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# Two examples in homotopy theory

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The examples are unconnected.

1. According to some published footnotes, examples are known of Serre fibrations which are not Hurewicz fibrations. However, as far as I know no such example has been published. The following example, which was stimulated by a more complicated example of G. Horrocks, seems worth mentioning, since it is extremely simple and does suggest some further questions.

Let E be the set of points (x, y) of  $R^2$  such that  $0 \le x \le 1$  and either y = 1/n for some positive integer n or y = x - 1. Let  $p: E \to [0, 1]$  be the projection  $(x, y) \sim x$ . It is easy to see that p has the covering homotopy property for all CW-complexes. However, p does not have the covering homotopy property for all spaces: in fact if  $F = p^{-1}(1)$ and  $i: F \to E$  is the inclusion, then a homotopy deforming pi from the constant map at 1 to the constant map at 0 cannot be covered by a homotopy of i.

In this example,  $p^{-1}(1)$  does not have the homotopy type of a CW-complex.

Question 1. If  $p: E \to B$  is a Serre fibration such that B and each  $p^{-1}(b)$  is a CW-complex (or is an ANR) is then p a Hurewicz fibration?

As an introduction to the second question, we recall that Dold(2) has shown that the property of invariance under fibre homotopy type is held by the weak covering homotopy property (w.c.H.P.) but not by the covering homotopy property. Let  $\mathscr{W}$  be the class of spaces of the homotopy type of a CW-complex. According to Stasheff(3) (although the proof has some gaps) if B is path-connected and  $F \to E \xrightarrow{p} B$  is a Hurewicz fibration such that  $F, B \in \mathscr{W}$ , then  $E \in \mathscr{W}$ .

Question 2. If  $F \to E \xrightarrow{p} B$  is a Serre fibration such that B is path-connected and  $F, B \in \mathcal{W}$ , then does  $E \in \mathcal{W}$ ? Does p have the w.c.h.p.?

2. Our second example is of a function space  $X^Y$  which admits an *H*-structure even though X admits no *H*-structure and Y admits no *H'*-structure. In fact, let Y be the complex projective plane. Then Y has non-trivial cup products and so admits no

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*H'*-structure. Let X have homotopy groups Z in dimensions m, 3m-1 (m > 4) with k-invariant  $k = \omega^3$ , the cube of the fundamental class. It is well known that X has no *H*-structure.

We calculate  $k^{Y}$ . Let u be a generator of  $H^{2}(Y; Z)$ . According to results of (1), we can determine  $k^{Y}$  by operating with a Künneth suspension on the cube of the element

$$e = \omega_{m-2} \times u + \omega_{m-4} \times u^2,$$

where  $\omega_r$  is the fundamental class of K(Z, r). But  $e^3$  is clearly 0. Therefore  $k^Y$  is zero, and  $X^Y$  is of the homotopy type of a product of Eilenberg-Maclane spaces.

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