Mathematics and Knots

Abstract

The exhibition `*Mathematics and Knots'* is intended to present some methods of mathematics to the general public. We explain these methods and the design underlying the presentation.

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1. Introduction

The Popularisation of Mathematics is a considerable challenge. The fascination of the subject is shown by the popularity of recent biographies of Wiles, of Erdos, and of Nash, as well as by the Royal Institution Christmas Lectures and books by Ian Stewart. Nonetheless, it is not clear if the biographies provide good rôle models or encourage students to take up the subject, and in all of these the nature of mathematics remains to some extent a mystery. It is not easy to find brief statements on: the objects of study of the subject; its methods; and its main achievements. Even a popular writer such as Deutsch [5] makes statements such as: `Mathematics is the study of absolutely necessary truths.', which to most people conveys nothing, and as a view of mathematics was discounted by Non Euclidean Geometry [7].

Instead of this fruitless philosophising, trying to make external justification for mathematics, it is worthwhile to show the practice of mathematics, and to relate it to the usual means by which we investigate and attempt to understand the world.

Through teaching the Theory of Knots to mathematics undergraduates at Bangor since about 1975 we have found its value for explaining some basic methods of the subject, and began to use some of the ideas in public presentations. For example, I gave a BAAS lecture at Sussex in 1983, a London Mathematical Popular Lecture in 1984, and a Mermaid Molecule Lecture in 1985. For these we accumulated a lot of visual material and in 1985 set about making this into a travelling exhibition.

The start was to discuss with a graphic designer, who gave us the basic format of mounted A2 boards with aluminium surround, and a travelling case. Over the four years of the exhibition's gestation, we consulted with three greatly helpful graphics designers, and this input was essential for the successful production for the Pop Maths Roadshow which opened at Leeds University in 1989 and then toured the UK. Support from a number of organisations, including one of the first COPUS Grants, was essential for the costs of all this work. We were fortunate in

1988 to get an ESF Grant for training of young people in IT, which supported two students to implement the exhibition in the first version of Pagemaker

The Exhibition was put on the web in 1997, with further support [1].

We started out very naive and had not realised that the exhibition format is one of the hardest. The reasons are:

1. The impact has to be predominantly visual.
2. Each board has to tell its own story.
3. Each board has to be properly related in content to the other boards.
4. Each board has to be properly related visually to the other boards.

In particular, a grid design has to be used so that there is a certain visual rhythm. A basic fault is also to try to put much on one board. The initial content of one board on Knots and Numbers was finally spread over three boards. The final graphic design, including the hand drawing of all the knots, was done by John Round.

In determining the content of each board according to these gradually realised principles, we also found that our views on the structure of the presentation and the nature of mathematics were changing. The emphasis developed in terms of the methodology of mathematics, rather than its nature. Indeed a full treatment of mathematics would have to involve understanding on matters of psychology, language and neurology way beyond current possibilities. What we can do is show how the mathematicians go about their business and how they use standard methods of investigation to advance their subject. In this way we demythologise the subject, and also we hope make it more exciting.

The theory of knots has many advantages for our purposes. The major one is that the objects of study are familiar to all. So also are its basic problems, as anyone who has tried to untangle string will know. The long history of knots is also an advantage: the oldest known pierced object is a wolf's tooth, presumably part of a necklace, and dates at 300,000 BP [8]. Perhaps the Stone Age should be called the Age of String!

The mathematics of knots begins in 1867 with the now forgotten Vortex Theory of the Atom. A theory of the atom had to explain:

- The stability of atoms.
- The variety of atoms, as shown by the periodic table of elements.
- The vibrational properties of atoms, as shown by their spectral lines.

Lord Kelvin had seen smoke rings of his physicist friend P.G. Tait, and was impressed by their stability, and vibrational properties. He had a vision of atoms as vortices in the aether, an imaginary substance which was supposed to fill all space. How to explain the variety of atoms? In 1867, Kelvin presented a paper to the Royal Society of Edinburgh, part of which read:

> Models of knotted and linked vortex atoms were presented to the Society, the infinite variety of which is more than sufficient to explain the allotropies and affinities of all known matter.

The first job was to compare a list of knots with the periodic table of the elements, and so Tait set about preparing a list of knots. The vortex theory of the atom soon disappeared, but Tait's 10 years of work on his list of knots of up to 10 crossings and the conjectures he made (some of which have been proved only recently) have been an inspiration ever since. Further, to determine what is meant by `a list of knots' required solving difficult conceptual problems.

The solution to these problems is basic to our presentation, and gave the underlying structure of our exhibition.

2. Analysis of the methodology

The objects with which mathematics deals may be said to be `structures'. We do not define this precisely, but this term conveys two impressions:

1) The objects have parts, which are related.

2) Mathematics deals with *abstract structures*, which means we have a notion of an instance of a general idea; for example a knot in this piece of string is an instance of the general notion of a knot. This abstractness is a basic aspect of language.

The first problem with examining a species of structure is that of:

2.1 Representation.

We have to find some way of showing, describing, presenting, or whatever, for the

structure under consideration. In the case of knots, we can in a lecture bring a piece of string with us, but on paper we resort to diagrams of knots.

We start with a piece of string as on the left below and tie a knot in it as on the right:

Assuming you are holding both ends, the right hand string cannot be changed to the left by any kind of manipulation of the string, but only by cutting and retying, or letting go of one end.

This shows the basic mathematical problem: how do you *prove* that the string cannot be untied? This may sound a silly question because some minutes' experiment shows it cannot be done. However a mathematician is asking for more certainty, and is asking for methods that can be applied not just to this problem but to more complex knots where the situation would not be so intuitively clear.

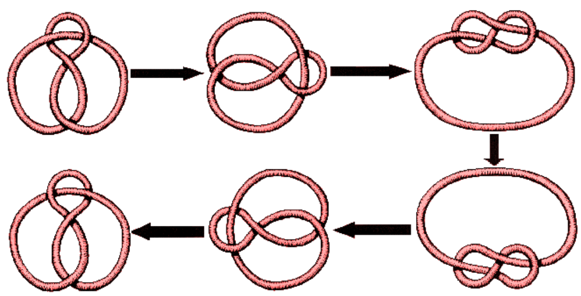
As a start, we find it bothersome holding both ends of the string, so we join them. In this way we get the unknot, and our simplest knots, the trefoil and its mirror image:

Thus our representation of knots is by these *knot diagrams*, in which at each crossover only two parts of the string cross.

2.2 Classification

A basic urge to make sense of the world is to classify. For example, we do not list all the insects in a piece of jungle but we do try to list all the insect species.

So we need to know when two knot diagrams represent the same knot. A knotted loop of string has essentially the same `knottiness', however it is pulled, twisted or



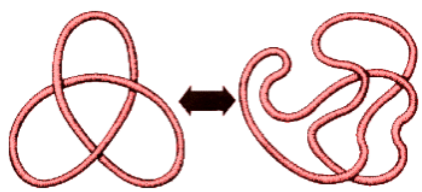
crumpled. This kind of change has to be shown in terms of knot diagrams. We will say more on this later. However the idea is illustrated by the following diagram, which shows how the figure eight knot is the same knot as its mirror image.

2.3 Invariants

To prove two knots are the same, that is, to prove two knot diagrams represent the same knot, you only have to move one diagram into the other. This is not easy as it looks: Tait's table of knots contained two 10 crossing knots that were proved the same only in 1974 by Perko.

A considerably harder problem is to prove two knots are not the same, because you have to prove that no possible movement can move one into the other, and there is no way of examining all the infinite number of possible movements. For example, the trefoil knot is not the same as its mirror image. This is a central problem in knot theory, and there is still no complete solution. The method for partial solutions is to find *knot invariants* which can be defined in terms of the diagram, which give the same result for equivalent knot diagrams, and for which there is some method of calculation. The exhibition gives details of: *crossing number, unknotting number, bridge number, three colouring.* For example, the trefoil knot can be coloured in three colours in a precise sense, but this would not be possible if the trefoil knot were an unknot. This gives a reasonably easy proof that the trefoil, and a number of other knots, are in fact knotted.

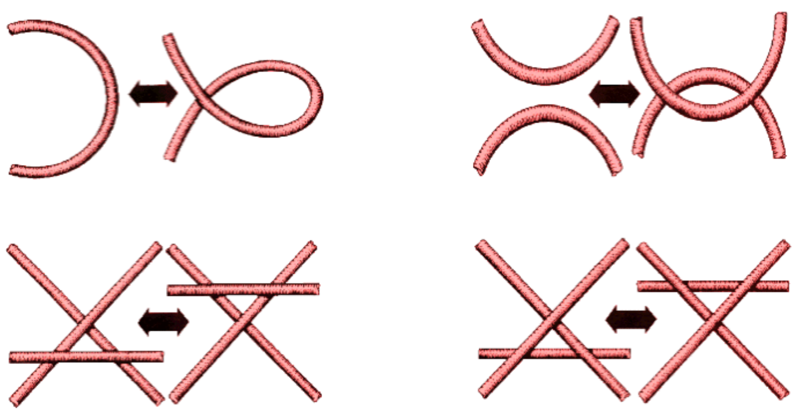
The *crossing number* of a knot is defined as the smallest number of crossings which can occur in a diagram of the knot. This illustrates a standard mathematical procedure, namely choose the least of a set of whole numbers, but is any case standard practice, since in drawing knot diagrams you tend to try to give the one which seems the simplest. The crossing number is easy to define but hard to determine for a complicated knot, since the definition is in terms of the infinite number of possible diagrams.



2.4 Decomposition into simple elements: Reidemeister moves

The process to be decomposed into simple basic elements is that of changing one knot diagram into another without changing the knot.

Reidemeister showed in the 1920s that

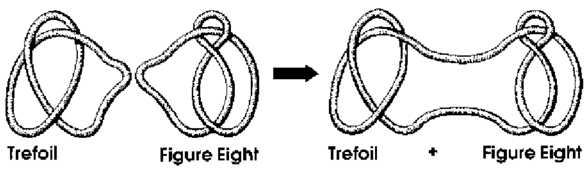


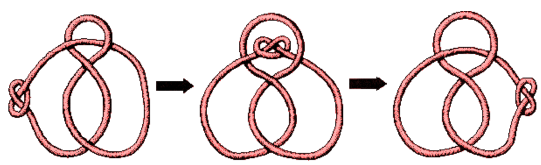
two knot diagrams define the same knot if and only if one can be changed to the other by a sequence of five basic moves: the first is to distort the diagram without changing the crossings, as in the diagram on the left. The other four moves are to change crossings in one of the following ways:

These moves are an important tool. For example, to prove a proposed knot invariant is invariant, all you have to do is show it is unchanged by the Reidemeister moves. Invariance under distortion is often easily verified, and we have only four other moves to check. This reduction to four cases is a considerable advance on studying an infinite number of cases, and is the method used to show 3-colourability is an invariant. Try it!

2.5 Analogy

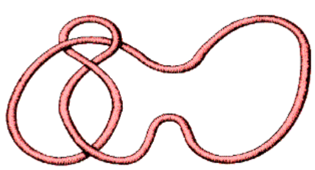
Although the word is rarely used in this context, analogy is in fact central to mathematical practice. The abstract nature of mathematics is precisely because it deals with `structures', and we want to see how a particular structure occurs in many situations. This gives us the excitement of `that reminds me of", and allows for the transfer of knowledge from one situation to another. Such a transfer often leads to the solution of problems, and is indeed sought for this purpose, in the style of: `If I could apply these techniques to that problem then ....!' The more surprising the analogy the better, and the finding of such a new analogy may be called an *insight*.



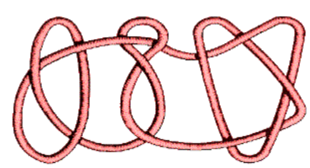


$K+L$ $L+K$

The analogy we can show here relates knots and numbers, and relies on a method of combining knots which we here call *addition*. This is illustrated above: pull a



$L+0=L$



$K+L+M$

piece out of both knots and join them as shown. It is important that that this process is independent of where on each knot one starts to join them. This is proved by the type of diagram on the left, which also shows that addition of knots is *commutative*: $K+L=L+K$

We can prove additional laws. If we write the unknot as 0 then it is easy to see that for any knot L we have $L+0=0+L=L$. Another useful rule is *associativity:* $K+(L+M)=(K+L)+M$. These rules, or laws, are shown by the above diagrams.

In formulating these laws we are using two analogies.

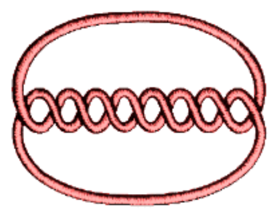
One of them is between the behaviour of knots and the behaviour of numbers. In fact we make the analogy between the *addition* of knots and the *product* of numbers. So the associativity rule for the product of numbers has the instance $3\times(4\times 5)=(3\times 4)\times 5$, that is $3\times 20=12\times 5$. The important feature is that a relation between numbers has analogies with a relation between knots: there are common structural features when you consider all knots and all numbers. There are also differences: there is no negative of a knot and no subtraction for knots. It is true that if $K+L=K+M$ then $L=M$, but this needs some ideas for its proof that we cannot give here.

The other analogy is between laws in different situations. By drawing attention to the commutative laws for addition and multiplication of numbers

$$m+n=n+m, \quad m\times n=n\times m$$

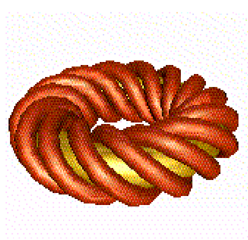
we are making an analogy between addition and multiplication. Mathematics is indeed *abstract*, and this abstractness has a clear purpose, to allow for analogies.

There are two reasons why we have called this `composition' of knots addition rather than multiplication, as is common in the literature on knots. One is that the notation 0 for the unknot is more intuitive than the notation 1. The other is to emphasise that we can have analogies between structures with different names.



2.6 Decomposition into simple elements: Prime knots

Now we have another example of decomposition into simple elements. We say that a knot K is *prime* if it is not the sum of simpler knots, that is if whenever we try to express K as a sum of knots $K=L+M$ then $L=0$ or $M=0$. The trefoil and the figure eight knot are prime knots, and so are all the knots in the family illustrated on the right. These are the *torus knots* because they can all be wrapped around an inner tube, a shape mathematicians call a torus.



On the left is an example of a torus knot with the torus shown. This idea is not developed in the exhibition, but is on the web site related to sculptures of John Robinson, since four of them can be described as torus knots.

In any case, the prime knots are the simple elements in the whole family of knots. The example of torus knots shows that there are infinitely many prime knots, though the proof that torus knots are prime is hard.

A remarkable similarity between the addition of knots and the product of numbers is that there is essentially only one way of writing a knot as a sum of prime knots. Again, this result needs for its proof ideas not given here. There is no algorithm for finding the decomposition of a knot into the sum of prime knots. So the analogy between knots and numbers is not complete. On the other hand, the factorisation of large numbers, with say 200 digits, is beyond the reach of current computers in reasonable time, a fact that is the basis of a form of cryptography, and so in a way the analogy resumes for large numbers.

From all this we see that one of the uses of analogy is to formulate questions. We wish to know in what ways two systems are analogous, and what ways not.

2.7 Laws

The laws obeyed by the addition of knots have already been discussed, but the theme needs some elaboration. These laws can be taken as the *axioms* of an algebraic system. A lot of mathematics is concerned with developing the consequences of some chosen axioms. This has led to the view that `Mathematics is the subject where we don't know what we are talking about and where we don't know whether what we are talking about is true.' Related views are that:

`The method of "postulating" what we want has many advantages; they are the same as the advantages of theft over honest toil.' [6,p.71]

`Mathematics is non creative because it is concerned only with the consequences of given rules.' [Heard in a lecture for young people by an established scientist.]

These views miss the point. Axioms (postulates) are tools for defining the structures we wish to study. The finding and choosing of these axioms for their relevance to the structures we wish to study is a key part of the creative process. Conjecturing and proving interesting consequences of axioms, that is, formulating and proving theorems, is a basic part of the creation of new mathematics, and often requires new concepts to state the theorems.

The Nobel Prize winning physicist Wigner [8] had a clear view of mathematics:

`Mathematics is the science of skilful operations with concepts and rules invented just for this purpose. [this purpose being the skilful operation ....] '

`The principal emphasis is on the invention of concepts.'

`The depth of thought which goes into the formation of mathematical concepts is later justified by the skill with which these concepts are used. '

2.8 Generalisation

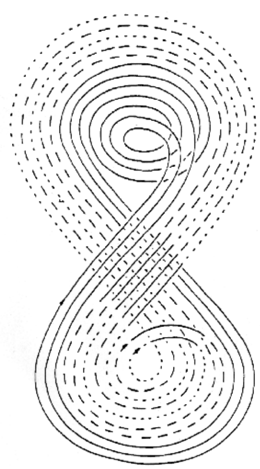
Now we reach into areas not touched on in the exhibition. It was pointed out in the 19th century by Klein that a knot can be untied in 4 dimensions. To see why this is so we again use analogy. A beetle on a table may be blocked in its passage by a vertical wall. If it is allowed the third dimension, for example by flying, then it can easily move over the barrier.

It is the crossings in a knot diagram which give the barrier to untying the knot. If we are allowed a 4th dimension, then it is easy to see we can then move one portion of string `over' another, and so change any crossing. In this why, it is easy to untie any knot in 4 dimensions.

The generalisation is to ask what we can tie in 4 dimensions? The answer is the surface of a balloon, which mathematicians call a 2-sphere. As expected, such a 2-knot can be untied in 5 dimensions.

More generally, an n-sphere can be tied in $(n+2)$-dimensions and untied in $(n+3)$-dimensions. The proof of this is rather hard, and was carried out by E.C. Zeeman. The general situation cannot be properly visualised. The formulation and representation of the abstract properties which describe the situation, and the logical argument with these, is where we have to rely, rather than on visualisation and interpretation, which is simply a starting point for our intuition. The problem is indeed that of building up our intuition of what is going on, and what might happen, in these dimly grasped complicated structures.

2.9 Applications

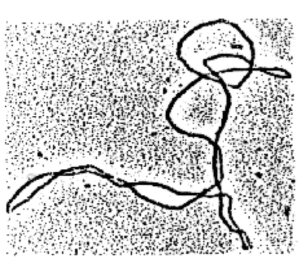


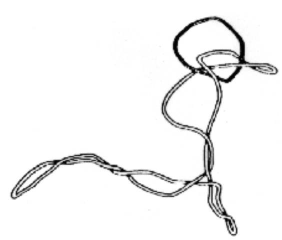
The picture on the left shows some knotted flow lines arising in chaotic flows resulting from some differential equations related to weather.

One of the points we wished to show in the exhibition is that many applications rely on all the above aspects to be effective. Even the original, and abandoned, idea of Vortex Atoms required the formulation of the subtle concepts of classification of knots, and their arithmetic, to decide that it was not going to work.

The two modern applications we mention are to chaotic flows, as above, and to DNA.

The second results from developments started as recently as 1985 and which have had remarkable effects on the theory of knots and its applications. This is a new theory of knot polynomials. It started at a seminar by Vaughan Jones in Geneva on a branch of mathematics called operator algebra theory. He obtained some laws for certain elements of this algebra, and a member of the audience remarked that these rules also arose in another

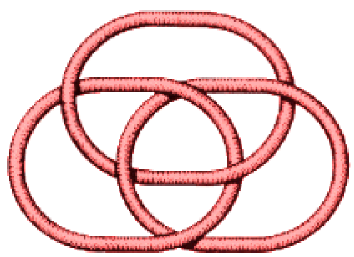




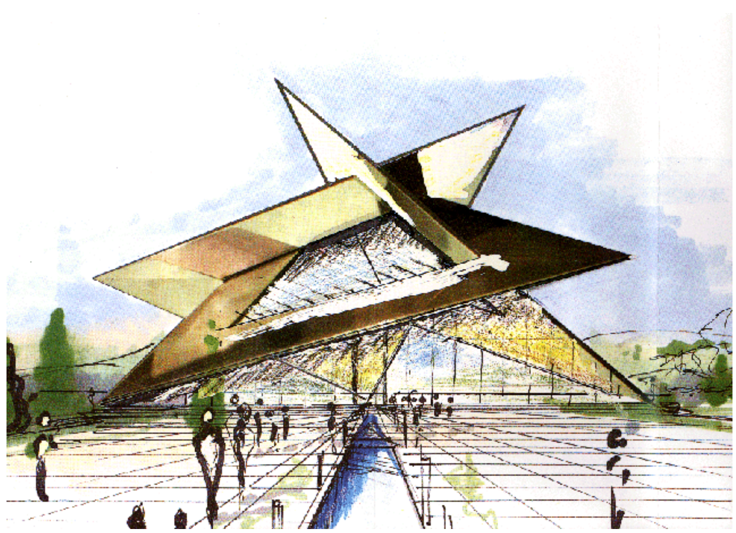
branch of mathematics, closely related to knot theory, called braid theory. In pursuing this idea with experts, a new theory of knot polynomials was born. These have been applied in studying the way DNA untangles itself when it divides. On the previous page is a micrograph and sketch of knotted and linked DNA (due to N. Cozarelli).

3. The link with art

One original aim of the exhibition was to show knots in history, art and technology. This was gradually seen to be too ambitious, but the opportunity came to ask John Robinson to exhibit his sculptures at the Pop Maths Roadshow in 1989. This exhibition and its catalogue [2] became the start of an extensive collaboration in opening the academic world to knowledge of his work.



The Borromean Rings, as on the left, is called a `link', rather than a knot, since it has three loops, whereas a knot has, by definition, only one loop. In this link, no two of the circles are linked, but the whole cannot be pulled apart. Such links, of which this is one of the simplest, show ways in which the whole is more than the sum of its pieces, that is the parts are placed together to form a *structure*. It is part of the job of mathematics to invent language to describe and determine the properties of structures, and to find interesting and extraordinary ones. Some of thesestructures model aspects of the world, and often have been developed for this reason. The peculiar properties of mathematical language are its rigour and exactness, and the way the



consistency of the structures developed has been tested by thousands of mathematicians and scientists, particularly over the last 200 years.

Robinson has made various sculptures based on the Borromean Rings. In view of the architects present at this meeting, I would like to end with an example of Robinson's sculpture *Intuition,* which he saw could be a central structure to a building proposed as a Pantheon of Mathematics, and sketched here by Ove Arup.

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WWW production: Cara Quinton.

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