Some strict higher homotopy groupoids: intuitions, examples, applications, prospects.

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Transpennine Topology Triangle- TTT74
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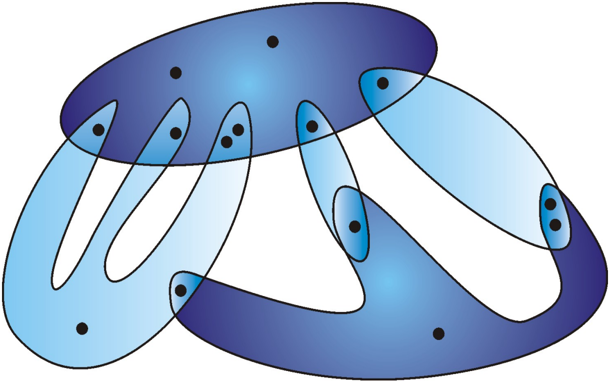
Origin of these ideas: van Kampen theorem for the fundamental groupoid on a set of base points:

$$\begin{tikzcd} \pi_1(W, W_0) \arrow[r] \arrow[d] & \pi_1(U, W_0) \arrow[d] \\ \pi_1(V, W_0) \arrow[r] & \pi_1(X, W_0) \end{tikzcd}$$

Pushout of groupoids if $X = IntU \cup IntV$, $W = U \cap V$ $W_0 \subseteq W$ meets each path component of W

This allows the complete computation of $\pi_1(X, x)$ as a small part of the larger structure $\pi_1(X, W_0)$.

Such computation involves choices and may not be algorithmic.



This success is contrary to the general philosophy of homological algebra.

Nonabelian cohomology yields only exact sequences.

It seems the success is because

groupoids have structure in dimensions 0 and 1

and so can model the geometry of the interactions of W_0, W, U, V allowing integration of homotopy 1-types.

Alexander Grothendieck

......people are accustomed to work with fundamental groups and generators and relations for these and stick to it, even in contexts when this is wholly inadequate, namely when you get a clear description by generators and relations only when working simultaneously with a whole bunch of base-points chosen with care - or equivalently working in the algebraic context of groupoids, rather than groups. Choosing paths for connecting the base points natural to the situation to one among them, and reducing the groupoid to a single group, will then hopelessly destroy the structure and inner symmetries of the situation, and result in a mess of generators and relations no one dares to write down, because everyone feels they won't be of any use whatever, and just confuse the picture rather than clarify it. I have known such perplexity myself a long time ago, namely in Van Kampen type situations, whose only understandable formulation is in terms of (amalgamated sums of) groupoids.

Conclusion: All of 1-dimensional homotopy theory is better expressed in terms of groupoids rather than groups.
van Kampen Theorem,
covering spaces,
orbit spaces and orbit groupoids.
R. Brown, 'Topology and Groupoids', 2006. (First edition, 1968)
Question: Can groupoids be useful in higher homotopy theory?
Can the use of groupoids allow for
nonabelian higher homotopy groupoids,
thus achieving the aims of the workers in topology of the early 20th century to find higher dimensional nonabelian versions of the fundamental group?
People have been overawed by the Eckmann-Hilton argument to suppose higher homotopy theory has to be abelian, and anything else is a mirage.
That argument does not apply to partial compositions.

Can one do analogous things in higher dimensions using homotopically defined objects with structure in dimensions $0, 1, \ldots, n$?
Can there be homotopy invariants with universal properties in dimensions > 1?
Clue: Whitehead's Theorem (1941-1948):

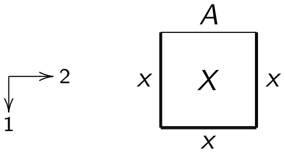
$$\pi_2(A \cup \{e^2_\lambda\}, A, x) \to \pi_1(A, x)$$

second relative homotopy group of A union 2-cells is a free crossed $\pi_1(A, x)$-module.

This freeness looks like a universal property in dimension 2!

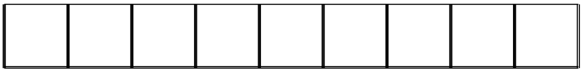
What are the 2nd relative homotopy groups

$$\pi_2(X, A, x) \to \pi_1(A, x)?$$



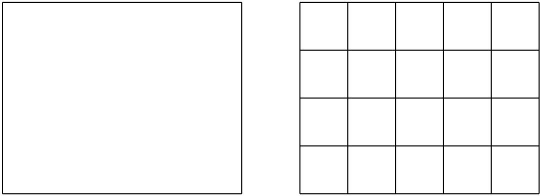
where thick lines show constant paths.

Compositions are as follows:



Whole construction involves choices, which is unaesthetic.

Consider the figures:



From left to right gives subdivision.
From right to left should give composition.
What we need for local-to-global problems is:
Algebraic inverses to subdivision.
We know how to cut things up, but how to control algebraically putting them together again?

Brown-Higgins 1974 $\rho_2(X, A, C)$:
homotopy classes rel vertices of maps $[0,1]^2 \to X$
with edges to A and vertices to C

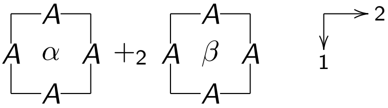
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & 2 \\ 1 & \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, dash, "A"] \arrow[d, dash, "A"'] \arrow[dr, phantom, "X"] & C \arrow[d, dash, "A"] \\ C \arrow[r, dash, "A"'] & C \end{tikzcd}$$

$$\rho_2(X, A, C) \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \pi_1(A, C) \rightrightarrows C$$

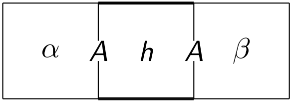
Childish idea: glue two squares if, for example, the right side of one is the same as the left side of the other. Thus these are partial algebraic compositions defined under geometric conditions.
That is my definition of higher dimensional algebra.

We would like to make a horizontal composition of classes:

$$\langle\!\langle \alpha \rangle\!\rangle +_2 \langle\!\langle \beta \rangle\!\rangle$$



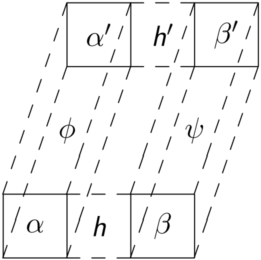
But the condition for the composition $+_2$ to be defined on classes in ρ_2 gives at least one homotopy h in A.
So we can form



where thick lines show constant paths, and define

$$\langle\!\langle \alpha \rangle\!\rangle +_2 \langle\!\langle \beta \rangle\!\rangle = \langle\!\langle \alpha +_2 h +_2 \beta \rangle\!\rangle$$

To show $+_2$ well defined, let $\phi : \alpha \equiv \alpha'$ and $\psi : \beta \equiv \beta'$, and let $\alpha' +_2 h' +_2 \beta'$ be defined. We get a picture in which dash-lines denote constant paths.



Can you see why the middle 'hole' can be filled appropriately? Thus $\rho(X, A, C)$ has in dimension 2 compositions in directions 1,2 satisfying the interchange law and is a double groupoid, containing as a substructure $\pi_2(X, A, x), x \in C$ and $\pi_1(A, C)$.

In dimension 1, we still need the 2-dimensional notion of commutative square:

$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "c"'] & \cdot \arrow[d, "b"] \\ \cdot \arrow[r, "d"'] & \cdot \end{tikzcd} \qquad ab = cd \quad a = cdb^{-1}$$

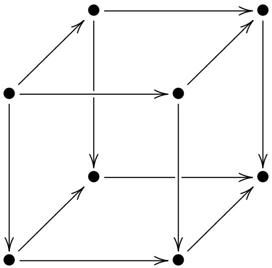
Easy result: any composition of commutative squares is commutative.

In ordinary equations:

$$ab = cd, ef = bg \text{ implies } aef = abg = cdg.$$

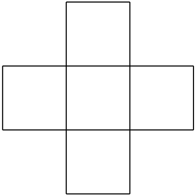
The commutative squares in a category form a double category!
Compare Stokes' theorem! Local Stokes implies global Stokes.

What is a commutative cube?

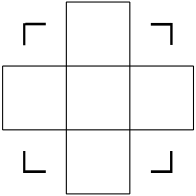


We want the faces to commute!

We might say the top face is the composite of the other faces: so fold them flat to give:



which makes no sense! Need fillers:



To resolve this, we need some special squares called thin:
First the easy ones:

$$\begin{pmatrix} 1 & \begin{matrix}1\\1\end{matrix} & 1\end{pmatrix} \qquad \begin{pmatrix} a & \begin{matrix}1\\1\end{matrix} & a\end{pmatrix} \qquad \begin{pmatrix} 1 & \begin{matrix}b\\b\end{matrix} & 1\end{pmatrix}$$

$$\square \qquad \overline{\underline{\quad}} \text{ or } \varepsilon_2 a \qquad |\,| \text{ or } \varepsilon_1 b$$

laws $\qquad \begin{bmatrix} a & \overline{\underline{\quad}} \end{bmatrix} = a \qquad \begin{bmatrix} b \\ |\,| \end{bmatrix} = b$

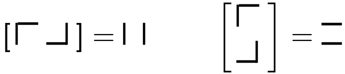
Then we need some new ones:

$$\begin{pmatrix} a & \begin{matrix}a\\1\end{matrix} & 1\end{pmatrix} \qquad \begin{pmatrix} 1 & \begin{matrix}1\\a\end{matrix} & a\end{pmatrix}$$

These are the connections

$$\lrcorner \qquad \ulcorner$$

What are the laws on connections?



(cancellation)

(transport)

These are equations on turning left or right, and so are a part of 2-dimensional algebra.

The term transport law and the term connections came from laws on path connections in differential geometry.

It is a good exercise to prove that any composition of commutative cubes is commutative.

One needs extra structure of connections, or thin structure:

double groupoids (with connection)	$\simeq$	crossed modules over groupoids
$\rho(X, A, C)$ as double groupoid	$\simeq$	$\pi_2(X, A, C) \to \pi_1(A, C)$
van Kampen theorem for the double groupoid $\rho(X, A, C)$	$\simeq$	van Kampen theorem for the crossed module over groupoid $\pi_2(X, A, C)$

So you can calculate some nonabelian crossed modules, i.e. some homotopy 2-types!

Calculation of the corresponding $\pi_2(X, x)$ may be tricky!

Computer calculations of the induced crossed module $\delta : \iota_*(P) \to S_4$ representing the 2-type of the mapping cone Γ of $B\iota : BP \to BS_4$ for various subgroups P of S_4, and of the kernel $\pi_2(\delta) \cong \pi_2(\Gamma)$ of δ.

P	$\iota_* P$	$\pi_2(\delta)$
C_2	$GL(2,3)$	C_2
C_3	$C_3 \ SL(2,3)$	C_6
S_3	$GL(2,3)$	C_2
C_2'	$C_2^3 \ H_8^+$	$C_2^3 C_4$
C_2^2	$S_4 C_2$	C_2
C_4	$SL(2,3) \rtimes C_4$	C_4
D_8	$S_4 C_2$	C_2

Here $C_2 = \langle (1,2) \rangle$, $C_2' = \langle (1,2)(3,4) \rangle$, $C_2^2 = \langle (1,2),(3,4) \rangle$; $SL(2,3)$, $GL(2,3)$ and $GL(3,2)$ are linear groups of orders 24, 48 and 168 respectively;
H_n is the holomorph of C_n, and H_n^+ is its positive subgroup in degree n.
π_1, π_2 give only a pale shadow of the 2-type, which is essentially nonabelian, but can be calculated in some cases.

Contrast with determining the k-invariant in $H^3(\pi_1(X), \pi_2(X))$.
It is almost impossible to determine the k-invariant of a union.
It is (under some conditions) possible to determine the crossed module of a union, as a pushout of crossed modules!
But this is not in the current 'canon' of algebraic/geometric topology.

Higher dimensions?

Category FTop of filtered spaces:

$$X_* : X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X_\infty = X$$

Example: n-cube I^n_*
$(RX_*)_n = \mathrm{FTop}(I^n_*, X_*)$
RX_* = cubical set with connections and compositions
$p : RX_* \to \rho X_* = (RX_*)/\equiv$
where $\equiv$ is thin homotopy, i.e. homotopy through filtered maps rel vertices of I^n
Amazing facts:
1) The natural structure on RX_* of cubical set with compositions and connections is inherited by ρX_*, the chief problem being the compositions,
making ρX_* a strict ω-groupoid.
2)$p : RX_* \to \rho X_* = (RX_*)$ is a Kan fibration of cubical sets.
The last fact gives the strong link between
the lax structures on RX_* and the strict structures on ρX_*.

3) We also need the notion of ΠX_*, the fundamental crossed complex of a filtered space, defined using the well known properties of the fundamental groupoid

$$(\Pi X_*)_1 = \pi_1(X_1, X_0),$$

the relative homotopy groups

$$(\Pi X_*)_n(x) = \pi_n(X_n, X_{n-1}, x)$$

for $n \geqslant 2, x \in X_0$, and the associated boundary maps and operations of $\Pi X_*)_1$.

4) Strict cubical ω-groupoids with connections are equivalent to crossed complexes and ρX_* is in this equivalent to ΠX_*.

5) This gives a different foundation for algebraic topology whose full consequences have yet to be worked out. See 'Nonabelian algebraic topology: filtered spaces, crossed complexes, cubical homotopy groupoids' R. Brown, P.J. Higgins, R. Sivera, EMS Tracts in Mathematics 15, xxxiii+640 pages, (autumn 2010).

We need both ρ and Π to develop theory and applications.
Sample application of the HHvKT for ρ and so for Π:
As a special case of calculating the excision map

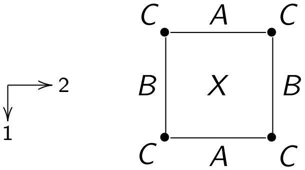
$$\pi_n(X, A, x) \to \pi_n(X \cup Y, Y, x)$$

when $A = X \cap Y$ we get:
If (X, A) is pointed and $(n-1)$-connected, then the natural map

$$\pi_n(X, A, x) \to \pi_n(X \cup CA, CA, x) \cong \pi_n(X \cup CA, x)$$

is, algebraically, factoring by the action of $\pi_1(A, x)$.
i.e. Relative Hurewicz Theorem is a consequence of a HHvKT!
The proof does not use homology or simplicial approximation.
Can also handle the many-pointed case.
Other applications, e.g. homotopy classification of maps, make strong use of monoidal closed structures.
Philosophy: spaces often come with structure, or are replaced by spaces with structure, so it is reasonable to base algebraic topology on spaces with structure rather than just bare spaces.

Tri-ads : $A, B \subseteq X$; set of base points $C \subseteq A \cap B$.
Consider the set $\Phi_2(X; A, B; C)$ of maps $I^2 \to X$



This forms a lax double category with the obvious compositions.
Not generally inherited by homotopy classes rel vertices.
Amazing fact: these compositions are inherited by the fundamental group

$$\pi_1(\Phi_2(X; A, B; x), \bar{x})$$

making it a
strict double groupoid internal to groups, i.e. a cat^2-group.

This generalises to $(n+1)$-ads, or even n-cubes of spaces, and so to cat^n-groups.
There is a HHvKT for the fundamental cat^n-group of an n-cube of spaces (Brown-Loday, 1987) allowing some new calculations and opening up new areas e.g. Higher Hopf formulae for homology.
Recent paper by Ellis-Mikhailov.
Strict n-fold groupoids model weak homotopy n-types, so there is still a lot to be said for studying the relations between strict and non strict structures.

Pushouts and Cubical Tricks

Suppose we have a homotopical functor Π of pairs which preserves certain pushouts of pairs of spaces- HHvKT.

If $X = A \cup B$, $C = A \cap B$, we get a pushout square

$$\begin{tikzcd} C \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & X \end{tikzcd}$$

which can be turned into a pushout square of pairs

$$\begin{tikzcd} (C, C) \arrow[r] \arrow[d] & (A, A) \arrow[d] \\ (B, C) \arrow[r, "\varepsilon"'] & (X, A) \end{tikzcd}$$

where ε is the excision map.

Applying Π gives an excision theorem for Π.

This is how we got a strong generalisation of Whitehead's theorem involving induced crossed modules and so bifibrations of categories. Three papers by Brown-Wensley include some group computation to do the sums: we obtain specific groups and numbers.

Suppose now we have a homotopical functor Π of squares of spaces which preserves certain pushouts of squares of spaces-HHvKT.

Consider again the first pushout square:

$$\begin{tikzcd} C \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & X \end{tikzcd}$$

this gives rise to a new square

$$\begin{tikzcd} \substack{CC \\ CC} \arrow[r] \arrow[d] & \substack{CA \\ CA} \arrow[d] \\ \substack{CC \\ BB} \arrow[r] & \substack{CA \\ BX} \end{tikzcd}$$

which is a pushout of squares of spaces.

By applying Π to this pushout, we got the nonabelian tensor product of groups which act on each other.

Computes certain nonabelian triad homotopy groups $\pi_3(X; A, B; x)$ as built up by generalised Whitehead products from lower relative homotopy groups.

If $X = X_1 \cup X_2 \cup X_3$ we get a pushout 3-cube X_{***} of spaces.
Like to know what is excision in this situation.
But X_{***} can be regarded as a map $x : X_{-**} \to X_{+**}$ of squares, and so
as a map of squares of squares, and so
as a 3-cube of squares of spaces
which is a 3-pushout of squares of spaces!

This is how we got a totally new triadic Hurewicz Theorem, essentially conjectured by Loday, and proved as a consequence of our van Kampen theorem for n-cubes of spaces.

Theorem

Suppose for the pointed triad $(X; A, B)$ that $A, B, A \cap B$ are connected, $(A, A \cap B), (B, A \cap B)$ are 1-connected, and $(X; A, B)$ is 2-connected. Then $X \cup CA \cup CB$ is 2-connected and the Hurewicz map

$$\pi_3(X; A, B) \to H_3(X; A, B)$$

factors the action of $\pi_1(A \cap B)$ and the generalised Whitehead product.

All these tricks extend easily to n-cubes of spaces, and the consequences have been largely unexplored, or merely scratched the surface.

Example: prove the n-ad connectivity theorem and determine in principle the critical group.

Conclusion: There are some advantages in using strict higher homotopical groupoids; we know they can be defined for certain structured spaces, and give new nonabelian information underlying homotopy theory.

Is this 'postmodern homotopy theory'?

Higher dimensional category theory contrasted with higher dimensional group theory.

Nonabelian methods in homotopy theory.

Alexander Grothendieck: Extract from Letter 02.05.1983

Don't be amazed at my supposed efficiency in digging out the right kind of notions- I have just been following, rather let myself be pulled along, by that very strong thread (roughly: understand noncommutative cohomology of topoi!) which I kept trying to sell for about ten or twenty years now, without anyone ready to "buy" it, namely, to do the work. So finally I got mad and decided to work out at least an outline by myself.

Yours very cordially,

Alexander

Prospects: Colimit theorems in applications of higher groupoids to algebraic topology,differential geometry, stacks, algebraic geometry, algebraic number theory.!!!???