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COPRODUCTS OF CROSSED P-MODULES: APPLICATIONS TO SECOND HOMOTOPY GROUPS AND TO THE HOMOLOGY OF GROUPS

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§1. INTRODUCTION

THE RELEVANCE of crossed modules to problems on second homotopy groups, and to some difficult problems in combinatorial group theory, is well known (see [5]). The difficulties are essentially those of understanding free crossed modules, and, more generally, colimits of crossed modules.

The algebraic purpose of this paper is to give a simple description of the *coproduct* of two crossed P-modules.

The application of this algebra to homotopy theory comes from the generalisation of the van Kampen theorem to dimension two given by Brown and Higgins [3]. This theorem shows that certain unions of pairs of spaces give rise to pushouts of crossed modules.

A simple special case of our main result (Corollary 3.2) concerns the union of Eilenberg-MacLane spaces. Suppose given a homotopy pushout

$$\begin{array}{ccc} K(P, 1) & \xrightarrow{\ i\ } & K(Q, 1) \\ {\scriptstyle j}\downarrow & & \downarrow \\ K(R, 1) & \longrightarrow & X. \end{array}$$

Then we have immediately a long exact Mayer–Vietoris homology sequence:

$$\cdots \to H_n(P) \to H_n(Q) \oplus H_n(R) \to H_n(X) \to H_{n-1}(P) \to \cdots.$$

The problem is to describe $H_n(X)$ in terms of group theoretic invariants of P, Q, R and the induced maps $i_*: P \to Q, j_*: P \to R$.

If i_*, j_* are injective, a well-known result of J. H. C. Whitehead implies $X \simeq K(Q *_P R, 1)$. From Corollary 3.2 we obtain:

THEOREM. *If* $i_*: P \to Q, j_*: P \to R$ *are surjective with kernels* M, N, *respectively, then*

$$\pi_2 X \cong (M \cap N)/[M, N].$$

As an application we obtain, if $P = MN$ and i_*, j_* are surjective, an exact homology sequence

$$H_2P \to H_2Q \oplus H_2R \to (M \cap N)/[M, N] \to H_1P \to H_1Q \oplus H_1R \to 0.$$

This reduces to a well-known exact sequence of Stallings if $M = P$.

§2. COPRODUCTS OF CROSSED P-MODULES

Let P be a group. Recall that a *crossed P-module* (X, χ) consists of a group X on which P acts on the right $(x, p) \mapsto x^p$, together with a morphism $\chi: X \to P$ of groups satisfying since the second component of (*) is

$$\begin{aligned} \bar{a}^{\bar{x}\bar{a}xa} a^{\bar{a}xa} &= (\bar{a}^{\bar{x}\bar{a}x} a^x)^a && \text{as } a^{\bar{a}} = a, \\ &= (a^x \bar{a})^a && \text{by CM (2),} \\ &= \bar{a} a^x. \end{aligned}$$

Also these elements $\{x, a\}$ generate the Peiffer group, since their conjugates are of the same form, as is shown by the equations (which the reader may verify)

$$\begin{aligned} (1, b)^{-1}\{x, a\}(1, b) &= \{x, a\} \\ (y, 1)^{-1}\{x, a\}(y, 1) &= \{x^y, a^y\}. \quad \square \end{aligned}$$

We write $\{X, A\}$ for the Peiffer subgroup of (XA, ∂'), and write $(X \circ A, \partial)$ for the induced crossed P-module with $X \circ A = (XA)/\{X, A\}$. Let $i: X \to X \circ A$, $j: A \to X \circ A$ be induced by the inclusions $i: X \to XA$, $j: A \to XA$, respectively.

2.4. THEOREM. *The crossed P-module* $(X \circ A, \partial)$ *with the two morphisms i, j above is the coproduct of the crossed P-modules* (X, χ) *and* (A, α).

Proof. This is immediate from Propositions 2.1, 2.3. $\square$

Our next aim is to identify $\mathrm{Ker}(\partial: X \circ A \to P)$. To this end, form the pull-back square

$$\begin{array}{ccc} X \times_P A & \xrightarrow{\alpha'} & A \\ {\scriptstyle \chi'}\downarrow & & \downarrow{\scriptstyle \alpha'} \\ X & \xrightarrow[\chi]{} & P \end{array}$$

so that $X \times_P A = \{(x, a) \in X \times A : \chi x = \alpha a)\}$. Let P operate diagonally on $X \times_P A$, and let X, A operate on $X \times_P A$ via χ and α, respectively. For (x, a), $(y, b) \in X \times_P A$

$$\begin{aligned} (x, a)(y, b) &= (xy, ab) \\ &= (yx^y, ba^b) \\ &= (y, b)(x, a)^b \qquad \text{xince } \chi y = \alpha b. \end{aligned}$$

Hence $X \times_P A$ is a crossed module over each of X, A and P (the latter via $\kappa = \chi\chi' = \alpha\alpha'$).
Define the function

$$\begin{aligned} h: X \times A &\to X \times_P A \\ (x, a) &\mapsto (x^{-1}x^a, (a^{-1})^x a), \end{aligned}$$

and write $\langle x, a \rangle$ for $h(x, a)$. We write $\langle X, A \rangle$ for the subgroup of $X \times_P A$ generated by the elements $\langle x, a \rangle$ for $x \in X$, $a \in A$.

2.5. PROPOSITION. *There is an exact sequence of P-groups*

$$1 \to X \times_P A \xrightarrow{\phi'} XA \xrightarrow{\partial'} P \tag{2.6}$$

in which $\phi'\colon (x, a) \mapsto (x, a^{-1})$. *Further*

$$\varphi'\langle X, A\rangle = \{X, A\}$$

so that there is an induced exact sequence

$$1 \mapsto (X \times_P A)/\langle X, A\rangle \xrightarrow{\phi} X \circ A \xrightarrow{\partial} P. \tag{2.7}$$

Also $\langle X, A\rangle$ *contains the commutator subgroup of* $X \times_P A$.

Proof. The check that ϕ' is a P-morphism is easy. It is clear that ϕ' is injective and has image equal to Ker ∂'. Also $\phi'\langle x, a\rangle = \{x, a\}$, $x \in X$, $a \in A$. Hence $\phi'\langle X, A\rangle = \{X, A\}$, and it follows that $\langle X, A\rangle$ is normal in $X \times_P A$. The exact sequence (2.7) is immediate. The last statement of the Proposition follows from the fact that $(X \circ A, \partial)$ is a crossed module, and so Ker ∂ is abelian. (A direct verification is easy.) □

Let $M = \chi X$, $N = \alpha A$. Then $\kappa\colon X \times_P A \to P$ satisfies

$$\kappa(X \times_P A) = M \cap N,$$

$$\kappa\langle X, A\rangle = [M, N].$$

2.8. PROPOSITION. *Let* $U = \operatorname{Ker}\chi \oplus \operatorname{Ker}\alpha$. *Then there is an exact sequence of P-groups*

$$1 \to U \to X \times_P A \xrightarrow{\kappa} M \cap N \to 1 \tag{2.9}$$

and an induced exact sequence of P-modules

$$0 \to U \cap \langle X, A\rangle \to U \to (X \times_P A)/\langle X, A\rangle \xrightarrow{\kappa} (M \cap N)/[M, N] \to 0, \tag{2.10}$$

Proof. This is immediate. □

2.11. COROLLARY. *The morphism* $\partial\colon X \circ A \to P$ *is injective if and only if*

(i) $\operatorname{Ker}\chi \oplus \operatorname{Ker}\alpha \subseteq \langle X, A\rangle$, *and*

(ii) $[M, N] = M \cap N$. □

2.12. EXAMPLE. Let $X = P$, $\chi = 1_P$ and let $\alpha = 0$, so that A is a P-module. Then $M \cap N = [M, N] = \{1\}$.

If $p \in P$, $a \in A$, then

$$\langle p, a\rangle = (p^{-1}p^a, (a^{-1})^p a)$$

$$= (1, (a^{-1})^p a),$$

and $\operatorname{Ker}\chi \oplus \operatorname{Ker}\alpha = A$. So the conditions of (2.11) for $\partial\colon P \circ A \to P$ to be injective are here satisfied if and only if A is generated by the elements $(a^{-1})^p a$, $a \in A$, $p \in P$. Note also that the composite $\partial j\colon A \to P \circ A \to P$ is just α, which is zero. So if $\partial\colon P \circ A \to P$ is injective then $j = 0$: $A \to P \circ A$.

We now write A additively. An example where A is generated by the elements $a - a^p$,

$a \in A$, $p \in P$ is when A is obtained from a P-module B by factoring out the submodule generated by elements $2b - b^{t(b)}$ where b ranges over a set of generators of B as P-module, and $t(b) \in P$. In particular, if P is the infinite (multiplicative) cyclic group on a generator t, and $B = \mathbb{Z}P$ is the group-ring of P considered as P-module, we can factor B by the submodule generated by $2 - t(= 2b - b^t$ where $b = 1)$ to obtain a P-module A. Then A is isomorphic to the additive group of rational numbers $m/2^n$, $m \in Z$, $n \geqslant 0$, so that A is non-zero (This special case is essentially due to Adams[1] p. 483.)

2.13. *Remark.* The pull-back diagram for $X \times_P A$ together with the map $h: X \times A \to X \times_P A$, $(x, a) \mapsto \langle x, a \rangle$, is (with due allowance for the change from left to right actions) a crossed square in the sense of [7] §5.

2.14. *Remark.* The construction of the coproduct $X \circ A$ as a quotient of $X * A$ may be found in [9], p. 428.

§3. APPLICATIONS

Let (K, K_0) be a pair of pointed spaces. It is standard that the second relative homotopy group $\pi_2(K, K_0)$, with the usual action of $\pi_1 K_0$ and the usual boundary $\pi_2(K, K_0) \to \pi_1 K_0$, is a crossed $\pi_1 K_0$-module. Further, we have the following special case of the pushout theorem for crossed modules in [3].

3.1. THEOREM (Brown–Higgins). *If the connected* CW-*complex K is the union of connected subcomplexes K_1, K_2 with connected intersection K_0, and (K_1, K_0), (K_2, K_0) are* 1-*connected, then there is an isomorphism of crossed $\pi_1 K_0$-modules*

$$\pi_2(K, K_0) \cong \pi_2(K_1, K_0) \circ \pi_2(K_2, K_0).$$

Proof. Apply Theorem C of [3] to the diagram of inclusions

$$\begin{array}{ccc} (K_0, K_0) & \longrightarrow & (K_1, K_0) \\ \downarrow & & \downarrow \\ (K_2, K_0) & \longrightarrow & (K, K_0). \end{array} \quad \square$$

3.2. COROLLARY. *Suppose, in addition to the assumptions of* (3.1), *that $\pi_2 K_0 = 0$. Let $P = \pi_1 K_0$, and let X, A denote the crossed P-modules $\pi_2(K_1, K_0)$, $\pi_2(K_2, K_0)$, respectively. Then there is an isomorphism of P-modules*

$$\pi_2 K \cong (X \times_P A)/\langle X, A \rangle$$

and hence an exact sequence:

$$0 \to (\pi_2 K_1 \oplus \pi_2 K_2) \cap \langle X, A \rangle \to \pi_2 K_1 \oplus \pi_2 K_2 \to \pi_2 K \to (M \cap N)/[M, N] \to 0$$

where M, N are the kernels of $\pi_1 K_0 \to \pi_1 K_1$, $\pi_1 K_0 \to \pi_1 K_2$ respectively.

Proof. The assumption that $\pi_2 K_0 = 0$ implies that

$$\pi_2 K_i = \mathrm{Ker}(\pi_2(K_i, K_0) \to \pi_1 K_0) \quad \text{for} \quad i = 1, 2, -. \quad \square$$

3.3. *Remark.* The exact sequence of (3.2) strengthens and generalises Theorem 1 of [6], which assumes that K is 2-dimensional and K_0 is the 1-skeleton of K, and does not determine the kernel of $\pi_2K_1 \oplus \pi_2K_2 \to \pi_2K$.

We now give an application to the homology of groups.

3.4. THEOREM. *Let M, N be normal subgroups of a group and let $L = M \cap N$. Then there is an exact sequence*

$$H_2(MN) \to H_2(M/L) \oplus H_2(N/L) \to L/[M, N] \to H_1(MN) \to H_1(M/L) \oplus H_1(N/L) \to 0.$$

Proof. Let $P = MN$, $Q = P/M = N/L$, $R = P/N = M/L$.

Let $K_0 = K(P, 1)$, $K_1 = K(Q, 1)$, $K_2 = K(R, 1)$ be Eilenberg–MacLane CW-complexes, and let the maps $i_1: K_0 \to K_1$, $i_2: K_0 \to K_2$ realise the morphisms $P \to Q$, $P \to R$, respectively. By homotopies and use of mapping cylinders, we may assume i_1, i_2 are cellular inclusions. Let K be the pushout of i_1, i_2. Part of the Mayer–Vietoris homology sequence for $K = K_1 \cup K_2$ is

$$H_2K_0 \to H_2K_1 \oplus H_2K_2 \to H_2K \to H_1K_0 \to H_1K_1 \oplus H_1K_2 \to H_1K \to 0.$$

Now $H_iK_0 = H_iP$, $H_iK_1 = H_iQ$, $H_iK_2 = H_iR$. Also $\pi_1K \cong P/MN = 0$. Hence $H_1K = 0$ and $H_2K \cong \pi_2K$. By Corollary 3.2, $H_2K = (M \cap N)/[M, N]$ (since $\pi_2K_1 = \pi_2K_2 = \pi_2 K_0 = 0$). □

3.5. *Remark.* The exact sequence of Theorem 3.4 reduces to a well-known exact sequence of Stallings in the case $M \subset N$, so that $L = M$ ([2] p. 47). This latter sequence was deduced in [3] by a similar method to the above.

3.6. *Remark.* Let M, N be normal subgroups of a group P, and let $Q = P/M$, $R = P/N$, $G = P/MN$. The method of proof of Theorem 3.4 yields an exact sequence

$$H_2P \to H_2Q \oplus H_2R \to H_2K \to H_1P \to H_1Q \oplus H_1R \to H_1G \to 0$$

(where K is as in the proof). By Exercise 6 on p. 175 of [2], there is an exact sequence

$$H_3K \to H_3G \to (\pi_2K) \otimes_{\mathbb{Z}G} \mathbb{Z} \to H_2K \to H_2G \to 0,$$

and by Corollary 3.2, $\pi_2K = (M \cap N)/[M, N]$.

3.7. *Remark.* A subsequent paper with Loday will extend the sequence (3.4) to the left, by identifying H_3K (where K is as in the proof) in terms of M, N, P as a kind of "Ganea term" [10].

3.8. *Remark.* Theorem 3.4 has applications to presentations of the trivial group, for example the presentation (in which $[a, b] = a^{-1}b^{-1}ab$)

$$\mathbf{P} = (x, y : x^{-1}[x^m, y^n], y^{-1}[y^p, x^q])$$

where m, n, p, $q \in \mathbb{Z}$. (This presentation was found by Gordon, and was communicated to me by Lickorish. I am grateful to Professor Gordon for permission to include it here.) Let P be the free group $\{x, y\}$ and let M, N be the normal closures in P of each of the relators. Then $P = MN$, since $\mathbf{P}$ presents the trivial group (see 3.9 below). Now $Q = P/M$, $R = P/N$ are one-relator groups whose relators are not proper powers, so that H_2Q, H_2R

are trivial,by Lyndon's Identity Theorem. Also one verifies easily that $H_1P \to H_1Q \oplus H_1R$ is an isomorphism. It follows from Theorem 3.4 that $M \cap N = [M, N]$.

3.9. *Remark*. For completeness we include a proof (due to Holt but similar to Gordon's proof) that **P** of 3.8 presents the trivial group. We work in P/MN, first by a change of convention, writing the relations as

$$x = x^{-n}y^m x^n y^{-m} \tag{1}$$

$$y = x^{-p}y^q x^p y^{-q}. \tag{2}$$

Then (1) implies

$$x^{n+1} = y^m x^n y^{-m}$$

whence

$$x^{(n+1)^q p} = y^{mq} x^{n^q p} y^{-mq}. \tag{3}$$

Also (2) implies

$$x^p y = y^q x^p y^{-q}$$

whence

$$x^p y^m = y^{mq} x^p y^{-mq}$$

and

$$(x^p y^m)^{n^q} = y^{mq} x^{n^q p} y^{-mq}. \tag{4}$$

From (3) and (4) we deduce $x^{(n+1)^q p}$ commutes with $x^p y^m$ and hence with y^m. But y^{-m} conjugates $x^{(n+1)^p q}$ to $x^{(n+1)^{q-1} np}$ and so $x^{(n+1)^{p-1} p} = 1$. Conjugating repeatedly by y^{-m} gives $x^p = 1$, and then $y = 1$ from (2) and $x = 1$ from (1).

3.10. *Remark*. Special cases (e.g. $m = p = 2$, $n = q = 1$) of the example have been considered as possible counter examples to the Andrews–Curtis conjecture[8], and this is one of the reasons for presenting the example in detail.

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