Moore hyperrectangles on a space form a strict cubical omega-category

Ronald Brown

September 21, 2009

Abstract

A question of Jack Morava is answered by generalising the notion of Moore paths to that of Moore hyperrectangles, so obtaining a strict cubical ω-category. This also has the structure of connections in the sense of Brown and Higgins, but cancellation of connections does not hold.

Introduction

We recall in Section 1 the notion of the space of Moore paths on a topological space X. A variant of the definition is given in [Bro06]. Moore paths have the advantage of giving a category of paths, with an associative composition, and identities, rather than the common description in terms of maps $I \to X$.

However, whereas in higher dimensions the appropriate and analogous operations on maps of cubes $I^n \to X$ have been well used, see for example [BH81b], there seems to have been in higher dimensions no definition analogous to that of Moore paths.

In this paper we give such a definition in Section 2 and in Section 3 we give the laws that this structure satisfies. The formulation of these is taken from [AABS02], but for a large part they go back to [BH77].

The cubical laws were given in [Kan55]. The cubical approach in that paper was abandoned in favour of simplicial sets once the problems of the geometric realisation of the cartesian product were found, and Milnor had written on the geometric realisation of the simplicial sets.

The introduction of 'connections' in all dimensions was in [BH77, BH81a, BH81b] for the purpose of discussing 'commutative shells'. This was extended to the category case in [Mos87, AA89, AABS02]. The general theory of cubical sites is developed in [GM03]. Maltsiniotis has shown in [Mal09] that cubical sets with connections, in contrast to the standard case, have good realisations of cartesian products. The thesis [Pat08] uses cubical sets with what he calls *pseudo connections* for the theory of derived functors, analogously to the simplicial case.

The paper [Gra02] uses an analogous procedure to this for the definition of a higher categorical structure, with cubes indexed on Z^n and constant on each variable outside of a certain 'support', but does not take the support as part of the structure.

An application of the cubical classifying space of a crossed complex is in [FRS95].

It is suggested in recent work that the notion of Kan simplicial set can be regarded as an ∞-groupoid, see for example [Lur09]. In some ways this is curious as this is regarded as the start of such an idea is the fundamental group or groupoid, which is made of classes of paths under homotopy relative to the end points. One would expect on the same principle to take some form of homotopy classes of maps of m-paths. The difficulty in this is shown that by the fact that an absolute strict homotopy m-groupoid has been defined only for $m = 1, 2$, in [BHKP02]. The paper [BH81b] shows that successful higher homotopy groupoids can be defined for filtered spaces. This allows a route into algebraic topology without setting up singular homology theory.

In any case, the construction $M_*(X)$ can be seen as another candidate for a weak form of ∞-groupoid.

1 Moore paths

Let $\mathbb{R}^+ = [0, \infty)$ be the nonnegative real line. For a space X let $M(X)$ be the subspace of $X^{\mathbb{R}^+} \times \mathbb{R}^+$ of pairs (f, r) such that f is constant on $[r, \infty)$. There are two maps

$$\partial^-, \partial^+ : M(X) \to X,$$
$$\partial^-(f, r) = f(0),$$
$$\partial^+(f, r) = f(r).$$

Now composition $\circ$ of Moore paths on $M(X)$ is given by the composition

$$M(X)_{\,\partial^+} \times_{\partial^-} M(X) \xrightarrow{\ \phi\ } X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \xrightarrow{\ 1 \times +\ } X^{\mathbb{R}^+} \times \mathbb{R}^+$$

where the first term is the pullback, and ϕ sends pairs $(f, r), (g, s) \in M(X)$ such that $f(r) = g(0)$ to triples $(h, r, s) \in X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+$ such that h is constant on $[r + s, \infty)$, $h|[0, r] = f|[0, r]$ and $h(t) = g(t - r)$ for $t \geqslant r$, and $+$ is the addition function. So composition is continuous.

We also have an identity function $\varepsilon : X \to M(X)$ given by $\varepsilon(x) = (\hat{x}, 0)$ where $\hat{x}$ is the constant map on $\mathbb{R}^+$ with value x.

This composition gives, as is well known, a category structure $(M(X), \partial^\pm, \circ, \varepsilon)$. This structure also has a 'reverse' $- : M(X) \to M(X)$ given by $-(f, r) = (g, r)$ where

$$g(t) = \begin{cases} f(r - t) & \text{if } 0 \leqslant t \leqslant r, \\ f(0) & \text{if } t \geqslant r. \end{cases}$$

Thus $\partial^-(-a) = \partial^+(a), \partial^+(-a) = \partial^- a$.

We now discuss the relation with the fundamental groupoid on a set C of base points in X.

By a homotopy H of elements $a^0 = (f^0, r^0), a^1 = (f^1, r^1)$ of $M(X)$ we mean a continuous map $H : [0,1] \to M(X)$ such that $H(0) = a^0, H(1) = a^1$, or, equivalently, a map $H : [0,1] \times \mathbb{R}^+ \to X$ such that $H(0,t) = a^0(t), H(1,t) = a^1(t)$ for $t \in \mathbb{R}^+$ and there is a continuous function $s \mapsto r(s)$ where $0 \leqslant s \leqslant 1, r(s) \in \mathbb{R}^+$, $r(0) = r^0, r(1) = r^1$ and $H(s,t) = H(s,r(s))$ for $t \geqslant r(s), 0 \leqslant s \leqslant 1$. This homotopy is *rel end points* if $H(s,0) = f^0(0), H(s,r(s)) = f^0(r^0)$ for all $0 \leqslant s \leqslant 1$. The fundamental groupoid $\pi_1(X,C)$ on the set of base points $C \subseteq X$ is the the set of homotopy classes rel end points of elements of $M(X)$ with source and target in C. For more information on the use of $\pi_1(X,C)$, but with a slightly different construction, see [Bro06].

2 Moore hyperrectangles

Let $M_n(X)$ be the subspace of $X^{(\mathbb{R}^+)^n} \times (\mathbb{R}^+)^n$ of pairs $(f, (r))$ where $(r) = (r_1, \ldots, r_n)$ such that
$$f(t_1, \ldots, t_i, \ldots, t_n) = f(t_1, \ldots, r_i, \ldots, t_n) \text{ for } t_i \geqslant r_i, i = 1, \ldots, n.$$
We call (r) the *shape* and f the *action* of the n-path $(f, (r))$. We have

$$\partial_i^-, \partial_i^+ : M_n(X) \to M_{n-1}(X)$$

given by evaluating at 0 or r_i in the ith position and omitting the r_i. More precisely, $\partial_i^\alpha(f, (r)) = (f', (r'))$ where $(r') = (r_1, \ldots, \hat{r}_i, \ldots, r_n)$ and $f'(r') = f(r_1, \ldots, \alpha', \ldots, r_n)$ where $\alpha' = 0$ or r_i according as $\alpha = -$ or $+$.

To define the degeneracies $\varepsilon_i : M_{n-1}(X) \to M_n(X)$ we set $\varepsilon_i(f', (r')) = (f, (r))$ where (r) is obtained from (r') by putting 0 in the ith place, and $f(t_1, \ldots, t_n) = f'(t_1, \ldots, \hat{t}_i, \ldots, t_n)$.

To define the connections $\Gamma_i^- : M_{n-1}(X) \to M_n(X)$ we set $\Gamma_i^-(f', (r')) = (f, (r))$ where (r) is obtained from (r') by repeating r_i (in the ith and $(i+1)$th place, and moving the others along), and setting
$$f(t_1, \ldots, t_n) = f'(t_1, \ldots, t_{i-1}, \max(t_i, t_{i+1}), t_{i+2}, \ldots, t_n).$$
Similarly we get Γ_i^+ using min instead of max. (This follows the conventions of [AABS02].)

For $i = 1, \ldots, n$ the category structure $(M_n(X), , \partial_i^-, \partial_i^+, \circ_i, \varepsilon_i)$ is simply that given in section 1 but in the ith place.

In this way we give the family $M_*(X) = \{M_n(X)\}$ for $n \geqslant 0$ the structure of cubical ω-category: the laws for this and the connections are given in Section 3. The paper [BH81c] also shows how to obtain what we now call a globular ω-category from this cubical structure, as a substructure in which certain faces of a cube have various levels of degeneracy. However this globular structure is not equivalent to the cubical structure, as the proof in [AABS02] requires the cancellation law for connections, which does not hold here: see Remark 3.2.

We refer also to [Bro08] for the construction of a fundamental globular ω-groupoid $\rho(X_*)$ of a filtered space X_*.

3 Laws

In this section we give the full structure and laws on the cubical set with connections and compositions $M_*(X)$. We take these from [AABS02].

Let K be a cubical set, that is, a family of sets $\{K_n; n \geqslant 0\}$ with for $n \geqslant 1$ face maps $\partial_i^\alpha : K_n \to K_{n-1}$ $(i = 1, 2, \ldots, n; \alpha = +, -)$ and degeneracy maps $\varepsilon_i : K_{n-1} \to K_n$ $(i = 1, 2, \ldots, n)$ satisfying the usual cubical relations:

$$\partial_i^\alpha \partial_j^\beta = \partial_{j-1}^\beta \partial_i^\alpha \qquad\qquad (i < j), \qquad\qquad (3.1)(\text{i})$$

$$\varepsilon_i \varepsilon_j = \varepsilon_{j+1} \varepsilon_i \qquad\qquad (i \leqslant j), \qquad\qquad (3.1)(\text{ii})$$

$$\partial_i^\alpha \varepsilon_j = \begin{cases} \varepsilon_{j-1} \partial_i^\alpha & (i < j) \\ \varepsilon_j \partial_{i-1}^\alpha & (i > j) \\ \text{id} & (i = j) \end{cases} \qquad\qquad (3.1)(\text{iii})$$

We say that K is a *cubical set with connections* if for $n \geqslant 0$ it has additional structure maps (called *connections*) $\Gamma_i^+, \Gamma_i^- : K_n \to K_{n+1}$ $(i = 1, 2, \ldots, n)$ satisfying the relations:

$$\Gamma_i^\alpha \Gamma_j^\beta = \Gamma_{j+1}^\beta \Gamma_i^\alpha \qquad\qquad (i < j) \qquad\qquad (3.2)(\text{i})$$

$$\Gamma_i^\alpha \Gamma_i^\alpha = \Gamma_{i+1}^\alpha \Gamma_i^\alpha \qquad\qquad (3.2)(\text{ii})$$

$$\Gamma_i^\alpha \varepsilon_j = \begin{cases} \varepsilon_{j+1} \Gamma_i^\alpha & (i < j) \\ \varepsilon_j \Gamma_{i-1}^\alpha & (i > j) \end{cases} \qquad\qquad (3.2)(\text{iii})$$

$$\Gamma_j^\alpha \varepsilon_j = \varepsilon_j^2 = \varepsilon_{j+1} \varepsilon_j, \qquad\qquad (3.2)(\text{iv})$$

$$\partial_i^\alpha \Gamma_j^\beta = \begin{cases} \Gamma_{j-1}^\beta \partial_i^\alpha & (i < j) \\ \Gamma_j^\beta \partial_{i-1}^\alpha & (i > j + 1), \end{cases} \qquad\qquad (3.2)(\text{v})$$

$$\partial_j^\alpha \Gamma_j^\alpha = \partial_{j+1}^\alpha \Gamma_j^\alpha = id, \qquad\qquad (3.2)(\text{vi})$$

$$\partial_j^\alpha \Gamma_j^{-\alpha} = \partial_{j+1}^\alpha \Gamma_j^{-\alpha} = \varepsilon_j \partial_j^\alpha. \qquad\qquad (3.2)(\text{vii})$$

The connections are to be thought of as extra 'degeneracies'. (A degenerate cube of type $\varepsilon_j x$ has a pair of opposite faces equal and all other faces degenerate. A cube of type $\Gamma_i^\alpha x$ has a pair of adjacent faces equal and all other faces of type $\Gamma_j^\alpha y$ or $\varepsilon_j y$.) Cubical complexes with this, and other, structures have also been considered by Evrard [Evr].

The prime example of a cubical set with connections is the singular cubical complex KX of a space X. Here for $n \geqslant 0$ K_n is the set of singular n-cubes in X (i.e. continuous maps $I^n \to X$) and the connection $\Gamma_i^\alpha : K_n \to K_{n+1}$ is induced by the map $\gamma_i^\alpha : I^{n+1} \to I^n$ defined by

$$\gamma_i^\alpha(t_1, t_2, \ldots, t_{n+1}) = (t_1, t_2, \ldots, t_{i-1}, A(t_i, t_{i+1}), t_{i+2}, \ldots, t_{n+1})$$

where $A(s, t) = \max(s, t), \min(s, t)$ as $\alpha = -, +$ respectively. Here are pictures of $\gamma_1^\alpha : I^2 \to I^1$ where the internal lines show lines of constancy of the map on I^2.

The complex KX has some further relevant structure, namely the composition of n-cubes in the n different directions. Accordingly, we define a *cubical complex with connections and compositions* to be a cubical set K with connections in which each K_n has n partial compositions $\circ_j$ $(j = 1, 2, \ldots, n)$ satisfying the following axioms. If $a, b \in K_n$, then $a \circ_j b$ is defined if and only if $\partial_j^- b = \partial_j^+ a$, and then

$$\begin{cases} \partial_j^-(a \circ_j b) = \partial_j^- a \\ \partial_j^+(a \circ_j b) = \partial_j^+ b \end{cases} \qquad \partial_i^\alpha(a \circ_j b) = \begin{cases} \partial_i^\alpha a \circ_{j-1} \partial_i^\alpha b & (i < j) \\ \partial_i^\alpha a \circ_j \partial_i^\alpha b & (i > j), \end{cases} \tag{3.3}$$

The interchange laws. If $i \neq j$ then

$$(a \circ_i b) \circ_j (c \circ_i d) = (a \circ_j c) \circ_i (b \circ_j d) \tag{3.4}$$

whenever both sides are defined. (The diagram

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{array}{c} \rightarrow i \\ \downarrow \\ j \end{array}$$

will be used to indicate that both sides of the above equation are defined and also to denote the unique composite of the four elements.)

If $i \neq j$ then

$$\varepsilon_i(a \circ_j b) = \begin{cases} \varepsilon_i a \circ_{j+1} \varepsilon_i b & (i \leqslant j) \\ \varepsilon_i a \circ_j \varepsilon_i b & (i > j) \end{cases} \tag{3.5}$$

$$\Gamma_i^\alpha(a \circ_j b) = \begin{cases} \Gamma_i^\alpha a \circ_{j+1} \Gamma_i^\alpha b & (i < j) \\ \Gamma_i^\alpha a \circ_j \Gamma_i^\alpha b & (i > j) \end{cases} \tag{3.6}(i)$$

$$\Gamma_j^+(a \circ_j b) = \begin{bmatrix} \Gamma_j^+ a & \varepsilon_j a \\ \varepsilon_{j+1} a & \Gamma_j^+ b \end{bmatrix} \quad \begin{array}{c} \rightarrow j \\ \downarrow \\ j+1 \end{array} \tag{3.6}(ii)$$
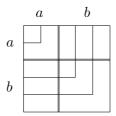
$$\Gamma_j^-(a \circ_j b) = \begin{bmatrix} \Gamma_j^- a & \varepsilon_{j+1} b \\ \varepsilon_j b & \Gamma_j^- b \end{bmatrix} \quad \begin{array}{c} \rightarrow j \\ \downarrow \\ j+1 \end{array} \tag{3.6}(iii)$$

These last two equations are the *transport laws*[1].

---

[1] Recall from [BS76] that the term *connection* was chosen because of an analogy with path-connections in differential geometry. In particular, the transport law is a variation or special case of the transport law for a path-connection.

5

It is easily verified that the cubical Moore complex M_*X of a space X satisfies these axioms with our above definitions. In this context the transport law for $\Gamma_1^-(a \circ b)$ can be illustrated by the picture

Remark 3.1 That the above laws for these structures apply to $M_*(X)$ is easy to check. It is important that the shape tuples $(r_1, \ldots, r_n)$ are part of the structure. Thus if $\partial_1^+(f, (r)) = \partial_1^-(g, (s))$ then this implies that $r_i = s_i, 2 \leqslant i \leqslant n$ as well as

$$f(r_1, t_2, \ldots, t_n) = g(0, t_2, \ldots, t_n) \text{ for } 0 \leqslant t_i \leqslant r_i, \ 2 \leqslant i \leqslant n.$$

This may seem a strong condition, but 'composition is the inverse of subdivision', and this enables one to obtain multiple compositions as the inverse of 'subdividing' an element $(f, (r)) \in M_n(X)$. $\qquad\square$

Remark 3.2 In [AABS02] a cubical ω-category with connections $G = \{G_n\}$ is defined as a cubical set with connections and compositions such that each $\circ_j$ is a category structure on G_n with identity elements $\varepsilon_j y$ $(y \in G_{n-1})$, and in addition

$$\Gamma_i^+ x \circ_i \Gamma_i^- x = \varepsilon_{i+1} x, \quad \Gamma_i^+ x \circ_{i+1} \Gamma_i^- x = \varepsilon_i x. \tag{2.7}$$

However this cancellation law does not hold for $M_*(X)$. Thus the equivalence between globular and cubical categories developed in [AABS02] does not apply to $M_*(X)$, nor does the exact relations between 'commutative shells' and 'thin elements' developed in [Hig05]. $\qquad\square$

Remark 3.3 There are also *reverses* $-_i : M_n(X) \to M_n(X), 1 \leqslant i \leqslant n$ defined as in Section 1. A problem with our construction is that a path $[0, 1] \to M_n(X)$ is not necessarily an element of $M_{n+1}(X)$. In particular the easily defined homotopy rel end points of paths $a \circ -a \simeq 0_{\partial^- a}$ is not an element of $M_2(X)$. $\qquad\square$

4 Tensor products

The tensor product $K \otimes L$ of cubical sets is also defined in [Kan55] and extended to cubical sets with connections in [BH87, AABS02]. We see the convenience of cubes in the current account since since if $a = (f, (r)) \in M_m(X)$ and $b = (g, (s)) \in M_n(Y)$ then their tensor product $a \otimes b \in M_{m+n}(X \times Y)$. It is given by $a \otimes b = (h, ((r) \circ (s)))$ where $(r) \circ (s) = (r_1, \ldots, r_m, s_1, \ldots, s_n)$ and $h = f \times g : (\mathbb{R}^+)^m \times (\mathbb{R}^+)^n \to X \times Y$ with the usual identification of $(\mathbb{R}^+)^m \times (\mathbb{R}^+)^n$ and $(\mathbb{R}^+)^{m+n}$.

References

[AA89]   Al-Agl, F. *Aspects of multiple categories*. Ph.D. thesis, University of Wales, Bangor (1989).

[AABS02]   Al-Agl, F. A., Brown, R. and Steiner, R. 'Multiple categories: the equivalence of a globular and a cubical approach'. *Adv. Math.* **170** (1) (2002) 71–118.

[Bro06]   Brown, R. *Topology and Groupoids*. Printed by Booksurge LLC, Charleston, S. Carolina, third edition (2006).

[Bro08]   Brown, R. 'A new higher homotopy groupoid: the fundamental globular ω-groupoid of a filtered space'. *Homology, Homotopy Appl.* **10** (1) (2008) 327–343.

[BHKP02]   Brown, R., Hardie, K. A., Kamps, K. H. and Porter, T. 'A homotopy double groupoid of a Hausdorff space'. *Theory Appl. Categ.* **10** (2002) 71–93 (electronic).

[BH77]   Brown, R. and Higgins, P. J. 'Sur les complexes croisés, ω-groupoïdes, et T-complexes'. *C. R. Acad. Sci. Paris Sér. A-B* **285** (16) (1977) A997–A999.

[BH81a]   Brown, R. and Higgins, P. J. 'On the algebra of cubes'. *J. Pure Appl. Algebra* **21** (3) (1981) 233–260.

[BH81b]   Brown, R. and Higgins, P. J. 'Colimit theorems for relative homotopy groups'. *J. Pure Appl. Algebra* **22** (1) (1981) 11–41.

[BH81c]   Brown, R. and Higgins, P. J. 'The equivalence of ∞-groupoids and crossed complexes'. *Cahiers Topologie Géom. Différentielle* **22** (4) (1981) 371–386.

[BH87]   Brown, R. and Higgins, P. J. 'Tensor products and homotopies for ω-groupoids and crossed complexes'. *J. Pure Appl. Algebra* **47** (1) (1987) 1–33.

[BS76]   Brown, R. and Spencer, C. B. 'Double groupoids and crossed modules'. *Cahiers Topologie Géom. Différentielle* **17** (4) (1976) 343–362.

[Evr]   Evrard, M. 'Homotopie des complexes simpliciaux et cubiques'. Preprint (1976).

[FRS95]   Fenn, R., Rourke, C. and Sanderson, B. 'Trunks and classifying spaces'. *Appl. Categ. Structures* **3** (4) (1995) 321–356.

[Gra02]   Grandis, M. 'An intrinsic homotopy theory for simplicial complexes, with applications to image analysis'. *Appl. Categ. Structures* **10** (2) (2002) 99–155.

[GM03]   Grandis, M. and Mauri, L. 'Cubical sets and their site'. *Theory Applic. Categories* **11** (2003) 185–201.

[Hig05]   Higgins, P. J. 'Thin elements and commutative shells in cubical ω-categories'. *Theory Appl. Categ.* **14** (2005) No. 4, 60–74 (electronic).

[Kan55]   Kan, D. M. 'Abstract homotopy. I'. *Proc. Nat. Acad. Sci. U.S.A.* **41** (1955) 1092–1096.

[Lur09]   Lurie, J. *Higher topos theory, Annals of Mathematics Studies*, Volume 170. Princeton University Press, Princeton, NJ (2009).

[Mal09]   Maltsiniotis, G. 'La catégorie cubique avec connections est une catégorie test stricte'. *(preprint)* (2009) 1–16.

[Mos87]   Mosa, G. H. *Higher dimensional algebroids and crossed complexes.* Ph.D. thesis, University of Wales, Bangor (1987).

[Pat08]   Patchkoria, I. *Cubical resolutions and derived functors.* Diplome thesis, Ivane Javakhishvili Tbilisi State University (2008), arXiv: 0907.1905.