids is unsolved. The difficulty is shown by the complexity of the arguments in [AABS02] compared with those of Chapter 13 of this book. □

Problem 16.1.28. A programme set by Grothendieck in 'Pursuing Stacks' is related to the previous problem. We quoted on p. xiv his aim to understand noncommutative cohomology of topoi. Earlier in the same letter he writes:

> For the last three weeks I haven't gone on writing the notes, as what was going to follow next is presumably so smooth that I went out for some scratchwork on getting an idea about things more obscure still, particularly about understanding the basic structure of '(possibly non-commutative) "derived categories", and the internal homotopy-flavoured properties of the "basic modelizer" (Cat), namely of functors between "small" categories,

> modelled largely on work done long time ago about étale cohomology properties of maps of schemes. I am not quite through yet but hope to resume work on the notes next week.

For work of Grothendieck on 'Modelizers' and 'Derivateurs', see [Gro89], [MaltDer]. □

Problem 16.1.29. The last problem is possibly related to the problem of relating the methods of this book to those of the modern theory of sheaves, as discussed in [Ive86], with applications to generalised Poincaré duality, known as Verdier duality. A related area is that of stratified spaces, on which a recent paper using higher order categories is [Woo10]. Stratifications are referred to in [Gro97], Section 5, while in 1983 Grothendieck wrote to Brown, see [GroPS2]:

> It seems to me, in any case, that this $\underrightarrow{\lim}$-operation ["higher order van Kampen theorem"] in the context of homotopy types is of a very fundamental character, with wide range of theoretical applications. To give just one example, relying on the existence of such a formalism, it is possible to give a very simple explicit algebraic description of the full homotopy types of the Mumford–Deligne compactifications of the modular topoi for complex curves of given genus g, say, with ν "marked" points, in terms essentia1ly of such a (finite) direct limit of $K(\pi, 1)$-spaces, where π ranges over certain "elementary" Teichmüller groups (those, roughly, corresponding to modular dimension $\leqslant 2$), and to give analogous descriptions, too, of all those subtopoi of the previous one, deducible from its canonical "stratification" at infinity by taking unions of strata. In fact, such descriptions should apply to any kind of "stratified" space or topos, as it can be expressed (in an essentially canonical way, which apparently was never made explicit yet in this literature) as a (usually finite) direct limit of simpler spaces, namely the "strata", and "tubes" around strata, and "junctions" of tubes, etc. Such a formalism was alluded to in one of my letters to Larry, in connection with so-called "tame topology" – a framework which has yet to be worked out – and I was more or less compelled lately to work it out heuristically in some detail, in order to get precise clues for working out a description of the fundamental groupoids of Mumford–Deligne–Teichmüller modular topoi (namely, essentially, of the standard Teichmüller groups), suitable for the arithmetic aspects I had in mind (namely, for a grasp of the action of the Galois group $\mathrm{Gal}_{\bar{Q}/Q}$ on the profinite completion).

However the methods of this book have not yet been applied in this area, and much work on 'tame topology' has been done since 1983. Relations between the 'crossed' techniques of this book and profinite theory are developed in the monograph [Por12]. □

Problem 16.1.30. A work on monoidal categories, Hopf algebras, species and related areas, and which strongly uses the Eilenberg–Zilber Theorem for chain complexes, is [AM10]. There are possibilities of relating their work to that done here, or bringing in crossed complexes into the areas studied in that book. □

Problem 16.1.31. Can this area of crossed complexes be helpfully related to that of complexes of groups, which generalises graphs of groups, as initiated by Haefliger in [Hae92]?

Problem 16.1.32. There is an extensive theory of quantum groups and of quantum groupoids. Can this be extended to 'quantum crossed complexes' using the methods of [Chi11], and related papers referenced there? □

Other problems in crossed complexes and related areas are given in [Bro90].

Appendices

Appendix A
A resumé of some category theory

Introduction

A categorical approach is basic to this book, and we use freely notions of category, functor, natural transformation, pushout, product category covered in the book [Bro06].

The classic paper, [EML45a], of Eilenberg and Mac Lane initiated the development of category theory. This theory gradually became in one role a great unifying theme for mathematics, giving an abstract basis for analogy and comparison across different fields and also for simplification. The benefits of abstraction are as usual: (i) covering many known examples at the same time; (ii) getting to the essence of an argument; (iii) allowing easy application to new examples. We see all of these in for example our Section B.1 on fibrations of categories.[250]

Some of our key proofs, for example of the HHSvKT, follow the pattern of: *we verify the universal property*. One importance of this is that we prove in this way that for example a particular fundamental groupoid $\pi_1(X, X_0)$ is given as a pushout of groupoids; but the proof makes no claim as to the general existence of pushouts of groupoids, nor does it show how to construct pushouts of groupoids in general. So the theorem raises questions as to how to prove existence of pushouts of groupoids, and how to construct them in practical ways.

In addition to the above topics, we need at various stages limits and colimits, equalisers and coequalisers, adjoint functors, ends and coends, cartesian closed categories, monoidal closed categories.

The other role of category theory has been to allow new algebraic structures. This is partly because a category is in the first instance a set with a *partial* algebraic structure, in which the domain of a composition is specified by a geometric condition. This allows for a definition of *higher dimensional algebra* as concerning algebraic structures with operations whose domains are specified by geometric conditions; such powerful structures can combine intuitions from algebra and geometry, and so are able to model complex underlying processes, and hence aid our understanding.

We find it difficult to give an adequate and complete coverage of what we need here, since that would be too large a task. Further, there is a considerable amount of information freely available online, including downloadable texts, or partial texts, and also web encyclopedia. Therefore the aim of this appendix is to indicate the necessary background and to supply more detail only when we can present or highlight a particular viewpoint or the material is not so accessible in the format we need. So this appendix should be supplemented with texts and downloadable material, for example Mac Lane, [ML71], and our presentation is intended to be in line with the title of that book, Adamek–Herrlich–Strecker, [AHS06] (downloadable), and many others. There is also the encyclopaedic Handbook of Categorical Algebra in 3 volumes, of which volume I is [Bor94]. Readers will also profit from accounts of these topics on Wikipedia, Planet Math, and the ncatlab.

A.1 Notation for categories

Recall that a category C has a class of *objects* $\mathrm{Ob}\,\mathsf{C}$ whose elements are denoted x, y, z, a class of *morphisms* $\mathrm{Mor}\,\mathsf{C}$ whose elements are denoted f, g, h, two maps $s, t\colon \mathrm{Mor}\,\mathsf{C} \to \mathrm{Ob}\,\mathsf{C}$ called *source* and *target* that divide the class of morphisms into disjoint sets $\mathsf{C}(x, y)$ and a partial composition $\mathsf{C}(x, y) \times \mathsf{C}(y, z) \to \mathsf{C}(x, z)$ satisfying associativity and existence of unit.[251] These *units* give another map $1\colon \mathrm{Ob}\,\mathsf{C} \to \mathrm{Mor}\,\mathsf{C}$ where $1(x) = 1_x$ is the identity on the object x.

An important point which has to be noticed is that the composition may be represented using either of two conventions, i.e. if $f \in \mathsf{C}(x, y)$ and $g \in \mathsf{C}(y, z)$ the composite may be represented either gf or fg. The first notation is taken from the composition of maps and the second one is more algebraic. The fact is that we have used both in the book! We use the first convention whenever we are working in a general setting, but we stick to the second one when working in particular groupoids.

Thus if C is a category and $f \in \mathsf{C}(x, y)$, for each $z \in \mathrm{Ob}\,\mathsf{C}$ using the composition of morphisms we may define $f_*\colon \mathsf{C}(z, x) \to \mathsf{C}(z, y)$ post-composing with f and $f^*\colon \mathsf{C}(y, z) \to \mathsf{C}(x, z)$ pre-composing with f.

The second element of interest is a *functor* $F\colon \mathsf{C} \to \mathsf{D}$ between categories. It is given by two maps $F_0\colon \mathrm{Ob}\,\mathsf{C} \to \mathrm{Ob}\,\mathsf{D}$ and $F_1\colon \mathrm{Mor}\,\mathsf{C} \to \mathrm{Mor}\,\mathsf{D}$. We denote functors by capital letters such as $F, G, H, \ldots$.

Example A.1.1. It is also important to have in mind a collection of categories and functors as examples.

1. In the definition we have stated that both objects and morphisms are classes of elements. If they are both sets, the category is called *small*. The category of small categories and functors is written Cat. If S is a small category and C is any category, then we write $\mathsf{Cat}(\mathsf{S}, \mathsf{C})$ for the class of functors $\mathsf{S} \to \mathsf{C}$ and $\mathsf{CAT}(\mathsf{S}, \mathsf{C})$ for the category of functors $\mathsf{S} \to \mathsf{C}$ and natural transformations between them.

2. The category of sets and maps is written Set.

3. The category of compactly generated topological spaces and continuous maps is written Top.

4. The category of groups and homomorphisms is written Groups.

5. A groupoid G is a small category where all morphisms are invertible.

6. The category of groupoids and morphisms of groupoids is written Gpds.

7. The category of directed graphs is written Grphs. □

The final element we are assuming is that of *natural transformation* between two functors $F, G\colon \mathsf{C} \to \mathsf{D}$. A natural transformation $\alpha\colon F \Rightarrow G$ is given by a map $\alpha\colon \operatorname{Ob}\mathsf{C} \to \operatorname{Mor}\mathsf{D}$ such that for every $x \in \operatorname{Ob}\mathsf{C}$, $\alpha(x) \in \mathsf{D}(Fx, Gx)$ and for each $f \in \mathsf{C}(x, y)$ the diagram

$$\begin{tikzcd} Fx \arrow[r, "Ff"] \arrow[d, "\alpha x"'] & Fy \arrow[d, "\alpha y"] \\ Gx \arrow[r, "Gf"'] & Gy \end{tikzcd}$$

commutes. The class of natural transformations between two functors is written $\mathsf{CAT}(F, G)$ for reasons that will become clear in Appendix C.

The natural transformation α is called a *natural equivalence* if $\alpha(x)$ is an isomorphism for each $x \in \operatorname{Ob}\mathsf{C}$.

A.2 Representable functors

We now give an introduction to the notion of representable functor: this is simple but the main result includes a pattern of argument which may not be so familiar to those not used to category theory.

Let C be a category. Then for each $y \in \operatorname{Ob}\mathsf{C}$ there is a functor $\mathsf{C}_{(y)}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ given by $\mathsf{C}_{(y)}(x) = \mathsf{C}(x, y)$ for each $x \in \operatorname{Ob}\mathsf{C}$ and $\mathsf{C}_{(y)}(x) = f_*$ for each $f \in \mathsf{C}(y, y')$ (we mention that this is a specialised notation for our purposes). An important property of such functors is the following. If $h\colon y \to z$ is a morphism in C then h induces a natural transformation

$$\mathsf{C}_{((h))}\colon \mathsf{C}_{(y)} \to \mathsf{C}_{(z)},$$

given by $\mathsf{C}_{((h))}(x) = h_*$. Thus if $f : x \to x'$ in C, we need to verify the commutativity of the diagram

$$\begin{tikzcd} \mathsf{C}(x, y) \arrow[r, "h_*"] & \mathsf{C}(x, z) \\ \mathsf{C}(x', y) \arrow[u, "f^*"] \arrow[r, "h_*"'] & \mathsf{C}(x', z). \arrow[u, "f^*"'] \end{tikzcd}$$

Indeed for any $g\colon x \to y$ the evaluation of both ways round the diagram yields hgf, so that the proof of naturality follows from associativity of the composition in C.

The converse of this result is easy to prove but turns out to be significant.[252]

Proposition A.2.1. *If $y, z \in C$ then there is a natural bijection*

$$\mathsf{CAT}(\mathsf{C}_{(y)}, \mathsf{C}_{(z)}) \to \mathsf{C}(y, z).$$

Proof. Suppose $\alpha\colon \mathsf{C}_{(y)} \to \mathsf{C}_{(z)}$ is a natural transformation, yielding for each $x \in \operatorname{Ob}\mathsf{C}$ a function $\alpha(x)\colon \mathsf{C}(x, y) \to \mathsf{C}(x, z)$. The naturality condition states that for each $f : x \to x'$ in C the first of the following diagrams is commutative:

$$\begin{tikzcd} \mathsf{C}(x,y) \arrow[r, "\alpha_x"] & \mathsf{C}(x,z) \\ \mathsf{C}(x',y) \arrow[u, "f^*"] \arrow[r, "\alpha'_x"'] & \mathsf{C}(x',z) \arrow[u, "f^*"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{C}(x,y) \arrow[r, "\alpha_x"] & \mathsf{C}(x,z) \\ \mathsf{C}(y,y) \arrow[u, "f^*"] \arrow[r, "\alpha_y"'] & \mathsf{C}(y,z) \arrow[u, "f^*"'] \end{tikzcd} \tag{A.2.1}$$

Now choose x' to be y, and set $g = \alpha_y(1_y)\colon y \to z$. In order to evaluate $\alpha_x(f)$ where $f : x \to y$ we use the second commutative diagram. Then $\alpha_x(f^*(1_y)) = \alpha_x(f)$, while $f^*\alpha_y(1_y) = f^*(g) = gf$. □

The idea can be extended.

Definition A.2.2. A functor $T : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ is called *representable* if it is naturally equivalent to a functor $\mathsf{C}_{(x)}$ for some object x of C. Then x is called a *representing object* for T, or we say T is *represented* by x.

Proposition A.2.3. *If functors $T, U : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ are represented by objects x, y of C, then there is a bijection*

$$\mathsf{CAT}(T, U) \cong \mathsf{C}(x, y).$$

In particular, a natural equivalence $T \cong U$ is determined completely by an isomorphism $x \cong y$.

The proof is easy from Proposition A.2.1.

A.3 Slice and comma categories

In this section we define some categories associated to a given category C.

Let C be a category, and let $x \in \operatorname{Ob}\mathsf{C}$.

Definition A.3.1. The *slice category* C/x or *category of objects over* x is defined as follows: its objects are morphisms $f : y \to x$ of C, i.e. elements of $\mathsf{C}(y, x)$ for all $y \in \operatorname{Ob} C$, and the elements of $\mathsf{C}/c(f, f')$ are commutative diagrams

$$\begin{tikzcd} y \arrow[rr, "g"] \arrow[dr, "f"'] & & y' \arrow[dl, "f'"] \\ & x & \end{tikzcd}$$

with the composition induced from that of C. □

This concept is in some sense behind the discussion of free crossed modules.

Let $F : \mathsf{C} \to \mathsf{B}$ be a functor of small categories. Let $y \in \operatorname{Ob} \mathsf{B}$.

Definition A.3.2. The *comma category* C/y is the category whose objects are pairs (x, s) where $x \in \operatorname{Ob} \mathsf{C}$ and $s\colon F(x) \to y$ in B. A morphism $(x, s) \to (x', s')$ in C/y is a morphism $f : x \to x'$ in C such that $s = s'F(f)$. □

Another category associated to a category C is the morphism category used in the derived module Section 7.4.ii.

Definition A.3.3. For any category C we define the category C^2, sometimes called the *morphism category* of C, to have objects the morphisms of C and morphisms $(u, v)\colon f \to g$ to be the commutative squares in C

$$\begin{tikzcd}
x \arrow[r, "u"] \arrow[d, "f"'] & y \arrow[d, "g"] \\
z \arrow[r, "v"'] & w
\end{tikzcd}$$

with composition of such morphisms the obvious horizontal one. □

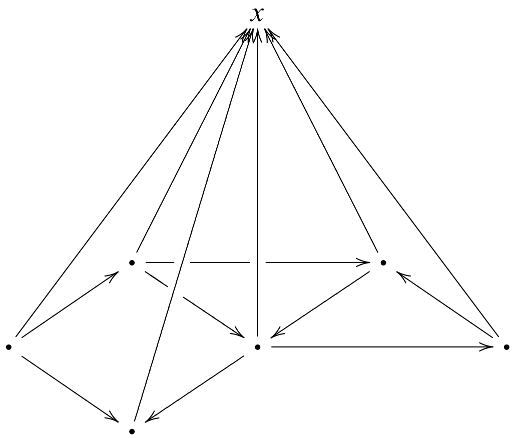
A.4 Colimits and limits

We concentrate first on the notion of colimit since this is a general concept closely related to the formulation of local-to-global properties. The idea is to give a general formulation of ‘gluing’, of putting together, a complex object using smaller pieces and rules for the gluing, to give what is called a *colimit*.

The ‘input data’ for a colimit is a diagram D, that is a collection of some objects in a category C and some morphisms between them, together with some ‘relations’, i.e. a specification that some parts of the diagram, for example the triangle τ below, are to be commutative:

$$D = \begin{tikzcd}
 & \bullet \arrow[rr] \arrow[dr] \arrow[rr, phantom, "\tau"', shift right=4] & & \bullet \arrow[dl] & \\
\bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[rr] \arrow[dl] & & \bullet \arrow[ul] \\
 & \bullet & & &
\end{tikzcd}$$

To describe the colimit output and its properties we need the following notion. A *cocone* with the diagram D as base and with vertex x consists of morphisms in C from all the objects of the diagram to the vertex x, for example as in

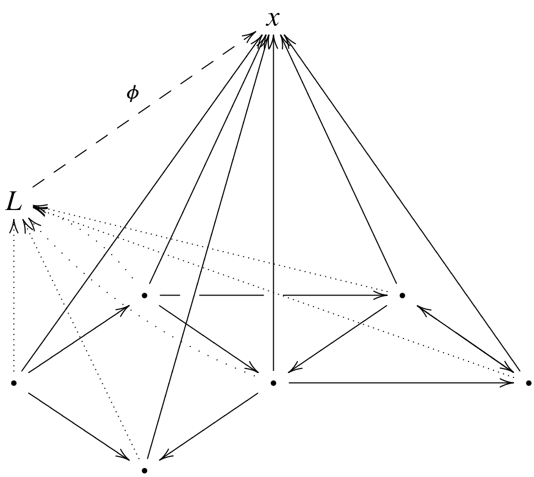


(A)

satisfying the ‘commutativity’ condition:

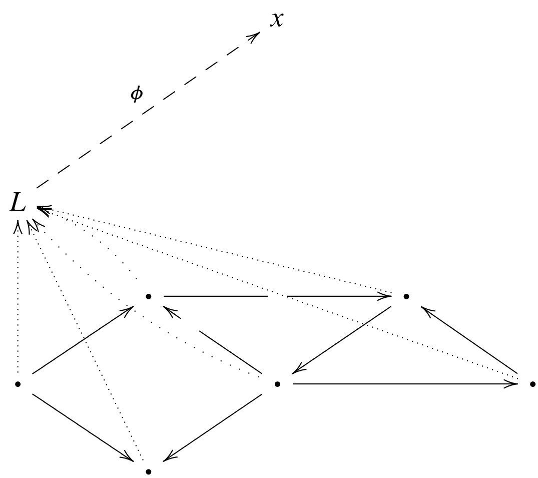
(c) any paths from a given object in D to x compose to give, subject also to the commutativity given in D, the same composite morphism.

In particular the colimit of the diagram D is an object $L = \operatorname{colim} D$ in C with a cocone with vertex L, called the colimiting cocone, shown in dotted arrows in the following diagram, with the universal property that any cocone with vertex x say factors through the colimiting cocone:



(B)

Thus any morphism $\phi : L \to x$ as in



(C)

is entirely determined by the cocone to x as in diagram (A).

Intuitions. Thus the morphism ϕ operates in a ‘distributed’ fashion, analogously to the way an email starting in L has to be split up into packets which are sent separately via the cocone to x, and then recombined at the destination x.

Example A.4.1. The lcm of two positive integers a, b can be seen as the colimit of the diagram

$$\begin{tikzcd} a & & b \\ & \mathrm{gcd}(a,b) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

in which an arrow $x \to y$ means simply that x, y are positive integers such that x divides y. The gcd, from a lower level of the hierarchy, ‘measures’ the interaction of a and b. □

Some have viewed biological organs as colimits of the diagrams of interacting cells within them.[253]

Remark A.4.2. Warning. Often colimits do not exist in a given category C for some diagrams. However, one can add colimits in a completion process, i.e. freely for a class of diagrams, and then compare these ‘virtual colimits’ with any that happen to exist. □

It is important to note that a colimit has more structure than merely the disjoint union of its individual parts, since it depends on the arrows of the diagram D as well as the objects. Thus the specification for a colimit object of the morphisms which define it can be thought of as a ‘subdivision’ of the colimit object. That is why the notion is of importance in local-to-global questions.

We now give a more formal definition. First note that it is convenient to consider not a diagram D but a small category, say D. This category can be obtained from D as the free category on the graph D factored out by relations given by the commutative cells of D. So we consider a colimit in C as defined by a functor $T\colon \mathsf{D} \to \mathsf{C}$. The colimit of T, if it exists, is an object of C, say L, but it is convenient to think of this as a constant functor $\bar{L}\colon \mathsf{D} \to \mathsf{C}$; the relation between T and $\bar{L}$ is defined to be a natural transformation $\alpha\colon T \Rightarrow \bar{L}$, called the *cocone*. The condition of being a natural transformation encompasses the commutativity condition (c) above: thus α gives for each morphism $f : x \to y$ of D a commutative diagram

$$\begin{tikzcd} Tx \arrow[rr, "Tf"] \arrow[dr, "\alpha x"'] & & Ty \arrow[dl, "\alpha y"] \\ & \bar{L}x = \bar{L}y = L. & \end{tikzcd}$$

Now we give our definition, in which we drop the bar on $\bar{L}$:

Definition A.4.3. Let D be a small category and let $T : \mathsf{D} \to \mathsf{C}$ be a functor to a category C. A *colimit* $\operatorname{colim} T$ of T is a natural transformation $\alpha\colon T \Rightarrow L$ to a constant functor, which is universal for natural transformations to constant functors: that is, if $\xi\colon T \Rightarrow L'$ is a natural transformation to a constant functor L', then there is a unique natural transformation $\phi\colon L \Rightarrow L'$ such that $\phi \circ \alpha = \xi$. (Note that a natural transformation between constant functors to C reduces to a morphism of C between their values.) If a colimit of T exists then it is unique up to natural equivalence, and is written $\operatorname{colim} T$; it is thought of either as a constant functor to C or as an object of C and it always comes with its universal cocone $T \Rightarrow \operatorname{colim} T$. Sometimes the colimit is written as $\operatorname{colim}_x T(x)$ where x ranges over the objects of D; this is useful but is an abuse of language since the morphisms of D are crucial to the definition. □

Example A.4.4. (i) A special case is the coproduct. In this case, D is the discrete category on a set of objects.

(ii) Another example is the pushout: here the diagram D has three objects, say 0, 1 and 2, and two arrows from 0, namely $0 \to 1$, $0 \to 2$.

(iii) Another example is the coequaliser: here the diagram D has two objects say 1 and 2 and two arrows $1 \rightrightarrows 2$. □

Definition A.4.5. A category C is said to be *cocomplete*, or *admits colimits*, if the colimit exists for any small category D and any functor $T : \mathsf{D} \to C$. Coproducts (coequalisers) are special cases of colimits, and if these exist we say that C admits coproducts (resp. coequalisers). □

The next proposition states that colimits may be constructed from coproducts and coequalisers.

Proposition A.4.6. *The category* C *is cocomplete if and only if it admits coproducts and coequalisers.*

Proof. We refer the reader to one of [Hig71], [ML71], [AHS06], or to web pages on limits and colimits in category theory. □

In a similar spirit, we define limits of a functor.

Definition A.4.7. Let D be a small category, and $T : \mathsf{D} \to \mathsf{C}$ a functor. A *limit* of T is a constant functor $L : \mathsf{D} \to \mathsf{C}$ and a natural transformation $\varepsilon : L \Rightarrow T$ (called the *cone* on T) with the universal property: for any natural transformation $\xi : L' \Rightarrow T$ from a constant functor L' to T, there exists a unique natural transformation $\phi : L' \Rightarrow L$ such that $\varepsilon \circ \phi = \xi$. Then L is also written $L = \lim T = \lim_x T(x)$.

A.5 Generating objects and dense subcategories

In the category of groups the infinite cyclic group C_∞ plays a key role. This suggests the following definition.

Definition A.5.1. A set S of objects in a category C is said to be *generating* C if for all pairs of morphisms $f, g : x \to y$ on objects of C, $f = g$ if and only if $fh = gh$ for all $z \in S$ and morphisms $h : z \to x$ in C. □

Example A.5.2. In the category Set of sets, any singleton is a generator. In the category Groups of groups the infinite cyclic group C_∞ is a generator. In the category Gpds of groupoids the unit interval groupoid $\mathcal{I}$ is a generator. Note that in these examples the generator identifies the elements of a group (or a groupoid in the second case), but gives no further structural information. This leads to our next definition. □

Definition A.5.3. An inclusion $K : \mathsf{D} \to \mathsf{C}$ of a subcategory of a category is called *dense in* C if D is small and for all objects x, y of C the canonical function

$$\mathsf{CAT}(\mathsf{C}(K(-), x), \mathsf{C}(K(-), y) \to \mathsf{C}(x, y) \tag{A.5.1}$$

is a bijection. □

Remark A.5.4. The meaning of this is that we can recover the morphisms $x \to y$ in C from information on the way the dense subcategory maps to x and y. Note that a universal property is defined by relating to all objects of a category: the advantage of a dense subcategory is that in principle, and for some purposes, we need look only at the objects of that dense subcategory.[254] □

Example A.5.5. The full subcategory of Groups on the object $F\{x, y\}$, the free group on the elements x, y, is dense in the category of groups. The essential part of the argument is to show that if G, H are groups, then a function $h\colon G \to H$ is a morphism of groups if and only if $hg\colon F\{x, y\} \to H$ is a morphism for every morphism $g\colon F\{x, y\} \to G$ of groups. The proof of this is a nice little exercise, as is working out the analogous example for groupoids. □

A.6 Adjoint functors

The notion of a pair of *adjoint functors* has proved a common and fruitful concept, both for its relation to universal properties, and also for its relation to preservation of limits and colimits.[255]

Definition A.6.1. To define this concept we consider two categories C and D and two functors $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{C}$. We say that F is *left adjoint* of G, which is sometimes written $F \dashv G$, (or that G is *right adjoint* of F) if there is an adjunction between them, i.e. a natural equivalence

$$\phi\colon \mathsf{D}(F(-), -) \cong \mathsf{C}(-, G(-)). \qquad \square$$

Note that $\mathsf{D}(F(-), -)$ is a functor of two variables, contravariant in the first and covariant in the second. We clarify exactly what naturality means for ϕ.

Thus we require that ϕ gives a map

$$\phi\colon \ \mathrm{Ob}\,\mathsf{C} \times \mathrm{Ob}\,\mathsf{D} \to \mathsf{Set}$$

such that for any $x \in \mathrm{Ob}\,\mathsf{C}$ and $y \in \mathrm{Ob}\,\mathsf{D}$, the map

$$\phi(x, y)\colon \mathsf{D}(F(x), y) \to \mathsf{C}(x, G(y))$$

is a bijection which is natural in both x and y, i.e. for any $f \in \mathsf{C}(x, x')$ the diagram

$$\begin{tikzcd}
\mathsf{D}(F(x), y) \arrow[r, "\phi(x,y)"] & \mathsf{C}(x, G(y)) \\
\mathsf{D}(F(x'), y) \arrow[u, "F(f)^*"] \arrow[r, "\phi(x',y)"'] & \mathsf{C}(x', G(y)) \arrow[u, "f^*"']
\end{tikzcd}$$

commutes, and for any $g \in \mathsf{D}(y, y')$ the diagram

$$\begin{tikzcd}
\mathsf{D}(F(x), y) \arrow[r, "\phi(x,y)"] \arrow[d, "g_*"'] & \mathsf{C}(x, G(y)) \arrow[d, "G(g)_*"] \\
\mathsf{D}(F(x), y') \arrow[r, "\phi(x,y')"'] & \mathsf{C}(x, G(y'))
\end{tikzcd} \qquad (*)$$

also commutes.

Example A.6.2. There are many examples of adjoint pairs coming from algebra and topology:[256]

(i) Free constructions, such as free group, free R-module, free groupoid over a directed graph, etc, are usually left adjoint of the corresponding forgetful functors (but we have offended against this by the abuse of language in referring to a *free crossed complex* in Section 7.3);

(ii) The functor $\mathrm{Ob}\colon \mathsf{Gpds} \to \mathsf{Set}$ has a left adjoint the *discrete* groupoid on a set, and a right adjoint the *indiscrete* (or *tree*) groupoid on the set.

(iii) the field of quotients of an integral domain is left adjoint to the inclusion of the category of fields in that of integral domains;

(iv) the completion of a metric space is left adjoint to the inclusion of the category of complete metric spaces in that of metric spaces;

(v) the abelianisation of a group is left adjoint to the inclusion of the category of abelian groups in that of groups.

(vi) there are two possible generalisations for the abelianisation of groupoids (see Section A.8) and both are left adjoints to the inclusion of a category in Gpds.

(vii) Another important class of examples comes from exponential laws (see Appendix C). Thus for a crossed complex C the functor $- \otimes C$ is left adjoint to the functor $\mathsf{CRS}(C, -)$ (see Equation (9.3.1)). □

We now consider some functors that are associated to any adjunction and, under some conditions, determine it. The first construction is the *unit of the adjunction*, a natural transformation

$$\eta\colon 1_{\mathsf{C}} \Rightarrow GF$$

defined by: for any $x \in \mathrm{Ob}\,\mathsf{C}$, $\eta(x)\colon x \to GF(x)$ is $\phi(1_{F(x)})$. It is easy to prove naturality. Moreover the unit is universal in the following sense:

Proposition A.6.3. *For any $x \in \mathrm{Ob}\,\mathsf{C}$, $\eta(x)$ is universal with respect to G, i.e. for any morphism $h\colon x \to G(y)$ there is a unique morphism $h'\colon F(x) \to y$ such that the diagram*

$$\begin{tikzcd}
x \arrow[r, "\eta(x)"] \arrow[d, "h"'] & GF(x) \arrow[dl, "G(h')"] \\
G(y) &
\end{tikzcd}$$

commutes.

Proof. We consider the following special case of the naturality diagram $(*)$, with $k\colon F(x) \to y$:

$$\begin{tikzcd}
\mathsf{D}(F(x), F(x)) \arrow[r, "\phi(x,F(x))"] \arrow[d, "k_*"'] & \mathsf{C}(x, GF(x)) \arrow[d, "G(k)_*"] \\
\mathsf{D}(F(x), y) \arrow[r, "\phi(x,y)"'] & \mathsf{C}(x, G(y)).
\end{tikzcd}$$

Since ϕ is an isomorphism, and $\phi(1_{F(x)}) = \eta(x)$, we deduce that $k\eta(x) = h$ if and only if $\phi(k) = h$. □

It is easy to see that we can recover the adjunction ϕ from its unit η since, for any $x \in \operatorname{Ob}\mathsf{C}$, $y \in \operatorname{Ob}\mathsf{D}$, $a \in \mathsf{D}(F(x), y)$, $\phi^{-1}(x, y)(a)$ is $\eta(x)G(a)$.

There is an analogous dual result. The *counit* of the adjoint pair is a natural transformation

$$\varepsilon\colon FG \Rightarrow 1_{\mathsf{D}}$$

For any $y \in \operatorname{Ob}\mathsf{D}$, $\varepsilon(y)\colon FG(y) \to y$ is $\phi 1_{G(y)}$. It is easy to prove naturality. Moreover the counit is universal in the following sense:

Proposition A.6.4. *For any $y \in \operatorname{Ob}\mathsf{D}$, $\varepsilon(y)$ is universal with respect to F, i.e. for any morphism $h\colon F(x) \to y$ there is a unique morphism $h'\colon G(y) \to x$ so that the diagram*

$$\begin{tikzcd}
FG(y) \arrow[r, "F(h')"] \arrow[d, "\varepsilon(y)"'] & F(x) \arrow[dl, "h"] \\
y &
\end{tikzcd}$$

commutes.

It is easy to see that we can recover the adjunction ϕ from its counit ε since $\phi(x, y)(f)$ is $\varepsilon(y)F(f)$ for any $x \in \operatorname{Ob}\mathsf{C}$, $y \in \operatorname{Ob}\mathsf{D}$ and $f \in \mathsf{C}(x, G(y))$.

A.7 Adjoint functors, limits and colimits

One of the most useful results about adjoint functors in this section is Theorem A.7.5 on preservation of limits and colimits. The result is especially important for this book because of our emphasis on the use of colimits for some calculations of homotopy invariants.

In order to obtain this main result on adjoint functors, we need a preliminary Proposition A.7.3 for which the understanding of the language is helped by simple examples, which you should verify for yourself.

Example A.7.1. Let $i_1\colon x_1 \to y$, $i_2\colon x_2 \to y$ be morphisms in a category C. Then these determine y as the coproduct in C of x_1, x_2 if and only if for all objects z of C the map

$$\mathsf{C}(y,z) \cong \mathsf{C}(x_1,z) \times \mathsf{C}(x_2,z)$$

given by $f \mapsto (f i_1, f i_2)$ is a bijection. □

Example A.7.2. Let

$$\begin{tikzcd} x_0 \arrow[r] \arrow[d] & x_1 \arrow[d] \\ x_2 \arrow[r] & y \end{tikzcd}$$

be a commutative square of morphisms in a category C. Then this diagram is a pushout if and only if for all objects z of C the induced square

$$\begin{tikzcd} \mathsf{C}(y,z) \arrow[r] \arrow[d] & \mathsf{C}(x_1,z) \arrow[d] \\ \mathsf{C}(x_2,z) \arrow[r] & \mathsf{C}(x_0,z) \end{tikzcd}$$

is a pullback of sets.

The following proposition in essence gives a restatement of the universal property for limits and colimits, and you should prove it yourself.

Proposition A.7.3. *Let A, C be categories such that A is small, and let $T\colon \mathsf{A} \to \mathsf{C}$ be a functor.*

(i) *A natural transformation $\gamma\colon T \to X$ where X is a constant functor $\mathsf{A} \to \mathsf{C}$ makes $X \cong \operatorname{colim} T$ if and only if for all $x \in C$ the induced natural transformation*

$$\gamma^*\colon \mathsf{C}(X,x) \to \mathsf{C}(T(-),x)$$

makes

$$\mathsf{C}(X,x) \cong \lim \mathsf{C}(T(-),x).$$

(ii) *A natural transformation $\lambda\colon X \to T$ where X is a constant functor $\mathsf{A} \to \mathsf{C}$ makes $X \cong \lim T$ if and only if for all $x \in C$ the induced natural transformation*

$$\lambda_*\colon \mathsf{C}(x,T(-)) \to \mathsf{C}(x,X)$$

makes

$$\mathsf{C}(x,X) \cong \lim \mathsf{C}(x,T(-)).$$

Remark A.7.4. This proposition is often stated simply as giving natural bijections

$$\mathsf{C}(\operatorname{colim} T, x) \cong \lim \mathsf{C}(T,x), \quad \mathsf{C}(x, \lim T) \cong \lim \mathsf{C}(x,T). \qquad □$$

Theorem A.7.5. *Let* $\phi\colon \mathsf{D}(F(-),-) \cong \mathsf{C}(-,G(-))$ *be an adjunction between the functors* $F\colon \mathsf{C} \to \mathsf{D}$, $G\colon \mathsf{D} \to \mathsf{C}$. *Then* F *preserves colimits, and* G *preserves limits.*

Proof. We first prove F preserves colimits. Let X be a small category and $T\colon \mathsf{X} \to \mathsf{C}$ a functor. We use the following set of natural equivalences for $y \in \mathsf{D}$:

$$\begin{aligned}
\mathsf{D}(F \operatorname{colim} T, y) &\cong \mathsf{C}(\operatorname{colim} T, Gy) && \text{by adjointness}\\
&\cong \lim \mathsf{C}(T, Gy) && \text{by Proposition A.7.3 (i)}\\
&\cong \lim \mathsf{D}(FT, y) && \text{by adjointness}\\
&\cong \mathsf{D}(\operatorname{colim} FT, y) && \text{by Proposition A.7.3 (i).}
\end{aligned}$$

By the representability Proposition A.2.3, there is a natural isomorphism $F \operatorname{colim} T \cong \operatorname{colim} FT$.

A similar argument, using $\mathsf{D}(y, \lim S) \cong \lim \mathsf{D}(y, S)$, proves that G preserves limits. □

Remark A.7.6. This result is very useful in quite basic constructions in topology and algebra. For example, it is standard that the forgetful functor $U\colon \mathsf{Top} \to \mathsf{Set}$ giving the underlying set of a topological space has left and right adjoints, given respectively by the discrete, and the indiscrete topologies on a set. Hence the underlying set of the product of topological spaces is the product of the underlying sets. The property we want of the product of spaces is the universal property, since this enables one to construct continuous functions into the product. Thus the categorical approach is not a luxury but a practical tool.

Again, when we have an hierarchical mathematical structure, we can often use Theorem A.7.5 to calculate limits and colimits at a given level in terms of those of lower levels, provided the relevant adjoint functors exist. □

A.8 Abelianisations of groupoids

We use at several points the well-known abelianisation of a group. This construction gives a functor

$$^{ab}\colon \mathsf{Groups} \to \mathsf{Ab}$$

which is left adjoint to the inclusion of categories

$$\mathsf{Ab} \to \mathsf{Groups}.$$

This fits with the following definition.

Definition A.8.1. A subcategory D of a category C is a *reflexive subcategory* (or *reflective subcategory*) if the inclusion $\mathsf{D} \to \mathsf{C}$ has a left adjoint. □

We define a groupoid G to be *abelian* if all its vertex groups are abelian groups. The *abelianisation of a groupoid* G, which we write G^{ab}, is obtained by quotienting G with the normal subgroupoid generated by the commutators of all vertex groups.

This left adjoint to the inclusion of abelian groupoids into groupoids is applied at several places in the book, from Chapter 7 onwards: in Chapter 7 it is used in Proposition 7.1.8 and the definition of modulisation of a crossed module of groupoids; in Definition 7.1.13 and the restriction functor in dimension 2 and many places in Section 7.4 where we explain the construction of the derived chain complex ∇C of a crossed complex C.

Nonetheless, once in this same Section 7.4, in Exercise 7.4.26, and later on in Section 14.7, we use another kind of abelianisation that associates to each groupoid not an abelian groupoid but an abelian group and a morphism $\upsilon : G \to G^{\mathrm{totab}}$ which is universal for morphisms to abelian groups. We call G^{totab} the *universal abelianisation* of the groupoid G.

Let Ab, Groups, Gpds denote respectively the categories of abelian groups, groups, and groupoids. Each of the inclusions

$$\mathsf{Ab} \to \mathsf{Groups} \to \mathsf{Gpds} \tag{A.8.1}$$

has a left adjoint. That from groupoids to groups is called the *universal group* UG of a groupoid G and is described in detail in [Hig71]and [Bro06], Section 8.1. In particular, the universal group of a groupoid G is the free product of the universal groups of the transitive components of G. Any transitive groupoid G may be written in a non-canonical way as the free product $G(a_0) * T$ of a vertex group $G(a_0)$ and an indiscrete or tree groupoid T.[257] Then

$$UG \cong G(a_0) * UT$$

and UT is the free group on the elements $x : a_0 \to a$ in T for all $a \in \mathrm{Ob}(T)$, $a \neq a_0$.

It follows that the universal abelianisation G^{totab} is isomorphic to the usual abelianisation of the group UG and also that it is isomorphic to the direct sum of the G_i^{totab} over all components G_i of G. So for a transitive groupoid G with $a_0 \in \mathrm{Ob}\, G$

$$G^{\mathrm{totab}} \cong G(a_0)^{\mathrm{ab}} \oplus F$$

where F is the free abelian group on the elements $x : a_0 \to a$ in T for all $a \in \mathrm{Ob}(T)$, $a \neq a_0$, for T a wide tree subgroupoid of G.

A.9 Coends and ends

For this account we need the following notion, in which we assume C is a small category.[258]

Definition A.9.1. Let $S, T\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{A}$ be two functors. A *dinatural transformation* from S to T, written $\alpha\colon S \stackrel{\cdot\cdot}{\to} T$, is a function which to every object x of C associates an morphism $\alpha(x)\colon S(x,x) \to T(x,x)$ of A with the property that for every morphism $f : x \to y$ of C the following diagram commutes:

$$\begin{tikzcd}
 & S(x,x) \arrow[rr, "\alpha_x"] & & T(x,x) \arrow[dr, "T(1,f)"] & \\
S(y,x) \arrow[ur, "S(f,1)"] \arrow[dr, "S(1,f)"] & & & & T(x,y) \\
 & S(y,y) \arrow[rr, "\alpha_y"'] & & T(y,y) \arrow[ur, "T(f,1)"'] &
\end{tikzcd} \tag{A.9.1}$$

□

Definition A.9.2. Let $S\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{A}$ be a functor, and let $a \in \mathrm{Ob}\,\mathsf{A}$. A *dicocone* from S to a is a dinatural transformation $\alpha\colon S \stackrel{\cdot\cdot}{\to} \bar{a}$ where $\bar{a}\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{A}$ is the constant functor with value a. □

Definition A.9.3. Let $S\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{A}$ be a functor, and let $a \in \mathrm{Ob}\,\mathsf{A}$. We say *$a$ is a coend of S* and write

$$a = \int^{\mathsf{C},x} S(x,x)$$

if there is a couniversal dicocone $u\colon S \stackrel{\cdot\cdot}{\to} \bar{a}$. This means that if $v\colon S \stackrel{\cdot\cdot}{\to} \bar{b}$ is any other dicocone, then there is a unique morphism $f : a \to b$ in A such that $\bar{f}u = v$. □

The following proposition is easy to prove.

Proposition A.9.4. *If a is a coend of S, and A admits coproducts then a is isomorphic to the coequaliser of the diagram*

$$\coprod_{x,y \in \mathrm{Ob}\,\mathsf{C}} S(x,y) \underset{j}{\overset{i}{\rightrightarrows}} \coprod_{x \in \mathrm{Ob}\,\mathsf{C}} S(x,x)$$

where i, j are induced by

$$S(f, 1_x)\colon S(y,x) \to S(x,x), \quad S(1_y, f)\colon S(y,x) \to S(y,y)$$

respectively for $f \in \mathsf{C}(x,y)$. □

Dually we have the notion of end.

Definition A.9.5. Let $S\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{A}$ be a functor, and let $a \in \mathrm{Ob}\,\mathsf{A}$. We say *$a$ is an end of S* and write

$$a = \int_{\mathsf{C},x} S(x,x)$$

if there is a universal dicone $u\colon \bar{a} \stackrel{\cdot\cdot}{\to} S$. This means that is if $v\colon \bar{b} \stackrel{\cdot\cdot}{\to} S$ is any other dicone, then there is a unique morphism $h\colon b \to a$ in A such that $u\bar{h} = v$. □

The following proposition is easy to prove.

Proposition A.9.6. *If b is an end of S, and* A *admits products then b is isomorphic to the equaliser of the diagram*

$$\prod_{x \in \mathrm{Ob}\,\mathsf{C}} S(x,x) \overset{i}{\underset{j}{\rightrightarrows}} \prod_{x,y \in \mathrm{Ob}\,\mathsf{C}} S(x,y)$$

where i, j are induced by

$$S(f, 1_x) \colon S(y,y) \to S(x,y), \quad S(1_x, f) \colon S(x,x) \to S(x,y)$$

respectively for $f \in \mathsf{C}(x,y)$. □

Example A.9.7. Let $F, G \colon \mathsf{C} \to \mathsf{A}$ be functors. Then the set $\mathsf{CAT}(F,G)$ of natural transformations $F \Rightarrow G$ can be described as the end

$$\int_{\mathsf{C},x} \mathsf{C}(F(x), G(x)).$$ □

Example A.9.8. Both the geometric realisation of a cubical set K studied in Section 11.1.iii and the two geometric realisations of simplicial sets described in the next section are nice examples of coends. □

Remark A.9.9. The interplay between ends, coends and morphism sets is well shown in the proof of Proposition 11.4.9 on the adjointness of the functors Π and N between Cub and Crs. □

A.10 Simplicial objects

The notion of simplicial object is fundamental in homology theory and in algebraic topology. Here we state some standard definitions, partly to fix the notation, and refer the reader to other texts and downloadable material for more information.

Definition A.10.1. A *simplicial object in a category* C is a family $K = \{K_n\}_{n \geqslant 0}$ of objects of C together with face operations $\partial_i \colon K_n \to K_{n-1}$ for $n \geqslant 1$ and *degeneracy operations* $\varepsilon_i \colon K_n \to K_{n+1}$ for $i = 0, \ldots, n$ and $n \geqslant 0$, satisfying the usual simplicial relations.

$$\begin{aligned}
\partial_i \partial_j &= \partial_{j-1} \partial_i && \text{if } i < j,\\
\varepsilon_i \varepsilon_j &= \varepsilon_{j+1} \varepsilon_i && \text{if } i \leqslant j,\\
\partial_i \varepsilon_j &= \begin{cases} \varepsilon_{j-1} \partial_i & \text{if } i < j,\\ 1 & \text{if } i = j,\ j+1,\\ \varepsilon_j \partial_{i-1} & \text{if } i > j+1. \end{cases}
\end{aligned}$$

It is standard to consider K also as a functor $K\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{C}$, where $\mathbf{\Delta}$ is the category called the *simplicial site*: its objects are the sets

$$[n] = \{0, 1, \dots, n\}$$

for all $n \geqslant 0$ and the morphisms $[m] \to [n]$ are the nondecreasing maps. The category of simplicial sets, i.e. when $\mathsf{C} = \mathsf{Set}$, is written Simp.[259]

Example A.10.2. The standard example is the singular simplicial set $S^{\Delta}X$ of a topological space X, which in dimension n is the set of continuous maps $\Delta^n \to X$. □

We shall also need the notion of simplicial set without degeneracies, namely a functor $U\colon \Upsilon^{\mathrm{op}} \to \mathsf{Set}$ where Υ is the wide subcategory of $\mathbf{\Delta}$ consisting of injective, and so strictly increasing, maps. Clearly any simplicial set K determines by means of the inclusion $i\colon \Upsilon^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}}$ a simplicial set without degeneracies, $K \circ i$, which we shall call a *presimplicial set*.[260] The category of presimplicial sets is written Υ-Set.

A *cosimplicial object* in a category C is a functor $\mathbf{\Delta} \to \mathsf{C}$. As an example, the Yoneda Lemma implies that the functor $\Delta\colon \mathbf{\Delta} \to \mathsf{Simp}$ whose value on $[n]$ is the functor $\Delta[n] = \mathbf{\Delta}(-, [n])$ gives a full embedding such that any object of Simp is a colimit of objects from the image of Δ. The standard geometric simplex may be realized as the set of points $(x_1, \dots, x_{n+1})$ in $\mathbb{R}^{n+1}$ satisfying

$$x_i \geqslant 0, \quad \text{and} \quad x_1 + \cdots + x_{n+1} = 1,$$

or alternatively, as is convenient for some purposes, such as products, as the set of points $(x_1, \dots, x_n)$ in $\mathbb{R}^n$ such that

$$0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant 1.$$

This may be given as a cosimplicial space $\Delta\colon \mathbf{\Delta} \to \mathsf{Top}$.

There is a *geometric realisation functor* on simplicial sets

$$|\ \ |\colon \mathsf{Simp} \to \mathsf{Top}$$

given as a coend

$$|K| = \int^{\mathbf{\Delta}, n} K_n \times \Delta^n.$$

This is a CW-complex with one cell for each nondegenerate element of K. In particular, we note that $|\Delta[n]| = \Delta^n$. It follows that $|\Delta[n]|$ is contractible.

A very convenient feature of simplicial sets is that the geometric realisation functor behaves well with respect to products: if K, L are simplicial sets, there is a natural homeomorphism

$$|K \times L| \cong |K| \times |L|,$$

where $\times$ denotes the categorical product in their respective categories.

There is also a *geometric realisation functor* on presimplicial sets

$$\| \ \| \colon \Upsilon\text{-}\mathsf{Set} \to \mathsf{Top}$$

given as a coend

$$\|U\| = \int^{\Upsilon,n} U_n \times \Delta^n,$$

which is a CW-complex with one cell for each element of U. Hence a simplicial set K has also *thick geometric realisation*, namely $\|K \circ i\|$, which is also written $\|K\|$, and which is the geometric realisation of the corresponding presimplicial set.

One of the problems with presimplicial sets is that while there is an n-simplex presimplicial set which we write $\Upsilon[n] = \Upsilon(-, [n])$, the presimplicial set $U \times \Upsilon[1]$ using the cartesian product does not give a useful model for homotopies.[261] However let f and g be two morphisms $U \to V$ of $\Upsilon\text{-}\mathsf{Set}$, and suppose there exists, for every p and every $i \leqslant p$, functions $k_i : U_p \to V_{p+1}$ verifying the following equalities:

$$\begin{aligned} \partial_i k_j &= \begin{cases} k_{j-1}\partial_i & \text{if } i < j, \\ k_j \partial_{i-1} & \text{if } i > j+1, \end{cases} \\ \partial_{i+1} k_{i+1} &= \partial_{i+1} k_i, \\ \partial_0 k_0 &= g_p, \\ \partial_{p+1} k_p &= f_p. \end{aligned}$$

These equalities ensure that the k_i for all i, p fit together as if they were a subdivision of a cylinder.

Lemma A.10.3. *The two maps $\|f\|$ and $\|g\| \colon \|U\| \to \|V\|$ are homotopic.*

Proof. The proof is by induction on the skeleta and passing to the colimit. □

We apply the lemma to the following situation. Let $U = V = \Upsilon[n]$, and if $\phi \colon [p] \to [q]$ is an element of $\Upsilon[n]_p$ we put

$$k_i(\phi)(j) = \begin{cases} 0 & \text{if } j \leqslant i, \\ \phi(j-1) & \text{if } j > i. \end{cases}$$

Then $k_i(\phi) \in \Upsilon[n]_{p+1}$ and it is easy to show that the aforesaid relations hold if one takes $g = \mathrm{id}$ and f being defined by $f(\phi)(j) = 0$ for all j. By the lemma, this shows that $\|\Delta[n]\|$ and $\|\Delta[0]\|$ have the same homotopy type. To show that $\|\Delta[n]\|$ is contractible it suffices therefore to prove the case $n = 0$.

Now $\Upsilon[0]$ contains only one simplex in each degree and its geometric realisation is a CW-complex V with only one cell in each dimension. One sees easily that the skeleta of V are $V_0 = *$, a point, $V_1 = S^1$, $V_2 \simeq *$ since the attaching maps of the 2-cell is homotopic to the identity $\mathrm{id} \colon S^1 \to S^1$. Then one convinces oneself that the even skeleta have the homotopy type of a point, and the odd skeleta the homotopy type of a sphere, more precisely, $V_{2p+1} \simeq S^{2p+1}$. So V is contractible. One can also give explicitly a homotopy which shows directly that $\|\Upsilon[0]\|$ retracts by deformation to a point.

Simplicial sets also play an important role in the theory of model categories for homotopy theory, see for example [Hir03], [Mal05].

A.10.i Crossed complexes, ω-groupoids and simplicial sets

A simplicial version of ω-groupoids was developed by Keith Dakin, [Dak77], and Nick Ashley, [Ash88], in their Bangor doctoral theses, using the notion of simplicial T-complex. However an explicit description of the tensor product of simplicial T-complexes, equivalent to that for crossed complexes, is unknown, and simplicial sets have not had formulated notions of multiple compositions, analogous to those we have given for cubical sets.

Nonetheless, simplicial sets play an important role in discussing nerves of categories and in formulating notions of weak ω-categories, see for example [Ver08b], [JT07].

The geometric realisations of simplicial or presimplicial sets are CW-complexes and so have skeletal filtrations. In particular the skeletal filtration of $|K|$ for a simplicial set K is written $|K|_*$. So the fundamental crossed complex functor defines a functor $\mathsf{Simp} \to \mathsf{Crs}$, also written Π. We have evaluated $\Pi(\Delta^n_*)$ as a crossed complex $a\Delta^n$ in Theorem 9.9.4, which gives the Homotopy Addition Lemma for a simplex. Thus for a simplicial set K the crossed complex ΠK is freely generated by the nondegenerate simplices of K with boundary given by the Homotopy Addition Lemma.

Notes

250 p. 555 The greatest contributions to the advance of category theory, and indeed of mathematics in the 20th century, have been made by Alexander Grothendieck. Although he apparently retired from mathematical contacts in 1970, he continued to write and advance his thoughts, and in the period 1982–1991 made these thoughts available to a number of people who have distributed and developed them, see for example [GroPS1], [GroPS2], [Gro89]. As an example of his attitude we give the following quotation from a letter to Ronnie Brown dated 12/04/1983:

> The question you raise "how can such a formulation lead to computations" doesn't bother me in the least! Throughout my whole life as a mathematician, the possibility of making explicit, elegant computations has always come out by itself, as a byproduct of a thorough conceptual understanding

> of what was going on. Thus I never bothered about whether what would come out would be suitable for this or that, but just tried to understand – and it always turned out that understanding was all that mattered.

For an elementary discussion of the role of category theory, see [BP06].

251 p. 556 The reader should turn to other sources such as books or the web for a discussion of the distinction between classes and sets.

252 p. 557 J. H. C. Whitehead once remarked in a seminar in response to a statement that 'The proof is trivial.': 'It is the snobbery of the young to suppose that a theorem is trivial because the proof is trivial!'

253 p. 561 Such an account of colimits in system theory is developed in [EV08]. See also [BP03]; our account of colimits is a modification of that in this paper.

254 p. 563 For more discussion on this notion and relation to the Yoneda embedding, see [Pra09].

255 p. 564 This concept was defined by Kan in [Kan58a] and has proved a central concept of category theory and its applications. In particular, it gave further background to the notion of universal property, and has even allowed a strong generalisation of Galois Theory, see [BJ01].

256 p. 565 For examples of adjoint functors, in fact exponential laws, in analysis see [KM97].

257 p. 569 This classification of transitive groupoids has been used to suggest that 'groupoids reduce to groups'; but this is analogous to suggesting that vector spaces reduce to numbers!

258 p. 569 This brief introduction could be supplemented by accounts in [ML71] and other category theory books, and include the discussion on the ncatlab.

259 p. 572 There is a large literature on simplicial objects and simplicial sets: see for example [ML63], [May67], [GZ67], [ML71], [GJ99], [Ina97]. There is also much downloadable material. Simplicial sets also play an important role in the theory of model categories for homotopy theory, see for example [Hir03], [Mal05].

260 p. 572 The term Δ-set is used in [RS71] for what we call an presimplicial set. The problem is that the symbol Δ is overused.

261 p. 573 The following argument is due to M. Zisman (private communication, 2009), and we are grateful for this information. There is further discussion of a different monoidal structure for their Δ-sets (our presimplicial sets) in [RS71]. They also

discuss the relation between the normalised and unnormalised realisations, but state only that the natural map induces isomorphisms of homology and of fundamental groups. The normalisation theorem was also proved directly in [BS07], following an analogue of the proof for chain complexes in [ML63]. Our presimplicial sets have also been called *presimplicial sets*.

Appendix B
Fibred and cofibred categories

B.1 Fibrations of categories

The notion of fibration of categories is intended to give a general background to constructions analogous to pullback by a morphism. It seems to be a very useful notion for dealing with hierarchical structures. A functor which forgets the top level of structure is often usefully seen as a fibration or cofibration of categories.

Definition B.1.1. Let $\Phi\colon \mathsf{C} \to \mathsf{B}$ be a functor. A morphism $\phi\colon Y \to X$ in C is called *cartesian over* $u = \Phi(\phi)$, or simply *cartesian*, if and only if for all $v\colon K \to J$ in B and $\theta\colon Z \to X$ with $\Phi(\theta) = uv$ there is a unique morphism $\psi\colon Z \to Y$ with $\Phi(\psi) = v$ and $\theta = \phi\psi$.[262]

This is illustrated by the following diagram:

$$\begin{tikzcd}
Z \arrow[r, dotted, "\psi"'] \arrow[rr, bend left, "\theta"] & Y \arrow[r, "\phi"'] & X & {} \arrow[d, "\Phi"] \\
K \arrow[r, "v"'] \arrow[rr, bend left, "uv"] & J \arrow[r, "u"'] & I & {}
\end{tikzcd}$$

□

It is straightforward to check that cartesian morphisms are closed under composition, and that ϕ is an isomorphism if and only if ϕ is a cartesian morphism over an isomorphism.

A morphism $\alpha\colon Z \to Y$ is called *vertical* (with respect to Φ) if and only if $\Phi(\alpha)$ is an identity morphism in B. We use a special notation here and for $I \in \operatorname{Ob}\mathsf{B}$ write C_I, called the *fibre over* I, for the subcategory of C consisting of those morphisms α with $\Phi(\alpha) = 1_I$. Thus $X \in \operatorname{Ob}\mathsf{C}_I$ if and only if $\Phi(X) = I$.

Definition B.1.2. The functor $\Phi\colon \mathsf{C} \to \mathsf{B}$ is a *fibration* or *category fibred over* B if and only if for all $u\colon J \to I$ in B and $X \in \operatorname{Ob}\mathsf{C}_I$ there is a cartesian morphism $\phi\colon Y \to X$ over u: such a ϕ is called a *cartesian lifting* of X along u.

Notice that cartesian liftings of $X \in \mathsf{C}_I$ along $u\colon J \to I$ are unique up to vertical isomorphism: if $\phi\colon Y \to X$ and $\phi'\colon Y' \to X$ are cartesian over u, then there exist vertical morphisms $\alpha\colon Y' \to Y$ and $\beta\colon Y \to Y'$ with $\phi\alpha = \phi'$ and $\phi'\beta = \phi$ respectively, from which it follows by cartesianness of ϕ and ϕ' that $\alpha\beta = 1_Y$ and $\beta\alpha = 1_Z$ as $\phi'\beta\alpha = \phi\alpha = \phi' = \psi 1_Y$ and similarly $\phi\beta\alpha = \phi 1_Y$.

Example B.1.3. The forgetful functor, Ob: $\mathsf{Gpds} \to \mathsf{Set}$, from the category of groupoids to the category of sets is a fibration. We can for a groupoid G over I and function $u\colon J \to I$ define the cartesian lifting $\phi\colon H \to G$ as follows: for $j, j' \in J$ set

$$H(j, j') = \{(j, g, j') \mid g \in G(uj, uj')\}$$

with composition

$$(j_1, g_1, j_1')(j, g, j') = (j_1, g_1 g, j'),$$

with ϕ given by $\phi(j, g, j') = g$. The universal property is easily verified. The groupoid H is usually called the *pullback* of G by u and denoted u^*G. This is a well-known construction (see for example [Mac05], §2.3, where the pullback by u is written $u^{\downarrow\downarrow}$). □

Definition B.1.4. If $\Phi\colon \mathsf{C} \to \mathsf{B}$ is a fibration, then using the axiom of choice for classes we may select for every $u\colon J \to I$ in B and $X \in \mathsf{C}_I$ a cartesian lifting of X along u

$$u^X\colon u^*X \to X.$$

Such a choice of cartesian liftings is called a *cleavage* or *splitting* of Φ.[263]

If we fix the morphism $u\colon J \to I$ in B, the splitting gives a so-called *reindexing functor*

$$u^*\colon \mathsf{C}_I \to \mathsf{C}_J$$

defined on objects by $X \mapsto u^*X$ and the image of a morphism $\alpha\colon X \to Y$ is $u^*\alpha$ the unique vertical morphism commuting the diagram:

$$\begin{tikzcd}
u^*X \arrow[r, "u^X"] \arrow[d, dashed, "u^*\alpha"'] & X \arrow[d, "\alpha"] \\
u^*Y \arrow[r, "u^Y"'] & Y
\end{tikzcd}$$

□

We can use this reindexing functor to get an adjoint situation for each $u\colon J \to I$ in B.

Proposition B.1.5. *Suppose $\Phi\colon \mathsf{C} \to \mathsf{B}$ is a fibration of categories, $u\colon J \to I$ in B, and a reindexing functor $u^*\colon \mathsf{C}_I \to \mathsf{C}_J$ is chosen. Then there is a bijection*

$$\mathsf{C}_J(Y, u^*X) \cong \mathsf{C}_u(Y, X)$$

natural in $Y \in \mathsf{C}_J$, $X \in \mathsf{C}_I$ where $\mathsf{C}_u(Y, X)$ consists of those morphisms $\alpha \in \mathsf{C}(Y, X)$ with $\Phi(\alpha) = u$.

Proof. This is just a restatement of the universal properties concerned. □

In general for composable maps $u\colon J \to I$ and $v\colon K \to J$ in B it does not hold that

$$v^*u^* = (uv)^*$$

as may be seen with the fibration of Example B.1.3. Nevertheless there is a natural equivalence $c_{u,v}\colon v^*u^* \simeq (uv)^*$ as shown in the following diagram in which the full arrows are cartesian and where $(c_{u,v})_X$ is the unique vertical morphism making the diagram commute:

$$\begin{tikzcd} v^*u^*X \arrow[r, "v^{u^*X}"] \arrow[d, dashed, "(c_{u,v})_X"', "\cong"] & u^*X \arrow[d, "u^X"] \\ (uv)^*X \arrow[r, "(uv)^X"'] & X\,. \end{tikzcd}$$

Let us consider this phenomenon for our main examples:

Example B.1.6. 1. Typically, for $\Phi_{\mathsf{B}} = \partial_1\colon \mathsf{B}^2 \to \mathsf{B}$, the fundamental fibration for a category with pullbacks, we do not know how to choose pullbacks in a functorial way.

2. In considering the functor $\mathrm{Ob}\colon \mathsf{Gpds} \to \mathsf{Set}$ form groupoids to sets as a fibration, if $u\colon J \to I$ is a map, we have a reindexing functor, the pullback $u^*\colon \mathsf{Gpds}_I \to \mathsf{Gpds}_J$ of Example B.1.3. We notice that v^*u^*G is naturally isomorphic to, but not identical to, $(uv)^*G$. □

A result which aids understanding of our calculation of pushouts and some other colimits of groupoids, modules, crossed complexes and higher categories is the following. Recall that a category C is *connected* if for any $c, c' \in \mathrm{Ob}\, C$ there is a sequence of objects $c_0 = c, c_1, \dots, c_{n-1}, c_n = c'$ such that for each $i = 0, \dots, n-1$ there is a morphism $c_i \to c_{i+1}$ or $c_{i+1} \to c_i$ in C. The sequence of morphisms arising in this way is called a *zig-zag* from c to c' of length n.

Theorem B.1.7. *Let $\Phi\colon \mathsf{C} \to \mathsf{B}$ be a fibration and let $J \in \mathrm{Ob}\,\mathsf{B}$. Then the inclusion $i_J\colon \mathsf{C}_J \to \mathsf{C}$ preserves colimits of connected diagrams.*

Proof. This result is proved in the paper [BS09]. We include here a short proof due to G. Janelidze in the case of most interest to us, when Φ has a right adjoint and C is cocomplete.

Let $j\colon \mathsf{C}_J \to \mathsf{C}$ be the inclusion. Let T be a functor from a small connected category S to C_J. Then $\Phi j T$ is the constant functor with value $\{J, 1_J\}$. Since S is connected this implies that $\operatorname{colim} \Phi j T = J$. Since C is cocomplete, $\operatorname{colim} jT$ exists. Since Φ has a right adjoint, it preserves colimits; and so $\Phi(\operatorname{colim} jT) = \operatorname{colim} \Phi T = J$. So $\operatorname{colim} jT$ is also in C_J and therefore is $\operatorname{colim} T$. □

Remark B.1.8. The connectedness assumption is essential in the theorem. Any small category C is the disjoint union of its connected components. If $T\colon \mathsf{S} \to \mathsf{C}$ is a functor, and C has colimits, then colim T is the coproduct (in C) of the colim T_i where T_i is the restriction of T to a component S_i. But given two objects in the same fibre of $\Phi\colon \mathsf{C} \to \mathsf{B}$, their coproduct in that fibre is in general not the same as their coproduct in X. For example, the coproduct of two groups in the category of groups is the free product of groups, while their coproduct as groupoids is their disjoint union. □

Remark B.1.9. A common application of the theorem is that the inclusion $\mathsf{C}_J \to \mathsf{C}$ preserves pushouts. This is relevant to our applications of pushouts in Section B.3. Pushouts are relevant to coproducts of crossed P-modules, see Proposition 4.3.1, are used to construct free crossed modules as a special case of induced crossed modules, as explained in Section 5.2, and to construct free crossed complexes as explained in Section 7.3. □

Exercise B.1.10. Let $\Phi\colon \mathsf{C} \to \mathsf{B}$ be a fibration and let $u\colon I \to J$ in B. Let morphisms $a_\lambda\colon X_\lambda \to X, \lambda \in \Lambda$ in C_I determine X as the coproduct in C_I of the X_λ. Prove the universal property: if $f_\lambda\colon X_\lambda \to Y$ is a family of morphisms of C over u, then there is a unique morphism $f\colon X \to Y$ over u such that $fa_\lambda = f_\lambda$ for all λ. Extend this result to colimits in C_I rather then just coproducts. Discuss the application of this property to coproducts of crossed P-modules. □

Exercise B.1.11. Let $\Phi\colon \mathsf{C} \to \mathsf{B}$ be a fibration and let $u\colon I \to J$ be an morphism of B. Let D be a small category and $T\colon \mathsf{D} \to \mathsf{C}$ a functor with image in C_I and let $\eta\colon T \Rightarrow X$ be a cocone in C such that:

(i) $\Phi(\eta d) = u$ for all $d \in \operatorname{Ob} \mathsf{D}$;

(ii) η is couniversal for this property, i.e. if $\eta'\colon T \Rightarrow X'$ is any cocone in C such that $\Phi(\eta' d) = u$ for all $d \in \operatorname{Ob} \mathsf{D}$, then there is a unique morphism $f\colon X \to X'$ in C_J such that $f\eta = \eta'$.

Prove that the following property holds: if further $v\colon J \to K$ in B, and $\gamma\colon T \Rightarrow Y$ is a cocone in C such that $\Phi(\eta d) = vu$ for all $d \in \operatorname{Ob} \mathsf{D}$, then there is a unique morphism $g\colon X \to Y$ over v such that $g\eta = \gamma$. Discuss the application of this property to induced crossed modules. □

B.2 Cofibrations of categories

We now give the duals of the above results.

Definition B.2.1. Let $\Phi\colon \mathsf{C} \to \mathsf{B}$ be a functor. A morphism $\psi\colon Z \to Y$ in C over $v := \Phi(\psi)$ is called *cocartesian* if and only if for all $u\colon J \to I$ in B and $\theta\colon Z \to X$ with $\Phi(\theta) = uv$ there is a unique morphism $\phi\colon Y \to X$ with $\Phi(\phi) = u$ and $\theta = \phi\psi$.

This is illustrated by the following diagram:

$$\begin{tikzcd}
Z \arrow[r, "\psi"'] \arrow[rr, bend left, "\theta"] & Y \arrow[r, dotted, "\phi"'] & X \arrow[d, phantom, ""] & {} \arrow[dd, "\Phi"] \\
& & & \\
K \arrow[r, "v"'] \arrow[rr, bend left, "uv"] & J \arrow[r, "u"'] & I & {}
\end{tikzcd}$$

□

It is straightforward to check that cocartesian morphisms are closed under composition, and that $\psi: Z \to Y$ is an isomorphism if and only if ψ is a cocartesian morphism over an isomorphism.

Definition B.2.2. The functor $\Phi: \mathsf{C} \to \mathsf{B}$ is a *cofibration* or *category cofibred over* B if and only if for all $v: K \to J$ in B and $Z \in \mathsf{C}_K$ there is a cocartesian morphism $\psi: Z \to Z'$ over v: such a ψ is called a *cocartesian lifting* of Z along v. □

The cocartesian liftings of $Z \in \mathsf{C}_K$ along $v: K \to J$ are also unique up to vertical isomorphism.

Remark B.2.3. As in Definition B.1.4, if $\Phi: \mathsf{C} \to \mathsf{B}$ is a cofibration, then using the axiom of choice for classes we may select for every $v: K \to J$ in B and $Z \in \mathsf{C}_K$ a cocartesian lifting of Z along v

$$v_Z: Z \to v_*Z.$$

Under these conditions, the functor v_* is commonly said to give the objects *induced* by v. Examples of induced crossed modules of groups are developed in Chapter 5. □

We now have the dual of Proposition B.1.5.

Proposition B.2.4. *For a cofibration $\Phi: \mathsf{C} \to \mathsf{B}$, a choice of cocartesian liftings of $v: K \to J$ in B yields a functor $v_*: \mathsf{C}_K \to \mathsf{C}_J$, and an adjointness*

$$\mathsf{C}_J(v_*Z, Y) \cong \mathsf{C}_v(Z, Y)$$

for all $Y \in \mathsf{C}_J$, $Z \in \mathsf{C}_K$.

We now state the dual of Theorem B.1.7.

Theorem B.2.5. *Let $\Phi: \mathsf{C} \to \mathsf{B}$ be a category cofibred over B. Then the inclusion of each fibre of Φ into C preserves limits of connected diagrams.*

Many of the examples we are interested in are both fibred and cofibred. For them we have an adjoint situation.

Proposition B.2.6. *For a functor $\Phi\colon \mathsf{C} \to \mathsf{B}$ which is both a fibration and cofibration, and a morphism $u\colon J \to I$ in B, a choice of cartesian and cocartesian liftings of u gives an adjointness*

$$\mathsf{C}_J(Y, u^*X) \cong \mathsf{C}_I(u_*Y, X)$$

for $Y \in \mathsf{C}_J$, $X \in \mathsf{C}_I$.

Remark B.2.7. There are interesting circumstances where u^* has a right adjoint: this is discussed in the case of the forgetful functor $\mathsf{Top} \to \mathsf{Set}$ in [BB78], Section 5, and is related to the 'fundamental theorem of topos theory', [Joh02], A2.3. □

It is interesting to get a characterisation of the cofibration property for a functor that already is a fibration. The following is a useful weakening of the condition for cocartesian in the case of a fibration of categories.

Proposition B.2.8. *Let $\Phi\colon \mathsf{C} \to \mathsf{B}$ be a fibration of categories. Then $\psi\colon Z \to Y$ in C over $v\colon K \to J$ in B is cocartesian if only if for all $\theta'\colon Z \to X'$ over v there is a unique morphism $\psi'\colon Y \to X'$ in C_J with $\theta' = \psi'\psi$.*

Proof. The 'only if' part is trivial. So to prove 'if' we have to prove that for any $u\colon J \to I$ and $\theta\colon Z \to X$ such that $\Phi(\theta) = uv$, there exists a unique $\phi\colon Y \to X$ over u completing the diagram

$$\begin{tikzcd}
Z \arrow[r, "\psi"'] \arrow[rr, bend left, "\theta"] & Y \arrow[r, dotted, "\phi"'] & X & {} \arrow[dd, "\Phi"] \\
 & & & \\
K \arrow[r, "v"'] & J \arrow[r, "u"'] & I\,. & {}
\end{tikzcd}$$

Since Φ is a fibration there is a cartesian morphism $\kappa\colon X' \to X$ over u. By the cartesian property, there is a unique morphism $\theta'\colon Z \to X'$ over v such that $\kappa\theta' = \theta$, as in the diagram

$$\begin{tikzcd}
Z \arrow[r, dashed, "\theta'"'] \arrow[rr, bend left, "\theta"] & X' \arrow[r, "\kappa"'] & X\,.
\end{tikzcd}$$

Now, suppose $\phi\colon Y \to X$ over $u\colon J \to I$ satisfies $\phi\psi = \theta$, as in the diagram

$$\begin{tikzcd}
Z \arrow[r, "\psi"'] \arrow[rr, bend left, "\theta"] \arrow[dr, dashed, "\theta'"'] & Y \arrow[r, dotted, "\phi"'] & X \\
 & X'\,. \arrow[ur, "\kappa"'] &
\end{tikzcd}$$

By the given property of ψ there is a unique morphism $\psi'\colon Y \to X'$ in C_J such that $\psi'\psi = \theta'$. By the cartesian property of κ, there is a unique morphism ϕ' in C_J such that $\kappa\phi' = \phi$. Then

$$\kappa\psi'\psi = \kappa\theta' = \theta = \phi\psi = \kappa\phi'\psi.$$

By the cartesian property of κ, and since $\psi'\psi, \phi'\psi$ are over uv, we have $\psi'\psi = \phi'\psi$. By the given property of ψ, and since ϕ', ψ' are in C_J, we have $\phi' = \psi'$. So $\phi = \kappa\psi'$, and this proves uniqueness.

But we have already checked that $\kappa\psi'\psi = \theta$, so we are done. □

Corollary B.2.9. *Let $\Phi\colon \mathsf{C} \to \mathsf{B}$ be a fibration of categories and suppose that for every morphism $v\colon K \to J$ in B there is chosen a reindexing functor $v^*\colon \mathsf{C}_J \to \mathsf{C}_K$ and that each v^* has a left adjoint. Then Φ is a cofibration.*

Proof. Proposition B.2.8 applies to show that the adjointness gives the lifting required for the cocartesian property. □

Remark B.2.10. Let $\Phi\colon \mathsf{C} \to \mathsf{B}$ be fibred and cofibred, and let I be an object of B. In general it is not true that the inclusion $\mathsf{C}_I \to \mathsf{C}$ preserves coproducts, as we have already said. However if $u\colon I \to J$ in B then $u_*\colon \mathsf{C}_I \to \mathsf{C}_J$ has a right adjoint and so preserves colimits and in particular preserves coproducts. □

To end this section, we give a useful result on compositions.

Proposition B.2.11. *The composition of fibrations* (*cofibrations*)*, is also a fibration* (*cofibration*)*.*

Proof. We leave this as an exercise. □

B.3 Pushouts and cocartesian morphisms

Here is a small result which we use in Section 7.2.iii and Section B.4, as it applies to many examples, such as the fibration $\mathrm{Ob}\colon \mathsf{Gpds} \to \mathsf{Set}$. The functor D in the following is thought of as 'discrete'.

Proposition B.3.1. *Let $\Phi\colon \mathsf{C} \to \mathsf{B}$ be a functor that has a left adjoint D. Then for each $K \in \mathrm{Ob}\,\mathsf{B}$, $D(K)$ is initial in C_K. In fact if $u\colon K \to J$ in B, then for any $X \in \mathsf{C}_J$ there is a unique morphism $\varepsilon_K\colon DK \to X$ over u.*

Proof. This follows immediately from the adjoint relation $\mathsf{C}_u(DK, X) \cong \mathsf{B}(K, \Phi X)$ for all $X \in \mathrm{Ob}\,\mathsf{C}_J$. □

Special cases of cocartesian morphisms are used in [Bro06], [Hig71], and in Chapter 5 and Chapter 8, Section 8.3.iii. A construction which arises naturally from an application of the Higher Homotopy Seifert–van Kampen, see Theorem 8.3.7, is given a general setting as follows:

Theorem B.3.2. *Let $\Phi\colon \mathsf{C} \to \mathsf{B}$ be a fibration of categories which has a left adjoint D. Suppose that C admits pushouts. Let $v\colon K \to J$ be a morphism in B, and let $Z \in \mathsf{C}_K$. Then a cocartesian lifting $\psi\colon Z \to Y$ of v is given precisely by the pushout in C:*

$$\begin{tikzcd} D(K) \arrow[r, "D(v)"] \arrow[d, "\varepsilon_K"'] & D(J) \arrow[d, "\varepsilon_J"] \\ Z \arrow[r, "\psi"'] & Y \end{tikzcd} \qquad (*)$$

Proof. Suppose first that diagram $(*)$ is a pushout in C. Let $u\colon J \to I$ in B and let $\theta\colon Z \to X$ satisfy $\Phi(\theta) = uv$, so that $\Phi(X) = I$. Let $f\colon D(J) \to X$ be the adjoint of $u\colon J \to \Phi(X)$.

$$\begin{tikzcd} D(K) \arrow[r, "D(v)"] \arrow[d, "\varepsilon_K"'] & D(J) \arrow[d, "\varepsilon_J"] \arrow[ddr, "f"] & \\ Z \arrow[r, "\psi"] \arrow[drr, "\theta"'] & Y \arrow[dr, dotted, "\phi"] & \\ & & X \end{tikzcd} \qquad (**)$$

$$K \xrightarrow[v]{} J \xrightarrow[u]{} I$$

Then $\Phi(fD(v)) = uv = \Phi(\theta\varepsilon_K)$ and so by Proposition B.3.1, $fD(v) = \theta\varepsilon_K$. The pushout property implies there is a unique $\phi\colon Y \to X$ such that $\phi\psi = \theta$ and $\phi\varepsilon_J = f$. This last condition gives $\Phi(\phi) = u$ since $u = \Phi(f) = \Phi(\phi\varepsilon_J) = \Phi(\phi)1_J = \Phi(\phi)$.

For the converse, we suppose given $f\colon D(J) \to X$ and $\theta\colon Z \to X$ such that $\theta\varepsilon_K = fD(v)$. Then $\Phi(\theta) = uv$ and so there is a cocartesian lifting $\phi\colon Y \to X$ of u. The additional condition $\phi\varepsilon_J = f$ is immediate by Proposition B.3.1. □

Corollary B.3.3. *Let $\Phi\colon \mathsf{C} \to \mathsf{B}$ be a fibration which has a left adjoint and suppose that C admits pushouts. Then Φ is also a cofibration.*

In view of the construction of hierarchical homotopical invariants as colimits from the HHSvKT in Chapter 8, the following is worth recording, as a consequence of Theorem B.1.7.[264]

Theorem B.3.4. *Let $\Phi\colon \mathsf{C} \to \mathsf{B}$ be fibred and cofibred. Assume B, C and all fibres C_I are cocomplete. Let $T\colon \mathsf{S} \to \mathsf{C}$ be a functor from a small connected category. Then* colim T *may be calculated as follows:*

(i) *First calculate $I = \operatorname{colim}(\Phi T)$, with cocone $\gamma\colon \Phi T \Rightarrow I$;*

(ii) *for each* $X \in \mathrm{Ob}\,\mathsf{S}$ *choose cocartesian morphisms* $\gamma'(X) \colon T(X) \to F(X)$, *over* $\gamma(X)$ *where* $F(X) \in \mathsf{C}_I$;

(iii) *make* $X \mapsto F(X)$ *into a functor* $F \colon \mathsf{S} \to \mathsf{C}_I$, *so that* γ' *becomes a natural transformation* $\gamma' \colon T \Rightarrow F$;

(iv) *form* $Y = \operatorname{colim} F \in \mathsf{C}_I$ *with cocone* $\mu \colon F \Rightarrow Y$.

Then Y *with* $\mu\gamma' \colon T \Rightarrow Y$ *is* $\operatorname{colim} T$.

Proof. We first explain how to make F into a functor by building in stages the following diagram:

$$\begin{tikzcd}
{} \arrow[dd, "\Phi"] & T(X) \arrow[rr, "\gamma'(X)"] \arrow[dr, "T(f)"'] \arrow[rrrr, bend left, "\eta"] & & F(X) \arrow[r, "\mu(X)"] \arrow[d, dashed, "F(f)"] & Y \arrow[r, dotted, "\tau"'] \arrow[d, "1"] & Z \\
& & T(X') \arrow[r, "\gamma'(X')"'] & F(X') \arrow[r, "\mu(X')"'] & Y \arrow[r, dotted, "\tau'"'] & Z \\
{} & K \arrow[r, "\Phi T(f)"'] & J \arrow[r, "\gamma(X')"'] & I \arrow[r, "1"'] & I \arrow[r, "w"'] & H
\end{tikzcd} \qquad \text{(B.3.1)}$$

Let $f \colon X \to X'$ be a morphism in S, $K = \Phi T(X)$, $J = \Phi T(X')$. By cocartesianness of $\gamma'(X)$, there is a unique vertical morphism $F(f) \colon F(X) \to F(X')$ such that $F(f)\gamma'(X) = \gamma'(X')T(f)$. It is easy to check, again using cocartesianness, that if further $g \colon X' \to X''$, then $F(gf) = F(g)F(f)$, and $F(1) = 1$. So F is a functor and the above diagram shows that γ' becomes a natural transformation $T \Rightarrow F$.

Let $\eta \colon T \Rightarrow Z$ be a natural transformation to a constant functor Z, and let $\Phi(Z) = H$. Since $I = \operatorname{colim}(\Phi T)$, there is a unique morphism $w \colon I \to H$ such that $w\gamma = \Phi(\eta)$.

By the cocartesian property of γ', there is a natural transformation $\eta' \colon F \Rightarrow Z$ such that $\eta'\gamma' = \eta$.

Since Y is also a colimit in C of F, we obtain a morphism $\tau \colon Y \to Z$ in C such that $\tau\mu = \eta'$. Then $\tau\mu\gamma' = \eta'\gamma' = \eta$.

Let $\tau' \colon Y \to Z$ be another morphism such that $\tau'\mu\gamma' = \eta$. Then $\Phi(\tau) = \Phi(\tau') = w$, since I is a colimit. Again by cocartesianness, $\tau'\mu = \tau\mu$. By the colimit property of Y, $\tau = \tau'$. □

This with Theorem B.3.4 shows how to compute colimits of connected diagrams in the examples we discuss in Sections B.4 to B.5, and in all of which a Seifert–van Kampen type theorem is available giving colimits of algebraic data for some glued topological data.

Corollary B.3.5. *Let* $\Phi \colon \mathsf{C} \to \mathsf{B}$ *be a functor satisfying the assumptions of Theorem B.3.4. Then* C *is connected cocomplete, i.e. admits colimits of all connected diagrams.*

B.4 Crossed squares and triad homotopy groups

In this section we give a brief sketch of the theory of triad homotopy groups, including the exact sequence relating them to homotopical excision, and show that the third triad group forms part of a *crossed square* which, as an algebraic structure with links over several dimensions, in this case dimensions 1, 2, 3, fits our criteria for a HHSvKT. Further, crossed squares model pointed weak homotopy 3-types. Finally we indicate a bifibration from crossed squares, so leading to the notion of *induced crossed square*, which is relevant to a triadic Hurewicz theorem in dimension 3.

A *triad of spaces* $(X\colon A, B; x)$ consists of a pointed space (X, x) and two pointed subspaces $(A, x), (B, x)$. Then $\pi_n(X\colon A, B; x)$ is defined for $n \geqslant 2$ as the set of homotopy classes of maps

$$(I^n\colon \partial_1^- I^n, \partial_2^- I^n; J_{1,2}^{n-1}) \to (X\colon A, B; x)$$

where $J_{1,2}^{n-1}$ denotes the union of the faces of I^n other than $\partial_1^- I^n, \partial_2^- I^n$. For $n \geqslant 3$ this set obtains a group structure, using the direction 3, say, and this group structure is abelian for $n \geqslant 4$. Further there is an exact sequence

$$\cdots \to \pi_{n+1}(X\colon A, B; x) \to \pi_n(A, C, x) \xrightarrow{\varepsilon} \pi_n(X, B, x) \to \pi_n(X\colon A, B; x) \to \cdots \tag{B.4.1}$$

where $C = A \cap B$, and ε is the excision map. The main interest of these sets and groups was that they measure the failure of excision. However they do not shed full light on the Homotopical Excision Theorem 8.3.7: exact sequences contain less information than colimits.[265]

The third triad homotopy group fits into a diagram of possibly nonabelian groups

$$\Pi(X; A, B, x)\colon = \begin{tikzcd} \pi_3(X; A, B, x) \arrow[r] \arrow[d] & \pi_2(B, C, x) \arrow[d] \\ \pi_2(A, C, x) \arrow[r] & \pi_1(C, x) \end{tikzcd} \tag{B.4.2}$$

in which $\pi_1(C, x)$ operates on the other groups and there is also a function

$$\pi_2(A, C, x) \times \pi_2(B, C, x) \to \pi_3(X : A, B; x)$$

known as the generalised Whitehead product.

This diagram has structure and properties which are known as those of a *crossed square*, [GWL81], [Lod82], explained below, and so this gives a homotopical functor

$$\Pi\colon (\text{based triads}) \to (\text{crossed squares}). \tag{B.4.3}$$

A *crossed square* is a commutative diagram of morphisms of groups

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd} \tag{B.4.4}$$

together with right actions of P on L, M, N and a function $h\colon M \times N \to L$ satisfying a number of axioms which we do not give in full here. Suffice it to say that the morphisms in the square preserve the action of P, which acts on itself by conjugation; M, N act on each other and on L via P; λ, λ', μ, ν and $\mu\lambda$ are crossed modules; and h satisfies axioms reminiscent of commutator rules, summarised by saying it is a *biderivation*. Morphisms of crossed squares are defined in the obvious way, giving a category XSq of crossed squares.

Let XMod^2 be the category of pairs of crossed modules $\mu\colon M \to P$, $\nu\colon N \to P$ (with P and μ, ν variable), and with the obvious notion of morphism. There is a forgetful functor $\Phi\colon \mathsf{XSq} \to \mathsf{XMod}^2$. This functor has a right adjoint F which completes the pair $\mu\colon M \to P, \nu\colon N \to P$ with $L = M \times_P N$ and λ, λ' given by the projections and $h\colon M \times N \to L$ given by

$$h(m,n) = (m^{-1}m^n, (n^{-1})^m n), \quad m \in M,\ n \in N$$

More interestingly, it has a left adjoint which to the above pair of crossed P-modules yields the 'universal crossed square'

$$\begin{tikzcd} M \otimes N \arrow[r] \arrow[d] & N \arrow[d, "\nu"] \\ M \arrow[r, "\mu"'] & P \end{tikzcd} \tag{B.4.5}$$

where $M \otimes N$, is the nonabelian tensor product of groups which act on each other (on the right).[266] This is the group generated by elements $m \otimes n$, $m \in M$, $n \in N$ with the relations

$$\begin{aligned} mm' \otimes n &= (m \otimes n)^{m'} (m' \otimes n), \\ m \otimes nn' &= (m \otimes n')(m \otimes n)^{n'} \end{aligned}$$

for all $m, m' \in M$, $n, n' \in N$, which may be expanded to

$$\begin{aligned} mm' \otimes n &= (m'^{-1}mm' \otimes n^{m'})(m' \otimes n), \\ m \otimes nn' &= (m \otimes n')(m^{n'} \otimes n'^{-1}nn'). \end{aligned}$$

Then Φ is a fibration of categories and also a cofibration. Thus we have a notion of *induced crossed square*, which according to Proposition B.3.2 is given by a pushout of the form

$$\begin{tikzcd}
\begin{pmatrix} M\otimes N & N \\ M & P \end{pmatrix} \arrow[r, "\left(\begin{smallmatrix} \alpha\otimes\beta & \beta \\ \alpha & \gamma \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} u & 1 \\ 1 & 1 \end{smallmatrix}\right)"'] & \begin{pmatrix} R\otimes S & S \\ R & Q \end{pmatrix} \arrow[d, "\left(\begin{smallmatrix} v & 1 \\ 1 & 1 \end{smallmatrix}\right)"] \\
\begin{pmatrix} L & N \\ M & P \end{pmatrix} \arrow[r, "\left(\begin{smallmatrix} \delta & \beta \\ \alpha & \gamma \end{smallmatrix}\right)"'] & \begin{pmatrix} T & S \\ R & Q \end{pmatrix}
\end{tikzcd}$$

in the category of crossed squares, given morphisms

$$(\alpha,\gamma)\colon (M\to P)\to(R\to Q), \quad (\beta,\gamma)\colon (N\to P)\to(S\to Q)$$

of crossed modules.[267]

B.5 Groupoids bifibred over sets

We have already seen in Example B.1.3 that the functor $\mathrm{Ob}\colon \mathsf{Gpds}\to\mathsf{Set}$ is a fibration. It also has a left adjoint F assigning to a set I the discrete groupoid on I, and a right adjoint assigning to a set I the indiscrete groupoid on I.

It follows from general theorems on algebraic theories that the category Gpds is cocomplete, and in particular admits pushouts, and so it follows from previous results that $\mathrm{Ob}\colon \mathsf{Gpds}\to\mathsf{Set}$ is also a cofibration. A construction of the cocartesian liftings of $u\colon I\to J$ for G a groupoid over I is given in terms of words, generalising the construction of free groups and free products of groups, in [Hig71], [Bro06]. In these references the cocartesian lifting of u to G is called a *universal morphism*, and is written $u_*\colon G\to U_u(G)$. This construction is of interest as it yields a normal form for the elements of $U_u(G)$, and hence u_* is injective on the set of non-identity elements of G.

A homotopical application of this cocartesian lifting is the following theorem on the fundamental groupoid. It shows how identification of points of a discrete subset of a space can lead to 'identifications of the objects' of the fundamental groupoid:

Theorem B.5.1. *Let (X,A) be a pair of spaces such that A is discrete and the inclusion $A\to X$ is a closed cofibration. Let $f\colon A\to B$ be a function to a discrete space B. Then the induced morphism*

$$\pi_1(X,A)\to\pi_1(B\cup_f X, B)$$

is the cocartesian lifting of f.

This theorem immediately gives the fundamental group of the circle S^1 as the infinite cyclic group C_∞, since S^1 is obtained from the unit interval $[0,1]$ by identifying

0 and 1. The theorem is a translation of [Bro06], 9.2.1, where the words 'universal morphism' are used instead of 'cocartesian lifting'. Section 8.2 of [Bro06] shows how free groupoids on directed graphs are obtained by a generalisation of this example.

The calculation of colimits in a fibre Gpds_I is similar to that in the category of groups, since both categories are protomodular, [BB04]. Thus a colimit is calculated as a quotient of a coproduct, where quotients are themselves obtained by factoring by a normal subgroupoid. Quotients are discussed in [Hig71], [Bro06].

Theorem B.3.4 now shows how to compute general colimits of groupoids.

We refer again to [Hig71], [Bro06] for further developments and applications of the algebra of groupoids. We generalise some aspects of the above to modules, crossed modules and crossed complexes in Chapter 7.

The following sections cover some aspects of groupoid theory needed earlier.

B.6 Free groupoids

We explain the notion of free groupoid on a graph – this is used implicitly in combinatorial group theory, for example in paths in a Cayley graph, and is required for combinatorial groupoid theory.[268] We will exploit free groupoids in a later chapter, when calculating crossed resolutions.

Definition B.6.1. By a *graph* $\Gamma = (E(\Gamma), V(\Gamma), s, t)$ we mean a set $E(\Gamma)$ of *edges*, a set $V(\Gamma)$ of *vertices* and two functions $s, t\colon E(\Gamma) \to V(\Gamma)$ called the *source and target maps*.

A *morphism* $f : \Gamma \to \Gamma'$ *of graphs* is a pair of functions

$$E(f)\colon E(\Gamma) \to E(\Gamma'), \quad V(f)\colon V(\Gamma) \to V(\Gamma')$$

which commute with the source and target maps. This gives the category Grphs of graphs. □

Remark B.6.2. This is commonly called a *directed graph*, but we shall use only these. Also we shall, in keeping with the terminology for categories and groupoids, use also the term objects of the graph instead of vertices. As for groupoids, we write $a : x \to y$ if a is an edge and $sa = x$, $ta = y$, say a is from x to y, and we write $\Gamma(x, y)$ for the set of edges from x to y in Γ.

Proposition B.6.3. *The forgetful functor* $U : \mathsf{Gpds} \to \mathsf{Grphs}$ *which forgets about the composition has a left adjoint, whose morphisms can be seen as paths in the graph.*

Proof. We outline a proof and leave the details as an exercise.

Let $\mathsf{2}$ denote the graph with two vertices $0, 1$ and one edge $\iota\colon 0 \to 1$. A given graph Γ can be regarded as obtained from a disjoint union of copies of the graph $\mathsf{2}$ by an appropriate identification of the vertices. The same identification for a similar disjoint union of copies of the groupoid $\mathcal{I}$ gives the free groupoid $F(\Gamma)$ on Γ.

Then $F(\Gamma)(x, y)$ can be seen as made up of classes of 'paths' from x to y in Γ, where such a path is a sequence of edges or a formal inverse of an edge, which are composable, and which starts at x and ends at y. (See also Section 8.2 in [Bro06].) □

Exercise B.6.4. Use various universal morphisms to construct and verify the properties of a free groupoid on a directed graph. In particular give the universal property, and show that the graph morphism $\Gamma \to F(\Gamma)$ is injective on vertices and on edges. □

Exercise B.6.5. Formulate and discuss the notion of generating graph for a groupoid. □

Problem B.6.6. A well-known result for a free group F on a finite set of n generators is that if a subset B of F with n elements generates F, then it generates F freely. This is a consequence of Grushko's theorem. Could there be an analogous result for free groupoids? □

B.7 Covering morphisms of groupoids

For the convenience of readers, and to fix the notation, we recall here the basic facts on covering morphisms of groupoids. Proofs can be found in the books [Bro06], [Hig71].[269] However we find it convenient to adopt different conventions, focussing on costars rather than stars, which ensure that some of our formulae in Section 10.3.ii work out in a nice way, see Equation (B.7.4).

Let G be a groupoid. For each object a_0 of G the *Costar* of a_0 in G, denoted by $\mathrm{Cost}_G\, a_0$, is the union of the sets $G(a, a_0)$ for all objects a of G, i.e.

$$\mathrm{Cost}_G\, a_0 = \{g \in G \mid tg = a_0\}.$$

A morphism $p\colon \widetilde{G} \to G$ of groupoids is a *covering morphism* if for each object $\tilde{a}$ of $\widetilde{G}$ the restriction of p

$$\mathrm{Cost}_{\widetilde{G}}\, \tilde{a} \to \mathrm{Cost}_G\, p\tilde{a} \tag{B.7.1}$$

is bijective. In this case $\widetilde{G}$ is called a *covering groupoid of* G. More generally p is called a *fibration of groupoids* if the restrictions of p to the Costars as in Equation (B.7.1) is surjective.

A basic result for covering groupoids is *unique path lifting*. That is, let $p\colon \widetilde{G} \to G$ be a covering morphism of groupoids, and let $(g_1, g_2, \ldots, g_n)$ be a sequence of composable elements of G. Let $\tilde{a} \in \mathrm{Ob}(\widetilde{G})$ be such that $p\tilde{a}$ is the target of g_n. Then there is a unique composable sequence $(\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_n)$ of elements of $\widetilde{G}$ such that $\tilde{g}_n$ ends at $\tilde{a}$ and $p\tilde{g}_i = g_i, i = 1, \ldots, n$.

If G is a groupoid, the slice category $\mathsf{GpdsCov}/G$ of coverings of G has as objects the covering morphisms $p\colon H \to G$ and has as morphisms the commutative diagrams of morphisms of groupoids, where p and q are covering morphisms,

$$\begin{tikzcd} H \arrow[rr, "f"] \arrow[dr, "p"'] & & K \arrow[dl, "q"] \\ & K\,. & \end{tikzcd}$$

By a standard result on compositions and covering morphisms ([Bro06], 10.2.3), f also is a covering morphism. It is convenient to write such a diagram as a triple (f, p, q). The composition in $\mathsf{GpdsCov}/G$ is then given as usual by

$$(g, q, r)(f, p, q) = (gf, p, r).$$

It is a standard result (see for example [Hig71] or [Bro70]) that the category $\mathsf{GpdsCov}/G$ is equivalent to the category of operations of the groupoid G on sets. We give the definitions and notations which we will use for this equivalence.

Recall we are writing composition of $g : p \to q$ and $h : q \to r$ in a groupoid as $gh : p \to r$. This is the opposite of the notation for functions in the category Set; the composite of a function $f : X \to Y$ and $g : Y \to Z$ is $gf : X \to Z$ with value $(gf)(x) = g(f(x))$.[270] Because of this 'opposite' nature of our conventions we have to make the following definition.

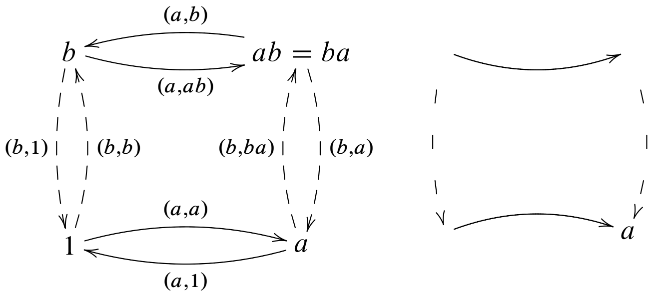
Definition B.7.1. A *left operation of a groupoid G on sets* is a functor $X : G^{\mathrm{op}} \to \mathsf{Set}$. If $p \in \mathrm{Ob}(G)$, $g : p \to q$ in G, and $x \in X(p)$, then $X(g)(x) \in X(q)$ may also be written gx.

Thus if $X : G^{\mathrm{op}} \to \mathsf{Set}$ is a functor, then $\widetilde{G} = G \ltimes X$ has object set the disjoint union of the sets $X(p))$ for $p \in \mathrm{Ob}(G)$ and morphisms $x \to y$ the pairs (g, x) such that $x \in X(tg)$ and $y = X(g)x$; in operator notation: $(g, x) : gx \to x$. The composition is $(g', gx)(g, x) = (g'g, x)$. The projection morphism $G \ltimes X \to G$, $(g, x) \mapsto g$, is a covering morphism.

This 'semidirect product' or 'Grothendieck construction' is fundamental for constructing covering morphisms to the groupoid G.[271] For example, if a_0 is an object of the transitive groupoid G, and A is a subgroup of the object group $G(a_0)$ then the groupoid G operates on the family of cosets $\{gA \mid g \in \mathrm{Cost}_G\, a_0\}$, by $g'(gA) = g'gA$ whenever $g'g$ is defined, and the associated covering morphism $\widetilde{G} \to G$ defines the covering groupoid $\widetilde{G}$ of the groupoid G determined by the subgroup A. When A is trivial this gives the *universal cover* at a_0 of the groupoid G. In particular, this gives the universal covering groupoid of a group, whose objects are the elements of G and morphisms are pairs $(g, h) : gh \to h$ for all $g, h \in G$. Then G operates on the right of the universal cover by $(g, h)^k = (g, hk)$. This operation preserves the map p and is called a *covering transformation*.

Example B.7.2. Here is a simple example: the universal covering groupoid $\widetilde{K}$ of the Klein 4-group $K = \mathsf{C}_2 \times \mathsf{C}_2$ with elements say $1, a, b, ab$. The group is generated by

a, b with the relations $a^2, b^2, aba^{-1}b^{-1}$, which we write respectively r, s, t. Then $\widetilde{K}$ has the elements of K as vertices and a morphism $(g, x) \colon gx \to x$ for each $g, x \in K$. The covering morphism $p \colon \widetilde{K} \to K$ is $(g, x) \mapsto g$. In terms of the generators a, b we obtain a diagram of $\widetilde{K}$ as the left-hand diagram in the following picture:



Note that for example $(a, ab) \colon b \to ab$ because $a^2 = 1$. The right-hand diagram illustrates a lift of the path $b^{-1}a^{-1}ba$ in K to a path starting and ending at a in the diagram of $\widetilde{K}$. You should draw the similar loops starting in turn at $1, b, ab$. We show in Section 10.3.ii that in the context of covering morphisms of crossed complexes these four loops form boundaries of four 'lifts' of the relation t. □

Example B.7.3. Given a morphism $\phi \colon F \to G$ of groups, let $q \colon \widehat{F} \to F$ be the pullback by ϕ of the universal covering morphism $p \colon \widetilde{G} \to G$ giving a commutative diagram

$$\begin{tikzcd} \widehat{F} \arrow[r, "\bar{\phi}"] \arrow[d, "q"'] & \widetilde{G} \arrow[d, "p"] \\ F \arrow[r, "\phi"'] & G. \end{tikzcd} \tag{B.7.2}$$

Note that a morphism in $\widehat{F}$ is a pair $(u, (\phi u, g)) \colon (\phi u)g \to g$, $u \in F$, $g \in G$. Since u determines ϕu, we can write a morphism of $\widehat{F}$ as $(u, g) \colon (\phi u)g \to g$. Again, G operates on the right of $\widehat{F}$ by $(u, g)^k = (u, gk)$, $k \in G$.

If X is a set of generators of the group G, we have an epimorphism $\phi \colon F \to G$ where F is the free group on the set X. Let $\widehat{X}$ be the graph $q^{-1}(X)$ in $\widehat{F}$. This is called the *Cayley graph* of the set of generators X of G. Its vertices are the elements of G and the morphisms are pairs $(x, g) \colon (\phi x)g \to g$. For our particular example with generators of the Klein group K this Cayley graph is often drawn in an abbreviated form as

$$\begin{tikzcd} b \arrow[d, "b"'] & ab \arrow[l, "a"'] \arrow[d, "b"] \\ 1 & a. \arrow[l, "a"] \end{tikzcd} \tag{B.7.3}$$

□

Exercise B.7.4. Carry out a similar analysis to the above for the universal cover of the symmetric group S_3, whose Cayley graph is drawn in Example 3.1.6.

The following is a key result.

Proposition B.7.5. *Given the epimorphism $\phi\colon F \to G$ where $F = F(X)$ is the free group on the set X of generators of G, then $\widehat{F}$ is the free groupoid on the graph $\widehat{X}$, whose morphisms can be written $(x, g)\colon (\phi x)g \to g$.*

Proof. This is 10.8.1, Corollary 1, in [Bro06]. See also [Hig71], Theorem 8, p. 112. The proofs use the solution of the word problem. □

This construction is used in Section 10.3.ii for computing resolutions, and is also relevant to Section 8.4, Exercise 7.4.26.

Remark B.7.6. The main reason for our choice of conventions on covering morphisms is the following. Let G be a group and $p\colon \widetilde{G} \to G$ its universal covering morphism. Then G operates on the right of the groupoid $\widetilde{G}$ by $(g, h)^k = (g, hk)$, $(g, h) \in \widetilde{G}$, $k \in G$. Let $e\colon G \to \widetilde{G}$ be the function $g \mapsto (g, 1)\colon g \to 1$. Then one easily checks that

$$e(gh) = e(g)^h\, e(h).$$

Thus e is a (nonabelian) derivation. □

Exercise B.7.7. In the circumstances of the last remark, prove that the composition

$$G \xrightarrow{e} \widetilde{G} \to \widetilde{G}^{\text{totab}}$$

is isomorphic to the universal derivation $G \to IG$. □

Exercise B.7.8. Also if $\phi\colon F \to G$ is a morphism of groups and $q\colon \widehat{F} \to F$ is the pullback of $p\colon \widetilde{G} \to G$ by ϕ, then G again operates on the groupoid $\widehat{F}$ and $d: F \to \widehat{F}$ given by $u \mapsto (u, 1)$ satisfies

$$d(uv) = d(u)^{\phi v}\, d(v), \tag{B.7.4}$$

i.e. d is a (nonabelian) ϕ-derivation. □

Remark B.7.9. It is also useful to note that in the situation of the last remark, if

$$(x_1, g_1), (x_2, g_2), \ldots, (x_n, g_n)$$

is a sequence of composable morphisms of $\widehat{F}$, so that $(\phi x_i)g_i = g_{i-1}$, $1 < i \leqslant n$, and each x_i or its inverse belongs to X, then their composite is $(x_1 x_2 \ldots x_n, g_n)$. □

B.8 Model categories for homotopy theory

The purpose of this section is to direct attention of the reader to the literature on model categories, though we do not have the space here to expand on the homotopy aspects of crossed complexes given in Chapters 9 and 11. We mention also the thesis [Sau03], although we feel this has a gap in the proof that the model category of crossed complexes is *proper* in the sense that a pushout of weak equivalences by a cofibration is necessarily a weak equivalence. The article [BG89b] gives an account of a model category structure for crossed complexes, and some of our account in Chapter 11 is taken from this paper and from [BH91]. See also [AM11] for a relation of this structure with a known structure on strict globular ∞-categories.

The general notion of *model category for homotopy theory* was necessitated by the variety of situations in which the notion of homotopy arose, and so the need to obtain one theory instead of several, as well as to be able to adapt homotopical methods to new situations.

The initial reference for model categories was Quillen's [Qui67]. Later references are [Bau89], [Hov99], [DS95], [Hir03], [KP97]. The last reference is appropriate for this book because of: its emphasis on starting with cylinder objects; it also relates the subject to extension properties on cubical sets; it uses crossed complexes as an example, and treats them in detail. The approach through cylinder objects also should seem familiar if the notion of cofibration of topological spaces has been learned from [Bro06]. A new approach to this area has developed ideas of Grothendieck in [Mal05], [Cis06]. The article [Mal09] develops the theory for cubical sets with connections.

Books on model categories for homotopy such as Hovey's deal with the model category of chain complexes over a fixed ring say R. However the crossed complex category is more analogous to the category of chain complexes of *modules over variable rings*, and this case seems not to be well studied.

For our purposes the easy starting point is the notion of a category with a cylinder object, of which specific examples are given earlier in terms of: topological spaces, with cylinder $I \times X$; filtered spaces with cylinder $I_* \otimes X_*$; groupoids with cylinder $\mathcal{I} \times G$; crossed complexes with cylinder $\mathcal{I} \otimes C$; cubical sets with cylinder $\mathbb{I} \otimes K$. It is possible to develop homotopy theory in terms of a *cofibration category with a cylinder object*, and from this to deduce notions of cofibration, fibration, and weak equivalence. One then ends up with what is called a model category, which can be proved to satisfy the following.

A 'model structure' on a category C consists of three distinguished classes of morphisms (equivalently subcategories): *weak equivalences, fibrations, and cofibrations*, and two functorial factorisations (α, β) and (γ, δ) subject to the following axioms. Note that a fibration that is also a weak equivalence is called an *acyclic* (or *trivial*) fibration and a cofibration that is also a weak equivalence is called an *acyclic* (or *trivial*) cofibration (or sometimes called an *anodyne morphism*). (Some authors find the term 'trivial' ambiguous and so prefer to use 'acyclic', and this is the modern trend.)

Axioms

1. **Retracts:** each of the distinguished classes is closed under retracts.
2. **2 of 3:** if f and g are maps in C such that f, g, and gf are defined and any two of these are weak equivalences then so is the third.
3. **Lifting:** acyclic cofibrations have the left lifting property with respect to fibrations and cofibrations have the left lifting property with respect to acyclic fibrations.
4. **Factorization:** for every morphism f in C, $\alpha(f)$ is a cofibration, $\beta(f)$ is an acyclic fibration, $\gamma(f)$ is an acyclic cofibration, and $\delta(f)$ is a fibration.

A *model category* is a category that has a model structure and all (small) limits and colimits, i.e. a complete and cocomplete category with a model structure. This structure is not unique; in general there can be many model category structures on a given category.

Examples. The category of topological spaces, Top, admits a standard model category structure with the usual (Serre) fibrations and cofibrations and with weak equivalences as weak homotopy equivalences. For the category of topological spaces, another such structure is given by Hurewicz fibrations and cofibrations. Recall from [Bro06] that a continuous map $f: X \to Y$ of topological spaces has a factorisation through the mapping cylinder $M(f)$ of f,

$$X \xrightarrow{i} M(f) \xrightarrow{p} Y,$$

and i is a cofibration, p is a homotopy equivalence. More work is needed to get a factorisation of the type required by the axioms for a model category, and we do not do that here.

Some constructions. Every closed model category has a terminal object by completeness and an initial object by cocompleteness, since these objects are the limit and colimit, respectively, of the empty diagram. Given an object X in the model category, if the unique map from the initial object to X is a cofibration, then X is said to be *cofibrant*. Analogously, if the unique map from X to the terminal object is a fibration then X is said to be *fibrant*.

If Z and X are objects of a model category such that Z is cofibrant and there is a weak equivalence from Z to X then Z is said to be a *cofibrant replacement* for X. Similarly, if Z is fibrant and there is a weak equivalence from X to Z then Z is said to be a *fibrant replacement* for X. In general, not all objects are fibrant or cofibrant, though this is sometimes the case. For example, all objects are cofibrant in the standard model category of simplicial sets and all objects are fibrant for the standard model category structure given above for topological spaces.

Left homotopy is defined with respect to cylinder objects and right homotopy is defined with respect to path objects. These notions of homotopy coincide when the domain is cofibrant and the codomain is fibrant. In that case, homotopy defines an equivalence relation on the hom sets of the model category giving rise to homotopy classes.

As an example of theorems which can be proved in a general model category we mention:

Proposition B.8.1 (Whitehead's theorem on homotopy equivalences). *If a morphism of cofibrant objects is a weak equivalence, then it is also a homotopy equivalence.*

Remark B.8.2. Another good example is the abstract version of the gluing theorem for homotopy equivalences, which was published first in [Bro68] in the case of spaces, and is in [Bro06], (7.5.7). Several uses of this lemma are in [Koz08].

For further information, look also for model categories on the following web source: the ncatlab http://ncatlab.org/nlab/show/HomePage.

Note also that the papers [BGPT97], [BGPT01] find that the theory of model categories is not a strong enough abstract homotopy theory for describing the equivariant analogues of the results of Chapter 11 on spaces of maps to a classifying space. Instead the basis needed in those papers is that of simplicially enriched categories and of homotopy coherence, [CP97]. The reason for this is that the aim of model category theory is to give a general basis for homotopy categories and their relationships, whereas in many situations one wants also to study higher homotopies, i.e. homotopies, homotopies of homotopies, and so on, and for this one does not want to pass to the homotopy category. The background here is that of higher dimensional algebra. See for example [Lur09]. See also [Shu09] for a discussion of enriched categories and homotopy limits and colimits.

Notes

262 p. 577 This notion of fibration was defined in [Gro68]. Further development was in [Gir64], [Gra66]. This exposition is based on [Str99], which was strongly influenced by notes of J. Benabou.

263 p. 578 Some writers define a fibration of categories so as to include a splitting, and others, for example J. Benabou, have argued strongly against this.

264 p. 584 Some uses of colimit calculations in homotopy theory other than those in this book are shown in [BL87], [BL87a], [ES87], [KFM08], [FM09], [EM10].

265 p. 586 The notion of 3-fold groupoid arises in this triadic situation as follows. Let S be the space of maps $f: I^2 \to X$ such that f maps the faces of I^2 in direction 1 in to A and those in direction 2 into B, and the vertices of I^2 to the base point of X. Then the usual composition of squares gives S two compositions $+_1$, $+_2$.

However these compositions are not in general inherited by homotopy classes of these maps. Let s_0 be the constant map in S. Then of course $G = \pi_1(S, s_0)$ has the structure of group. The remarkable fact is that the compositions $+_1$, $+_2$ are inherited by G to give it the structure of 2-fold groupoid in the category of groups. You are invited to prove this directly. The proof is given in [BL87], [Gil87] for the n-fold case. These methods led to a triadic Hurewicz theorem, see [BL87a], [Bro89]. For the use of crossed squares in some specific homotopy calculations up to level 3 see [Bro92], [Ell93].

266 p. 587 This was defined in [BL87] (for left actions). A bibliography on the non-abelian tensor product contains to date 114 items, see

http://www.bangor.ac.uk/r.brown/nonabtens.html

267 p. 588 The functor Π is exploited in [BL87] for an HHSvKT implying some calculations of the nonabelian group $\pi_3(X : A, B; x)$. Earlier results had used homological methods to obtain some abelian values. The applications are developed in [BL87a] for a triadic Hurewicz Theorem, and for the notion of free crossed square, both based on 'induced crossed squares'. Free crossed squares are exploited in [Ell93] for homotopy type calculations; see also [AU06]. In fact the HHSvKT works in all dimensions and in the more general setting of n-cubes of spaces, although not in a 'many base point' situation. For a recent application, see [EM10]. Crossed squares also occur in considering the notion of homotopy 3-type, and of automorphisms of crossed modules, see [BG89a], and the references there.

268 p. 589 For more details on free groupoids see [Hig71], [Bro06]. For a discussion on categories of graphs, and the distinction between directed and undirected graphs, see [BMSW08], and the references there.

269 p. 590 The earliest definition of covering morphism of groupoids seems to be in [Smi51a], [Smi51b], where such a morphism is called a 'regular' morphism. Generalisation of covering morphisms from groupoids to categories are seen in [BN00], in terms of a *unique factorisation lifting functor*.

270 p. 591 It is possible to resolve this confusion by writing functions on the right of their argument as $(x)f$. This 'algebraist's' convention is followed successfully in [Hig71], and contrasts with the usual 'analyst's' convention.

271 p. 591 This so called 'Grothendieck construction' has also been developed by C. Ehresmann in [Ehr57], in which he defines both an action of a category and the associated 'category of hypermorphisms', and also what in the case of local groupoids he calls the complete enlargement of a species of structures.

Appendix C
Closed categories

In Section 9.1 we have given an account of various exponential laws. Here we give a sketch of some of the underlying categorical ideas.

In specialising to the category of groupoids, we get some indication of possible notions of 'higher order symmetry'.

C.1 Products of categories and coherence

Let Cat be the category of all small categories with morphisms being the functors. This category is known to be complete and cocomplete. The product of categories is constructed in for example [Bro06], Section 6.4, and has the universal property of a product in a category.

Let C, D be categories. The *product* $\mathsf{C} \times \mathsf{D}$ is defined to have objects all pairs (x, y) for $x \in \mathrm{Ob}\,\mathsf{C}$, $y \in \mathrm{Ob}\,\mathsf{D}$ and to have as morphisms the pairs (f, g), for $f \in \mathsf{C}$, $g \in \mathsf{D}$ – thus the set $\mathsf{C} \times \mathsf{D}$ is just the cartesian product of the two sets. Also, if $f : x \to x'$ in C, $g : y \to y'$ in D, then we take in $\mathsf{C} \times \mathsf{D}$

$$(f, g) : (x, y) \to (x', y').$$

The composition is defined as one would expect by

$$(f', g')(f, g) = (f'f, g'g)$$

whenever $f'f$, $g'g$ are defined. It is very easy to show that $\mathsf{C} \times \mathsf{D}$ is a category.

Notice also that if f, g have inverses f^{-1}, g^{-1} then (f, g) has inverse (f^{-1}, g^{-1}). It follows that if C, D are both groupoids then so also is $\mathsf{C} \times \mathsf{D}$.

Let $p_1 : \mathsf{C} \times \mathsf{D} \to \mathsf{C}$, $p_2 : \mathsf{C} \times \mathsf{D} \to \mathsf{D}$ be the obvious projection functors. Then we have the universal property: *if $F : \mathsf{E} \to \mathsf{C}$, $G : \mathsf{E} \to \mathsf{D}$ are functors then there is a unique functor $(F, G) : \mathsf{E} \to \mathsf{C} \times \mathsf{D}$ such that $p_1(F, G) = F$, $p_2(F, G) = G$.* The proof is easy and is left to the reader. As usual, this property characterises the product up to isomorphism.

Note that this is how product is defined in elementary category theory. So, in an interesting kind of self reference, we use category theory to discuss category theory itself. This is partly because of the dual role of categories and groupoids in mathematics – on the one hand for metamathematical considerations, and on the other as algebraic objects in their own right.

Let $F : \mathsf{C} \times \mathsf{D} \to \mathsf{E}$ be a functor, where $\mathsf{C}, \mathsf{D}, \mathsf{E}$ are categories. If 1_x is the identity at x in C, then let us write $F(x, g)$ for $F(1_x, g)$ where g is any morphism in D. Similarly, let us write $F(f, y)$ for $F(f, 1_y)$ for any object y of D and any morphism f of C. Then, as is easily verified, $F(x, \sim)$ is a functor $\mathsf{D} \to \mathsf{E}$ (called the x-*section* of F) and $F(\sim, y)$ is a functor $\mathsf{C} \to \mathsf{E}$ (called the y-*section* of F). These two families of functors determine F. If $f : x \to x'$, $g : y \to y'$ are morphisms in C, D respectively then we have a commutative diagram

$$\begin{tikzcd} F(x, y) \arrow[r, "F(f,y)"] \arrow[d, "F(x,g)"'] \arrow[dr, "F(f,g)"] & F(x', y) \arrow[d, "F(x',g)"] \\ F(x, y') \arrow[r, "F(f,y')"'] & F(x', y') \end{tikzcd} \tag{C.1.1}$$

since $F(1_{x'} f, g 1_y) = F(f, g) = F(f 1_x, 1_{y'} g)$.

Proposition C.1.1. *Suppose for each x in* $\operatorname{Ob} \mathsf{C}$ *and y in* $\operatorname{Ob} \mathsf{D}$ *we are given functors*

$$F(x, \sim) : \mathsf{D} \to \mathsf{E}, \quad F(\sim, y) : \mathsf{C} \to \mathsf{E}$$

such that $F(x, y)$ is a unique object of E. *Suppose for each $f : x \to x'$ in* C *and $g : y \to y'$ in* D *the outer square of* (C.1.1) *commutes. Then the diagonal composite $F(f, g)$ makes F a functor* $\mathsf{C} \times \mathsf{D} \to \mathsf{E}$. *All functors* $\mathsf{C} \times \mathsf{D} \to \mathsf{E}$ *arise in this way.*

Proof. The verification of the preservation of the identity for F is easy since

$$\begin{aligned} F(1_x, 1_y) &= F(1_x, y) F(x, 1_y) \\ &= 1_{F(x,y)} 1_{F(x,y)} \\ &= 1_{F(x,y)}. \end{aligned}$$

The verification of the composition rule involves four commutative squares:

$$\begin{tikzcd} F(x, y) \arrow[r, "F(f,y)"] \arrow[d, "F(x,g)"'] \arrow[dr, "F(f,g)"] & F(x', y) \arrow[r, "F(f',y)"] \arrow[d, "F(x',g)"] & F(x'', y) \arrow[d, "F(x'',g)"] \\ F(x, y') \arrow[r, "F(f,y')"'] \arrow[d, "F(x,g')"'] & F(x', y') \arrow[r, "F(f',y')"] \arrow[d, "F(x',g')"'] \arrow[dr, "F(f',g')"] & F(x'', y') \arrow[d, "F(x'',g')"] \\ F(x, y'') \arrow[r, "F(f,y'')"'] & F(x', y'') \arrow[r, "F(f',y'')"'] & F(x'', y''). \end{tikzcd}$$

The last statement of the proposition is clear from the discussion preceding its statement. □

C.2 Cartesian closed categories

We have already given in Section 9.1 some background to the fundamental notion of an 'exponential law'. Here we will sketch the ideas for one aspect of that, and how the category Cat of small categories comes into this framework with an exponential law of the form of a natural bijection

$$\mathsf{Cat}(\mathsf{C} \times \mathsf{D}, \mathsf{E}) \cong \mathsf{Cat}(\mathsf{C}, \mathsf{CAT}(\mathsf{D}, \mathsf{E})) \tag{C.2.1}$$

for all small categories $\mathsf{C}, \mathsf{D}, \mathsf{E}$. The small category $\mathsf{CAT}(\mathsf{D}, \mathsf{E})$ has objects the functors $\mathsf{D} \to \mathsf{E}$ and morphisms the natural transformations.

We will not give a proof of this, but sketch some of the ideas in a way related to previous work.

In Section 6.1 we have defined the notion of double category and given the example of the double category $\square \mathsf{E}$ of commuting squares in a category E. This double category has two compositions which were there written $+_1$, $+_2$ and here we will write $\circ_1$, $\circ_2$. This gives rise to two categories $\square_1 \mathsf{E}$, $\square_2 \mathsf{E}$ in which the morphisms are the commutative squares in E but in which the compositions are respectively $\circ_1$, $\circ_2$.

Proposition C.2.1. *The natural transformations of functors* $\mathsf{D} \to \mathsf{E}$ *are bijective with the elements of* $\mathsf{Cat}(\mathsf{D}, \square_2 \mathsf{E})$.

That is, instead of saying that a natural transformation $\phi\colon F \to G$ assigns to each object d of D a morphism $\phi(d)\colon F(d) \to G(d)$ in E such that for every morphism $f : d \to d'$ in D a certain square diagram in E commutes, we say that a natural transformation ϕ is a functor $\mathsf{D} \to \square_2 \mathsf{E}$, and the composition of natural transformations is determined by the composition $\circ_1$ in $\square \mathsf{E}$.[272]

C.3 The internal hom for categories and groupoids

Let us prove that the category of small categories (and that of groupoids) is closed. Thus, for any couple of small categories (groupoids) C, D, we need to construct the small category (groupoid) of internal morphisms from C to D that we are going to denote as $\mathsf{CAT}(\mathsf{C}, \mathsf{D})$ ($\mathsf{GPDS}(\mathsf{C}, \mathsf{D})$).

The objects of $\mathsf{CAT}(\mathsf{C}, \mathsf{D})$ are $\mathsf{Cat}(\mathsf{C}, \mathsf{D})$, all functors (morphisms) between the given categories.

Its morphisms are all the natural transformations between such functors.

The source, target and identity of $\mathsf{CAT}(\mathsf{C}, \mathsf{D})$ are the obvious one. For any two natural transformations $\phi\colon F \Rightarrow F'$ and $\phi'\colon F' \Rightarrow F''$, we define the composition $\phi'\phi$ by $\phi'\phi(x) = \phi'(\phi(x))$. It is clear that this composition completes the structure of category over $\mathsf{CAT}(\mathsf{C}, \mathsf{D})$.

It is immediate to see that when C and D are groupoids, any natural transformation $\phi\colon F \Rightarrow F'$ has inverse ϕ^{-1} defined by $\phi^{-1}(x) = (\phi(x))^{-1}$. Thus $\mathsf{CAT}(\mathsf{C},\mathsf{D})$ is a groupoid that we denote by $\mathsf{GPDS}(\mathsf{C},\mathsf{D})$.

The construction of internal morphisms $\mathsf{CAT}(\mathsf{C},\mathsf{D})$ is natural in C and D. Let us check that it is the adjoint of the cartesian product using essentially the same procedure as in Set.

Theorem C.3.1. *If* C, D, E *are small categories, there is a natural bijection of sets*

$$\theta\colon\ \mathsf{Cat}(\mathsf{C}\times\mathsf{D},\mathsf{E}) \cong \mathsf{Cat}(\mathsf{C},\mathsf{CAT}(\mathsf{D},\mathsf{E})).$$

Proof. To define θ, let us start with any functor $F : \mathsf{C}\times\mathsf{D} \to \mathsf{E}$ and we are going to construct the functor $\theta(F) = \widehat{F} : \mathsf{C} \to \mathsf{CAT}(\mathsf{D},\mathsf{E})$.

For an $x \in \operatorname{Ob}\mathsf{C}$, its image is the functor $\widehat{F}(x)\colon \mathsf{D} \to \mathsf{E}$ given by the x-section of F, i.e. $\widehat{F}(x) = F(x,\sim)$

Now, let $f : x \to x'$ be a morphism in C. The natural transformation

$$\widehat{F}(f)\colon\ F(x,\sim) \Rightarrow F(x',\sim)$$

is defined by assigning to each object y in D a morphism $\widehat{F}(f)(y) = F(f,y)$. It is clear that for any morphism $g\colon\ y \to y'$ in D the following square commutes:

$$\begin{tikzcd}
F(x,y) \arrow[r, "F(f{,}y)"] \arrow[d, "F(x{,}d)"'] & F(x',y) \arrow[d, "F(x'{,}d)"] \\
F(x,y') \arrow[r, "F(f{,}y')"'] & F(x',y').
\end{tikzcd}$$

To prove bijectivity, we construct $\phi = \theta^{-1}$. Thus, for any functor $G\colon \mathsf{C} \to \mathsf{CAT}(\mathsf{D},\mathsf{E})$ we define a functor $\phi(G) = \widehat{G}\colon \mathsf{C}\times\mathsf{D} \to \mathsf{E}$ using Proposition C.1.1 by giving its sections $\widehat{G}(x,\sim)\colon \mathsf{D} \to \mathsf{E}$, and $\widehat{G}(\sim,y)\colon \mathsf{C} \to \mathsf{E}$, and verifying the commutativity of the appropriate diagram.

For any x object in C, we define the x-section $\widehat{G}(x,\sim) = G(x)\colon \mathsf{D} \to \mathsf{E}$. Then, on objects $\widehat{G}(x,y) = F(x)(y)$.

For any y object in D, the functor $\widehat{G}(\sim,y)$ is clear on objects. Let $f : x \to x'$ be an morphism of C. The natural transformation $G(f)\colon G(x) \Rightarrow G(y)$ is given by $G(f)(y)\colon G(x,y) \to G(x',y)$. We take $\widehat{G}(f,y) = G(f)(y)$.

These sections give a functor $\mathsf{C} \to \mathsf{E}$ because the commutativity of the square is a direct consequence of naturality. $\square$

Corollary C.3.2. *There is a natural isomorphism of categories*

$$\Theta\colon\ \mathsf{CAT}(\mathsf{C}\times\mathsf{D},\mathsf{E}) \cong \mathsf{CAT}(\mathsf{C},\mathsf{CAT}(\mathsf{D},\mathsf{E}))$$

that on objects is θ.

Our interest lies not so much in general small categories but mainly in groupoids. We have seen that if G and H are groupoids, the category $\mathsf{CAT}(G,H)$ is also a groupoid that we represent by $\mathsf{GPDS}(G,H)$. The same bijection above proves that this internal morphisms make Gpds a cartesian closed category.

Corollary C.3.3. *If G, H, K are groupoids, there is a natural bijection of sets*

$$\mathsf{Gpds}(G\times H,K)\cong\mathsf{Gpds}(G,\mathsf{GPDS}(H,K))$$

and hence a natural isomorphism of groupoids

$$\mathsf{GPDS}(G\times H,K)\cong\mathsf{GPDS}(G,\mathsf{GPDS}(H,K)).$$

The reader will have noticed that since groups are special cases of groupoids, this corollary applies to the case when G,H,K are all groups and then yields a bijection of sets

$$\mathsf{Groups}(G\times H,K)\cong\mathsf{Gpds}(G,\mathsf{GPDS}(H,K))$$

natural with respect to morphisms of G, H, K. Thus to obtain an adjoint to the cartesian product of groups, we have to go outside the category of groups since $\mathsf{GPDS}(H,K)$ has, in general, more than one object. We shall come back to this case in Section C.6.

The applications of this exponential law confirm again that the sensible approach is to study the algebraic objects which arise in a given geometric situation, and to examine their uses in order to see how their algebraic properties match up to the formal requirements of the geometric situation. An important aspect of the properties of the algebraic objects is the properties of the category of these objects. As we see above, the category of groups has limitations, in that it is not cartesian closed. On the other hand, the category of groupoids is cartesian closed. We will obtain an application of this in the next section.[273]

In order to use the preceding results we have to make some deductions from them and get familiar with the deductions of some standard operations. Some of these arguments work in a general cartesian closed category, but it is important to become familiar with a particular example other than the standard category of sets, in which it is possible to proceed in an *ad hoc* basis.

C.4 The monoid of endomorphisms in the case of groupoids

It is well known that in the case of a cartesian closed category C, for any object E the internal endomorphisms E^E may be given a monoid structure. We are going to study the case of the category of groupoids. For the general case see [Kel82].

As we have seen for any groupoids, G, H, and K there are natural bijections

$$\mathsf{Gpds}(G\times H,K)\cong\mathsf{Gpds}(G,\mathsf{GPDS}(H,K)).$$

In particular, for any groupoids G and H there is a bijection

$$\theta\colon \mathsf{Gpds}(\mathsf{GPDS}(G,H)\times G, H) \to \mathsf{Gpds}(\mathsf{GPDS}(G,H), \mathsf{GPDS}(G,H))$$

with inverse ϕ.

We are going to study the *evaluation morphism*,

$$\varepsilon_{GH} = \phi(1_{\mathsf{GPDS}(G,H)})\colon \mathsf{GPDS}(G,H)\times G \to H$$

i.e. the functor corresponding to the identity $1_{\mathsf{GPDS}(G,H)}$ under the above bijection.

Remark C.4.1. Let us see the action of the evaluation morphism recalling the definition of ϕ. So, to define ε_{GH}, we give its sections.

For any functor $f: G \to H$, the section $\varepsilon_{GH}(f,\sim)\colon G \to H$ is defined to be f. Then, on objects, we have $\varepsilon_{GH}(f,x) = f(x)$, for any functor $f: G \to H$ and object $x \in G$.

For any object x in G, the section $\varepsilon_{GH}(\sim,x)\colon \mathsf{GPDS}(G,H) \to H$ is defined on objects as before, and for any natural transformation $\phi\colon f \Rightarrow f'$, $\varepsilon_{GH}(\sim,x)(\phi) = \phi(x)$.

Then, for any natural transformation $\phi\colon f \Rightarrow f'$ and morphism $a\colon x \to y$, $\varepsilon_{GH}(\phi,a)$ is the common composition of the commutative square

$$\begin{tikzcd}
f(x) \arrow[r, "f(a)"] \arrow[d, "\phi(x)"'] \arrow[dr, "\phi(a)"] & f(x') \arrow[d, "\phi(x')"] \\
f'(x) \arrow[r, "f'(a)"'] & f'(x')
\end{tikzcd}$$

which we write $\phi(a)$. Notice that for $a = 1_x$, we have $\phi(1_x) = \phi(x)$. □

Using the evaluation maps ε_{GH} we can define the map

$$\alpha\colon \mathsf{GPDS}(H,K)\times\mathsf{GPDS}(G,H)\times G \xrightarrow{1\times\varepsilon_{GH}} \mathsf{GPDS}(H,K)\times H \xrightarrow{\varepsilon_{HK}} K.$$

Now, using the bijection above, we get a functor

$$* = \theta(\alpha)\colon \mathsf{GPDS}(H,K)\times\mathsf{GPDS}(G,H) \to \mathsf{GPDS}(G,K)$$

which we call the *composition* of internal morphisms.

Remark C.4.2. To study the composition functor, it is better to have a more explicit description of α. On objects, for any two morphisms of groupoids $g\colon H \to K$, $f: G \to H$ and an object x in G, we have $\alpha(g,f,x) = g(f(x))$. On morphisms, given two natural transformations $\psi: g \Rightarrow g'$, $\phi\colon f \Rightarrow f'$ and a morphism $a\colon x \to x'$, we have

$$\alpha(\psi,\phi,a) = \psi\phi(a).$$

Now, we construct $\theta(\alpha)$ following Theorem C.3.1. Thus on objects $\theta(\alpha)$ is $(g, f) \mapsto gf$ and on morphisms, for any two natural transformations $\psi\colon g \Rightarrow g'$ and $\phi\colon f \Rightarrow f'$, $\psi * \phi\colon gf \Rightarrow g'f'$ is the natural transformation given by

$$\psi * \phi(x) = \alpha((\psi, \phi), 1_x) = \psi\phi(1_x) = \psi\phi(x),$$

i.e. the common composition of the arrows of the diagram

$$\begin{tikzcd} gf(x) \arrow[r, "g\phi(x)"] \arrow[d, "\psi f(x)"'] & gf'(x) \arrow[d, "\psi f'(x)"] \\ g'f(x) \arrow[r, "g'\phi(x)"'] & g'f'(x). \end{tikzcd}$$

Notice that $\psi * \phi$ may be seen as the common composition $(\psi f')(g\phi) = (g'\phi)(\psi f)$. It is easy to see that the composition is natural. □

Thus, for any groupoid G, the set of morphisms of the groupoid $\mathsf{END}(G) = \mathsf{GPDS}(G, G)$ is a monoid with respect to the composition just defined:

$$*\colon\ \mathsf{END}(G) \times \mathsf{END}(G) \to \mathsf{END}(G).$$

Moreover, the source, target and identity are homomorphisms between $\mathsf{END}(G)$ and $\mathsf{End}(G)$. To check that those compositions make $\mathsf{END}(G)$ a monoid on the category of groupoids it remains to prove the following:

Proposition C.4.3. *The composition of morphisms in* $\mathsf{END}(G)$*,*

$$\mathsf{END}(G) \times_{\mathsf{End}(G)} \mathsf{END}(G) \to \mathsf{END}(G),$$

is a homomorphism with respect to the composition $*$*, i.e. we have*

$$(\psi'\psi) * (\phi'\phi) = (\psi' * \phi')(\psi * \phi)$$

for any natural transformations $\phi\colon f \Rightarrow f', \phi'\colon f' \Rightarrow f'', \psi\colon g \Rightarrow g'$ *and* $\psi'\colon g' \Rightarrow g''$.

Proof. This is direct from the definition and the commutative diagram

$$\begin{tikzcd} & & g'f' \arrow[ddr, "\psi' f'"] \arrow[ddrr, "\psi' * \phi'"] & & \\ & & & & \\ gf \arrow[uurr, "\psi * \phi"] \arrow[r, "\psi f"] \arrow[ddrr, "\psi'\psi f"'] & g'f \arrow[uur, "g'\phi"] \arrow[ddr, "\psi' f"] & & g''f' \arrow[r, "g''\phi'"] & g''f'' \\ & & & & \\ & & g''f \arrow[uur, "g''\phi"] \arrow[uurr, "g''\phi'\phi"'] & & \end{tikzcd}$$

since the composition of the two morphisms on the bottom is $(\psi'\psi) * (\phi'\phi)$. □

C.5 The symmetry groupoid and the actor of a groupoid

It is a reasonable expectation that the symmetry of an object of type T should in some sense be a 'group object' of type T, and so some kind of higher order structure than T itself.

When looking for a structure reflecting the symmetries of a groupoid G, it is logical to consider all invertible elements of $\mathsf{END}(G)$. Let us write as usual $\mathrm{Aut}(G)$ for the subgroup of $\mathsf{End}(G)$ consisting of all automorphisms of the groupoid G and $\mathsf{AUT}(G)$ the full subcategory of $\mathsf{END}(G)$ having $\mathrm{Aut}(G)$ as objects. Clearly $\mathsf{AUT}(G)$ is a submonoid and a subgroupoid. Let us check that it is also a group with respect to $*$. The group-groupoid $\mathsf{AUT}(G)$ is called the *symmetry groupoid* of the groupoid G.

Proposition C.5.1. *The category* $\mathsf{AUT}(G)$ *is a group internal to groupoids.*

Proof. Let $\phi\colon f \Rightarrow f'$ be natural transformation from f to f', both being automorphisms of the groupoid G. Then the natural transformation $f'^{-1}\phi^{-1}f^{-1}\colon f^{-1} \Rightarrow f'^{-1}$ is the inverse of ϕ with respect to $*$. □

Now, we are going to define an equivalent structure. Let us consider the source map

$$s\colon \mathsf{AUT}(G) \to \mathrm{Aut}(G).$$

It is a homomorphism and the identity homomorphism is a right inverse. Thus the short exact sequence of groups and homomorphisms

$$1 \to \mathrm{Ker}\, s \to \mathsf{AUT}(G) \to \mathrm{Aut}(G) \to 1$$

splits, i.e. there is a bijection

$$\mathsf{AUT}(G) \cong \mathrm{Aut}(G) \times \mathrm{Ker}\, s$$

that maps any natural transformation of automorphisms $\phi\colon f \Rightarrow f'$ to the pair $(f, 1_{f^{-1}} * \phi)$ where the latter is a natural transformation $1 \to f^{-1}f'$, i.e. an element in $\mathrm{Ker}\, s$.

This bijection is an isomorphism when we endow the cartesian product with appropriate structure. This is the semidirect product with respect to the action of $\mathrm{Aut}(G)$ on $\mathrm{Ker}\, s$ on the right given by the identity and conjugation, i.e.

$$\mathsf{AUT}(G) \cong \mathrm{Aut}(G) \ltimes \mathrm{Ker}\, s$$

where the semidirect product $G \ltimes M$ of a group G and a G-group M is the cartesian product with the product given by $(g, m)(g', m') = (gg', m^{g'}m')$.

Thus, given $\mathrm{Ker}\, s$ and the action of $\mathrm{Aut}(G)$ on it, the source homomorphism is recovered directly since it is the identity on $\mathrm{Aut}(G)$ and the constant map on $\mathrm{Ker}\, s$ and the target homomorphism is determined once we know its restriction

$$t|\colon \mathrm{Ker}\, s \to \mathrm{Aut}(G)$$

since it is also the identity on $\mathrm{Aut}(G)$. This morphism $t|$ is called the *actor* of the groupoid. We shall see that it is an example of crossed module and that it is equivalent to the group-groupoid $\mathsf{AUT}(G)$. Thus these ideas are related to the notions of cat^1-group in Section 2.5.

Let us now consider yet another group that is equivalent to the actor. For any groupoid G, we define $\mathbb{M}(G)$ to be the set of sections $\sigma\colon \operatorname{Ob} G \to G$ of the target map $t\colon G \to \operatorname{Ob} G$, i.e. such sections σ which assign to each object $x \in \operatorname{Ob} G$ an morphism $\sigma(x)\colon s(\sigma(x)) \to x$. Then there is a map

$$\Delta\colon \mathbb{M}(G) \to \mathsf{END}(G)$$

such that to any section $\sigma\colon \operatorname{Ob} G \to G$ maps the natural transformation $\Delta(\sigma)\colon \delta(\sigma) \to 1$ where $\delta(\sigma)$ is the functor defined on objects by $\delta(\sigma)(x) = s\sigma(x)$ and on morphisms by $\delta(\sigma)(g) = \sigma(t(g))^{-1} g \sigma(s(g))$. The natural transformation $\Delta(\sigma)$ is then given by $\sigma(x)$.

It is clear from the definition that Δ is a bijection onto $\operatorname{Ker} t$. It is an isomorphism once we give the appropriate definition to the product of sections. For any two sections σ, τ their product $\tau * \sigma$ is defined as the section that for any $x \in \operatorname{Ob} G$, $\tau * \sigma(x)$ is the composition

$$s\tau(s\sigma(x)) \xrightarrow{\tau s\sigma(x)} s\sigma(x) \xrightarrow{\sigma(x)} x.$$

It is not difficult to prove that Δ is a homomorphism with this product. Let us consider the restriction to the group of units $\mathbb{M}^*(G)$.

Proposition C.5.2. *For any section $\sigma \in \mathbb{M}(G)$, the following are equivalent:*

(i) *σ is a unit;*

(ii) *$\Delta(\sigma)$ is bijective on objects;*

(iii) *$\Delta(\sigma)$ is bijective on morphisms;*

(iv) *$\Delta(\sigma)$ is an automorphism.*

Proof. We leave the proof to the reader. □

Thus the restriction gives an isomorphism $\Delta\colon \mathbb{M}^*(G) \cong \operatorname{Ker} t \subseteq \mathsf{AUT}(G)$. Using this isomorphism, the map s becomes $\delta\colon \mathbb{M}^*(G) \to \mathrm{Aut}(G)$ and the action of $\mathrm{Aut}(G)$ on $\mathsf{AUT}(G)$ induces an action on $\mathbb{M}^*(G)$ given by $\sigma^f(x) = f^{-1}\sigma f(x)$. This provides another possible interpretation of the actor of a groupoid.[274]

C.6 The case of a group

As we have seen, a group G may be regarded as a category, also written G, with one object $*_G$ and G as set of morphisms. The composition of morphisms is given by the product in G. This gives a full embedding of categories

$$\mathsf{Groups} \hookrightarrow \mathsf{Cat}$$

which also preserves products.

Thus a potential internal hom structure for the category Groups of groups should correspond to that of CAT, i.e. the internal hom between two groups G, H should be

$$\mathsf{CAT}(G, H) = \mathsf{GPDS}(G, H).$$

We know that this is a groupoid and we shall see that, in general, this groupoid has more than one object.

The set of objects of $\mathsf{GPDS}(G, H)$ is $\mathsf{Gpds}(G, H) = \mathsf{Hom}(G, H)$, i.e. the set of homomorphisms between the two groups. Clearly this set has many elements in general, thus $\mathsf{GPDS}(G, H)$ lies outside the category of groups and the category of groups is not a closed category, as is well known.

Let us see a characterisation of the morphisms of $\mathsf{GPDS}(G, H)$. Recall than an morphism of Groups is just a group homomorphism $f : G \to H$. A natural transformation $\phi\colon f \Rightarrow f'$ is given by a unique morphism $\phi(*_G) = y \in H$ corresponding to the object $*_G$, such that for any $x \in G$ satisfies the naturality condition, i.e. the diagram

$$\begin{tikzcd}
*_H \arrow[r, "f(x)"] \arrow[d, "y"'] & *_H \arrow[d, "y"] \\
*_H \arrow[r, "f'(x))"'] & *_H
\end{tikzcd}$$

commutes, giving $f'(x)y = yf(x)$ for all $x \in G$. Thus f' may be recovered from f and y since $f'(x) = yf(x)y^{-1}$ for all $x \in G$. We write $y\colon f \Rightarrow f'$. Thus a natural transformation $f \Rightarrow f'$ is just conjugation by an element $y \in H$.

We can also compute the evaluation and composition maps in this case.

Following the Remark C.4.1 the evaluation map $\varepsilon_{GH}\colon \mathsf{GPDS}(G, H) \times G \to H$ may be easily described in the case of groups. Since both G and H have a unique object, the functor ε_{GH} is trivial on objects. To describe the action on morphisms, we use the above characterisation of the elements of $\mathsf{GPDS}(G, H)$ as elements of H. Thus for any $y\colon f \Rightarrow f'\colon G \to H$ and any $x \in G$, we define $\varepsilon_{GH}(y, x) \in H$ by

$$\varepsilon_{GH}(y, x) = yf(x) = f'(x)y.$$

Following the Remark C.4.2, the composition

$$c\colon \mathsf{GPDS}(H, K) \times \mathsf{GPDS}(G, H) \to \mathsf{GPDS}(G, K)$$

for groups G, H, K can be easily described. It is clear that on objects c is just the composition of homomorphisms. Let us study the morphisms using the same characterisation as before. Let us consider $y\colon f \Rightarrow f'\colon G \to H$ and $z\colon g \Rightarrow g'\colon H \to K$, its composite $z * y\colon gf \Rightarrow g'f'\colon G \to K$ is the common value of the product $z * y = zg(y) = g'(y)z$.

Thus $\mathsf{END}(G) = \mathsf{GPDS}(G, G)$ is a monoid with the product just described.

Let us study the *symmetry groupoid* of the group G. As before, $\mathsf{AUT}(G)$ is the full subcategory of $\mathsf{END}(G)$ having as set of objects $\mathrm{Aut}(G)$ the group of all automorphisms of the group G. Its elements are $x\colon f \Rightarrow f'$ where $x \in G$, $f, f' \in \mathrm{Aut}(G)$ and $f'(x') = xf(x')x^{-1}$ for all $x' \in G$, i.e. f' is the 'left conjugate of f by x'. As seen, it is a groupoid and a group with respect to $*$. In this case the inverse with respect to $*$ of an element $x\colon f \Rightarrow f'$ may be easily computed to be $f'^{-1}(x\acute{}-1)\colon f^{-1} \Rightarrow f'^{-1}$.

Now, let us consider $\mathrm{Ker}\, s$, the kernel of the source map. Its elements are $x\colon 1 \Rightarrow f$, where f is left conjugation by x, i.e. $f(x') = xxx^{-1}$ for all $x' \in G$. The $*$ product in this subgroup is $x * x' = xx'$, thus $\mathrm{Ker}\, s$ is naturally isomorphic to G.

The action of $\mathrm{Aut}(G)$ on $\mathrm{Ker}\, s$ by the identity and conjugation, in this case is $x^{f'} = f'^{-1}(x)$ for any natural transformation $x\colon 1 \Rightarrow f$ and automorphism f'. Notice that $x^{f'}\colon 1 \Rightarrow f'^{-1} f f'$. With this action

$$\mathsf{AUT}(G) \cong \mathrm{Aut}(G) \ltimes \mathrm{Ker}\, s \cong \mathrm{Aut}(G) \ltimes G.$$

Remark C.6.1. There is the beginnings of a rough analogy between symmetry objects and homotopy types. Thus discrete sets model homotopy 0-types. The automorphisms of a set form a group, and groups model pointed homotopy 1-types. The automorphisms of a group may be formed into a crossed module, and these model pointed homotopy 2-types. The next stage involves crossed squares, which model pointed homotopy 3-types, see [Nor90], [BG89a], [AW10]. It is not known how to continue this process! □

C.7 Monoidal and monoidal closed categories

The aim of this section, which is joint work of R. Brown and S. V. Soloviev, is to give definitions and pointers to the literature on this important area.

A *monoidal category* is a category which has roughly speaking the structure of a monoid with respect to the usual product of categories.[275] Specifically, a *monoidal* structure on a category C consists of:

(i) a bifunctor $\otimes\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$, where the images of object (A, B) and morphism (f, g) are written $A \otimes B$ and $f \otimes g$ respectively,

(ii) an associativity isomorphism $a_{ABC}\colon (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$, for arbitrary objects A, B, C in C, such that a_{ABC} is natural in A, B and C. In other words,

- $a_{-BC}\colon (- \otimes B) \otimes C \Rightarrow - \otimes (B \otimes C)$ is a natural transformation for arbitrary objects B, C in C,
- $a_{A-C}\colon (A \otimes -) \otimes C \Rightarrow A \otimes (- \otimes C)$ is a natural transformation for arbitrary objects A, C in C,
- $a_{AB-}\colon (A \otimes B) \otimes - \Rightarrow A \otimes (B \otimes -)$ is a natural transformation for arbitrary objects A, B in C,

(iii) there is given an object I in C called the *unit object* (or simply the *unit*) together with for any object A in C natural isomorphisms:

$$l_A \colon I \otimes A \cong A \quad \text{and} \quad r_A \colon A \otimes I \cong A;$$

that is, both $l \colon I \otimes - \Rightarrow -$ and $r \colon - \otimes I \Rightarrow -$ are natural transformations.

The axioms are that the following diagrams commute:

- *unit coherence law*

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "a_{AIB}"] \arrow[dr, "r_A \otimes 1_B"'] & & A \otimes (I \otimes B) \arrow[dl, "1_A \otimes r_B"] \\
& A \otimes B &
\end{tikzcd}$$

- *associativity coherence law*

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "a_{A\otimes B,C,D}"] \arrow[d, "a_{ABC} \otimes 1_D"'] & (A \otimes B) \otimes (C \otimes D) \arrow[dd, "a_{A,B,C\otimes D}"] \\
(A \otimes (B \otimes C)) \otimes D \arrow[d, "a_{A,B\otimes C,D}"'] & \\
A \otimes ((B \otimes C) \otimes D) \arrow[r, "1_A \otimes a_{BCD}"'] & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

The bifunctor $- \otimes -$ is called the *tensor product* on C, and the natural isomorphisms a, l, r are called the *associativity isomorphism*, the *left unit isomorphism*, and the *right unit isomorphism* respectively.

The unit I defines a functor

$$U = \mathsf{C}(I, -) \colon \mathsf{C} \to \mathsf{Set}.$$

The important consequence of these axioms is the coherence theorem, which roughly speaking asserts the commutativity of all diagrams built up from the isomorphisms α, l, r given in the definition. For a precise statement we refer to [ML71].

It is also possible to add structure involving symmetry, which involve a natural equivalence of functors, called a *braiding*, $\tau_{AB} \colon A \otimes B \to B \otimes A$, with appropriate relations to the previous structure. Then C is called a *braided monoidal category*. If $\tau_{BA} = \tau_{AB}^{-1}$, then τ is called a *symmetry*, and the monoidal category is called *symmetric*.

A monoidal category $(\mathsf{C}, \otimes, \alpha, l, r)$ is called *closed* if for all objects B the functor $- \otimes B \colon \mathsf{C} \to \mathsf{C}$ has a right adjoint, written $\mathsf{HOM}(B, -)$, and sometimes $B \multimap -$, so that we have an adjunction

$$\mathsf{C}(A \otimes B, C) \cong \mathsf{C}(A, \mathsf{HOM}(B, C)) \tag{C.7.1}$$

with unit and counit (the latter called *evaluation*) say

$$\eta_{AB} \colon A \to \mathsf{HOM}(B, A \otimes B), \quad \varepsilon_{BC} \colon \mathsf{HOM}(B, C) \otimes B \to C.$$

Example C.7.1. In the monoidal closed category of vector spaces η_{AB} maps $a \in A$ to the linear mapping which sends $b \mapsto a \otimes b$, and ε_{BC} sends $f \otimes b \mapsto f(b)$. □

Putting $B = I$ in (C.7.1) and using the natural isomorphism $r \colon A \otimes I \cong A$ we deduce a natural isomorphism

$$i \colon C \cong \mathsf{HOM}(I, C). \tag{C.7.2}$$

Again, applications of associativity give natural bijections for any objects A, B, C, D

$$\mathsf{C}(D, \mathsf{HOM}(A \otimes B, C)) \cong \mathsf{C}(D, \mathsf{HOM}(A, \mathsf{HOM}(B, C)))$$

from which is obtained, either by putting $D = I$ and using Equation (C.7.2), or by representability, the natural isomorphism:

$$\mathsf{HOM}(A \otimes B, C) \cong \mathsf{HOM}(A, \mathsf{HOM}(B, C)). \tag{C.7.3}$$

Notice also that the unit and counit satisfy some generalised naturality conditions which can be illustrated by the diagrams:

$$\begin{tikzcd}
\mathsf{HOM}(B', C) \otimes B \arrow[r, "\mathsf{HOM}(f,1)\otimes 1"] \arrow[d, "\mathsf{HOM}(1,1)\otimes f"'] & \mathsf{HOM}(B, C) \otimes B \arrow[d, "\varepsilon_{BC}"] \\
\mathsf{HOM}(B', C) \otimes B' \arrow[r, "\varepsilon_{B'C}"'] & C,
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[r, "\eta_{CB}"] \arrow[d, "\eta_{CB'}"'] & \mathsf{HOM}(B, C \otimes B) \arrow[d, "\mathsf{HOM}(1,1\otimes f)"] \\
\mathsf{HOM}(B', C \otimes B') \arrow[r, "\mathsf{HOM}(f,1\otimes 1)"'] & \mathsf{HOM}(B, C \otimes B')
\end{tikzcd}$$

with $f \colon B \to B'$. The theory of these kinds of naturality has been developed in [EK66].

The coherence problem for monoidal closed categories is of course more complicated than that for monoidal categories, and some important results are found in [KML71], [Sol97]. The general framework to present these results is as follows, in which we abbreviate 'symmetric monoidal closed category' to 'SMCC':

1) What is a free SMCC $F_{\text{smcc}}(\mathbf{A})$ should be defined (or at least explained).
2) It is necessary to explain that the morphisms in $F_{\text{smcc}}(\mathbf{A})$ can be always interpreted as natural transformations (in generalized sense) over any SMCC K (if the variables are interpreted as identity functors). Let us call this the *canonic interpretation*.

3) If a diagram is commutative in $F_{\text{smcc}}(\mathbf{A})$, then all its interpretations in any SMCC K will be commutative.
4) (*Mac Lane Conjecture*) Then the diagram in $F_{\text{smcc}}(\mathbf{A})$ is commutative if and only if its canonic interpretation in the category of functors and natural transformations over the category of vector spaces is commutative.

The last statement is the main theorem of Soloviev in [Sol97]. This work contains the first proof of the theorem, though the theorem was used without proof or with a very incomplete proof, and was mentioned – and even used – earlier.[276]

It is also possible to define a natural composition

$$c_{ABC} \colon \mathsf{HOM}(B, C) \otimes \mathsf{HOM}(A, B) \to \mathsf{HOM}(A, C)$$

as the adjoint of the composition

$$\mathsf{HOM}(B, C) \otimes \mathsf{HOM}(A, B) \otimes A \xrightarrow{1 \otimes \varepsilon_{AB}} \mathsf{HOM}(B, C) \otimes B \xrightarrow{\varepsilon_{BC}} C$$

(where we slur over associativity), and then verify that this composition c is coherently associative, and has, up to natural isomorphisms, left and right identities.

The above definition of natural composition can be justified by reference to the paper [KML71]. A particular case of the main theorem of that paper is that there exists at most one natural transformation $F \to G$ in a free SMCC if $F \to G$ does not contain the tensor unit I and is *balanced*, i.e. $\mathsf{HOM}(F, G)$ contains each variable exactly twice with opposite variances. Because the adjoint to the composition c_{ABC} exits, the morphism c_{ABC} is unique and may be called the *natural composition*.

Another interesting aspect of this area, though perhaps tangential to the matters of this book, is as follows. Let A^* denote the 'dual object' $\mathsf{HOM}(A, I)$. There is a classical natural morphism $d_A \colon A \to A^{**}$ defined as the composition, in which we abbreviate HOM to H:

$$A \xrightarrow{\eta_{AA^*}} \mathsf{H}(A^*, A \otimes A^*) \xrightarrow{\mathsf{H}(1_{A^*}, \tau_{AA^*})} \mathsf{H}(A^*, A^* \otimes A) \xrightarrow{\mathsf{H}(1, \varepsilon_{A^*A})} \mathsf{H}(A^*, I) = A^{**}.$$

Readers of the classical paper defining categories [EML45a] will recognise this as a key example of a natural morphism for the example of vector spaces. In that example, $d_A \colon A \to A^{**}$ is an isomorphism if and only if A is finite dimensional.

A classical example of noncommutative diagram is the 'triple dual diagram':

$$\begin{tikzcd} A^{***} \arrow[rr, "1"] \arrow[dr, "d_A^*"'] & & A^{***} \\ & A^* \arrow[ur, "d_{A^*}"'] & \end{tikzcd} \qquad (*)$$

It is commutative if and only if d_A^* and d_{A^*} are mutually inverse isomorphisms.

This diagram does not satisfy the conditions of the Kelly–Mac Lane coherence theorem. Its noncommutativity can be verified e.g. in the category of infinite dimensional vector spaces. There is an unsolved conjecture:

Conjecture. *For a given SMCC, all balanced diagrams commute if and only if for all A, $d_A^*\colon A^{***} \to A^*$ and $d_{A^*}\colon A^* \to A^{***}$ are mutually inverse isomorphisms.*

There is more on this conjecture in [Sol90], [BS90]. Note that these ideas are not relevant to the category of crossed complexes, where the 'unit object' for the monoidal structure is the terminal object, the trivial crossed complex on a point. On other hand, an interesting monoidal closed category is that of modules (M, G) over groupoids in which G is finite and M is a family of finitely generated abelian groups. The unit for the monoidal structure is the module $(\mathbb{Z}, 1)$ where 1 here denotes the trivial groupoid on a point. In this category, the triple dual diagram is commutative.

The monoidal closed category C is said to be *biclosed* if it is closed and for every object A the functor $A \otimes -$ has a right adjoint, say $b\,\mathsf{HOM}(A, -)$. When C is symmetric monoidal, then C is closed if and only if it is biclosed and then $\mathsf{HOM} = b\,\mathsf{HOM}$. Cubical sets give an example of a biclosed monoidal category which is not symmetric, as we have observed in Chapter 10.

The necessity for coherence considerations when dealing with a notion of 'monoidal functor' leads to much work on 'weak n-categories, of which we just give the sample [Bén67], [CG07]. Such studies are mostly on 'globular' notions, and there is at present little work on weak cubical n-categories, nor on axiomatisations of the structure held by the cubical singular complex.

Remark C.7.2. In a cartesian closed category C we can identify a 'symmetry object' of an object C of C as $\mathsf{AUT}(C)$, which is a subgroup object of $\mathsf{END}(C)$. In a monoidal closed category, by contrast, we cannot define a group object, and therefore no subgroup object, as the tensor product has no diagonal comparable to that for the cartesian product. Nonetheless, some progress can be made in some cases, see for example [BG89a], [Bro10b]. Thus for a crossed complex C with a strict monoid structure $C \otimes C \to C$, it should be of interest to examine those cases where the induced monoid structure on C_0 is that of a group. The background to such studies is of course the wide importance of symmetry in mathematics and science, and the possibilities of 'higher order symmetry'. □

C.8 Crossed modules and quotients of groups

We start with some very basic facts on group theory.

Let N be the kernel of a homomorphism $f\colon G \to H$ of groups. Then N is a normal subgroup of G. This is equivalent to saying that the group G acts on the group N by conjugation in G. This is why a normal subgroup is a special case of a crossed module. We can put the emphasis slightly differently by saying that the kernel of a homomorphism of groups is a *group with action*, and in fact a special case of a crossed module.

Now a normal subgroup is closely associated with the notion of *quotient group*. The notion of quotient structure is very important in mathematics and science since it is closely associated with the idea of *classification*. In looking at an insect in a rain forest, say, we do not try to list all insects, but we do try to list as many species as we can find. Similarly, in mathematics, we often want to consider sets of elements as objects in themselves, for example lines are considered as sets of points in a plane. The basic tool for this is the standard notion of equivalence relation R on a set X and the associated set X/R of equivalence classes.

In order to fit the notion of equivalence relation into the notion of quotient groups, it is convenient to use the fact that a subgroup N of a group G determines an equivalence relation $\sim_N$ on G by the rule $g \sim_N g'$ if and only if $Ng = Ng'$, for $(g, g') \in G \times G$. In general this subset $\sim_N$ of $G \times G$ is not a subgroup of $G \times G$, where the latter has its usual group structure (for example, considered as a product of categories).

Proposition C.8.1. *The equivalence relation $\sim_N$ is a subgroup of $G \times G$ if and only if the subgroup N is normal in G.* □

We omit the proof since this is exactly the kind of result you have to verify for yourself.

It is usual to call an equivalence relation on G which is a subgroup of $G \times G$ a *congruence* on the group G.

It was quite early observed that an equivalence relation R on a set X is a special case of a groupoid with object set X, in which the set of morphisms is R and $R(x, y)$ consists of the set $\{(y, x)\}$ with multiplication $(z, y)(y, x) = (z, x)$. That is, in thinking about a groupoid H, we realise that H defines an equivalence relation on Ob H whose classes are the connected components of H. For this equivalence relation the elements of $H(x, y)$ could be thought of 'reasons why' x is equivalent to y, or as 'proofs that' x is equivalent to y. This analogy leads naturally to the consideration of higher dimensional theories, such as 'proofs of proofs', and so on. The relations of this idea with homotopy theory is steadily becoming more apparent. From this basic approach, the utility of notions of higher dimensional groupoids also becomes clear.

Thus it is natural to consider the generalisation of a congruence on a group G to some kind of groupoid on the set G. Part of the reason is that the notion of 'free equivalence relation', and hence of presentation, of an equivalence relation is not well defined. However the notion of presentation of a groupoid (and more generally of a group-groupoid) is well defined, and so to use analogues of combinatorial group theory for equivalence relations it is convenient to widen the scope of combinatorial group theory to combinatorial groupoid theory. This also allows the discussion of presentations of group actions, by considering the corresponding covering groupoids.

We refer also to page 49 for a discussion of the relation of kernels of morphisms of crossed modules to crossed squares, and so to homotopy 3-types.

These ideas should also be related to those of 'internal crossed modules', see [Jan03].

Another question is the relation of these ideas to those of universal algebra, and a relevant paper here is [JMT02]. In particular, it is there asserted that the category Crs of crossed complexes is a semi-abelian category.

Notes

272 p. 600 This approach, given initially in [BE69], was followed in [BN79] where it has the advantage of applying to the topological case.

273 p. 602 These results on cartesian closure are special cases of the result that the category of categories or of groupoids internal to a cartesian closed category is also cartesian closed, [BE69]. See also [BN79] for the topological case.

274 p. 606 These ideas have been developed to consider the actor of a crossed module. The first work on this was by Whitehead in [Whi48]. Later work was by Lue [Lue79], and Norrie, [Nor90]. This was examined from the point of view of the monoidal closed category of crossed modules in [BG89a]. A further development of these ideas to crossed modules over groupoids is in [Bİ03a], but essentially directly modelling Whitehead's work. See also [AW10] for both the groupoid and crossed module case. For an analysis of automorphism structures for various cartesian closed categories of graphs, see [BMSW08].

275 p. 608 For more information on this topic, see [ML71]. Possibly the earliest discussion of such extra structure on a category is in [Bén63], [Bén64]. Monoidal categories are also called tensor categories, see [JS91]. For recent discussions see [GM10], [AM10], and the references there. The monograph [Kel82] (downloadable) on enriched categories deals with the subject of $\mathcal{V}$-categories, which are a notion analogous to that of a category but in which the hom-sets are now objects of a monoidal, or monoidal closed, category $\mathcal{V}$, which could for example be one of the monoidal categories discussed in this book. An early work on such structures is [Bén65].

Any monoidal closed category may be seen as enriched over itself, and the category FTop of filtered spaces may be enriched over the category Crs using the functor Π.

276 p. 611 Szabo in [Sza78] claimed that he had a proof (but never published) and Chu used it with reference to Szabo in his appendix to [Bar79] where he introduced the "Chu construction". The account in [Sza78] is analysed in [Sol97], p. 304.

Bibliography

The numbers at the end of each item refer to the pages on which the respective work is cited.

[AHS06] Adámek, J., Herrlich, H., and Strecker, G. E., Abstract and concrete categories: the joy of cats. *Repr. Theory Appl. Categ.* (17) (2006) 1–507. Reprint of the 1990 original (Wiley, New York). 555, 563

[Ada56] Adams, J. F., Four applications of the self-obstruction invariants. *J. London Math. Soc.* **31** (1956) 148–159. 511

[Ada72] Adams, J. F., *Algebraic topology—a student's guide*. London Math. Soc. Lecture Note Ser. 4, Cambridge University Press, London (1972). 276

[Ada78] Adams, J. F., *Infinite loop spaces*. Ann. Math. Stud. 90, Princeton University Press, Princeton, N.J. (1978). 12

[AH56] Adams, J. F., and Hilton, P. J., On the chain algebra of a loop space. *Comment. Math. Helv.* **30** (1956) 305–330. 548

[AM10] Aguiar, M., and Mahajan, S., *Monoidal functors, species and Hopf algebras*. CRM Monogr. Ser. 29, Amer. Math. Soc., Providence, RI (2010). 551, 614

[AA89] Al-Agl, F., *Aspects of multiple categories*. Ph.D. thesis, University of Wales, Bangor (1989). xxxi

[AABS02] Al-Agl, F. A., Brown, R., and Steiner, R., Multiple categories: the equivalence of a globular and a cubical approach. *Adv. Math.* **170** (1) (2002) 71–118. xxxi, 202, 203, 204, 322, 478, 541, 542, 549

[AF61] Aleksandrov, P. S., and Finikov, S. P., Eduard Čech: Obituary. *Uspehi Mat. Nauk* **16** (1 (97)) (1961) 119–126 (1 plate). 12

[AW00] Alp, M., and Wensley, C. D., Enumeration of cat^1-groups of low order. *Internat. J. Algebra Comput.* **10** (4) (2000) 407–424. 85

[AW10] Alp, M., and Wensley, C. D., Automorphisms and homotopies of groupoids and crossed modules. *Appl. Cat. Struct.* **18** (5) (2010) 473–504. 608, 614

[And78] Anderson, D. W., Fibrations and geometric realizations. *Bull. Amer. Math. Soc.* **84** (5) (1978) 765–788. 434

[And57] Ando, H., A note on Eilenberg-MacLane invariant. *Tôhoku Math. J.* (2) **9** (1957) 96–104. 61, 255, 395, 436

[AN09] Andruskiewitsch, N., and Natale, S., The structure of double groupoids. *J. Pure Appl. Algebra* **213** (6) (2009) 1031–1045. 202, 203

[Ant00] Antolini, R., Cubical structures, homotopy theory. *Ann. Mat. Pura Appl.* (4) **178** (2000) 317–324. 394, 511

[AM11] Ara, D., and Métayer, F., The Brown-Golasiński model structure on ∞-groupoids revisited. *Homology, Homotopy, Appl.* **13** (2011) 121–142. 434, 542, 594

[Art72] Artzy, R., Kurt Reidemeister, (13.10.1893- 8.7.1971). *Jahresberichte der Deutschen Mathematiker-Vereinigung* **74** (1972) 96–104. xxxv

[AU06] Arvasi, Z., and Ulualan, E., On algebraic models for homotopy 3-types. *J. Homotopy Relat. Struct.* **1** (1) (2006) 1–27. 597

[AU07] Arvasi, Z., and Ulualan, E., 3-types of simplicial groups and braided regular crossed modules. *Homology, Homotopy Appl.* **9** (1) (2007) 139–161. 546

[Ash88] Ashley, N., Simplicial T-complexes and crossed complexes: a nonabelian version of a theorem of Dold and Kan. With a preface by R. Brown, *Dissertationes Math.* (*Rozprawy Mat.*) **265** (1988) 1–61.
http://ehres.pagesperso-orange.fr/Cahiers/AshleyEM32.pdf
xxvii, 322, 353, 479, 510, 512, 574

[BS90] Babaev, A. A., and Soloviev, S. V., On conditions of full coherence in biclosed categories: a new application of proof theory. In *COLOG*-88 (Tallinn, 1988), Lecture Notes in Comput. Sci. 417, Springer-Verlag, Berlin (1990), 3–8. 612

[BL04] Baez, J. C., and Lauda, A. D., Higher-dimensional algebra. V. 2-groups. *Theory Appl. Categ.* **12** (2004) 423–491. 62, 63, 202

[BS09] Baez, J. C., and Stevenson, D., The classifying space of a topological 2-group. In *Algebraic topology*, Abel Symp. 4, Springer-Verlag, Berlin (2009), 1–31. 323, 436

[BM09] Barmak, J. A., and Minian, E. G., Strong homotopy types, nerves and collapses. *Discrete Comput. Geom.*, to appear. DOI10.1007/s00454-011-9357-5. 546

[BL91] Barnes, D. W., and Lambe, L. A., A fixed point approach to homological perturbation theory. *Proc. Amer. Math. Soc.* **112** (3) (1991) 881–892. 365, 547

[Bar79] Barr, M., $*$-*autonomous categories*. With an appendix by Po Hsiang Chu, Lecture Notes in Math. 752, Springer-Verlag, Berlin (1979). 614

[Bar02] Barr, M., *Acyclic models*. CRM Monograph Series 17, Amer. Math. Soc., Providence, RI (2002). 366

[BW56] Barratt, M. G., and Whitehead, J. H. C., The first nonvanishing group of an $(n+1)$-ad. *Proc. London Math. Soc.* (3) **6** (1956) 417–439. 276

[BE69] Bastiani, A., and Ehresmann, C., Catégories de foncteurs structurés. *Cahiers Topologie Géom. Différentielle* **11** (1969) 329–384. 614

[BE72] Bastiani, A., and Ehresmann, C., Categories of sketched structures. *Cahiers Topologie Géom. Différentielle* **13** (1972) 105–214. 541

[Bau89] Baues, H. J., *Algebraic homotopy*, Cambridge Studies in Advanced Mathematics 15, Cambridge University Press, Cambridge (1989). 256, 353, 366, 435, 510, 546, 547, 594

[Bau91] Baues, H. J., *Combinatorial homotopy and* 4-*dimensional complexes*. With a preface by Ronald Brown, de Gruyter Expositions in Mathematics 2. Walter de Gruyter & Co., Berlin (1991). 256, 437, 511

[BB93] Baues, H. J., and Brown, R., On relative homotopy groups of the product filtration, the James construction, and a formula of Hopf. *J. Pure Appl. Algebra* **89** (1-2) (1993) 49–61. 323, 542, 548

[BC90] Baues, H. J., and Conduché, D., The central series for Peiffer commutators in groups with operators. *J. Algebra* **133** (1) (1990) 1–34. 84

[BC92] Baues, H. J., and Conduché, D., On the tensor algebra of a nonabelian group and applications. *K-Theory* **5** (6) (1991/92) 531–554. 323, 542

[BT97] Baues, H.-J., and Tonks, A., On the twisted cobar construction. *Math. Proc. Cambridge Philos. Soc.* **121** (2) (1997) 229–245. 546, 548

[Bén63] Bénabou, J., Catégories avec multiplication. *C. R. Acad. Sci. Paris* **256** (1963) 1887–1890. 614

[Bén64] Bénabou, J., Algèbre élémentaire dans les catégories avec multiplication. *C. R. Acad. Sci. Paris* **258** (1964) 771–774. 614

[Bén65] Bénabou, J., Catégories relatives. *C. R. Acad. Sci. Paris* **260** (1965) 3824–3827. 614

[Bén67] Bénabou, J., Introduction to bicategories. In *Reports of the Midwest Category Seminar*, Springer-Verlag, Berlin (1967), 1–77. 612

[Ber02] Berger, C., A cellular nerve for higher categories. *Adv. Math.* **169** (1) (2002) 118–175. 478, 510, 549

[BL70] Birkhoff, G., and Lipson, J. D., Heterogeneous algebras. *J. Combinatorial Theory* **8** (1970) 115–133. 201

[Bjö92] Björner, A., The homology and shellability of matroids and geometric lattices. In Matroid applications. In *Encyclopedia Math. Appl.* 40, Cambridge University Press, Cambridge (1992), 226–283. 84

[Bla48] Blakers, A., Some relations between homology and homotopy groups. *Ann. of Math.* (2) **49** (1948) 428–461. xxvi, 255, 323, 395, 478, 511

[BM49] Blakers, A. L., and Massey, W. S., The homotopy groups of a triad. *Proc. Nat. Acad. Sci. U. S. A.* **35** (1949) 322–328. 276

[BJT10] Blanc, D., Johnson, M. W., and Turner, J. M., Higher homotopy operations and cohomology. *J. K-Theory* **5** (1) (2010) 167–200. xxxii, 394, 511

[Bog94] Bogley, W. A., Unions of Cockcroft two-complexes. *Proc. Edinburgh Math. Soc.* (2) **37** (2) (1994) 317–324. 104

[BG00] Bogley, W. A., and Gilbert, N. D., The homology of Peiffer products of groups. *New York J. Math.* **6** (2000) 55–71 (electronic). 104

[BG92] Bogley, W. A., and Gutiérrez, M. A., Mayer-Vietoris sequences in homotopy of 2-complexes and in homology of groups. *J. Pure Appl. Algebra* **77** (1) (1992) 39–65. 104

[BO93] Book, R. V., and Otto, F., *String-rewriting systems*. Texts and Monographs in Computer Science, Springer-Verlag, New York (1993). 365

[BB78] Booth, P. I., and Brown, R., On the application of fibred mapping spaces to exponential laws for bundles, ex-spaces and other categories of maps. *General Topology and Appl.* **8** (2) (1978) 165–179. 582

[BT80] Booth, P. I., and Tillotson, J., Monoidal closed categories and convenient categories of topological spaces. *Pacific J. Math.* **88** (1980) 33–53. 254

[Bor94] Borceux, F., *Handbook of categorical algebra. 1. Basic category theory*. Encyclopedia of Mathematics and its Applications 50, Cambridge University Press, Cambridge (1994). 556

[BB04] Borceux, F., and Bourn, D., *Mal'cev, protomodular, homological and semi-abelian categories*. Math. Appl. 566, Kluwer Academic Publishers, Dordrecht (2004). 589

[BJ01] Borceux, F., and Janelidze, G., *Galois theories*. Cambridge Stud. Adv. Math. 72, Cambridge University Press, Cambridge (2001). 63, 575

[Bou70] Bourbaki, N., *Éléments de mathématique. Théorie des ensembles*. Hermann, Paris (1970). 13

[Bou87] Bourn, D., The shift functor and the comprehensive factorization for internal groupoids. *Cahiers Topologie Géom. Différentielle Catég.* **28** (3) (1987) 197–226. 328

[Bou07] Bourn, D., Moore normalization and Dold-Kan theorem for semi-abelian categories. In *Categories in algebra, geometry and mathematical physics*, Contemp. Math. 431, Amer. Math. Soc., Providence, RI (2007), 105–124. 512

[Bra26] Brandt, H., Über eine Verallgemeinerung des Gruppenbegriffes. *Math. Ann.* **96** (4) (1926) 360–366. 13

[Bre94] Breen, L., On the classification of 2-gerbes and 2-stacks. *Astérisque* **225** (1994). 436

[Bro59] Brown, E. H., Jr., Twisted tensor products. I. *Ann. of Math.* (2) **69** (1959) 223–246. 367

[Bro94] Brown, K. S., *Cohomology of groups*. Grad. Texts in Math. 87, Springer-Verlag, New York (1994). Corrected reprint of the 1982 original. 66

[Bro62] Brown, R., *Some problems in algebraic topology: function spaces and FD-complexes*. Ph.D. thesis, University of Oxford (1962). 254, 433, 541

[Bro63] Brown, R., Ten topologies for $X \times Y$. *Quart. J. Math. Oxford Ser.* (2) **14** (1963) 303–319. 254

[Bro64a] Brown, R., Cohomology with chains as coefficients. *Proc. London Math. Soc.* (3) **14** (1964) 545–565. 434, 436, 541

[Bro64b] Brown, R., Function spaces and product topologies. *Quart. J. Math. Oxford Ser.* (2) **15** (1964) 238–250. 254

[Bro64c] Brown, R., On Künneth suspensions. *Proc. Cambridge Philos. Soc.* **60** (1964) 713–720. 434, 436, 541

[Bro65a] Brown, R., On a method of P. Olum. *J. London Math. Soc.* **40** (1965) 303–304. xxxiv, 15, 544

[Bro65b] Brown, R., The twisted Eilenberg-Zilber theorem. In *Simposio di Topologia* (Messina, 1964), Edizioni Oderisi, Gubbio (1965), 33–37. 367

[Bro66] Brown, R., Two examples in homotopy theory. *Proc. Cambridge Philos. Soc.* **62** (1966) 575–576. 434

[Bro67] Brown, R., Groupoids and van Kampen's theorem. *Proc. London Math. Soc.* (3) **17** (1967) 385–401. xiii, xxxiv, 14, 28, 201, 544

[Bro68] Brown, R., *Elements of modern topology*. McGraw-Hill Book Co., New York (1968). xxii, 16, 434, 510, 548, 596

[Bro70] Brown, R., Fibrations of groupoids. *J. Algebra* **15** (1970) 103–132. 434, 591

[Bro80] Brown, R., On the second relative homotopy group of an adjunction space: an exposition of a theorem of J. H. C. Whitehead. *J. London Math. Soc.* (2) **22** (1) (1980) 146–152. 32, 42, 140

[Bro82] Brown, R., Higher dimensional group theory. In *Low-dimensional topology* (Bangor, 1979), London Math. Soc. Lecture Note Ser. 48, Cambridge University Press, Cambridge (1982), 215–238. xiii, xxxiv, 62, 203

[Bro83] Brown, R., An introduction to simplicial T-complexes. *Esquisses Math.* **32** (1983) 1–27. http://ehres.pagesperso-orange.fr/Cahiers/brownEM32.pdf 479

[Bro84a] Brown, R., Coproducts of crossed P-modules: applications to second homotopy groups and to the homology of groups. *Topology* **23** (3) (1984) 337–345. 12, 103

[Bro84b] Brown, R., Nonabelian cohomology and the homotopy classification of maps. In *Algebraic homotopy and local algebra* (Luminy, 1982), *Astérisque* **113** (1984), Soc. Math. France, Paris, 167–172. 395

[Bro87] Brown, R., From groups to groupoids: a brief survey. *Bull. London Math. Soc.* **19** (2) (1987) 113–134. xxxiv, 14, 16

[Bro89] Brown, R., Triadic Van Kampen theorems and Hurewicz theorems. In *Algebraic topology* (Evanston, IL, 1988), Contemp. Math. 96, Amer. Math. Soc., Providence, RI (1989), 39–57. xxvii, 276, 597

[Bro90] Brown, R., Some problems in nonabelian homotopical and homological algebra. In *Homotopy theory and related topics* (Kinosaki, 1988), Lecture Notes in Math. 1418, Springer-Verlag, Berlin (1990), 105–129. 551

[Bro92] Brown, R., Computing homotopy types using crossed n-cubes of groups. In *Adams Memorial Symposium on Algebraic Topology, 1* (Manchester, 1990), London Math. Soc. Lecture Note Ser. 175, Cambridge University Press, Cambridge (1992), 187–210. 201, 549, 597

[Bro94] Brown, R., Higher order symmetry of graphs. *Irish Math. Soc. Bull.* **32** (1994) 46–59. 30

[Bro96] Brown, R., Higher dimensional group theory. (1996) http://www.bangor.ac.uk/r.brown/hdaweb2.htm. 62

[Bro99] Brown, R., Groupoids and crossed objects in algebraic topology. *Homology Homotopy Appl.* **1** (1999) 1–78. 61, 323

[Bro06] Brown, R., *Topology and groupoids*. Printed by Booksurge LLC, Charleston, S. Carolina, third edition (2006). xiv, xvii, xviii, xxi, xxv, 12, 14, 15, 16, 18, 19, 25, 26, 27, 28, 30, 35, 44, 60, 61, 62, 83, 84, 103, 135, 140, 200, 202, 204, 211, 212, 213, 227, 254, 256, 265, 269, 274, 281, 324, 325, 326, 327, 344, 397, 403, 434, 502, 510, 545, 546, 548, 555, 569, 583, 588, 589, 590, 591, 593, 594, 595, 596, 597, 598

[Bro07] Brown, R., Three themes in the work of Charles Ehresmann: Local-to-global; groupoids; higher dimensions. In *7th Conference on the geometry and topology of manifolds: The mathematical legacy of Charles Ehresmann*, Bedlewo, Poland, 8.05.2005–15.05.2005, Banach Centre Publications 76, Institute of Mathematics, Polish Academy of Sciences, Warsaw (2007), 51–63. 6, 14

[Bro08a] Brown, R., Exact sequences of fibrations of crossed complexes, homotopy classification of maps, and nonabelian extensions of groups. *J. Homotopy Relat. Struct.* **3** (1) (2008) 331–342. 435, 436

[Bro08b] Brown, R., A new higher homotopy groupoid: the fundamental globular ω-groupoid of a filtered space. *Homology, Homotopy Appl.* **10** (1) (2008) 327–343. 478, 511, 512

[Bro09a] Brown, R., 'Double modules', double categories and groupoids, and a new homotopy double groupoid. arXiv:0903.2627 [math.CT] (2009) 1–8. 202, 203

[Bro09b] Brown, R., Moore hyperrectangles on a space form a strict cubical omega-category. arXiv:0909.2212v4 [math.CT] (2009) 1–7. 202, 478, 510

[Bro10a] Brown, R., On the homotopy 2-type of a free loop space. arXiv:1003.5617 [math.AT] (2010) 1–5. 322

[Bro10b] Brown, R., Possible connections between whiskered categories and groupoids, Leibniz algebras, automorphism structures and local-to-global questions. *J. Homotopy Relat. Struct.* **5** (1) (2010) 305–318. 546, 612

[Bro11] Brown, R., Covering morphisms of groupoids, derived modules and a 1-dimensional Relative Hurewicz Theorem. arXiv:1012.2824 [math.AT] (2011) 1–17. 276, 511

[BE88] Brown, R., and Ellis, G. J., Hopf formulae for the higher homology of a group. *Bull. London Math. Soc.* **20** (2) (1988) 124–128. 140, 141

[BG89a] Brown, R., and Gilbert, N. D., Algebraic models of 3-types and automorphism structures for crossed modules. *Proc. London Math. Soc.* (3) **59** (1) (1989) 51–73. 60, 546, 597, 608, 612, 614

[BG89b] Brown, R., and Golasiński, M., A model structure for the homotopy theory of crossed complexes. *Cahiers Topologie Géom. Différentielle Catég.* **30** (1) (1989) 61–82. 400, 434, 594

[BGPT97] Brown, R., Golasiński, M., Porter, T. and Tonks, A. Spaces of maps into classifying spaces for equivariant crossed complexes. *Indag. Math.* (*N.S.*) **8** (2) (1997) 157–172. 367, 394, 395, 436, 547, 596

[BGPT01] Brown, R., Golasiński, M., Porter, T. and Tonks, A. Spaces of maps into classifying spaces for equivariant crossed complexes. II. The general topological group case. *K-Theory* **23** (2) (2001) 129–155. 367, 394, 395, 436, 547, 596

[BHKP02] Brown, R., Hardie, K. A., Kamps, K. H., and Porter, T., A homotopy double groupoid of a Hausdorff space. *Theory Appl. Categ.* **10** (2002) 71–93 (electronic). 478

[BH70] Brown, R., and Heath, P. R., Coglueing homotopy equivalences. *Math. Z.* **113** (1970) 313–325. 62, 487, 510

[BH87] Brown, R., and Heath, P. R., Lifting amalgamated sums and other colimits of groups and topological groups. *Math. Proc. Cambridge Philos. Soc.* **102** (2) (1987) 273–280. 363

[BHK83] Brown, R., Heath, P. R., and Kamps, K. H., Groupoids and the Mayer-Vietoris sequence. *J. Pure Appl. Algebra* **30** (2) (1983) 109–129. xxxv, 434, 545

[BH77] Brown, R., and Higgins, P. J., Sur les complexes croisés, ω-groupoïdes, et T-complexes. *C. R. Acad. Sci. Paris Sér. A-B* **285** (16) (1977) A997–A999. 255

[BH78a] Brown, R., and Higgins, P. J., On the connection between the second relative homotopy groups of some related spaces. *Proc. London Math. Soc.* (3) **36** (2) (1978) 193–212. xxiii, 43, 140, 201, 203, 204

[BH78b] Brown, R., and Higgins, P. J., Sur les complexes croisés d'homotopie associés à quelques espaces filtrés. *C. R. Acad. Sci. Paris Sér. A-B* **286** (2) (1978) A91–A93. 256

[BH81] Brown, R., and Higgins, P. J., On the algebra of cubes. *J. Pure Appl. Algebra* **21** (3) (1981) 233–260. xxvi, xxxi, 255, 477, 479, 512

[BH81a] Brown, R., and Higgins, P. J., Colimit theorems for relative homotopy groups. *J. Pure Appl. Algebra* **22** (1) (1981) 11–41. xiii, xxvi, 256, 394, 479, 510

[BH81b] Brown, R., and Higgins, P. J., The equivalence of ∞-groupoids and crossed complexes. *Cahiers Topologie Géom. Différentielle* **22** (4) (1981) 371–386. 201, 255, 322, 434, 478, 479, 512, 542

[BH81c] Brown, R., and Higgins, P. J., The equivalence of ω-groupoids and cubical T-complexes. *Cahiers Topologie Géom. Différentielle* **22** (4) (1981) 349–370. 477, 479, 512

[BH82] Brown, R., and Higgins, P. J., Crossed complexes and nonabelian extensions. In *Int. Conf. on Category theory* (Gummersbach, 1981), Lecture Notes in Math. 962, Springer-Verlag, Berlin (1982), 39–50. 364, 435, 436

[BH87] Brown, R., and Higgins, P. J., Tensor products and homotopies for ω-groupoids and crossed complexes. *J. Pure Appl. Algebra* **47** (1) (1987) 1–33. 541

[BH89] Brown, R., and Higgins, P. J., The classifying space of a crossed complex. UWB Math. Preprint **89.06** (1989) 30pp. 394, 395

[BH90] Brown, R., and Higgins, P. J., Crossed complexes and chain complexes with operators. *Math. Proc. Cambridge Philos. Soc.* **107** (1) (1990) 33–57. 256, 257, 322

[BH91] Brown, R., and Higgins, P. J., The classifying space of a crossed complex. *Math. Proc. Cambridge Philos. Soc.* **110** (1) (1991) 95–120. 61, 367, 368, 395, 435, 436, 541, 594

[BH03] Brown, R., and Higgins, P. J., Cubical abelian groups with connections are equivalent to chain complexes. *Homology Homotopy Appl.* **5** (1) (2003) 49–52 (electronic). 395, 479, 512

[BH82] Brown, R., and Huebschmann, J., Identities among relations. In *Low-dimensional topology* (Bangor, 1979), London Math. Soc. Lecture Note Ser. 48, Cambridge University Press, Cambridge (1982), 153–202. 32, 61, 83, 84, 85, 365, 366, 436

[Bİ03a] Brown, R., and İcen, İ., Homotopies and automorphisms of crossed modules of groupoids. *Appl. Categ. Structures* **11** (2) (2003) 185–206. 60, 256, 512, 614

[Bİ03b] Brown, R., and İcen, İ., Towards a 2-dimensional notion of holonomy. *Adv. Math.* **178** (1) (2003) 141–175. 256

[BJ99] Brown, R., and Janelidze, G., Galois theory of second order covering maps of simplicial sets. *J. Pure Appl. Algebra* **135** (1) (1999) 23–31. 436

[BJ04] Brown, R., and Janelidze, G., Galois theory and a new homotopy double groupoid of a map of spaces. *App. Cat. Struct.* **12** (2004) 63–80. 62, 63, 202, 549

[BKP05] Brown, R., Kamps, K. H., and Porter, T., A homotopy double groupoid of a Hausdorff space. II. A van Kampen theorem. *Theory Appl. Categ.* **14** (9) (2005) 200–220 (electronic). 204, 478

[BL87] Brown, R., and Loday, J.-L., Van Kampen theorems for diagrams of spaces. *Topology* **26** (3) (1987) 311–335. With an appendix by M. Zisman. xxxv, 38, 62, 63, 140, 276, 511, 545, 549, 596, 597

[BL87a] Brown, R., and Loday, J.-L., Homotopical excision, and Hurewicz theorems for n-cubes of spaces. *Proc. London Math. Soc.* (3) **54** (1) (1987) 176–192. xxvii, 105, 140, 276, 323, 545, 549, 596, 597

[BM92] Brown, R., and Mackenzie, K. C. H., Determination of a double Lie groupoid by its core diagram. *J. Pure Appl. Algebra* **80** (3) (1992) 237–272. 202, 203

[BMPW02] Brown, R., Moore, E. J., Porter, T., and Wensley, C. D., Crossed complexes, and free crossed resolutions for amalgamated sums and HNN-extensions of groups. Dedicated to Professor Hvedri Inassaridze on the occasion of his 70th birthday. *Georgian Math. J.* **9** (4) (2002) 623–644. 339, 353, 364, 365

[BMSW08] Brown, R., Morris, I., Shrimpton, J., and Wensley, C. D., Graphs of morphisms of graphs. *Electron. J. Combin.* **15** (1) (2008) Article 1, 28. 597, 614

[BM99] Brown, R., and Mosa, G. H., Double categories, 2-categories, thin structures and connections. *Theory Appl. Categ.* **5** (7) (1999) 163–175 (electronic). 203

[BM94] Brown, R., and Mucuk, O., Covering groups of nonconnected topological groups revisited. *Math. Proc. Cambridge Philos. Soc.* **115** (1) (1994) 97–110. 63, 435, 436

[BN79] Brown, R., and Nickolas, P., Exponential laws for topological categories, groupoids and groups, and mapping spaces of colimits. *Cahiers Topologie Géom. Différentielle* **20** (2) (1979) 179–198. 614

[BP96] Brown, R., and Porter, T., On the Schreier theory of non-abelian extensions: generalisations and computations. *Proc. Roy. Irish Acad. Sect. A* **96** (2) (1996) 213–227. 435, 436, 437

[BP03] Brown, R., and Porter, T., Category theory and higher dimensional algebra: potential descriptive tools in neuroscience. In *Proceedings of the International Conference on Theoretical Neurobiology* Delhi, February 2003, Volume 1 (2003), 80–92. 575

[BP06] Brown, R., and Porter, T., Category theory: an abstract setting for analogy and comparison. In *What is Category Theory?* Advanced Studies in Mathematics and Logic, Polimetrica Publisher, Monza (2006) 257–274. 575

[BRS84] Brown, R., and Razak Salleh, A., A van Kampen theorem for unions of nonconnected spaces. *Arch. Math.* (*Basel*) **42** (1) (1984) 85–88. xxxiv, 30, 204

[BRS99] Brown, R., and Razak Salleh, A., Free crossed resolutions of groups and presentations of modules of identities among relations. *LMS J. Comput. Math.* **2** (1999) 28–61 (electronic). 32, 365, 366

[BS07] Brown, R., and Sivera, R., Normalisation for the fundamental crossed complex of a simplicial set. *J. Homotopy and Related Structures* (*Special issue devoted to the memory of Saunders Mac Lane*(**2** (2) (2007) 49–79. 323, 366, 576

[BS09] Brown, R., and Sivera, R., Algebraic colimit calculations in homotopy theory using fibred and cofibred categories. *Theory App. Cat.* **22** (2009) 221–251. 579

[BS76a] Brown, R., and Spencer, C. B., Double groupoids and crossed modules. *Cahiers Topologie Géom. Différentielle* **17** (4) (1976) 343–362. xiii, xxx, 43, 202, 203

[BS76b] Brown, R., and Spencer, C. B., G-groupoids, crossed modules and the fundamental groupoid of a topological group. *Nederl. Akad. Wetensch. Proc. Ser. A* **79** = *Indag. Math.* **38** (4) (1976) 296–302. xiii, 43, 62, 63, 202, 203, 435

[BS10] Brown, R., and Street, R., Covering morphisms of crossed complexes and of cubical omega-groupoids are closed under tensor product. *Cahiers Topologie Géom. Différentielle*, to appear; arXiv:1009.5609v2 [math.AT] (2010) 1–15. 541, 542

[BW95] Brown, R., and Wensley, C. D., On finite induced crossed modules, and the homotopy 2-type of mapping cones. *Theory Appl. Categ.* **1** (3) (1995) 54–70 (electronic). 126, 140, 141, 364

[BW96] Brown, R., and Wensley, C. D., Computing crossed modules induced by an inclusion of a normal subgroup, with applications to homotopy 2-types. *Theory Appl. Categ.* **2** (1) (1996) 3–16 (electronic). 104, 140, 141

[BW03] Brown, R., and Wensley, C. D., Computation and homotopical applications of induced crossed modules. *J. Symbolic Comput.* **35** (1) (2003) 59–72. 140

[Bro92] Brown, R. A., Generalized group presentation and formal deformations of CW complexes. *Trans. Amer. Math. Soc.* **334** (2) (1992) 519–549. 547

[BP02] Buchstaber, V. M., and Panov, T. E., *Torus actions and their applications in topology and combinatorics*. University Lecture Series 24, Amer. Math. Soc., Providence, RI (2002). xxxii

[BFGM05] Bullejos, M., Faro, E., and García-Muñoz, M. A., Postnikov invariants of crossed complexes. *J. Algebra* **285** (1) (2005) 238–291. 437, 548

[BN00] Bunge, M., and Niefield, S., Exponentiability and single universes. *J. Pure Appl. Algebra* **148** (3) (2000) 217–250. 364, 597

[CC91] Carrasco, P., and Cegarra, A. M., Group-theoretic algebraic models for homotopy types. *J. Pure Appl. Algebra* **75** (3) (1991) 195–235. 512

[CGV06] Carrasco, P., Garzón, A. R., and Vitale, E. M., On categorical crossed modules. *Theory Appl. Categ.* **16** (22) (2006) 585–618 (electronic). 202

[CLV02] Casas, J. M., Ladra, M., and Vieites, A. M., Obstruction theory in crossed modules. *Comm. Algebra* **30** (8) (2002) 3589–3609. 435

[Čec32] Čech, E., *Höherdimensionale Homotopiegruppen*. Verhandlungen des Internationalen Mathematiker-Kongresses Zurich, Band 2 (1932). 10

[CGCO02] Cegarra, A. M., García-Calcines, J. M., and Ortega, J. A., On graded categorical groups and equivariant group extensions. *Canad. J. Math.* **54** (5) (2002) 970–997. 436

[CG07] Cheng, E., and Gurski, N., The periodic table of n-categories for low dimensions. I. Degenerate categories and degenerate bicategories. In *Categories in algebra, geometry and mathematical physics*. Contemp. Math. 431, Amer. Math. Soc., Providence, RI (2007), 143–164. 612

[Chi11] Chikhladze, D., A category of quantum categories. *Theory Appl. Cat.* **25** (1) (2011) 1–37. 551

[CCH81] Chiswell, I. M., Collins, D. J., and Huebschmann, J., Aspherical group presentations. *Math. Z.* **178** (1) (1981) 1–36. 365

[Cis06] Cisinski, D.-C., Les préfaisceaux commes modèles des types d'homotopie. *Astérisque* **308**, Soc. Math. France, Paris (2006). 394, 594

[Coa74] Coates, R. B., *Semantics of generalised algebraic structures*. Ph.D. thesis, University of London (1974). 541

[Coh89] Cohen, D. E., *Combinatorial group theory: a topological approach*. London Math. Soc. Student Texts 14, Cambridge University Press, Cambridge (1989). 83, 301

[Coh73] Cohen, M. M., *A course in simple-homotopy theory*. Grad. Texts in Math. 10. Springer-Verlag, New York (1973). 61, 276, 547

[Con84] Conduché, D., Modules croisés généralisés de longueur 2, *J. Pure Appl. Algebra*, **34** (1984) 155–178. 512

[CE89] Conduché, D., and Ellis, G. J., Quelques propriétés homologiques des modules précroisés. *J. Algebra* **123** (2) (1989) 327–335. 84

[Con72] Conduché, F., Au sujet de l'existence d'adjoints à droite aux foncteurs "image réciproque" dans la catégorie des catégories. *C. R. Acad. Sci. Paris Sér. A-B* **275** (1972) A891–A894. 364

[Con94] Connes, A., *Noncommutative geometry*. Academic Press Inc., San Diego, CA (1994). 14

[Cor87] Cordier, J.-M., Homologie de Steenrod-Sitnikov et limite homotopique algébrique. *Manuscripta Math.* **59** (1) (1987) 35–52. 547

[CP97] Cordier, J.-M., and Porter, T., Homotopy coherent category theory. *Trans. Amer. Math. Soc.* **349** (1) (1997) 1–54. 394, 395, 596

[Cra99] Crans, S. E., On combinatorial models for higher dimensional homotopies. PhD. Thesis, Utrecht (1995). http://home.tiscali.nl/secrans/papers/comb.html 322, 542, 543

[Cra99] Crans, S. E., A tensor product for **Gray**-categories. *Theory Appl. Categ.* **5** (2) (1999) 12–69 (electronic). 322, 541

[CP01] Crisp, J., and Paris, L., The solution to a conjecture of Tits on the subgroup generated by the squares of the generators of an Artin group. *Invent. Math.* **145** (1) (2001) 19–36. 365

[Cro59] Crowell, R. H., On the van Kampen theorem. *Pacific J. Math.* **9** (1959) 43–50. xxxiv, 30

[Cro61] Crowell, R. H., Corresponding group and module sequences. *Nagoya Math. J.* **19** (1961) 27–40. 257

[Cro71] Crowell, R. H., The derived module of a homomorphism. *Adv. Math.* **6** (1971) 210–238 (1971). 15, 257, 276

[CF63] Crowell, R. H., and Fox, R. H., *Introduction to knot theory*. Grad. Texts in Math. 57, Springer-Verlag, New York (1963). 13, 257

[CR06] Curtis, C. W., and Reiner, I., *Representation theory of finite groups and associative algebras*. AMS Chelsea Publishing, Providence, RI (2006). Reprint of the 1962 original. 112, 141

[Dak77] Dakin, M. K., *Kan complexes and multiple groupoid structures*. PhD. Thesis, University of Wales, Bangor (1977). http://ehres.pagesperso-orange.fr/Cahiers/dakinEM.pdf xiv, 445, 479, 574

[Dan91] Danas, G., *Group extensions and cohomology in cartesian closed categories*. PhD. thesis, City University of New York (1991). 437

[DP93a] Dawson, R., and Paré, R., Characterizing tileorders. *Order* **10** (2) (1993) 111–128. 478

[DP93b] Dawson, R., and Paré, R., General associativity and general composition for double categories. *Cahiers Topologie Géom. Différentielle Catég.* **34** (1) (1993) 57–79. 478

[Day70] Day, B. J., *Construction of Biclosed Categories*. PhD Thesis, University of New South Wales, 1970. http://www.math.mq.edu.au/~street/DayPhD.pdf 542

[Day70a] Day, B. J., On closed categories of functors. In *Reports of the Midwest Category Seminar IV*, Lecture Notes in Math. 137, Springer-Verlag, Berlin (1970), 1–38. 542

[Day72] Day, B. J., A reflection theorem for closed categories. *J. Pure Appl. Algebra* **2** (1972) 1–11. 542

[Ded58] Dedecker, P., Cohomologie de dimension 2 à coefficients non abéliens. *C. R. Acad. Sci. Paris* **247** (1958) 1160–1163. 255

[Ded60] Dedecker, P., Sur la cohomologie non abélienne. I. *Canad. J. Math.* **12** (1960) 231–251. 29, 255, 436

[Ded63] Dedecker, P., Sur la cohomologie non abélienne. II. *Canad. J. Math.* **15** (1963) 84–93. 255

[Ded64] Dedecker, P., Les foncteurs Ext_Π, $\mathrm{H}_\Pi{}^2$ et $H_\Pi{}^2$ non abéliens. *C. R. Acad. Sci. Paris* **258** (1964) 4891–4894. 436

[Die89] Dieudonné, J., *A history of algebraic and differential topology. 1900–1960*. Birkhäuser, Boston, MA (1989). 10

[Dol58] Dold, A., Homology of symmetric products and other functors of complexes. *Ann. of Math.* (2) **68** (1958) 54–80. 479, 512

[Dol95] Dold, A., *Lectures on algebraic topology*. Classics in Mathematics, Springer-Verlag, Berlin (1995). Reprint of the 1972 edition. 366

[Dus75] Duskin, J., Simplicial methods and the interpretation of "triple" cohomology. *Mem. Amer. Math. Soc.* **3** (issue 2, 163) (1975). 479

[DS95] Dwyer, W. G., and Spaliński, J., Homotopy theories and model categories. In *Handbook of algebraic topology*, North-Holland, Amsterdam (1995), 73–126. 434, 594

[DV73] Dyer, E., and Vasquez, A. T., Some small aspherical spaces. *J. Austral. Math. Soc.* **16** (1973) 332–352. Collection of articles dedicated to the memory of Hanna Neumann, III. 364, 436, 547

[EH62] Eckmann, B., and Hilton, P. J., Group-like structures in general categories. I. Multiplications and comultiplications. *Math. Ann.* **145** (1961/1962) 227–255. 11

[EP97] Ehlers, P. J., and Porter, T., Varieties of simplicial groupoids. I. Crossed complexes. *J. Pure Appl. Algebra* **120** (3) (1997) 221–233. xxvii

[EV08] Ehresmann, A., and Vanbremeersch, J. P., *Memory evolutive systems: hierarchy, emergence, cognition*. Studies in Multidisciplinarity 4, Elsevier, Amsterdam (2008). 575

[Ehr57] Ehresmann, C., Gattungen von lokalen Strukturen. *Jahresber. Deutsch. Math. Verein.* **60** (1) (1957) 49—77. 597

[Ehr65] Ehresmann, C., *Catégories et structures*. Dunod, Paris (1965). xx, 201

[Ehr83] Ehresmann, C., Œuvres complètes et commentées. I-1, 2. Topologie algébrique et géométrie différentielle. *Cahiers Topologie Géom. Différentielle* **24** (suppl. 1) (1983) [(1984)], with commentary by W. T. van Est, M. Zisman, G. Reeb, P. Libermann, R. Thom, J. Pradines, R. Hermann, A. Kock, A. Haefliger, J. Bénabou, R. Guitart, and A. C. Ehresmann, edited by A. C. Ehresmann. 6

[Eil44] Eilenberg, S., Singular homology theory. *Ann. of Math.* (2) **45** (1944) 407–447. xxvi

[Eil47] Eilenberg, S., Homology of spaces with operators. I. *Trans. Amer. Math. Soc.* **61** (1947) 378–417; errata, 62, 548 (1947). 276

[EK66] Eilenberg, S., and Kelly, G. M., Closed categories. In *Proc. Conf. Categorical Algebra* (La Jolla, Calif., 1965), Springer-Verlag, New York (1966), 421–562. 610

[EML45a] Eilenberg, S., and Mac Lane, S., General theory of natural equivalences. *Trans. Amer. Math. Soc.* **58** (1945) 231–294. 555, 611

[EML45b] Eilenberg, S., and MacLane, S., Relations between homology and homotopy groups of spaces. *Ann. of Math.* (2) **46** (1945) 480–509. xxvi, 13

[EML47] Eilenberg, S., and Mac Lane, S., Cohomology theory in abstract groups. I. *Ann. of Math.* (2) **48** (1947) 51–78. 65

[EML49] Eilenberg, S., and Mac Lane, S., Homology of spaces with operators. II. *Trans. Amer. Math. Soc.* **65** (1949) 49–99. 276

[EML50] Eilenberg, S., and Mac Lane, S., Relations between homology and homotopy groups of spaces. II. *Ann. of Math.* (2) **51** (1950) 514–533. 61, 436

[EML53a] Eilenberg, S., and Mac Lane, S., Acyclic models. *Amer. J. Math.* **75** (1953) 189–199. 366, 367, 511

[EML53b] Eilenberg, S., and Mac Lane, S., On the groups $H(\pi, n)$. I. *Ann. of Math.* (2) **58** (1953) 55–106. 367

[EZ53] Eilenberg, S., and Zilber, J. A., On products of complexes. *Amer. J. Math.* **75** (1953) 200–204. 366, 367

[Ell93] Ellis, G., Crossed squares and combinatorial homotopy. *Math. Z.* **214** (1) (1993) 93–110. 545, 597

[Ell04] Ellis, G., Computing group resolutions. *J. Symbolic Comput.* **38** (3) (2004) 1077–1118. 60, 365, 366, 437

[Ell08] Ellis, G., Homological algebra programming. (2008) http://hamilton.nuigalway.ie/Hap/www/ 365, 366

[EM10] Ellis, G., and Mikhailov, R., A colimit of classifying spaces. *Adv. Math.* **223** (6) (2010) 2097–2113. 549, 596, 597

[Ell88a] Ellis, G. J., The group $K_2(\Lambda; I_1, \dots, I_n)$ and related computations. *J. Algebra* **112** (2) (1988) 271–289. 62

[Ell88b] Ellis, G. J., Homotopy classification the J. H. C. Whitehead way. *Exposition. Math.* **6** (2) (1988) 97–110. 435

[EP86] Ellis, G. J., and Porter, T., Free and projective crossed modules and the second homology group of a group. *J. Pure Appl. Algebra* **40** (1) (1986) 27–31. 140

[ES87] Ellis, G. J., and Steiner, R., Higher-dimensional crossed modules and the homotopy groups of $(n+1)$-ads. *J. Pure Appl. Algebra* **46** (2-3) (1987) 117–136. 62, 276, 596

[Eps66] Epstein, D. B. A., Semisimplicial objects and the Eilenberg-Zilber theorem. *Invent. Math.* **1** (1966) 209–220. 367

[EW07] Evans, G. A., and Wensley, C. D., Complete involutive rewriting systems. *J. Symbolic Comput.* **42** (11-12) (2007) 1034–1051. 365

[EGVdL08] Everaert, T., Gran, M., and Van der Linden, T., Higher Hopf formulae for homology via Galois theory. *Adv. Math.* **217** (5) (2008) 2231–2267. 141

[EKVdL06] Everaert, T., Kieboom, R. W., and Van der Linden, T., Model structures for homotopy of internal categories. *Theory Appl. Categ.* **15** (3) (2005/06) 66–94 (electronic).

[Evr76] Evrard, M., Homotopie des complexes simpliciaux et cubiques. Preprint (1976). 478

[FM07] Faria Martins, J., A new proof of a realisation theorem of H.-J. Baues. Preprint (2007) 1–16. 353

[FM09] Faria Martins, J., The fundamental crossed module of the complement of a knotted surface. *Trans. Amer. Math. Soc.* **361** (9) (2009) 4593–4630. 61, 545, 596

[FM11] Faria Martins, J., The fundamental 2-crossed complex of a reduced CW-complex. Preprint (2011) 1–30. 512, 545

[FMP11] Faria Martins, J., and Picken, R., Surface holonomy for non-abelian 2-bundles via double groupoids. *Adv. Math.* **226** (2011) 3309–3366. xxxii, 436, 478, 546, 548

[FRS95] Fenn, R., Rourke, C., and Sanderson, B., Trunks and classifying spaces. *Appl. Categ. Structures* **3** (4) (1995) 321–356. 394, 395

[FR05] Forester, M., and Rourke, C., Diagrams and the second homotopy group. *Comm. Anal. Geom.* **13** (4) (2005) 801–820. 256, 276

[Fox53] Fox, R. H., Free differential calculus. I. Derivation in the free group ring. *Ann. of Math.* (2) **57** (1953) 547–560. 257

[Fre72] Freyd, P., Aspect of topoi. *Bull. Austral. Math. Soc.* **7** (1972) 1–76. 541

[GZ67] Gabriel, P., and Zisman, M., *Calculus of fractions and homotopy theory*. Ergeb. Math. Grenzgeb. 35, Springer-Verlag, New York (1967). 16, 367, 394, 575

[GAP08] The GAP Group. *GAP – Groups, algorithms, and programming, Version 4.4.12* (2008). 137

[Gar80] Gardner, R., Review of "Invariant theory", by J.A. Springer. *Bull. Amer. Math. Soc.* **2** (1980) 246–256. 64

[Gil87] Gilbert, N. D., On the fundamental cat^n-group of an n-cube of spaces. In *Algebraic topology* (Barcelona, 1986), Lecture Notes in Math. 1298, Springer-Verlag, Berlin (1987), 124–139. 38, 62, 63, 549, 597

[GH89] Gilbert, N. D., and Higgins, P. J., The nonabelian tensor product of groups and related constructions. *Glasgow Math. J.* **31** (1) (1989) 17–29. 103, 104, 323

[Gir64] Giraud, J., Méthode de la descente. *Bull. Soc. Math. France Mém.* **2** (1964) viii+150. 364, 596

[Gle82] Glenn, Paul G., Realization of cohomology classes in arbitrary exact categories. *J. Pure Appl. Algebra* **25** (1982) 33-–105. 437

[GJ99] Goerss, P. G., and Jardine, J. F., *Simplicial homotopy theory*. Progr. Math. 174, Birkhäuser, Basel (1999). 367, 394, 575

[GS06] Golubitsky, M., and Stewart, I., Nonlinear dynamics of networks: the groupoid formalism. *Bull. Amer. Math. Soc.* (*N.S.*) **43** (3) (2006) 305–364. 14

[Goo92] Goodwillie, T. G., Calculus. II. Analytic functors. *K-Theory* **5** (4) (1991/92) 295–332. 276

[Got69] Gottlieb, D. H., Covering transformations and universal fibrations. *Illinois J. Math.* **13** (1969) 432–437. 435

[Gou03] Goubault, E., Some geometric perspectives in concurrency theory. Algebraic topological methods in computer science (Stanford, CA, 2001), *Homology Homotopy Appl.* **5** (2) (2003) 95–136. xxxii

[GL01] Grabmeier, J., and Lambe, L. A., Computing resolutions over finite p-groups. In *Algebraic combinatorics and applications* (Gößweinstein, 1999), Springer-Verlag, Berlin (2001), 157–195. 364

[Gra92] Gramain, A., Le théorème de van Kampen. *Cahiers Topologie Géom. Différentielle Catég.* **33** (3) (1992) 237–251. xxxiv

[GM03] Grandis, M., and Mauri, L., Cubical sets and their site. *Theory Appl. Categ.* **11** (2003) 185–201. 394, 478

[Gra66] Gray, J. W., Fibred and cofibred categories. In *Proc. Conf. Categorical Algebra* (La Jolla, Calif., 1965), Springer-Verlag, New York (1966), 21–83. 596

[Gro68] Grothendieck, A., *Catégories cofibrées additives et complexe cotangent relatif*. Lecture Notes in Math. 79, Springer-Verlag, Berlin (1968). 257, 512, 596

[Gro89] Grothendieck, A., *Les Dérivateurs*. Edited by M. Künzer, J. Malgoire, G. Maltsiniotis (1989), c. 2000 pages.
http://www.math.jussieu.fr/ maltsin/groth/Derivateurs.html 550, 574

[Gro97] Grothendieck, A., Esquisse d'un programme. In *Geometric Galois actions*, 1. London Math. Soc. Lecture Note Ser. 242. Cambridge University Press, Cambridge (1997), 5–48 (with an English translation on pp. 243–283). xxv, 550

[GroPS1] Grothendieck, A., Pursuing stacks. Manuscrit, edited by G. Maltsiniotis, to appear in *Documents Mathématiques*. xiv, 394, 574

[GroPS2] Grothendieck, A., Pursuing stacks et correspondance. Manuscrits, edited by M. Künzer, G. Maltsiniotis and B. Toen, to appear in *Documents Mathématiques*. xiv, 394, 550, 574

[Gru57] Gruenberg, K., Residual properties of infinite soluble groups. *Proc. London Math. Soc.* (3) **7** (1957) 29–62. 303

[GR80] Gruenberg, K. W., and Roggenkamp, K. W., Extension categories of groups and modules. II. Stem extensions. *J. Algebra* **67** (2) (1980) 342–368. 257

[Gug57] Gugenheim, V. K. A. M., On supercomplexes. *Trans. Amer. Math. Soc.* **85** (1957) 35–51. 367, 395

[GM57] Gugenheim, V. K. A. M., and Moore, J. C., Acyclic models and fibre spaces. *Trans. Amer. Math. Soc.* **85** (1957) 265–306. 366, 546

[GNAPGP88] Guillén, F., Navarro Aznar, V., Pascual Gainza, P., and Puerta, F., *Hyperrésolutions cubiques et descente cohomologique*. Papers from the Seminar on Hodge-Deligne Theory held in Barcelona, 1982, Lecture Notes in Math. 1335, Springer-Verlag, Berlin (1988). 511

[GSNPR08] Guillén Santos, F., Navarro, V., Pascual, P., and Roig, A., Monoidal functors, acyclic models and chain operads. *Canad. J. Math.* **60** (2) (2008) 348–378. 366

[GWL81] Guin-Waléry, D., and Loday, J.-L., Obstruction à l'excision en K-théorie algébrique. In *Algebraic K-theory* (Evanston 1980), Lecture Notes in Math. 854. Springer-Verlag, Berlin (1981), 179–216. 62, 255, 586

[GM09] Guiraud, Y., and Malbos, P., Higher dimensional categories with finite derivation type. *Theory Appl. Cat.* **22** (18) (2009) 420–478. 202

[GM10] Guiraud, Y., and Malbos, P., Coherence in monoidal track categories. arXiv:1004.1055v2 [math.CT] (2010) 1–22. 614

[GH86] Gutiérrez, M. A., and Hirschhorn, P. S., Free simplicial groups and the second relative homotopy group of an adjunction space. *J. Pure Appl. Algebra* **39** (1-2) (1986) 119–123. 61, 140

[Hae92] Haefliger, A., Extension of complexes of groups. *Ann. Inst. Fourier* (*Grenoble*) **42** (1-2) (1992) 275–311. 551

[HW08] Haglund, F., and Wise, D. T., Special cube complexes. *Geom. Funct. Anal.* **17** (5) (2008) 1551–1620. xxxii

[Hat02] Hatcher, A., *Algebraic topology*. Cambridge University Press, Cambridge (2002). 276

[Hea05] Heath, P. R., Fibre techniques in Nielsen theory calculations. In *Handbook of topological fixed point theory*, Springer-Verlag, Dordrecht (2005), 489–554. 546

[HK81] Heath, P. R., and Kamps, K. H., Groupoids, stability cokernel sequences and duality. Third Colloquium on Categories (Amiens, 1980), Part II, *Cahiers Topologie Géom. Différentielle* **22** (1) (1981) 97–104. 434, 545

[HK82] Heath, P. R., and Kamps, K. H., On exact orbit sequences. *Illinois J. Math.* **26** (1) (1982) 149–154. 434

[HK89] Heath, P. R., and Kamps, K. H., Lifting colimits of (topological) groupoids and (topological) categories. In *Categorical topology and its relation to analysis, algebra and combinatorics* (Prague, 1988), World Sci. Publ., Teaneck, NJ (1989), 54–88. 363

[HR99] Heyworth, A., and Reinert, B., Applications of Gröbner bases to group rings and identities among relations. Bangor Maths. Preprint 99.09. 366

[HW03] Heyworth, A., and Wensley, C. D., Logged rewriting and identities among relators. In *Groups St. Andrews 2001 in Oxford*, Vol. I, London Math. Soc. Lecture Note Ser. 304, Cambridge University Press, Cambridge (2003), 256–276. 365, 437

[Hig63] Higgins, P. J., Algebras with a scheme of operators. *Math. Nachr.* **27** (1963) 115–132. 201

[Hig64] Higgins, P. J., Presentations of groupoids, with applications to groups. *Proc. Cambridge Philos. Soc.* **60** (1964) 7–20. xxxiv, 5, 15, 16, 30

[Hig71] Higgins, P. J., *Notes on categories and groupoids*. Mathematical Studies 32, Van Nostrand Reinhold Co. London (1971); Reprints in *Theory Appl. Categ.* **7** (2005), 1–195,. 5, 12, 14, 25, 28, 30, 60, 83, 84, 103, 201, 301, 324, 365, 563, 569, 583, 588, 589, 590, 591, 593, 597

[Hig76] Higgins, P. J., The fundamental groupoid of a graph of groups. *J. London Math. Soc.* (2) **13** (1) (1976) 145–149. 364

[Hig05] Higgins, P. J., Thin elements and commutative shells in cubical ω-categories. *Theory Appl. Categ.* **14** (4) (2005) 60–74 (electronic). 202, 203, 204, 478, 479

[HT82] Higgins, P. J., and Taylor, J., The fundamental groupoid and the homotopy crossed complex of an orbit space. In Category theory (Gummersbach, 1981), *Lecture Notes in Math.*, Volume 962. Springer-Verlag, Berlin (1982), 115–122. 546

[HW60] Hilton, P. J., and Wylie, S., *Homology theory, an introduction to algebraic topology*. Cambridge University Press, London (1960). 394

[Hin73] Hintze, S., *Polysets, □-sets and semi-cubical sets*. M.Phil thesis, University of Warwick (1973). 394

[HO09] Hirashima, Y., and Oda, N., Brown-Booth-Tillotson theory for classes of exponentiable spaces. *Topology Appl.* **156** (13) (2009) 2264–2283. 322

[Hir03] Hirschhorn, P. S., *Model categories and their localizations*. Math. Surveys Monogr. 99, Amer. Math. Soc., Providence, RI (2003). 574, 575, 594

[HAMS93] Hog-Angeloni, C., Metzler, W., and Sieradski, A. J. (eds.), *Two-dimensional homotopy and combinatorial group theory*. London Math. Soc. Lecture Note Ser. 197, Cambridge University Press, Cambridge (1993). 32, 42, 103, 140, 365, 366, 545

[Hol79] Holt, D. F., An interpretation of the cohomology groups $H^n(G, M)$. *J. Algebra* **60** (2) (1979) 307–320. 437

[Hop42] Hopf, H., Fundamentalgruppe und zweite Bettische Gruppe. *Comment. Math. Helv.* **14** (1942) 257–309. 65

[Hor79] Horadam, K. J., The mapping cylinder resolution for a groupnet diagram. *J. Pure Appl. Algebra* **15** (1) (1979) 23–40. 364

[Hov99] Hovey, M., *Model categories*. Math. Surveys Monogr. 63, Amer. Math. Soc., Providence, RI (1999). 594

[How79] Howie, J., Pullback functors and crossed complexes. *Cahiers Topologie Géom. Différentielle* **20** (3) (1979) 281–296. 328, 434

[Hu48] Hu, S.-T., On the Whitehead group of automorphisms of the relative homotopy groups. *Portugaliae Math.* **7** (1948) 181–206 (1950). 60

[Hu53] Hu, S.-T., The homotopy addition theorem. *Ann. of Math.* (2) **58** (1953) 108–122. 323

[Hu59] Hu, S.-T., *Homotopy theory*. Pure and Applied Mathematics VIII, Academic Press, New York (1959). 38, 256, 325

[Hue77] Huebschmann, J., *Verschränkte n-fache Erweiterungen von Gruppen und Cohomologie*. Ph.D. thesis, ETH Zurich (1977). 255, 437, 477

[Hue80a] Huebschmann, J., Crossed n-fold extensions of groups and cohomology. *Comment. Math. Helv.* **55** (2) (1980) 302–313. 255, 323, 364, 436, 437, 477

[Hue80b] Huebschmann, J., The first k-invariant, Quillen's space BG^+, and the construction of Kan and Thurston. *Comm. Math. Helv.* **55** (1980) 314–318. 255, 436

[Hue81a] Huebschmann, J., Automorphisms of group extensions and differentials in the Lyndon-Hochschild-Serre spectral sequence. *J. Algebra* **72** (2) (1981) 296–334. 255, 364

[Hue81b] Huebschmann, J., Group extensions, crossed pairs and an eight term exact sequence. *J. Reine Angew. Math.* **321** (1981) 150–172. 255, 364, 437

[Hue89] Huebschmann, J., Cohomology of nilpotent groups of class 2. *J. Algebra* **126** (2) (1989) 400–450. 367

[Hue91] Huebschmann, J., Cohomology of metacyclic groups. *Trans. Amer. Math. Soc.* **328** (1) (1991) 1–72. 364

[Hue99] Huebschmann, J., Extended moduli spaces, the Kan construction, and lattice gauge theory. *Topology* **38** (3) (1999) 555–596. 84

[Hue07] Huebschmann, J., Origins and breadth of the theory of higher homotopies. In *Higher structures in geometry and physics*. In honor of Murray Gerstenhaber and Jim Stasheff, Progr. Math. 287, Birkhäuser/Springer, New York (2011), 159–179. 255

[Hue09] Huebschmann, J., Braids and crossed modules. arXiv:0904.3895v2 [math.AT] (2009) 1–17. 61, 140

[Hur35] Hurewicz, W., Beiträge zur Topologie der Deformationen. *Nederl. Akad. Wetensch. Proc. Ser. A* **38** (1935) 112–119,521–528. 12

[Ina97] Inassaridze, H., *Non-abelian homological algebra and its applications*. Math. Appl. 421, Kluwer Academic Publishers, Dordrecht (1997). 575

[Isa11] Isaacson, S. B., Symmetric cubical sets. *J. Pure Appl. Algebra* **215** (2011), 1146–1173. 394, 511

[Ive86] Iversen, B., *Cohomology of sheaves*. Universitext, Springer-Verlag, Berlin (1986). 550

[Jac43] Jacobson, N., *The theory of rings*. Amer. Math. Soc. Math. Surveys I, Amer. Math. Soc., New York (1943). 13

[Jaj80] Jajodia, S., Homotopy classification of lens spaces for one-relator groups with torsion. *Pacific J. Math.* **89** (1980) (2) 301–311. 435

[Jan03] Janelidze, G., Internal crossed modules. *Georgian Math. J.* **10** (1) (2003) 99–114. 61, 62, 63, 613

[JMT02] Janelidze, G., Márki, L., and Tholen, W., Semi-abelian categories. Category theory 1999 (Coimbra), *J. Pure Appl. Algebra* **168** (2-3) (2002) 367–386. 546, 614

[Jar06] Jardine, J. F., Categorical homotopy theory. *Homology, Homotopy Appl.* **8** (1) (2006) 71–144 (electronic). 394

[Joh32] Johansson, I., Über die Invarianz der topologischen Wechselsumme $\alpha_0 - \alpha_1 + \alpha_2 - \alpha_3 + \dots$ gegenüber Dimensionsänderungen. *Avhand. Norske. Vidensk.-Akad.* (1) (1932) 2–8. 394

[JL01] Johansson, L., and Lambe, L., Transferring algebra structures up to homology equivalence. *Math. Scand.* **89** (2) (2001) 181–200. 367

[Joh97] Johnson, D. L., *Presentations of groups*. Second edition, London Math. Soc. Student Texts 15, Cambridge University Press, Cambridge (1997). 60, 83, 84

[Joh03] Johnson, F. E. A., Stable modules and Wall's $D(2)$-problem. *Comment. Math. Helv.* **78** (1) (2003) 18–44. 276

[Joh04] Johnson, F. E. A., Minimal 2-complexes and the D(2)-problem. *Proc. Amer. Math. Soc.* **132** (2) (2004) 579–586 (electronic). 257

[Joh09] Johnson, F. E. A., Homotopy classification and the generalized Swan homomorphism. *J. K-Theory* **4** (3) (2009) 491–536. 257, 276

[Joh02] Johnstone, P. T., *Sketches of an elephant: a topos theory compendium*. Vol. 1, 2, Oxford Logic Guides 43, 44, The Clarendon Press/Oxford University Press, New York, Oxford (2002). 542, 582

[Jon88] Jones, D. W., A general theory of polyhedral sets and the corresponding T-complexes. *Dissertationes Math.* (*Rozprawy Mat.*) **266** (1988) 110. 84, 512

[JS91] Joyal, A., and Street, R., The geometry of tensor calculus. I. *Adv. Math.* **88** (1) (1991) 55–112. 614

[JT07] Joyal, A., and Tierney, M., Quasi-categories vs Segal spaces. In *Categories in algebra, geometry and mathematical physics*, Contemp. Math. 431, Amer. Math. Soc., Providence, RI (2007), 277–326. 549, 574

[Kam33] Kampen, E. H. v., On the connection between the fundamental groups of some related spaces. *Amer. J. Math.* **55** (1933) 261–267. xxxiv

[Kam72] Kamps, K. H., Kan-Bedingungen und abstrakte Homotopiethoerie. *Math. Z.* **124** (1972) 215–236. 394

[KP97] Kamps, K. H., and Porter, T., *Abstract homotopy and simple homotopy theory*. World Scientific Publishing Co. Inc., River Edge, NJ (1997). 62, 323, 394, 510, 594

[KP02] Kamps, K. H., and Porter, T., 2-groupoid enrichments in homotopy theory and algebra. *K-theory* **25** (2002) 373–409. 322, 545

[Kan55] Kan, D. M., Abstract homotopy. I. *Proc. Nat. Acad. Sci. U.S.A.* **41** (1955) 1092–1096. 394, 478

[Kan58a] Kan, D. M., Adjoint functors. *Trans. Amer. Math. Soc.* **87** (1958) 294–329. 575

[Kan58b] Kan, D. M., Functors involving c.s.s. complexes. *Trans. Amer. Math. Soc.* **87** (1958) 330–346. 479

[KFM08] Kauffman, L. H., and Faria Martins, J., Invariants of welded virtual knots via crossed module invariants of knotted surfaces. *Compositio Math.* **144** (4) (2008) 1046–1080. 61, 545, 596

[Kel82] Kelly, G. M., *Basic concepts of enriched category theory*. London Math. Soc. Lecture Note Ser. 64, Cambridge University Press, Cambridge (1982); Reprints in *Theory Appl. Categ.* **10** (2005) 1–136. 379, 602, 614

[KML71] Kelly, G. M., and Mac Lane, S., Coherence in closed categories. *J. Pure Appl. Algebra* **1** (1) (1971) 97–140. 610, 611

[KOK+86] Kneser, M., Ojanguren, M., Knus, M.-A., Parimala, R., and Sridharan, R., Composition of quaternary quadratic forms. *Compositio Math.* **60** (2) (1986) 133–150. 13

[Koc10] Kock, A., Cubical version of combinatorial differential forms. *Appl. Categ. Structures* **18** (2) (2010) 165–183. xxxii

[Koz08] Kozlov, D., *Combinatorial algebraic topology*. Algorithms Comput. Math. 21, Springer-Verlag, Berlin (2008). 364, 596

[KM97] Kriegl, A., and Michor, P. W., *The convenient setting of global analysis*. Math. Surveys Monogr. 53. Amer. Math. Soc., Providence, RI (1997). 575

[Kün07] Künzer, M., Heller triangulated categories. *Homology, Homotopy Appl.* **9** (2) (2007) 233–320. 549

[Lab99] Labesse, J.-P., Cohomologie, stabilisation et changement de base. Appendix A by Laurent Clozel and Labesse, and Appendix B by Lawrence Breen. *Astérisque* **257** (1999). 202, 395

[Lac10] Lack, S., Note on the construction of free monoids. *Appl. Categ. Structures* **18** (1) (2010) 17–29. 542

[LMW10] Lafont, Y., Métayer, F., and Worytkiewicz, K., A folk model structure on omega-cat. *Adv. Math.* **224** (3) (2010) 1183–1231.

[LS87] Lambe, L. and Stasheff, J., Applications of perturbation theory to iterated fibrations. *Manuscripta Math.* **58** (3) (1987) 363–376. 367

[Law04] Lawvere, F. W., Functorial semantics of algebraic theories and some algebraic problems in the context of functorial semantics of algebraic theories. *Repr. Theory Appl. Categ.* **5** (2004), 1–121; reprinted from *Proc. Nat. Acad. Sci. U.S.A.* **50** (1963), 869–872, and *Reports of the Midwest Category Seminar. II*, Lecture Notes in Math. 61, Springer-Verlag, Berlin (1968), 41–61. 256, 542

[Lev40] Levi, F. W., The commutator group of a free product. *J. Indian Math. Soc.* (*N.S.*) **4** (1940) 136–144. 303

[Lev57] Levi, F. W., Darstellung der Komposition in einer Gruppe als Relation. *Arch. Math.* (*Basel*) **8** (1957) 169—170. 479

[Lod78] Loday, J.-L., Cohomologie et groupe de Steinberg relatifs. *J. Algebra* **54** (1) (1978) 178–202. 62, 255

[Lod82] Loday, J.-L., Spaces with finitely many nontrivial homotopy groups. *J. Pure Appl. Algebra* **24** (2) (1982) 179–202. 29, 38, 62, 255, 323, 435, 512, 586

[Lod00] Loday, J.-L., Homotopical syzygies. In *Une dégustation topologique: homotopy theory in the Swiss Alps* (Arolla, 1999), Contemp. Math. 265. Amer. Math. Soc., Providence, RI (2000), 99–127. 83, 437, 548

[Lom81] Lomonaco, S. J., Jr., The homotopy groups of knots. I. How to compute the algebraic 2-type. *Pacific J. Math.* **95** (2) (1981) 349–390. 277

[Lüc87] Lück, W., The geometric finiteness obstruction. *Proc. London Math. Soc.* (3) **54** (2) (1987) 367–384. 255

[Lue71] Lue, A. S.-T., Cohomology of algebras relative to a variety. *Math. Z.* **121** (1971) 220–232. 255, 437, 477, 549

[Lue79] Lue, A. S.-T., Semicomplete crossed modules and holomorphs of groups. *Bull. London Math. Soc.* **11** (1) (1979) 8–16. 60, 614

[Lue81] Lue, A. S.-T., Cohomology of groups relative to a variety. *J. Algebra* **69** (1) (1981) 155–174. 255, 437, 477

[Lur09] Lurie, J., *Higher topos theory*. Ann. of Math. Stud. 170, Princeton University Press, Princeton, NJ (2009). xxxv, 256, 478, 510, 549, 596

[Lyn50] Lyndon, R. C., Cohomology theory of groups with a single defining relation. *Ann. of Math.* (2) **52** (1950) 650–665. 436

[LS01] Lyndon, R. C., and Schupp, P. E., *Combinatorial group theory*. Reprint of the 1977 edition, Classics in Mathematics, Springer-Verlag, Berlin (2001). 60, 83, 103

[Mac87] Mackenzie, K., *Lie groupoids and Lie algebroids in differential geometry*. London Math. Soc. Lecture Note Ser. 124, Cambridge University Press, Cambridge (1987). 5

[Mac05] Mackenzie, K. C. H., *General theory of Lie groupoids and Lie algebroids*. London Math. Soc. Lecture Note Ser. 213, Cambridge University Press, Cambridge (2005). 14, 578

[ML49] Mac Lane, S., Cohomology theory in abstract groups. III. Operator homomorphisms of kernels. *Ann. of Math.* (2) **50** (1949) 736–761. 29, 255

[ML63] Mac Lane, S., *Homology*. Grundlehren Math. Wiss. 114, Springer-Verlag, Berlin (1963). 257, 364, 366, 435, 437, 575, 576

[ML71] Mac Lane, S., *Categories for the working mathematician*. Grad. Texts in Math. 5, Springer-Verlag, New York (1971). 82, 322, 520, 521, 542, 555, 563, 575, 609, 614

[ML78] Mac Lane, S., Origins of the cohomology of groups. *Enseign. Math.* (2) **24** (1-2) (1978) 1–29. 65

[ML79] Mac Lane, S., Historical note. *J. Algebra* **69** (1979) 319–320. 255, 437, 477

[MLM94] Mac Lane, S., and Moerdijk, I., *Sheaves in geometry and logic*. A first introduction to topos theory. Corrected reprint of the 1992 edition, Universitext, Springer-Verlag, New York (1994). 63

[MLM96] Mac Lane, S., and Moerdijk, I., Topos theory. In *Handbook of algebra, Vol.* 1, North-Holland, Amsterdam (1996), 501–528. 14

[MLW50] Mac Lane, S., and Whitehead, J. H. C., On the 3-type of a complex. *Proc. Nat. Acad. Sci. U. S. A.* **36** (1950) 41–48. xxiv, 61, 435

[Mal05] Maltsiniotis, G., La théorie de l'homotopie de Grothendieck. *Astérisque* **301** (2005). 394, 574, 575, 594

[Mal09] Maltsiniotis, G., La catégorie cubique avec connexions est une catégorie test stricte. *Homology, Homotopy Appl.* **11** (2) (2009) 309–326. xxxii, 394, 478, 511, 546, 594

[MaltDer] Maltsiniotis, G., Introduction à la théorie des dérivateurs (d'après Grothendieck. Preprint http://www.math.jussieu.fr/~maltsin/textes.html 550

[Man76] Manes, E. G., *Algebraic theories*. Grad. Texts in Math. 26, Springer-Verlag, New York (1976). 256, 542

[MM10] Mantovani, S., and Metere, G., Internal crossed modules and Peiffer condition. *Theory Appl. Categ.* **23** (2010) 113–135. 63

[Mas80] Massey, W. S., *Singular homology theory*. Grad. Texts in Math. 70, Springer-Verlag, New York (1980). 140, 394, 503, 511

[May67] May, J. P., *Simplicial objects in algebraic topology*. Van Nostrand Mathematical Studies 11, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London (1967). 394, 479, 575

[May72] May, J. P., *The geometry of iterated loop spaces*. Lectures Notes in Math. 271, Springer-Verlag, Berlin (1972). 12

[May77a] May, J. P., *E_∞ ring spaces and E_∞ ring spectra*. With contributions by F. Quinn, N. Ray, and J. Tornehave, Lecture Notes in Math. 577, Springer-Verlag, Berlin (1977). 12

[May77b] May, J. P., Infinite loop space theory. *Bull. Amer. Math. Soc.* **83** (4) (1977) 456–494. 12

[May82] May, J. P., Multiplicative infinite loop space theory. *J. Pure Appl. Algebra* **26** (1) (1982) 1–69. 12

[Maz65] Mazur, B., Morse theory. In *Differential and combinatorial topology* (A Symposium in Honor of Marston Morse), Princeton University Press, Princeton, N.J. (1965), 145–165. 547

[McC71] McClendon, J. F., Obstruction theory in fiber spaces. *Math. Z.* **120** (1971) 1–17. 436

[McP69] McPherson, J. M., On the nullity and enclosure genus of wild knots. *Trans. Amer. Math. Soc.* **144** (1969) 545–555. 257

[Met79] Metzler, W., Äquivalenzklassen von Gruppenbeschreibungen, Identitäten von Relationen und einfacher Homotopietyp in niederen Dimensionen. In *Homological group theory* (Durham, 1977), with an appendix by C. T. C. Wall, London Math. Soc. Lecture Note Ser. 36, Cambridge University Press, Cambridge (1979), 291–326. 84

[MW10] Mikhailov, R., and Wu, J., On homotopy groups of the suspended classifying spaces. *Algebr. Geom. Topol.* **10** (1) (2010) 565–625. 511, 549

[Mil71] Milnor, J., *Introduction to algebraic K-theory*. Ann. Math. Stud. 72, Princeton University Press, Princeton, N.J. (1971). 62

[MOt06] Minian, G., and Ottina, M., A geometric decomposition of spaces into cells of different types. *J. Homotopy Relat. Struct.* **1** (1) (2006) 245–271. 255

[Mit72] Mitchell, B., Rings with several objects. *Adv. Math.* **8** (1972) 1–161. 322

[Moo01] Moore, E., *Graphs of groups: word computations and free crossed resolutions*. Ph.D. thesis, University of Wales, Bangor (2001). 353, 364, 365

[Mos87] Mosa, G. H., *Higher dimensional algebroids and crossed complexes*. Ph.D. thesis, University of Wales, Bangor (1987). 203, 549

[MP02] Mukherjee, G., and Pandey, N., Equivariant cohomology with local coefficients. *Proc. Amer. Math. Soc.* **130** (1) (2002) 227–232 (electronic). 436

[MT07] Muro, F., and Tonks, A., The 1-type of a Waldhausen K-theory spectrum. *Adv. Math.* **216** (1) (2007) 178–211. 367

[MTW10] Muro, F., Tonks, A., and Witte, M., On determinant functors and K-theory. arXiv:1006.5399v1 [math.KT] (2010) 1–71. 549

[NT89a] Nan Tie, G., A Dold-Kan theorem for crossed complexes. *J. Pure Appl. Algebra* **56** (2) (1989) 177–194. xxvii, 479

[NT89b] Nan Tie, G., Iterated $\bar{W}$ and T-groupoids. *J. Pure Appl. Algebra* **56** (2) (1989) 195–209. 512

[Nee01] Neeman, A., *Triangulated categories*. Ann. Math. Stud. 148, Princeton University Press, Princeton, NJ (2001). 363, 549

[Nor90] Norrie, K., Actions and automorphisms of crossed modules. *Bull. Soc. Math. France* **118** (2) (1990) 129–146. 60, 62, 256, 608, 614

[Olu50] Olum, P., Obstructions to extensions and homotopies. *Ann. of Math.* (2) **52** (1950) 1–50. 435

[Olu53] Olum, P., On mappings into spaces in which certain homotopy groups vanish. *Ann. of Math.* (2) **57** (1953) 561–574. 435

[Olu58] Olum, P., Non-abelian cohomology and van Kampen's theorem. *Ann. of Math.* (2) **68** (1958) 658–668. xxxiv

[Pao09] Paoli, S., Weakly globular cat^n-groups and Tamsamani's model. *Adv. Math.* **222** (2) (2009) 621–727. 549

[Pap63] Papakyriakopoulos, C. D., Attaching 2-dimensional cells to a complex. *Ann. of Math.* (2) **78** (1963) 205–222. 61, 84, 276, 545

[PRP09] Pascual, P., and Rubió Pons, L., Algebraic K-theory and cubical descent. *Homology, Homotopy Appl.* **11** (2) (2009) 5–25. xxxii

[Pat08] Patchkoria, I., Cubical resolutions and derived functors. arXiv:0907.1905v1 [math.KT] (2008) 1–20. 394

[Pei49] Peiffer, R., Über Identitäten zwischen Relationen. *Math. Ann.* **121** (1949) 67–99. xxiv, 60, 83, 84, 85, 365

[PS85] Plotnick, S. P., and Suciu, A. I., k-invariants of knotted 2-spheres. *Comment. Math. Helv.* **60** (1) (1985) 54–84. 277

[Poi96] Poincaré, H., *Œuvres. Tome VI*. Géométrie. Analysis situs (topologie), reprint of the 1953 edition, Les Grands Classiques, Éditions Jacques Gabay, Gauthier-Villars Sceaux (1996). 8

[Poi10] Poincaré, H., *Papers on topology: analysis situs and its five supplements*. Translated by John C. Stillwell, History of Mathematics 37, Amer. Math. Soc., Providence, RI/London Math. Soc., London (2010). 8

[Pon09] Ponto, K., Relative fixed point theory. *Algebr. Geom. Topol.* **11** (2011) 839–886. 546

[PS09] Ponto, K., and Shulman, M., Shadows and traces in bicategories. arXiv:0910.1306v1 [math.CT] (2009) 1–63. 546

[Por87] Porter, T., Extensions, crossed modules and internal categories in categories of groups with operations. *Proc. Edinburgh Math. Soc.* (2) **30** (3) (1987) 373–381. 62

[Por93] Porter, T., n-types of simplicial groups and crossed n-cubes. *Topology* **32** (1) (1993) 5–24. 62

[Por07] Porter, T., Formal homotopy quantum field theories. II. Simplicial formal maps. In *Categories in algebra, geometry and mathematical physics*, Contemp. Math. 431, Amer. Math. Soc., Providence, RI (2007), 375–403. 394

[Por11] Porter, T., *Crossed menagerie*. (2011) 1–450. http://ncatlab.org/nlab/show/Crossed+Menagerie. 201, 323

[Por12] Porter, T., *Profinite algebraic homotopy*. Algebra and Applications, Springer-Verlag, to appear. 550

[PT08] Porter, T., and Turaev, V., Formal homotopy quantum field theories I: formal maps and crossed $\mathcal{C}$-algebras. *J. Homotopy Relat. Struct.* **3** (1) (2008) 113–159. 394

[Pra07] Pradines, J., In Ehresmann's footsteps: from group geometries to groupoid geometries. In *Geometry and topology of manifolds*, Banach Center Publ. 76, Polish Acad. Sci., Warsaw (2007), 87-–157. 6, 13

[Pra09] Pratt, V., The Yoneda Lemma as a foundational tool for algebra. Stanford preprint: http://boole.stanford.edu/pub/yon.pdf (2009) 1–18. 542, 575

[Pup97] Puppe, D., Nachruf auf H. Seifert. *Jahrbuch der Heidelberger Akademie der Wissenschaften* (1997) 139–144. xxxv

[Pup99] Puppe, D., Herbert Seifert (1907-1996). In *History of Topology*, North-Holland, Amsterdam (1999), 1021–1028. xxxv

[Qui67] Quillen, D. G., *Homotopical algebra*. Lecture Notes in Math. 43. Springer-Verlag, Berlin (1967). 594

[RW90] Ranicki, A., and Weiss, M., Chain complexes and assembly. *Math. Z.* **204** (2) (1990) 157–185. 547

[Rat80] Ratcliffe, J. G., Free and projective crossed modules. *J. London Math. Soc.* (2) **22** (1) (1980) 66–74. 61, 140

[RS76] Razak Salleh, A., *Union theorems for double groupoids and groupoids; some generalisations and applications*. Ph.D. thesis, University of Wales (1976). 61, 204

[Rei32] Reidemeister, K., *Einführung die kombinatorische Topologie*. F. Vieweg & Sohn, Braunschweig, Berlin (1932); Reprint Chelsea, New York (1950). 13

[Rei34] Reidemeister, K., Homotopiegruppen von Komplexen. *Abh. Math. Sem. Universität Hamburg* **10** (1934) 211–215. 29, 60, 255, 257, 478

[Rei49] Reidemeister, K., Über Identitäten von Relationen. *Abh. Math. Sem. Univ. Hamburg* **16** (1949) 114–118. xxiv, 60, 83, 84, 276, 365

[Rei50] Reidemeister, K., Complexes and homotopy chains. *Bull. Amer. Math. Soc.* **56** (1950) 297–307. 257, 478

[Rot08] Rota, G.-C., *Indiscrete thoughts*. Modern Birkhäuser Classics, Birkhäuser, Boston, MA (2008); Reprint of the 1997 edition, with forewords by Reuben Hersh and Robert Sokolowski, edited, with notes and an epilogue by Fabrizio Palombi. 30

[RS71] Rourke, C. P., and Sanderson, B. J., $\triangle$-sets. I. Homotopy theory. *Quart. J. Math. Oxford Ser.* (2) **22** (1971) 321–338. 366, 575

[Sau03] Sauvageot, O., *Stabilisation des complexes croisés*. Ph.D. thesis, École Polytechnique Fédérale de Lausanne (2003). 594

[Sch73] Schellenberg, B., The group of homotopy self-equivalences of some compact CW-complexes. *Math. Ann.* **200** (1973) 253–266. 435

[Sch76] Schön, R., Acyclic models and excision. *Proc. Amer. Math. Soc.* **59** (1) (1976) 167–168. 367

[Sch91] Schön, R., Effective algebraic topology. *Mem. Amer. Math. Soc.* **92** (451) (1991) vi+63. xxvi

[Sch37] Schumann, H.-G., Über Moduln und Gruppenbilder. *Math. Ann.* **114** (1) (1937) 385–413. 257

[Seg99] Segal, S., Topologists in Hitler's Germany. In *History of Topology*, North-Holland, Amsterdam (1999), 849–862. xxxv

[Seh03] Sehgal, S. K., Group rings. In *Handbook of algebra, Vol. 3*, North-Holland, Amsterdam (2003), 455–541. 255

[Sei31] Seifert, H., Konstruktion dreidimensionaler geschlossener Räume. *Berichte Sächs. Akad. Leipzig, Math.-Phys. Kl.* **83** (1931) 26–66. xxxiv

[ST80] Seifert, H., and Threlfall, W., *Seifert and Threlfall: a textbook of topology*. Pure Appl. Math. 89, Academic Press Inc. New York (1980), translated from the German edition of 1934 by Michael A. Goldman; with a preface by Joan S. Birman, with "Topology of 3-dimensional fibered spaces" by Seifert, translated from the German by Wolfgang Heil. xxxv, 546

[Sha93] Sharko, V. V., *Functions on manifolds. Algebraic and topological aspects*. Translated from the Russian by V. V. Minachin. Transl. Math. Monogr. 131. Amer. Math. Soc., Providence, RI (1993). xxv, 433, 548

[Shi62] Shih, W., Homologie des espaces fibrés. *Inst. Hautes Études Sci. Publ. Math.* (13) (1962) 88. 367

[Shu09] Shulman, M. A., Homotopy limits and colimits and enriched homotopy theory. arXiv:0610194v3 [math.AT] (2009) 1–79. 596

[Smi51a] Smith, P. A., The complex of a group relative to a set of generators. I. *Ann. of Math.* (2) **54** (1951) 371–402. 597

[Smi51b] Smith, P. A., The complex of a group relative to a set of generators. II. *Ann. of Math.* (2) **54** (1951) 403–424. 597

[Sol90] Soloviev, S. V., On the conditions of full coherence in closed categories. *J. Pure Appl. Algebra* **69** (3) (1990) 301–329. 612

[Sol97] Soloviev, S. V., Proof of a conjecture of S. Mac Lane. *Ann. Pure Appl. Logic* **90** (1-3) (1997) 101–162. 610, 611, 614

[Spe77] Spencer, C. B., An abstract setting for homotopy pushouts and pullbacks. *Cahiers Topologie Géom. Différentielle* **18** (4) (1977) 409–429. 204

[SW83] Spencer, C. B., and Wong, Y. L., Pullback and pushout squares in a special double category with connection. *Cahiers Topologie Géom. Différentielle* **24** (2) (1983) 161–192. 204

[Ste43] Steenrod, N. E., Homology with local coefficients. *Ann. of Math.* (2) **44** (1943) 610–627. 436

[Ste67] Steenrod, N. E., A convenient category of topological spaces. *Michigan Math. J.* **14** (1967) 133–152. 254

[Ste72] Steenrod, N. E., Cohomology operations, and obstructions to extending continuous functions. *Adv. Math.* **8** (1972) 371–416. 61

[Ste06] Steiner, R., Thin fillers in the cubical nerves of omega-categories. *Theory Appl. Categ.* (8) **16** (2006) 144–173 (electronic). 203, 479, 549

[Str87] Street, R., The algebra of oriented simplexes. *J. Pure Appl. Algebra* **49** (3) (1987) 283–335. 479

[Str88] Street, R., Gray's tensor product of 2-categories (1988). Handwritten notes. 542, 543

[SV10] Street, R., and Verity, D., The comprehensive factorization and torsors. *Theory Appl. Categ.* **23** (2010) 42–75. 328

[Str99] Streicher, T., Fibred categories à la Bénabou. (1999) 1–85. http://www.mathematik.tu-darmstadt.de/∼streicher/ 596

[Sza78] Szabo, M., *Algebra of proofs*. Studies in Logic and the Foundations of Mathematics 88, North-Holland P.C. (1978). 614

[Tab09] Tabuada, G., Postnikov towers, k-invariants and obstruction theory for dg categories. *J. Algebra* **321** (12) (2009) 3850–3877. 549

[Tab10] Tabuada, G., Homotopy theory of dg categories via localizing pairs and Drinfeld's dg quotient. *Homology, Homotopy Appl.* **12** (1) (2010) 187–219. 549

[Tay53] Taylor, R. L., Compound group extensions. I. Continuations of normal homomorphisms. *Trans. Amer. Math. Soc.* **75** (1953) 106–135. 435, 436

[Tay54] Taylor, R. L., Covering groups of nonconnected topological groups. *Proc. Amer. Math. Soc.* **5** (1954) 753–768. 435, 436

[Tay55] Taylor, R. L., Compound group extensions. III. *Trans. Amer. Math. Soc.* **79** (1955) 490–520. 435

[TW95] Thévenaz, J., and Webb, P., The structure of Mackey functors. *Trans. Amer. Math. Soc.* **347** (6) (1995) 1865–1961. 254

[tD08] tom Dieck, T., *Algebraic topology*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich (2008). 276, 367, 510, 546, 548

[tDKP70] tom Dieck, T., Kamps, K. H., and Puppe, D., *Homotopietheorie*. Lecture Notes in Math. 157, Springer-Verlag, Berlin (1970). 486

[Ton92] Tonks, A., Cubical groups which are Kan. *J. Pure Appl. Algebra* **81** (1) (1992) 83–87. 478, 512

[Ton94] Tonks, A., *Theory and applications of crossed complexes*. Ph.D. thesis, University of Wales (1994). 364, 367, 395, 436, 543, 547

[Ton03] Tonks, A. P., On the Eilenberg-Zilber theorem for crossed complexes. *J. Pure Appl. Algebra* **179** (1-2) (2003) 199–220. 323, 360, 364, 367, 395

[Tur38] Turing, A. M., The extensions of a group. *Compositio Math.* **5** (1938) 357–367. 437

[Ver08a] Verity, D., Complicial sets characterising the simplicial nerves of strict ω-categories. *Mem. Amer. Math. Soc.* **193** (905) (2008). 479

[Ver08b] Verity, D. R. B., Weak complicial sets. I. Basic homotopy theory. *Adv. Math.* **219** (4) (2008) 1081–1149. 479, 574

[Wal65] Wall, C. T. C., Finiteness conditions for CW-complexes. *Ann. of Math.* (2) **81** (1965) 56–69. 366

[Wei01] Weinstein, A., Groupoids: unifying internal and external symmetry. A tour through some examples. In *Groupoids in analysis, geometry, and physics* (Boulder, CO, 1999), Contemp. Math. 282, Amer. Math. Soc., Providence, RI (2001), 1–19. 12, 14

[Wei61] Weinzweig, A. E., The fundamental group of a union of spaces. *Pacific J. Math.* **11** (1961) 763–776. xxxiv

[WA97] Wensley, C., and Alp, M., Xmod, a gap share package for computation with crossed modules. *GAP Manual* (1997) 1355–1420. 365

[Whi78] Whitehead, G. W., *Elements of homotopy theory*. Grad. Texts in Math. 61, Springer-Verlag, New York (1978). 140, 323

[Whi39] Whitehead, J. H. C., Simplicial spaces, nuclei, and m-groups. *Proc. London Math. Soc.* **45** (1939) 243–327. 29, 394

[Whi41a] Whitehead, J. H. C., On adding relations to homotopy groups. *Ann. of Math.* (2) **42** (1941) 409–428. 29, 31, 42, 60, 104

[Whi41b] Whitehead, J. H. C., On incidence matrices, nuclei and homotopy types. *Ann. of Math.* (2) **42** (1941) 1197–1239. 29, 30, 255, 394, 478, 510, 547

[Whi46] Whitehead, J. H. C., Note on a previous paper entitled "On adding relations to homotopy groups.". *Ann. of Math.* (2) **47** (1946) 806–810. 28, 32, 42, 60, 255

[Whi48] Whitehead, J. H. C., On operators in relative homotopy groups. *Ann. of Math.* (2) **49** (1948) 610–640. 60, 256, 614

[Whi49a] Whitehead, J. H. C., Combinatorial homotopy. I. *Bull. Amer. Math. Soc.* **55** (1949) 213–245. 29, 61, 256, 394

[Whi49b] Whitehead, J. H. C., Combinatorial homotopy. II. *Bull. Amer. Math. Soc.* **55** (1949) 453–496. xxv, 29, 32, 42, 61, 116, 255, 256, 257, 276, 365, 394, 435, 478, 500, 511

[Whi50a] Whitehead, J. H. C., On group extensions with operators. *Quart J. Math., Oxford Ser.* (2) **1** (1950) 219–228. 436

[Whi50b] Whitehead, J. H. C., Simple homotopy types. *Amer. J. Math.* **72** (1950) 1–57. 29, 30, 31, 85, 366, 394, 510, 547

[Whi50c] Whitehead, J. H. C., A certain exact sequence. *Ann. of Math.* (2) **52** (1950) 51–110. 511

[Woo10] Woolf, J., Transversal homotopy theory. *Theory Appl. Categ.* **24** (7) (2010) 148–178. 550

[Zas49] Zassenhaus, H., *The Theory of Groups*. Translated from the German by Saul Kravetz, Chelsea Publishing Company, New York, N. Y. (1949). 436, 437

[Živ06] Živaljević, R. T., Groupoids in combinatorics—applications of a theory of local symmetries. In *Algebraic and geometric combinatorics*, Contemp. Math. 423, Amer. Math. Soc., Providence, RI (2006), 305–324. xxxii

Glossary of symbols

We begin with a list of symbols for category theory, since some of them are used throughout the book.

Category theory

C	a category	556
$\mathrm{Ob}\,\mathsf{C}$	objects in a category (also C_0)	556
$\mathrm{Mor}\,\mathsf{C}$	morphisms in a category (also C_1)	556
$t\colon\ \mathrm{Mor}\,\mathsf{C} \to \mathrm{Ob}\,\mathsf{C}$	target of a morphism (also ∂^+)	556
$s\colon\ \mathrm{Mor}\,\mathsf{C} \to \mathrm{Ob}\,\mathsf{C}$	source of a morphism (also ∂^-)	556
$\mathsf{C}(x, y)$	morphisms in C from x to y	556
$1\colon\ \mathrm{Ob}\,\mathsf{C} \to \mathrm{Mor}\,\mathsf{C}$	identity map (also ε)	556
$1(x) = 1_x$	the identity on the object x	556
$f_*\colon \mathsf{C}(z, x) \to \mathsf{C}(z, y)$	map got by post-composing with $f \in \mathsf{C}(x, y)$	556
$f^*\colon \mathsf{C}(y, z) \to \mathsf{C}(x, z)$	map got by pre-composing with $f \in \mathsf{C}(x, y)$	556
$F\colon \mathsf{C} \to \mathsf{D}$	functor between categories	556
$\alpha\colon\ F \Rightarrow G$	natural transformation between functors	557
Set	the category of sets and maps	556
Groups	the category of groups and homomorphisms	556
Cat	category of small categories and functors	556
$\mathsf{CAT}(F, G)$	internal hom in Cat	557
$\mathsf{C}_{(y)}$	functor $\mathsf{C} \to \mathsf{Set}$ determined by $y \in \mathrm{Ob}\,\mathsf{C}$	557
$\mathsf{C}_{((h))}$	natural transformation $\mathsf{C}_{(y)} \to \mathsf{C}_{(z)}$ determined by $h\colon\ y \to z \in \mathsf{C}$	557
C/x	slice category or category of objects over $x \in \mathrm{Ob}\,\mathsf{C}$	558
C/y	comma category over an $y \in \mathrm{Ob}\,\mathsf{B}$ and an $F\colon \mathsf{C} \to \mathsf{B}$	559
C^2	morphism category of C	559
$\operatorname{colim} T$	colimit of a functor $T\colon \mathsf{D} \to \mathsf{C}$	562
$\lim T$	limit of a functor $T\colon \mathsf{D} \to \mathsf{C}$	563
$F \dashv G$	adjoints: $\mathsf{D}(Fx, y) \cong \mathsf{C}(x, Gy)$	564
$\eta\colon 1_{\mathsf{C}} \Rightarrow GF$	unit of an adjunction $F\colon \mathsf{C} \leftrightarrows \mathsf{D}\colon G$	565
$\varepsilon\colon\ FG \Rightarrow 1_{\mathsf{D}}$	counit of an adjunction $F\colon \mathsf{C} \leftrightarrows \mathsf{D}\colon G$	566
$\alpha\colon\ S \overset{\cdot\cdot}{\to} \bar{a}$	a dicone from S to a	570
$a = \int^{\mathsf{C},x} S(x, x)$	a is a coend of $S\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{A}$	570
$a = \int_{\mathsf{C},x} S(x, x)$	a is an end of $S\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{A}$	570
$\Phi\colon \mathsf{C} \to \mathsf{B}$	a fibration between categories	577
C_I	fibre of $\Phi\colon \mathsf{C} \to \mathsf{B}$ over $I \in \mathrm{Ob}\,\mathsf{B}$	577
u^*X	pullback of X along $u\colon\ J \to \Phi(X)$	578

$\Phi\colon \mathsf{C} \to \mathsf{B}$	a cofibration between categories	581
u_*Z	object induced from Z along $u\colon \Phi(Z) \to J$	581

Groups

P	a group	9
BP	the classifying space of the group P	9
CS	centraliser of a set S in a group P	39
ZP	center of a group P	39
$[m,n]$	commutator of two elements $m,n \in P$	40
$[P,P]$	commutator subgroup of the group P	40
P^{ab}	abelianisation of the group P	40
C_n	cyclic group of order n	41
D_8	dihedral group of order 8	67
$F(X)$	free group on the set X	68
$\mathcal{P} = \langle X \mid R \rangle$	presentation of a group P, $R \subseteq F(X)$	69
$\mathrm{gp}\,\mathcal{P}$	group defined by a presentation $\mathcal{P}$	69
$\langle \omega R \rangle^P$	normal closure of R in the group P	69
$c \in \langle \omega R \rangle^P$	consequence of R in P	69
S_3	symmetric group on three letters	70
$\mathcal{P} = \langle X \mid \omega \rangle$	presentation of a group P, $\omega\colon R \to F(X)$	74
Q_8	quaternion group of order 8	71
M^{*T}	copower of the group M	120
$M^{\oplus T}$	cosum of the abelian group M	122
D_{2n}	dihedral group of order $2n$	123
$Q = G \wr C_2$	wreath product of G and C_2	132
C_n	cyclic group of order n	137
$\mathbb{Z}G$	group ring of a group G	240
IG	augmentation ideal of a group ring	240

Groupoids

G	a generic groupoid	12
Gpds	the category of groupoids and their morphisms	12
G_0 or $\mathrm{Ob}\,G$	objects of the groupoid G	25
$G(a)$	object group of the groupoid G at $a \in G_0$	25
$G(a,b)$	elements of G from a to b	25
$I(S)$	indiscrete groupoid on the set S	26
$\mathcal{I}$	unit interval groupoid	26
$\mathrm{Inn}(G)$	totally disconnected groupoid formed by the object groups of G	223
$\mathsf{GPDS}(G,H)$	Internal hom in Gpds	283
G^{ab}	abelianisation of the groupoid G	569

UG	universal group of the groupoid G	569
$F(\Gamma)$	free groupoid on a graph Γ	589
G^{totab}	universal abelianisation of the groupoid G	569
$\mathsf{Cost}_G\, a_0$	Costar of the groupoid G at a_0	590
$p\colon \widetilde{G} \to G$	covering morphism of groupoids	590
$\mathsf{GpdsCov}/G$	category of covering morphisms of the groupoid G	590
u_*G	groupoid induced by a function $u\colon \operatorname{Ob} G \to X$	588

Topological spaces

X	a generic topological space	8
$\pi_0(X)$	set of path components of the space X	9
$H_n(X)$	n-dimensional homology group of the space X	9
I	unit interval	19
∂I	boundary of the unit interval	19
S^1	unit circle	27
E^1_+	top semicircle	27
E^2_+	top hemisphere of the 2-sphere S^2	119
S^n	n-sphere	259
E^n	n-cell	270
I^n	n-dimensional cube	33
∂I^n	boundary of the n-dimensional cube	33
$A \cup \{e^2_i\}_{i\in I}$	space with 2-cells attached	31
$f \simeq g$	homotopy between f and g	33
$[f]$	homotopy class of a map f	33
$[X, Y]$	set of homotopy classes	33
$\mathcal{F}$	a fibration	52
$p\colon \widetilde{X}(v) \to X$	universal cover of X at v	274
Top	the category of compactly generated topological spaces	211
$\mathsf{TOP}(Y, Z)$	the internal hom in Top	281
TopCov	category of coverings of topological spaces	362

Topological based spaces

x	a point in a topological space	8
(X, x)	a based topological space	8
Top_*	the category of based topological spaces	8
α	path (or loop) in some space X	19
$[\alpha]$	homotopy class of the path α	19
$[X, Y]_*$	set of based homotopy classes	33
$\pi_1(X, x)$	fundamental group of the based space (X, x)	8

$\pi_1(X, A)$	fundamental groupoid on a set of base points $A \subseteq X$	13
$\pi_n(X, x)$	n-dimensional homotopy group of (X, x)	33
$\pi_n(X, A, x)$	n-th relative homotopy group of the based pair (X, A, x)	35
$\pi_n(X : A, B; x)$	n-th triadic homotopy group of the based triad	586
$A \vee B$	wedge, union with base points identified	259

Crossed modules and modules

M	a P-group	38
$\mathcal{M} = (\mu : M \to P)$	crossed module over a group P or crossed P-module	38
$B\mathcal{M}$	classifying space of a crossed module	47
$P \ltimes M$	semidirect product of groups	50
$\mathsf{Cat}^1\text{-}\mathsf{Groups}$	category of cat^1-groups	50
$F_P(R)$	free P-group on the set R	74
$\mathcal{M}$	precrossed module over a group P or precrossed P-module	75
$[\![m, m']\!]$	Peiffer commutator of m, m'	76
$[\![M, M]\!]$	Peiffer subgroup of a precrossed module	76
$\mathcal{M}^{\mathrm{cr}}$	crossed module associated to a precrossed one	77
$\mathrm{FX}(\omega)$	free crossed P-module on the function $\omega : R \to P$	78
$*_{t \in T} \mathcal{M}_t$	coproduct of precrossed P-modules (free product)	88
$\bigcirc_{t \in T} \mathcal{M}_t$	coproduct of crossed P-modules	88
$\mathcal{M} \ltimes \mathcal{N}$	semidirect product of precrossed P-modules	90
$\{M, N\}$	Peiffer subgroup of $\mathcal{M} \ltimes \mathcal{N}$	91
$n^{-1}n^m$	pseudo-commutators	98
$[N, M]$	displacement subgroup	98
N_M	trivialisation of N	98
$f^*\mathcal{N} = (f^*N \to P)$	pullback of a crossed module by $f : P \to Q$	108
$f_*\mathcal{M} = (f_*M \to Q)$	crossed module induced by $f : P \to Q$	109
$\mathcal{M}^{\circ T}$	copower of a crossed module	127
$\mathcal{M} = (\mu : M \to P)$	crossed module over a groupoid P	152
XMod	category of crossed modules over groupoids	153
γG	crossed module associated to the double groupoid G	154
(M, G)	module over a groupoid G	213
Mod	category of modules over groupoids	214
$\Phi_{\mathsf{M}} : \mathsf{Mod} \to \mathsf{Gpds}$	forgetful functor	230
$(F(\omega), Q)$	free Q-module on $\omega : B \to Q_0$	233

$\Phi_1 : \mathsf{XMod} \to \mathsf{Gpds}$	forgetful functor	231
$\partial \colon FX(\omega) \to P$	free crossed P-module on $\omega \colon B \to P_0$	235
$\mathsf{PCat}^1\!-\mathsf{Groups}$	category of pre-cat^1-groups	81
$\mathsf{MOD}((M,G),(N,H))$	internal hom in Mod	284
$(M \otimes N, G \times H)$	tensor product in Mod	286
$\vec{\mathbb{Z}}G$	adjoint module of a groupoid G	241
$\vec{\mathbb{Z}}$	trivial G-module	241
$\varepsilon \colon \vec{\mathbb{Z}}G \to \vec{\mathbb{Z}}$	augmentation map for a groupoid G	241
$\vec{I}G$	augmentation module of a groupoid G	241
$G \ltimes M$	semidirect product of a groupoid and a module	242
$P : \mathsf{Mod} \to \mathsf{Gpds}$	pull back groupoid of a module	243
$\kappa_{(M,G)}$	conjugate to the inclusion $\vec{I}G \to \vec{\mathbb{Z}}G$	243
D_ϕ	derived module of a morphism $\phi \colon H \to G$	245

Double categories and groupoids

$D = (D_2, D_1, D_0)$	a double category	145
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